

RAUL B. GONZALEZ DE PAZ

**On the optimal design of elastic shafts**

*Modélisation mathématique et analyse numérique*, tome 23, n° 4  
(1989), p. 615-625

[http://www.numdam.org/item?id=M2AN\\_1989\\_\\_23\\_4\\_615\\_0](http://www.numdam.org/item?id=M2AN_1989__23_4_615_0)

© AFCET, 1989, tous droits réservés.

L'accès aux archives de la revue « Modélisation mathématique et analyse numérique » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## ON THE OPTIMAL DESIGN OF ELASTIC SHAFTS (\*)

by Raul B. GONZALEZ DE PAZ <sup>(1)</sup>

Communicated by R. Teman

---

*Abstract. — In order to study the design of hollow shafts with maximal torsional rigidity, we define a functional associated with the shape of the shaft and investigate its minimization. Introducing a relaxation by means of a duality approach we are able to apply convex analysis techniques and prove the existence of the optimal design.*

*Résumé. — Afin d'étudier la forme de poutres creuses de rigidité maximale à la torsion, on définit une fonctionnelle associée à la forme de la section droite et on cherche à la minimiser. En introduisant une relaxation moyennant une certaine classe de multiplicateurs de Lagrange, on applique des techniques d'analyse convexe pour montrer l'existence de la section droite optimale.*

### 1. INTRODUCTION

We consider the problem of torsion of a hollow elastic shaft. We denote by  $\Omega$  the region occupied by the cross section in the  $x - y$  plane, which we shall assume to be doubly connected. We denote by  $\Gamma_0$  and  $\Gamma$  the interior and exterior boundary of the domain  $\Omega$ . The direction of the applied torque coincides with the  $z$ -axis. We assume that the shaft material is homogeneous and isotropic. We express the nonzero components of the stress tensor in terms of the stress function :

$$\tau_{xz} = G\theta u_y, \quad \tau_{yz} = G\theta u_x$$

---

(\*) Received in June 1987, revised in June 1988.

(<sup>1</sup>) Depto de Matemáticas, Universidad del Valle, Apartado Postal 82, 01901 Guatemala, Guatemala et Laboratoire d'Analyse Numérique, Université de Paris-Sud, Bâtiment 425, 91045 Orsay, France

where  $G$  is the shear modulus,  $\theta$  is the angle of twist per unit length of the shaft and  $u$  is the stress function satisfying :

$$u_x = \partial u / \partial x, \quad u_y = \partial u / \partial y .$$

It is well known that for this case, the torsion problem is reduced to finding the stress function  $u$  such that :

$$-\Delta u = 2 \quad \text{in} \quad \Omega \quad (1.1)$$

$$u = 0 \quad \text{on} \quad \Gamma \quad (1.2)$$

$$u = c \quad \text{on} \quad \Gamma_0 \quad (1.3)$$

$$-\int_{\Gamma_0} \partial u / \partial n \, d\sigma = 2 A_0 \quad (1.4)$$

where  $A_0$  is the area of the region bounded by the curve  $\Gamma_0$  and  $c$  is an unknown quantity whose value can be determined using (1.4). The torsional rigidity  $K_\Omega$  is given by

$$K_\Omega = 2 \left( \int_{\Omega} u \, d\omega + C A_0 \right) . \quad (1.5)$$

Let us assume that the boundary  $\Gamma_0$  and the following isoperimetric condition are given :

$$\text{meas} \quad \Omega = A \quad (1.6)$$

where  $A$  is a positive constant. We look for the shape of  $\Omega$  such that the rigidity  $K_\Omega$  is maximized. Among others, this problem has been studied by N. Banichuk [3], see also other related papers by Cea [6] and Cea-Malanowski [8].

We remark that by minimizing the functional

$$v \rightarrow J(v) = 1/2 \int_{\Omega} |\nabla v|^2 \, d\omega - 2 \int_{\Omega} v \, d\omega$$

on a suitable function space we obtain a solution for the problem (1.1)-(1.4). For the corresponding solution  $u_\Omega$  we have :

$$J(u_\Omega) = -1/2 \int_{\Omega} |\nabla u_\Omega|^2 \, d\omega = -1/2 K_\Omega .$$

Thus, the domain  $\Omega$  that maximizes  $K_\Omega$ , minimizes  $J(u_\Omega)$ . By using this property, we shall define a new « relaxed » problem and applying some convex analysis techniques we shall prove the existence of the optimal domain  $\Omega$ . In fact, the relaxed problem leads to the minimization of a

concave functional on a convex set of functions. The concavity structure will allow us to prove that there exists a characteristic function where the minimum is attained. This approach is similar to the one used by Gonzalez de Paz [13] for the study of the existence of a domain with minimal capacity when the interior boundary is unknown. In the appendix we show how our results can be applied to study capacity problems where the exterior boundary is unknown a priori.

