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SOME REMARKS ON THE OPTIMAL DESIGN OF PERIODICALLY REINFORCED STRUCTURES (*)

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Abstract. — *We present a natural optimality criterion for periodically reinforced structures. Some examples of optimal structures are given. More specifically we consider a non trivial optimal two dimensional honeycomb structure in dimension 2.*

Résumé. — *On donne un critère naturel d'optimalité pour les structures périodiques renforcées. Quelques exemples sont étudiés. En particulier on met en évidence une structure en nid d'abeille optimale non triviale en dimension 2.*

1. INTRODUCTION

The aim of this paper is to give an optimality criterion for the design of a periodically reinforced structure and to make explicit calculations in some simple cases.

Our work is based on the paper by H. Attouch and G. Buttazzo [2] concerning the homogenization of periodically reinforced structures, and we shortly indicate now the framework of this study. Consider a bounded open subset Ω of $\mathbb{R}^n$ ($n = 2$ or 3 in the applications) with a Lipschitz boundary,

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and assume that it is filled with many periodically distributed thin highly conducting layers. More precisely, let

$Y = [0, 1[^n$ be the unit cube of $\mathbb{R}^n$,
$S \subseteq Y$ be a piecewise smooth $n-1$ dimensional surface,

$$S_\varepsilon = \{\varepsilon(x+y) : x \in S, y \in \mathbb{Z}^n\},$$

$$S_{\varepsilon, r} = \left\{x \in \mathbb{R}^n : \text{dist}\,(x, S_\varepsilon) < \frac{r}{2}\right\},$$

$$a_{\varepsilon, r, \lambda}(x) = \begin{cases} \lambda & \text{if} \quad x \in S_{\varepsilon, r} \\ \alpha & \text{if} \quad x \in \mathbb{R}^n \setminus S_{\varepsilon, r} \end{cases}$$

where ε, r, λ, α are positive parameters.

The network S_ε (see *fig.* 1) will be called a periodically reinforced structure, and we want to study its behaviour (in terms of the potential obtained with a given charge density $g(x)$) as $\varepsilon \to 0$.

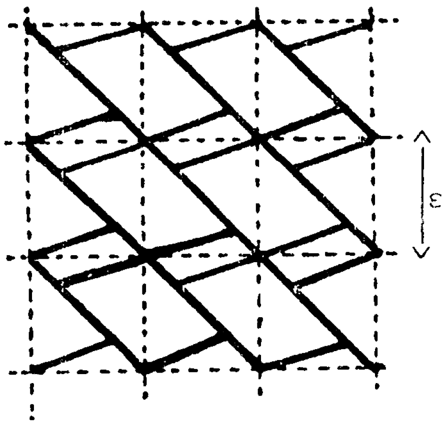

Figure 1.

To do this, define on the Sobolev space $H^1(\Omega)$ the functionals

$$F_{\varepsilon, r, \lambda}(u) = \int_\Omega a_{\varepsilon, r, \lambda}(x) |Du|^2 \, dx$$

and consider the solutions $u_{\varepsilon, r, \lambda}$ of the variational problems

$$\min \left\{F_{\varepsilon, r, \lambda}(u) + \int_\Omega gu \, dx : u \in H_0^1(\Omega)\right\} \tag{1.1}$$

where $g \in L^2(\Omega)$ is a given function. In [2] (see also [C5-S]) H. Attouch and G. Buttazzo studied the asymptotic behaviour, as $(\varepsilon, r, \lambda) \to (0, 0, +\infty)$, of the functions $u_{\varepsilon, r, \lambda}$; by using the Γ-convergence theory, they proved that if $\lambda r/\varepsilon \to k$, then $u_{\varepsilon, r, \lambda}$ tends in $L^2(\Omega)$ to the solution of the problem

$$\min \left\{ \int_\Omega f_\alpha(Du)\, dx + \int_\Omega gu\, dx : u \in H_0^1(\Omega) \right\} \tag{1.2}$$

where $f_\alpha(z)$ is the quadratic form in $\mathbb{R}^n$ defined by

$$f_\alpha(z) = \min \left\{ \alpha \int_Y |Dv|^2\, dy + \right. \\ \left. + k \int_S |D_\tau v|^2\, dH^{n-1} : v - \langle z, . \rangle \in W \right\} . \tag{1.3}$$

