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TIME-DISCRETIZATION AND INERTIAL MANIFOLDS

by F DEMENGEL ([1]) and J M GHIDAGLIA ([1])

INTRODUCTION

Studies of the long time behavior of solutions to nonlinear partial differential equations (p d e 's) have dramatically increased in the last ten years One of the goal of these researches was to establish connections between infinite dimensional dynamical systems generated by p d e 's and finite dimensional ones This has led to introduce new (with respect to the classical p d e theory) concepts such as attractors, global Lyapunov exponents, determining modes, and very recently that of inertial manifolds (Foias, Sell and Temam [3]) Attractors and inertial manifolds are invariant under time-evolution, they represent the long time behavior However the speed of convergence of trajectories towards an attractor can be very small, allowing complex transients and « simple » attractors The inertial manifolds do not have this disadvantage these sets which are obtained as graphs of Lipschitz mappings defined on an M-dimensional space, attract *exponentially* the trajectories It follows that the infinite dimensional dynamical system under consideration is reduced to M ordinary differential equations As it is known, attractors are very sensitive to perturbations (see e g the introduction of J Hale [6]) On the contrary, inertial manifolds are robust (see the paper by Luskin and Sell [4] in this volume) Such a property is very important if one has in mind to compute solutions to the original p d e by using the techniques of inertial manifolds

In a recent work [1], on which we report here, we have addressed questions pertaining to this matter Let us write the evolutionary p d e under consideration as follows

$$\frac{du}{dt} = N(u) \tag{0 1}$$

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where N denotes an unbounded and nonlinear operator on an infinite dimensional Banach space H in which the function u $t \to u(t)$ takes its values We consider a splitting $H = H_1 \oplus H_2$ where H_1 is finite dimensional An inertial manifold for (0 1) will be searched as a graph of a function ϕ $H_1 \to H_2$,

$$\mathcal{M} = \mathcal{M}(\phi) = \{(p_1, \phi(p_1)), p_1 \in H_1\}$$

Denoting by P_1 the projection on the first factor and assuming that $\mathcal{M}$ is invariant by (0 1), we see that if $u(0) \in \mathcal{M}$ then $u = p + \phi(p)$ where

$$\frac{dp}{dt} = P_1 N(p + \phi(p)) . \tag{0 2}$$

Investigations on the qualitative behavior of solutions to nonlinear (evolution) p d e 's are mainly computational Hence systems like (0 1) are time-discretized and replaced by iterations, for example

$$u^{n+1} = u^n + \tau N_\tau(u^n) \tag{0 3}$$

where $\tau > 0$ represents the time step and « $N_\tau \to N$ » as $\tau \to 0$ An inertial manifold for (0 3), $\mathcal{M}_\tau = \mathcal{M}(\phi_\tau)$, is the graph of a Lipschitz function ϕ_τ from H_1 into H_2 which is invariant by (0 3) and attracts exponentially its solutions If we take $u^0 \in \mathcal{M}_\tau$ i e $u^0 = p^0 + \phi_\tau(p^0)$, $p^0 \in H_1$, we have $u^n = p^n + \phi_\tau(p^n)$ and

$$p^{n+1} = p^n + \tau P_1 N_\tau(p^n + \phi_\tau(p^n)) \tag{0 4}$$

Now (0 4) is a discrete, finite dimensional dynamical system well suited for numerical investigations

The main goal was to represent accurately the long time behavior of solutions to (0 1) Provided $\mathcal{M}$ and $\mathcal{M}_\tau$ are close, the long time behavior (i e as $n \to +\infty$) of (0 4) will indeed represents that of (0 1) One of our results (Theorem 3 1) will answer positively this question

At this point, we notice that approximating (0 1) by (0 3) on *large time intervals* is not an easy task Indeed, recall that classical error estimates (even in the o d e case) are of the form

$$\|u(n\tau) - u^n\| \leq C\tau^\mu , \qquad 0 \leq n \leq N$$

where μ is the order of the method, but the constant C grows exponentially with N and therefore this estimate vanishes as $N \to \infty$ Moreover, in systems of interests, (0 1) presents sensitivity to initial conditions and then it is expected that the previous error estimate is sharp

Our applications (for which we refer to [1]) include complex amplitude equations and strongly dissipative perturbations of the Korteweg-de Vries equation. Other applications will be reported elsewhere.