This research was partly done while the author was visiting the Courant Institute of Mathematical Sciences in New York supported by a Fulbright Grant. The author wishes to thank R. Kohn and the referee for their advices and criticisms, they were very helpful for the achievement of this work.

## 2. THE RELAXED PROBLEM

Let  $\Omega_0$  be a star-shaped, connected, bounded domain in  $\mathbb{R}^2$  with Lebesgue measure  $A_0$  and boundary  $\Gamma_0$  which is Lipschitz continuous. Let  $B_R$  be an open disc with center at some point in the interior of  $\Omega_0$ ; in order to allow for the feasible domains to be contained in the disc, we choose the radius  $R$  large enough so that for  $d = \text{dist}(\partial B_R, \Gamma_0)$ , the annulus with outer boundary  $\partial B_R$  and width  $d$  has an area greater than the given constant  $A$ , we put  $D_R = B_R \setminus \bar{\Omega}_0$  and denote by  $\|\cdot\|$  the usual  $L^2$ -norm in  $B_R$ . Furthermore, let  $\mu$  be a positive, bounded function such that :

$$0 \leq \mu \leq 1 \quad \text{almost everywhere in } B_R \quad (2.1)$$

$$\int_{B_R} \mu \, d\omega = A_0 + A \quad (2.2)$$

$$\int_{\Omega_0} \mu \, d\omega = A_0. \quad (2.3)$$

Following the definitions introduced by Lanchon [16], we put

$$E_R = \{v \mid v \in H_0^1(B_R), v = \text{const. on } \Omega_0\};$$

here  $H_0^1(B_R)$  denotes the usual Sobolev space (see Nečas [18]).

We now define on the Sobolev space the functional

$$v \rightarrow J_\mu(v) = 1/2 \int_{B_R} |\nabla v|^2 \, d\omega - \int_{B_R} \mu f v \, d\omega. \quad (2.4)$$

We remark that for the special case of the elastic torsion, the function  $f$  is a given positive constant. This special case is contained in our framework if we suppose that  $f$  is strictly positive and bounded.

The problem  $P(\mu)$ : The minimization of  $v \rightarrow J_\mu(v)$  on  $E_R$  was treated by H. Lanchon [16]. This functional is convex and weakly lower semicontinuous, so that for each  $\mu \in L^\infty(B_R, \mathbb{R}^+)$  there exists a  $u_\mu \in E_R$  such that the functional is minimized (cf. Ekeland-Temam [10], Moreau [17]) and  $u_\mu$  is the weak solution of the following boundary value problem :

$$-\Delta u_\mu = \mu f \quad \text{in } D_R = B_R/\Omega_0 \quad (2.5)$$

$$u_\mu = c \quad \text{on } \Omega_0 \quad (2.6)$$

$$u_\mu = 0 \quad \text{on } \partial B_R \quad (2.7)$$

$$\int_{\Gamma_0} \partial u_\mu / \partial n \, d\sigma = - \int_{\Omega_0} \mu f \, d\omega. \quad (2.8)$$

In (2.8)  $n$  denotes the unit normal exterior to  $\Omega_0$  at each point of  $\Gamma_0$ . In the case  $f$  is a constant and  $\mu = 1$  on  $\Omega_0$ , this is the classical integral constraint (1.4).

*Remark 2.1* : The element  $u_\mu$  is a non-negative function. In fact define

$$u_\mu^+ = \max(u_\mu, 0).$$

This is an element of  $H_0^1(B_R)$  (cf. Kinderlehrer-Stampacchia [15]). Moreover, because of the extremality property of  $u_\mu$  we have :  $u_\mu^+ \in E_R$  and

$$\int_{B_R} \mu f u_\mu^+ \, d\omega = \int_{B_R} \mu f u_\mu \, d\omega.$$

If  $u_\mu$  were strictly negative on a set of positive measure, then

$$\|\nabla u_\mu^+\| < \|\nabla u_\mu\|$$

so that

$$J_\mu(u_\mu^+) < J_\mu(u_\mu).$$

This is a contradiction, so  $u_\mu^+ = u_\mu$ .