Here H^{n-1} denotes the $n-1$ dimensional Hausdorff measure, $D_\tau v$ is the tangential derivative

$$D_\tau v = Dv - \nu \langle Dv, \nu \rangle \quad (\nu \text{ is the unit normal vector to } S) ,$$

and W is the space of all functions in $H^1_{\text{loc}}(\mathbb{R}^n)$ which are Y-periodic. For the sake of simplicity, in the following we shall assume $k = 1$. The function $f(z)$ defined by

$$f(z) = \lim_{\alpha \to 0} f_\alpha(z) = \inf_{\alpha > 0} f_\alpha(z) \tag{1.4}$$

is a quadratic form on $\mathbb{R}^n$, and we have

$$f(z) = \inf \left\{ \int_S |D_\tau v|^2\, dH^{n-1} : v - \langle z, . \rangle \in W \right\} . \tag{1.5}$$

Denote by A the $n \times n$ symmetric matrix

$$A_{ij} = \frac{1}{2} \frac{\partial^2 f}{\partial z_i\, \partial z_j} , \tag{1.6}$$

by T its trace, and by $|S|$ the $n-1$ dimensional measure of S. The main result of this paper is the following.

THEOREM 1.1 : *We have* $T \leq (n-1)|S|$. *Moreover, the eigenvalues* λ_i *of* A *satisfy the inequalities*

$$0 \leq \lambda_i \leq |S| . \quad \blacksquare \tag{1.7}$$

In this way, we say that a periodically reinforced structure is optimal if

$$T = (n-1)|S| .$$

In Section 2 we shall prove Theorem 1.1, and in Section 3 we shall give some examples of optimal periodically reinforced structures in dimension 2 and 3.

2. PROOF OF THEOREM 1.1

With the notations of Section 1, we define, for every $\beta > 0$

$$f_{\alpha,\beta}(z) = \inf \left\{ \int_Y a_{1,\frac{1}{\beta},\beta}(y) |Dv|^2 \, dy \; ; v - \langle z, . \rangle \in W \right\} .$$

It is proved in [6], [3], [1], that the $n \times n$ symmetric matrix

$$A_{ij}^{\alpha\beta} = \frac{1}{2} \frac{\partial^2 f_{\alpha\beta}}{\partial z_i \, \partial z_j}$$

is the G-limit as $\varepsilon \to 0$ of the sequence of matrices

$$A_\varepsilon^{\alpha\beta}(x) = a_{1,\frac{1}{\beta},\beta}\left(\frac{x}{\varepsilon}\right) I .$$

Denote by $\lambda_i^{\alpha\beta}$ the eigenvalues of $A^{\alpha\beta}$. In order to obtain informations on the $\lambda_i^{\alpha\beta}$, we use a result of F. Murat and L. Tartar.

THEOREM 2.1 (*see* [7], [9]) : *The eigenvalues* $\lambda_i^{\alpha\beta}$ *verify the following estimates* (*for simplicity we omit the indices* α, β) :

$$\mu_- \leqslant \lambda_i \leqslant \mu_+ \tag{2.1}$$

$$\sum_{i=1}^n \frac{1}{\lambda_i - \alpha} \leqslant \frac{1}{\mu_- - \alpha} + \frac{n-1}{\mu_+ - \alpha} \tag{2.2}$$

$$\sum_{i=1}^n \frac{1}{\beta - \lambda_i} \leqslant \frac{1}{\beta - \mu_-} + \frac{n-1}{\beta - \mu_+} \tag{2.3}$$

where

$$\begin{aligned} \mu_- &= \left(\frac{1-\theta}{\alpha} + \frac{\theta}{\beta} \right)^{-1} \\ \mu_+ &= (1-\theta)\,\alpha + \theta\beta \\ \theta &= \text{meas}\,(Y \cap S_{1,1/\beta}) . \end{aligned}$$

Proof of theorem 1.1 : We remark that

$$\theta = \frac{|S|}{\beta} + \omega(\beta) \quad \text{with} \quad \lim_{\beta \to +\infty} \beta\omega(\beta) = 0 ,$$

so that

$$\lim_{\beta \to +\infty} \mu_- = \alpha \quad \text{and} \quad \lim_{\beta \to +\infty} \mu_+ = \alpha + |S| . \tag{2.4}$$