1. THE FUNCTIONAL SETTING, THE CONTINUOUS CASE

The continuous dynamical systems we consider are associated with evolution equations of the following type

$$\frac{du}{dt} + Au + Cu + F(u) = 0 , \qquad u(0) = u_0 \tag{1.1}$$

on a separable Hilbert space H. The linear operator A is closed unbounded positive self-adjoint with domain $D(A) \subset H$. We assume that $v \to |Av|$ is a norm on $D(A)$ equivalent to the graph-norm, A^{-1} being *compact* on H. Hence there exists a complete orthonormal family $\{w_j\}_{j=1}^{\infty}$ in H made of eigenfunctions of A :

$$Aw_j = \lambda_j w_j , \qquad j = 1, \dots$$
$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty \qquad \text{as} \qquad j \to \infty ,$$

where the λ_j's are the associated eigenvalues repeated according to their multiplicity. We denote by $\sigma(A) = \{\Lambda_k\}_{k=1}^{\infty}$, $\Lambda_1 < \Lambda_2 < \dots$ the set of distinct eigenvalues and by $m_k \in \mathcal{N}^*$ the finite multiplicity of Λ_k. The spectral projections R_Λ and P_Λ are defined as usual :

$$R_\Lambda = \sum_{j \;\; \lambda_j = \Lambda} (v, w_j) w_j , \qquad P_\Lambda = \sum_{\lambda \leq \Lambda} R_\lambda ,$$

where $|\,.\,|$ and $(,)$ denote the norm and the scalar product on H.

The linear operator C is bounded from $D(A^{s_0})$ into H (for some given $s_0 \in \mathcal{R}$), squew-symmetric and commutes with $A : AC = CA$. Concerning the function F, we assume that it is a Lipschitz function from $D(A^\alpha)$ into $D(A^{\alpha-\gamma})$ for some *given* $\alpha \in \mathcal{R}$ and $\gamma \in [0, 1/2]$:

$$|A^{-\gamma}(F(v) - F(w))|_\alpha \leq L_F |v - w|_\alpha , \qquad \forall v, w \in D(A^0) , \tag{1.2}$$

$$|A^{-\gamma} F(v)| \leq L_F(1 + |v|_\alpha) , \qquad \forall v \in D(A^\alpha) , \tag{1.3}$$

where we denote by

$$(v, w)_\alpha = (A^\alpha v, A^\alpha w) , \qquad |v|_\alpha = |A^\alpha v|$$

the scalar product and norm on $D(A^\alpha)$.

As it is well known, under these hypotheses, the Cauchy problem (1 1) is well posed on $D(A^{\alpha})$ and $D(A^{\alpha + 1/2})$ We denote by $S(t)$ the semi-group that solves (1 1), and introduce with [3] the following definition

DEFINITION 1 1 *An inertial manifold $\mathcal{M} \subset D(A^{\alpha})$ for (1 1) is a finite dimensional Lipschitz manifold which is invariant by $S(t)$*

$$S(t)\,\mathcal{M} \subset \mathcal{M}\,, \qquad \forall t \geqslant 0\,, \tag{1 4}$$

and attracts exponentially all its solutions

$$\forall R > 0\,, \quad \exists \sigma > 0\,, \quad C > 0 \text{ s t } \quad \forall t \geqslant 0\,, \quad u_0 \in D(A^{\alpha})\,, \quad |u_0|_{\alpha} \leqslant R\,,$$

$$d_{\alpha}(S(t)\,u_0, \mathcal{M}) = \operatorname*{Inf}_{m \in \mathcal{M}} |S(t)\,u_0 - m|_{\alpha} \leqslant C\,e^{-\sigma t} \tag{1 5}$$