*Remark 2.2* : The function  $u_\mu$  is an element of  $C^{1,1}(\bar{D}_R)$ . First we recall that  $u_\mu \in C^{1,K}(\bar{D}_R)$  for  $0 \leq \alpha < 1$  (see Kinderlehrer-Stampacchia [13]). It follows that  $u_\mu$  is a Lipschitz function. Besides,  $\Delta u_\mu \in L^\infty(D_R)$ . From these results and the boundary conditions (2.6) and (2.7) it follows :

$$u_\mu \in W^{2,\infty}(D_R)$$

(see C. Gebhardt [11] and R. Jensen [24]), this implies that  $\nabla u_\mu$  is a Lipschitz function (for the definition of  $W^{2,\infty}(D_R)$ , see J. Nečas [18]).

The optimization problem related to  $\mu$  : We now define the functional  $\Phi$  on  $L^\infty(B_R, \mathbb{R}^+)$  as follows :

$$\Phi(\mu) = J_\mu(u_\mu) = \min_{u \in E_R} J_\mu(u) .$$

We study the problem of minimization of  $\Phi$  in  $C \subset L^\infty(B_R, \mathbb{R}^+)$  where  $C$  denotes the convex set defined by the constraints (2.1), (2.2) and (2.3). The convex set  $C$  is compact for the topology  $\sigma(L^\infty, L^1)$ . We shall prove that the functional  $\Phi$  is continuous for the same topology in order to show the existence of the minimizing element.

**THEOREM 2.1 :** *The functional  $\Phi$  is  $\sigma(L^\infty, L^1)$ -continuous on  $C$ .*

*Proof:* Firstly we establish the following assertion : there exists a ball  $B_\rho$  in  $H_0^1(B_R)$  of radius  $\rho$  such that for every  $\mu \in C$  :

$$\min_{u \in E_R} J_\mu(u) = \min_{u \in E_R \cap B_\rho} J_\mu(u) .$$

Let  $\mu$  be given, and let  $u_\mu$  be the corresponding minimizing element of  $J_\mu$  in  $E_R$ , then for every  $v \in E_R$  we have :

$$(\nabla u_\mu, \nabla v) = (\mu f, v) .$$

Here the parentheses denote the usual scalar product in  $L^2(B_R)$ . For the special case  $v = u_\mu$  :

$$\|\nabla u_\mu\|^2 = (\mu f, u_\mu) \leq \|\mu f\|_L \|u_\mu\|_L$$

and by using the Cauchy-Schwarz and the Poincaré inequality :

$$\|u_\mu\|_L \leq \alpha \|u_\mu\| \leq \alpha' \|\nabla u_\mu\|$$

where  $\alpha$  and  $\alpha'$  are constants depending on the ball  $B_R$ , then we obtain for every  $\mu \in C$  :

$$\|\mu f\|_L \leq \|f\|_L$$

and finally,

$$\|\nabla u_\mu\| \leq \alpha' \|f\|_L$$

so the expected ball has radius  $\rho = \alpha' \|f\|_L$ .

Because of the Rellich-Kondrasov injection theorem, the set  $K = E_R \cap B_\rho$  is compact in  $L^1(B_R)$  (cf. Nečas [16]). Besides, it is well known that if a family of affine functions is equicontinuous, the lower

enveloppe of this family is a continuous function. So if we define the family  $\{J_u : \mu \rightarrow J_\mu(u), u \in K\}$ , we see that

$$\Phi(\mu) = \inf_{u \in K} J_\mu(u) = J_\mu(u_\mu).$$

Let  $\varepsilon > 0$  be given, and let  $\mu \in C$  be such that  $\mu - \mu_0 \in (1/\varepsilon K)^0$ , the polar set of  $(1/\varepsilon)K$ . The latter is strongly compact in  $L^1$ , so  $\mu$  is in a neighborhood of  $\mu_0$  for the topology of the uniform convergence of compact sets of  $L^1$ , so that we have for every  $u \in K$ :

$$|\langle \mu - \mu_0, u \rangle_{L^\infty, L^1}| \leq \varepsilon$$

where the brackets denote the  $(L^\infty, L^1)$  duality. Then, for every  $u \in K$ :

$$|J_\mu(u) - J_{\mu_0}(u)| \leq \varepsilon$$

which establishes the equicontinuity. We need only to remark that the topology used above is equivalent to the weak topology  $\sigma(L^\infty, L^1)$  on the unit ball of  $L^\infty(B_R)$ , so that the functional is continuous for this topology on  $C$  (cf. Bourbaki [4]). This gives our next result:

**THEOREM 2.2:** *There exists an element  $\mu_R \in C$  such that*

$$\Phi(\mu_R) = \min_{\mu \in C} \Phi(\mu).$$

*Remark 2.3:* The functional  $\Phi$  is the lower envelope of affine linear functions, so that it is concave. This implies that among the minimizing elements there are extremal points of  $C$ , and these are characteristic functions of sets with measure  $A + A_0$  (cf. Castaing-Valadier [5]). So there exists  $\mu_R = \chi_{\tilde{\Omega}}$  with  $\tilde{\Omega} = \Omega_0 \cup \Omega_R$  with  $\Omega_R$  an optimal set. We shall study the necessary conditions of optimality in order to obtain a description of the optimal domain as the solution of a free boundary value problem.

### 3. NECESSARY CONDITIONS OF OPTIMALITY AND THEIR CONSEQUENCES

**THEOREM 3.1:** *The functional  $\Phi$  has a weak derivative in the sense of Gateaux for every  $\mu \in L^\infty(B_R, \mathbb{R}^+)$ .*

*Proof:* Being  $\Phi$  the lower envelope of a family of affine functions, it follows from a theorem of Valadier [21]:

$$\Phi'(\mu; \alpha) = - \langle fu_\mu, \alpha \rangle_{L^1, L^\infty} \quad (3.1)$$

for every  $\alpha = \gamma - \mu$  with  $\gamma \in L^\infty(B_R, \mathbb{R}^+)$ .

*Remark 3.1 :*  $\Phi$  is concave and  $\sigma(L^\infty, L^1)$ -continuous, so it follows that its derivative is a Frechet-derivative also (cf. Valadier [21]).

*Remark 3.2 :* The first order necessary conditions of optimality give for every  $\alpha = \mu - \mu_R, \mu \in C :$

$$- \langle fu_R, \alpha \rangle \geq 0 \tag{3.2}$$

with  $u_R$  the corresponding solution for the boundary value problem  $P(\mu_R)$ .

If we restrict ourselves to characteristic functions, we obtain for every domain  $\Omega$  in  $D_R$  with measure equal to  $A$  and such that  $\Gamma_0$  is contained in  $\partial\Omega :$

$$\int_{\Omega_R} fu_R d\omega \geq \int_{\Omega} fu_R d\omega . \tag{3.3}$$

The inequality (3.3) states that the integrand  $fu_R$  must be « placed » in  $D_R$  so that the integral has a maximal value. We denote  $\Gamma$  the boundary of  $\Omega_R$  related to  $D_R, \Gamma$  can be interpreted as a free boundary and we have :

**THEOREM 3.2 :** *Let  $f$  be a constant function, then there exists a positive number  $p_R$  such that*

$$\Omega_R = \{x \in D_R | u_R(x) > p_R\}$$

where the equality is understood to hold a.e., and

$$\Gamma = \{x \in D_R | u_R(x) = p_R\} .$$

*Proof :* The existence of a Lagrange multiplier related to the constraint (2.2) for the functional  $\mu \rightarrow \int_{B_R} fu_R \mu d\omega$  is a classical fact (cf. Cea-Malanowski [8]). This means, there exists a constant  $p_R$  such that for all elements  $\gamma$  of the unit ball in  $L^\infty(B_R, \mathbb{R}^+)$  :

$$\int_{B_R} \mu u_R d\omega - p_R \int_{B_R} \mu d\omega \geq \int_{B_R} \gamma u_R d\omega - p_R \int_{B_R} \gamma d\omega .$$

Then we have for almost every  $x \in B_R :$

$$\begin{aligned} u_R(x) > p_R & \text{ implies } \mu(x) = 1 \\ u_R(x) < p_R & \text{ implies } \mu(x) = 0 . \end{aligned}$$

For  $\tilde{\Omega}$  as in Remark 2.3 we define  $G = B_R \setminus \tilde{\Omega}$ , and it follows :

$$\begin{aligned} \{x \in B_R | u_R(x) > p_R\} & \subset \tilde{\Omega} \\ \{x \in B_R | u_R(x) < p_R\} & \subset G . \end{aligned}$$



Both inclusions must be understood in the sense almost everywhere.