By (2.3) it follows

$$(\beta - \mu_+)(\beta - \mu_-) \sum_{i=1}^{n} \prod_{j \neq i} (\beta - \lambda_j) \leqslant [n\beta - \mu_+ - (n-1)\mu_-] \prod_{i=1}^{n} (\beta - \lambda_i)$$

which can be written in the form

$$\begin{aligned}(\beta - \mu_+)(\beta - \mu_-) \sum_{i=1}^{n} \left[\beta^{n-1} - \beta^{n-2} \sum_{j \neq i} \lambda_j + P_{n-3}(\beta)\right] \leqslant \\ \leqslant [n\beta - \mu_+ - (n-1)\mu_-]\left[\beta^n - \beta^{n-1} \sum_{i=1}^{n} \lambda_i + P_{n-2}(\beta)\right]\end{aligned} \tag{2.5}$$

where $P_{n-3}(\beta)$, $P_{n-2}(\beta)$ are polynomials of degree $n-3$, $n-2$ respectively.

After some simple calculations, (2.5) becomes

$$\beta^n[\operatorname{Tr} A^{\alpha\beta} - \mu_- - (n-1)\mu_+] \leqslant P_{n-1}(\beta) . \tag{2.6}$$

Multiplying both sides of (2.6) by β^{-n}, passing to the limit as $\beta \to +\infty$, and taking into account (2.1), (2.4) we get

$$\lim_{\beta \to +\infty} \operatorname{Tr}(A^{\alpha\beta}) \leqslant n\alpha + (n-1)|S| . \tag{2.7}$$

It is well known (see for instance [4], [8], [1]) that

$$f_\alpha(z) = \lim_{\beta \to +\infty} f_{\alpha\beta}(z) , \quad \text{for every} \quad \alpha > 0 \quad \text{and} \quad z \in \mathbb{R}^n , \tag{2.8}$$

where f_α is defined in (1.3).

Coming back to (2.7), by (2.8), we get

$$\operatorname{Tr}(A^\alpha) \leqslant n\alpha + (n-1)|S|$$

where A^α is the $n \times n$ symmetric matrix

$$A_{ij}^\alpha = \frac{1}{2} \frac{\partial^2 f_\alpha}{\partial z_i \, \partial z_j} .$$

Passing to the limit in (2.7) as $\alpha \to 0$, (1.4) yields

$$\operatorname{Tr} A \leq (n-1) |S| .$$

Finally, (1.7) follows from (2.1) and (2.4). ∎

Remark 2.2 : Theorem 1.1 provides only an estimate on the trace and on the eigenvalues of the matrix A. But in general, if $L > 0$ is fixed, the set M_L of all matrices given by formula (1.6), where S runs over all surfaces with $|S| = L$, is smaller than the set of all matrices with

$$\begin{cases} \operatorname{Tr} A \leq (n-1) L \\ 0 \leq \lambda_i \leq L . \end{cases}$$

For instance, if $n = 2$ and $L < 1$, it is easy to see that the set M_L reduces to the only null matrix, whereas the set M_1 consists of the two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

It would be interesting (and as far we know it is an open problem) to characterize explicitely the set M_L, or at least the matrices of M_L which are optimal.

3. SOME EXPLICIT CALCULATIONS

In this section we give some examples of optimal periodic reinforced structures in dimension 2 and 3. Let us assume first $n = 2$. If S is a curve $\gamma(s)$ parametrized by its curvilinear abscissa, then formula (1.5) becomes (see [2])

$$\langle Az, z \rangle = \inf \left\{ \int_0^L |w'(s) + \langle z, \gamma'(s) \rangle|^2 \, ds \right\} \tag{3.1}$$

where $L = |S|$ and the infimum is taken over all functions $w(s)$ satisfying the periodicity conditions. By the first order necessary conditions, it is easy to see that the solution $w(.)$ of (3.1) is such that $w'(s) + \langle z, \gamma'(s) \rangle$ is constant on $[0, L]$. A formula analogous to (3.1) holds if S is the union of finitely many curves $\gamma_i(s)$ $(i = 1, \dots, N)$

$$\langle Az, z\rangle = \inf \left\{ \sum_{i=1}^{N} \int_0^{L_i} |w'(s) + \langle z, \gamma_i'(s)\rangle|^2 \, ds \right\} . \tag{3.2}$$

Consider now the structure whose elementary cell is represented in figure 2.