Concerning existence of such manifolds we have the following result

THEOREM 1 1 *If N is such that*

$$\begin{aligned} \Lambda_{N+1} &\geqslant 3\,L_F^2\,\Lambda_1^{2\gamma - 1}/2\,, \\ \Lambda_{N+1} - \Lambda_N &\geqslant 30\,L_F(\Lambda_N^{\gamma} + \Lambda_{N+1}^{\gamma})\,, \end{aligned} \tag{1 6}$$

then there exists $\phi \in \mathscr{C}(P_{\Lambda_N} H, (I - P_{\Lambda_N})\,D(A^{\alpha}))$ whose graph is an inertial manifold for (1 4)

The first result in this direction is due to Foias, Sell and Temam [11] Theorem 1 1 generalizes Temam [5, Theorem p 436] in the sense that we do not assume that F has bounded support and we can consider the case where $C \neq 0$

Let us briefly mention some of the steps of the proof in order to introduce some notations which are useful in the analysis of the discrete case Given $\ell \geqslant 0$, we denote by $\mathscr{F}_{\ell}$ the following set of functions from PH into $QD(A^{\alpha})$, where $P = P_{\Lambda}$, $Q = I - P_{\Lambda}$

$$\mathscr{F}_{\ell} = \{\phi,\ \|\phi\|_{\alpha} \leqslant \ell\ ,\ \operatorname{lip}_{\alpha}(\phi) \leqslant \ell\} \tag{1 7}$$

We have denoted

$$\begin{aligned} \|\phi\|_{\alpha} &= \operatorname{Sup}\ \left\{|\phi(p)|_{\alpha}/(1 + |p|_{\alpha}), p \in PH\right\}\,, \\ \operatorname{Lip}_{\alpha}(\phi) &= \operatorname{Sup}\ \left\{|\phi(p_1) - \phi(p_2)|_{\alpha}/|p_1 - p_2|_{\alpha}, p_i \in PH\right\} \end{aligned} \tag{1 8}$$

Now, given $\phi \in \mathscr{F}_{\ell}$, we can solve the following o d e on PH

$$\frac{dp}{dt} + Ap + Cp + PF(p + \phi(p)) = 0\,, \quad p(0) = p_0 \tag{1 9}$$

We denote by $S_\phi(t)$ the mapping

$$S_\phi(t)\, p_0 = p(t)$$

and this produces a *group* acting on PH: $\{S_\phi(t), t \in \mathscr{R}\}$. We set

$$(\mathscr{T}\phi)(p_0) = -\int_{-\infty}^{0} e^{(A+C)\sigma} QF(S_\phi(\sigma)\, p_0 + \phi(S_\phi(\sigma)\, p_0))\, d\sigma\,, \quad (1.10)$$

which defines a mapping $\mathscr{T}$ from $\mathscr{F}_\ell$ into $\mathscr{C}(PH, QD(A^\alpha))$.

We notice that if $\mathscr{M} = \mathscr{M}(\phi)$, $\phi \in \mathscr{F}_\ell$, satisfies (1.4) then $S(t)\,\mathscr{M} = \mathscr{M}$, $\forall t \in \mathscr{R}$. Moreover, the evolution on $\mathscr{M}$ is given by S_ϕ, i.e. $S(t)(p + \phi(p)) = S_\phi(t)\,p + \phi(S_\phi(t)\,p)$. Then the invariance of $\mathscr{M} = \mathscr{M}(\phi)$ can also be expressed by the fact that ϕ is a fixed point of $\mathscr{T} : \mathscr{T}\phi = \phi$.

Due to (1.6), we have (see [1] and [2] for the details) the following properties :

$$\begin{gathered} \mathscr{T} \text{ maps } \mathscr{F}_{1/4} \text{ into itself}\,, \\ \mathscr{T} \text{ is a strict contraction on } \mathscr{F}_{1/4}\,. \end{gathered} \quad (1.11)$$

It follows that $\mathscr{T}$ possesses a unique fixed point $\phi \in \mathscr{F}_{1/4}$. Moreover using again (1.6), one shows a stronger property than (1.5), namely :

$$\begin{gathered} \text{There exist two positive constants } K \text{ and } \sigma \\ \text{such that for every } u_0 \in D(A^\alpha)\,, \\ d_\alpha(S(t)\,u_0, \mathscr{M}(\phi)) \leq K d_\alpha(u_0, \mathscr{M}(\phi))\, e^{-\sigma t}\,, \quad \forall t \geq 0\,. \end{gathered} \quad (1.12)$$

Comments :

(i) The method of construction of an inertial manifold we have briefly mentioned is known as the Lyapunov-Perron method and has been introduced in [3]. Other methods are available (see the review by Luskin and Sell [4] in this volume).