Furthermore we have

$$\tilde{\Omega} \subset \{x \in B_R | u_R(x) \geq p_R\} \quad \text{a.e.}$$

which implies

$$\Omega_R \subset \{x \in D_R | u_R(x) \geq p_R\} \quad \text{a.e.}$$

From the definition of  $\Omega_R$  it follows :

$$\{x \in D_R | u_R(x) < p_R\} \subset \Omega_R \quad \text{a.e.}$$

Besides, because of the regularity of  $f : u_R \in H^2(D_R)$ , so that the equation (2.5) is verified in the sense almost everywhere. This implies (see Zolesio [23]) :  $\text{meas} (\{x \in D_R | u_R(x) = p_R\} \cap \Omega_R) = 0$  and the first assertion of the theorem is proved.

The characterization of  $\Gamma$  follows from the fact that the function  $u_R$  is continuous and superharmonic in  $D_R$  (see Gonzalez de Paz [13]).

**COROLLARY 3.3 :** *The support of the measure  $\mu \, d\omega$  is the compact set :*

$$\tilde{\Omega} = \{x \in B_R | u_R(x) \geq p_R\} .$$

*Remark 3.3 :* For the boundary condition (2.6) we have :

$$u_R = c \geq p_R \quad \text{on} \quad \Omega_0 .$$

We shall omit for the rest of this paragraph the index  $R$ .

*Remark 3.4 :* The function  $u \in H_0^1(B_R)$  is a solution of the following free boundary value problem :

$$-\Delta u = f \quad \text{in} \quad \Omega \tag{3.4}$$

$$\Delta u = 0 \quad \text{in} \quad D_R \setminus \bar{\Omega} \tag{3.5}$$

$$u = p \quad \text{on} \quad \Gamma \tag{3.6}$$

$$u = c \quad \text{on} \quad \Gamma_0 . \tag{3.7}$$

*Remark 3.5 :* We should point out that in the case  $\Omega_0$  is not star-shaped,  $D_R \setminus \bar{\Omega}$  might have more than one connected component ; which should mean the existence of more « holes » in the cross section of the shaft.

*Remark 3.6 :* The gradient of  $u$  is continuous, so that

$$(\nabla u)^+ = (\nabla u)^- \quad \text{on} \quad \Gamma$$

where the plus sign denotes the limit at the boundary taken in the inward direction to  $\Omega$  and the minus sign denotes the limit in the outward direction.

Because of the regularity of  $u$ , it is known that free boundaries of this type are locally Lipschitz (cf. Kinderlehrer-Stampacchia [15]). If we recall the fact that the free boundary  $\Gamma$  is a level set of  $u$ , we have in the neighborhoods of points where  $|\nabla u| > 0$  on  $\Gamma$  :

$$\partial u / \partial n^+ = \partial u / \partial n^- \quad \text{on } \Gamma .$$

Some analog free boundary value problems have been studied by Zolesio [22] using other optimal design techniques.

4. APPENDIX

The method described in this paper can be applied to prove the following result :

**THEOREM :** *Let  $\Gamma_0$  be a given closed Lipschitz continuous curve, non intersecting itself so that the domain  $\Omega_0$  inclosed is star-shaped. Let  $W$  be the set of all doubly connected domains  $\Omega$  with a given measure and with  $\Gamma_0$  as inner boundary. Then there exists a domain  $\Omega^* \in W$  such that for all  $\Omega \in W$  :*

$$\text{Cap}_{\Omega^*} (\Omega_0) \leq \text{Cap}_{\Omega} (\Omega_0) .$$

*Proof :* We need only to remark that for the case  $f = \varepsilon$  with  $\varepsilon$  a positive constant, the stated results can be applied. Replacing the integral constraint (2.8) with the Dirichlet condition  $u = 1$  on  $\Omega_0$  all the main results remain unchanged. For a given domain  $\Omega$ , the corresponding solution  $u_\varepsilon$  of the boundary value problem has the form  $u_\varepsilon = u_0 + u^\varepsilon$ , where  $u_0$  is the capacity potential of the domain  $\Omega_0$  related to  $\Omega$  and  $u^\varepsilon$  the corresponding solution of the Poisson equation in  $\Omega$  with homogeneous Dirichlet conditions. So we have :

$$\begin{aligned} \text{En}_\varepsilon (\Omega) = & 1/2 \text{Cap}_\Omega (\Omega_0) + (\nabla u_0, \nabla u^\varepsilon) + \\ & + 1/2 \int_\Omega |\nabla u^\varepsilon|^2 d\omega - \varepsilon \int_\Omega u_\varepsilon d\omega . \end{aligned} \quad (4.1)$$