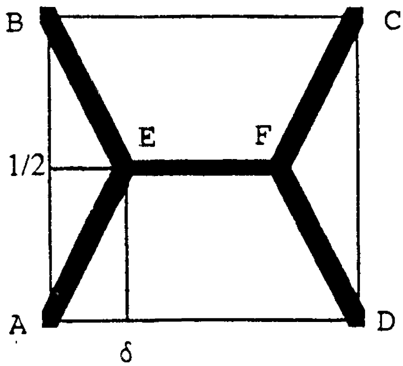

Figure 2.

It is easy to see that if $\delta = 0$ or $\delta = \frac{1}{2}$ the structure above is optimal ; in fact we have

$$\text{if } \delta = 0 \quad \langle Az, z\rangle = z_1^2 + 2 z_2^2 \quad \text{and} \quad |S| = 3$$

$$\text{if } \delta = \frac{1}{2} \quad \langle Az, z\rangle = \sqrt{2}\, z_1^2 + \sqrt{2}\, z_2^2 \quad \text{and} \quad |S| = 2\sqrt{2} .$$

PROPOSITION 3.1 : *For $0 < \delta < \frac{1}{2}$ the structure of figure 2 is optimal if and only if $\delta = \frac{\sqrt{3}}{6}$.*

Proof: We want to use (3.2) to compute the quadratic form $\langle Az, z\rangle$. Let us define

$$\lambda_0 = 1 - 2\delta \quad \text{and} \quad \lambda = \sqrt{\delta^2 + \frac{1}{4}} ;$$

in this way, if $\gamma_i(s)$ $(i = 1, 2, 3, 4, 5)$ are parametrizations of the segments EF, AE, BE, CF, DF respectively, we have

$$\begin{cases} \gamma_1' = (1, 0) \\ \gamma_2' = (\delta/\lambda, 1/2\lambda) \\ \gamma_3' = (\delta/\lambda, -1/2\lambda) \\ \gamma_4' = (-\delta/\lambda, -1/2\lambda) \\ \gamma_5' = (-\delta/\lambda, 1/2\lambda). \end{cases}$$

Analogously, for the functions $w_i(s)$ $(i = 1, 2, 3, 4, 5)$, denoting by a, b the values on E, F respectively, we obtain

$$\begin{cases} w_1' = (b-a)/\lambda_0 \\ w_2' = w_3' = a/\lambda \\ w_4' = w_5' = b/\lambda, \end{cases}$$

so that, by (3.2) we get

$$\langle Az, z\rangle = \inf_{a,b} \left\{ \frac{1}{\lambda_0}(b - a + \lambda_0 z_1)^2 + \frac{1}{\lambda}\left(a + \delta z_1 + \frac{1}{2} z_2\right)^2 + \frac{1}{\lambda}\left(a + \delta z_1 - \frac{1}{2} z_2\right)^2 + \frac{1}{\lambda}\left(b - \delta z_1 - \frac{1}{2} z_2\right)^2 + \frac{1}{\lambda}\left(b - \delta z_1 + \frac{1}{2} z_2\right)\right\}. \tag{3.3}$$

The optimality conditions for a, b give

$$a = -b = \lambda_0 z_1 \frac{\lambda - 2\delta}{2(\lambda + \lambda_0)},$$

hence, after some calculations, (3.3) becomes

$$\langle Az, z\rangle = \frac{1}{\lambda + \lambda_0} z_1^2 + \frac{1}{\lambda} z_2^2.$$

Since $|S| = \lambda_0 + 4\lambda$, the structure is optimal if

$$\lambda_0 + 4\lambda = \frac{1}{\lambda + \lambda_0} + \frac{1}{\lambda}. \tag{3.4}$$

Equation (3.4) can be written in the form

$$\lambda_0(\lambda^2 - 4\delta\lambda + 4\delta^2) = 0$$

that is $\lambda = 2\delta$, which gives $\delta = \dfrac{\sqrt{3}}{6}$. ■

Let us conclude by a simple example in dimension 3. Consider the case of the cross structure

$$S = \left\{ (x_1, x_2, x_3) \in Y : x_1 = \frac{1}{2} \text{ or } x_2 = \frac{1}{2} \text{ or } x_3 = \frac{1}{2} \right\} .$$

In this case we obtain

$$f(z) = 2\, z_1^2 + 2\, z_2^2 + 2\, z_3^2 ,$$

and this shows that the structure is optimal, because

$$T = 6 = 2\,|S| .$$

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