(ii) At first sight, the hypothesis (1.2) seems very restrictive. However, besides the fact that some equations with saturable nonlinearity ([1]) satisfy (1.2), many nonlinear p.d.e.'s can be reformulated using a truncation method in order that their nonlinear part satisfy (1.2). See for that purpose the book by Temam [5] and the references therein.

(iii) We notice that equation (1.1) is not necessarily dissipative (in the sense of existence of bounded absorbing sets) and applications of interests include cases where (1.1) possesses unbounded solutions (as $t \to +\infty$), see [1].

2. CONSTRUCTION OF AN INERTIAL MANIFOLD IN THE DISCRETE CASE

We consider the following time discretization of (1.1). Given $u^0 \in D(A^\alpha)$, we define u^n and $u^{n+1/2}$ by

$$\begin{aligned}(u^{n+1/2} - u^n)/\tau + Au^{n+1/2} + F(u^n) &= 0\,,\\ (u^{n+1} - u^{n+1/2})/\tau + C(u^{n+1} + u^{n+1/2})/2 &= 0\,,\end{aligned} \tag{2.1}$$

where $\tau > 0$ is the time-step. When $C \equiv 0$, (2.1) is a standard semi-implicit scheme, while $(2.1)_2$ can be seen as a leap frog scheme. Hence (2.1) is a fractional-step method.

For the sake of convenience in the notations we introduce the two linear and bounded operators on H:

$$R(\tau) = (I + \tau A)^{-1}\,, \quad U(\tau) = (I - \tau C/2)(I + \tau C/2)^{-1}\,, \tag{2.2}$$

and set

$$\bar{R}(\tau) \equiv U(\tau)\,R(\tau)\,, \quad \tau > 0\,. \tag{2.3}$$

Therefore (2.1) can also be written as

$$u^{n+1} = \bar{R}(\tau)(u^n - \tau F(u^n))\,, \quad \forall n \geq 0\,, \tag{2.4}$$

which motivates us to introduce the following mapping on $D(A^\alpha)$ ·

$$S^\tau v = \bar{R}(\tau)(v - \tau F(v))\,, \quad v \in D(A^\alpha)\,. \tag{2.5}$$

By mimicking Definition 1.1, we set

DEFINITION 2.1 : *An inertial manifold $\mathcal{M} \subset D(A^\alpha)$ for* (2.1) *is a finite dimensional Lipschitz manifold which is invariant by S^τ:*

$$S^\tau \mathcal{M} \subset \mathcal{M}\,, \tag{2.6}$$

and attracts exponentially all its solutions :

$$\begin{gathered}\forall R > 0\,, \quad \exists \sigma > 0\,, \quad C \geq 0 \text{ s.t. } \quad \forall n \geq 0\,, \quad u^0 \in D(A^\alpha)\,, \quad |u_0|_\alpha \leq R\\ d_\alpha((S^\tau)^n u^0, \mathcal{M}) = d_\alpha(u^n, \mathcal{M}) \leq C\, e^{-\sigma n \tau}\,.\end{gathered} \tag{2.7}$$

We are going to construct such a manifold as the graph of a function $\phi \in \mathcal{F}_\ell$ for some $\ell \geq 0$, and then (2.1) reads on $\mathcal{M} = \mathcal{M}(\phi)$:

$$p^{n+1} = S^\tau_\phi\, p^n\,, \quad n \geq 0 \tag{2.8}$$

where S_ϕ^τ is the Lipschitz continuous map on PH defined as follows

$$S_\phi^\tau p = \bar{R}(\tau)(p - \tau PF(p + \phi(p))) . \tag{2.9}$$

With these notations we can state

THEOREM 2.1 : *We assume that N is such that* (1.6) *holds true. For every τ satisfying*

$$\tau \Lambda_{N+1} \leqslant 1 , \tag{2.10}$$

the discrete infinite dimensional system (2.1) *possesses an inertial manifold $\mathcal{M}_\tau$ which is the graph of a Lipschitz function from $P_N H$ into $(I - P_N) D(A^\alpha)$.*

The proof of this result is similar to that of Theorem 1.1. One introduces the mapping $\mathcal{T}_\tau$ on $\mathcal{F}_{1/4}$: $P = P_{\Lambda_N}$, $Q = I - P$,

$$(\mathcal{T}_\tau \phi)(p^0) = -\tau \sum_{k=1}^{+\infty} \bar{R}(\tau)^k QF((S_\phi^\tau)^{-k} p^0 + \phi((S_\phi^\tau)^{-k} p^0)) . \tag{2.11}$$

And one checks easily that thanks to (1.6) and (2.10) the mapping S_ϕ^τ is invertible, and (2.11) makes sense. Then, one shows that $\mathcal{T}_\tau$ maps $\mathcal{F}_{1/4}$ into itself and is contracting :

$$\|\mathcal{T}_\tau \phi - \mathcal{T}_\tau \psi\|_\alpha \leqslant (7/10)\|\phi - \psi\|_\alpha , \quad \forall \phi, \psi \in \mathcal{F}_{1/4} . \tag{2.12}$$

It follows that $\mathcal{T}_\tau$ possesses a unique fixed point ϕ_τ and one proves that $\mathcal{M}_\tau = \mathcal{M}(\phi_\tau)$ is the desired manifold.

3. CONVERGENCE OF THE INERTIAL MANIFOLDS

Given N satisfying (1.6), we know according to Theorem 1.1 that (1.1) has an inertial manifold $\mathcal{M} = \mathcal{M}(\phi)$ which is a graph over $P_{\Lambda_N} H$. Then for $\tau \leqslant \Lambda_{N+1}^{-1}$, thanks to Theorem 2.1 we also have an inertial manifold $\mathcal{M}_\tau = \mathcal{M}(\phi_\tau)$ which is a graph over the same space. A natural question then is that of the convergence of the ϕ_τ towards ϕ as τ goes to zero. The following result answers positively this question and provide an error estimate between these manifolds.

THEOREM 3.1 : *For N satisfying* (1.6), *there exists a constant κ such that for every $\tau \in \left]0, \Lambda_{N+1}^{-1}\right]$, the previous inertial manifolds satisfy*

$$\|\phi - \phi_\tau\|_\alpha \leqslant \kappa \tau^\zeta |\mathrm{Log}\, \tau| \tag{3.1}$$

where $\zeta = 1 - \gamma$ for $s_0 \leqslant 1$ and $\zeta = (1 - \gamma)/(2 s_0 - 1)$ for $s_0 \geqslant 1$.

The proof of Theorem 3.1 is divided in two steps ; the first one consists in the error estimate between the finite dimensional parts, the second gives an estimate of the error of the infinite dimensional ones.

To deal with the first goal, we are given $\phi \in \mathcal{F}_\ell$, and consider the two following dynamical systems

$$\frac{dp}{dt} + Ap + Cp + PF(p + \phi(p)), \tag{3.2}$$

$$p^{n+1} = \bar{R}(\tau)(p^n - \tau PF(p^n + \phi(p^n))), \tag{3.3}$$

where $P = P_{\Lambda_N}$, N satisfying (1.6).

We do not assume for the moment that ϕ is the graph of an inertial manifold. Since (3.2) is a standard o.d.e., we know from classical results on one-step methods that the error $e_n = p(n\tau) - p^n$, $n \in \mathcal{Z}$ tends to zero with τ. More precisely we have the following estimate

PROPOSITION 3.1 : *Let p_0 be given in H and $p(t)$, $t \in \mathcal{R}$, (resp. p^n, $n \in \mathcal{Z}$) be the solution to* (3.2) *(resp.* (3.3)*) satisfying $p(0) = p^0$ (resp. $p_0 = p^0$). For every negative integer n, we have*

$$|e_n| \leq \frac{\tau^2 K(\lambda)}{1 - \tau L_F(1 + \ell)\lambda^\gamma}(\eta^{-n} e^{-n\tau\bar{\lambda}})(\eta e^{-\tau\bar{\lambda}} - 1)(1 + |p^0|_\alpha) \tag{3.4}$$

where $K(\lambda)$ is independent of τ and n, $\eta = (1 + \tau\lambda)(1 - \tau L_F(1 + \ell)\lambda^\gamma)^{-1}$ (hence $\eta > 1$), $\bar{\lambda} = \lambda + \lambda^\gamma L_F(1 + \ell)$ and $\lambda = \Lambda_N$.

The details of the proof are given in [1]. Let us remark that even in the linear case, the error estimate is not better than the previous one.

To deal with the infinite dimensional part, we evaluate in fact the norm of the difference $\mathcal{T}_\tau \phi - \mathcal{T}\phi$, for a given ϕ in $\mathcal{F}_\ell$;

PROPOSITION 3.2 : *Assume that $\tau\Lambda_{N+1} \leq 1$ and*

$$\Lambda_{N+1} - \Lambda_N \geq 2 L_F(1 + \ell)\Lambda_N^\gamma . \tag{3.5}$$

Then for every $p^0 \in PH$, $\phi \in \mathcal{F}_\ell$, we have

$$|(\mathcal{T}_\tau \phi - \mathcal{T}\phi)(p_0)|_\alpha \leq C_0(1 + |p^0|_\alpha)\tau^\zeta |\mathrm{Log}\,\tau| \tag{3.6}$$

where C_0 depends only on Λ_N, Λ_{N+1}, s_0 and γ but not on τ ; ζ is as in Theorem 3.1.

The proof of Proposition 3.2 is rather long and consists in evaluating the differences

$$\sum_{k=1}^{\infty} \int_{-k\tau-\tau}^{-k\tau} e^{(A+C)\sigma} Q(F(p+\phi(p))(\sigma)) - F(p_{-k}+\phi(p_{-k}))\,,$$

$$\sum_{k} \int_{-k\tau-\tau}^{-k\tau} (e^{(A+C)\sigma} - \bar{R}(\tau)^k)\, QG(p^{-k})\,,$$

and

$$\int_{-\tau}^{0} e^{(A+C)\sigma} QG(p(\sigma))\, d\sigma\,,$$

where we have set $G(p) = F(p + \phi(p))$. The reader is referred to [4] for the details.

The proof of Theorem 3.1 is now straightforward. Indeed let us take ϕ and ϕ_τ respectively the fixed point of $\mathcal{T}$ and $\mathcal{T}_\tau$, obtained by Theorems 1.1 and 2.1. We then have $\phi = \mathcal{T}\phi$, $\phi_\tau = \mathcal{T}_\tau\, \phi_\tau$ and $\phi, \phi_\tau \in \mathcal{F}_{1/4}$. Hence we can write

$$\phi - \phi_\tau = \mathcal{T}\phi - \mathcal{T}_\tau\, \phi_\tau = \mathcal{T}\phi - \mathcal{T}_\tau\, \phi + \mathcal{T}_\tau\, \phi - \mathcal{T}_\tau\, \phi_\tau\,.$$

Now according to (2.12)

$$\|\mathcal{T}_\tau\, \phi - \mathcal{T}_\tau\, \phi_\tau\|_\alpha \leq (7/10)\, \|\phi - \phi_\tau\|_\alpha\,.$$

It follows that

$$\|\phi - \phi_\tau\|_\alpha \leq (10/3)\, \|\tau\phi - \mathcal{T}_\tau\, \phi\|_\alpha \leq (10\, C_0/3)\, \tau^\zeta |\mathrm{Log}\, \tau|\,,$$

which is exactly (3.1).

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