By applying theorem 2.2 for a given  $\varepsilon$ , we know there exists a domain  $\Omega^* \in W$  such that for every  $\Omega \in W$  :

$$\text{En}_\varepsilon (\Omega^*) \leq \text{En}_\varepsilon (\Omega) .$$

Being  $u_0$  harmonic in  $\Omega$ , the second term of the right side in (4.1) vanishes. Besides, it is known that in the case  $\varepsilon \rightarrow 0$ , then  $u^\varepsilon \rightarrow 0$  strongly in  $H^1(\Omega)$ , this implies for every  $\Omega \in W$  :

$$\text{En}_\varepsilon (\Omega) \rightarrow 1/2 \text{Cap}_\Omega (\Omega_0)$$

which gives the result.

Other authors have proved that in the case the free boundary  $\Gamma$  is smooth enough :

$$|\nabla u| = \lambda \quad \text{on } \Gamma$$

where  $\lambda$  is a positive constant which can be interpreted as a Lagrange multiplier for the functional  $\Omega \rightarrow \int_{\Omega} |\nabla u_{\Omega}|^2 d\omega$  related to the measure constraint of the domain, here  $u_{\Omega}$  denotes the corresponding potential (see for example Banichuk [3]).

It should be mentioned that Alt-Cafarelli [2] study the following related problem : find  $v \in K$  which minimizes the functional

$$v \rightarrow J(v) = \int_{\Omega} |\nabla v|^2 d\omega + Q \int_{\Omega} \chi_{v>0} d\omega$$

where  $K = \{v \in L^1_{\text{loc}}(\Omega) | \nabla v \in L^2(\Omega), v = u^0 \text{ on } S\}$ , here  $u^0 > 0$ ,  $Q \geq 0$  and  $S \subset \partial\Omega$  are given. For the case that  $Q$  and  $u^0$  are constants, the solution of their problem solves ours for  $A = \int_{\Omega} \chi_{u>0} d\omega$ .

Besides, the stationary points of the functional  $J$  have the property

$$|\nabla u| = Q \quad \text{on } \Gamma = \Omega \cap \partial \{u > 0\} .$$

In their case,  $Q$  is given and the constant  $A$  is a result ; in ours,  $A$  is given and the constant  $\lambda$  is a consequence of the necessary conditions of optimality (see another related results using different techniques in Acker [1]).

#### REFERENCES

- [1] A. ACKER, *A free boundary optimization problem*. SIAM J. Math. Anal. 9 (1978) pp. 1179-1191.
- [2] H. W. ALT and L. A. CAFARELLI, *Existence and regularity for a minimum problem with free boundary*. J. Reine u. Angew. Mathematik 325 (1981) pp. 104-144.
- [3] N. V. BANICHUK, *Problems and methods of optimal structural design*. Plenum Press, London (1983).
- [4] N. BOURBAKI, *Espaces Vectoriels Topologiques*, Hermann, Paris (1973).
- [5] Ch. CASTAING and M. VALADIER, *Convex Analysis and Measurable Multifunctions*. Lect. Notes in Math. no. 580, Springer-Verlag, New York (1977).
- [6] J. CEA, *Optimisation, théorie et algorithmes*. Dunod, Paris (1971).

- [7] J CEA, *Problems of shape optimal design in « Optimization of distributed parameter structures, NATO Proceedings »* (Eds Haug, E J and Cea, J) Sijthoff and Noordhoff (1981)
- [8] J CEA and K MALANOWSKI, *An example of a max-min problem in partial differential equations* SIAM J Control and Optimization (1970) pp 305-316
- [9] D CHENAIS, *On the existence of a solution in a domain identification problem* J Math Anal and Appl 52 (1975) pp 189-219
- [10] I EKELAND and R TEMAM, *Analyse convexe et problemes variationnels* Dunod, Paris (1974)
- [11] Cl GEBHARDT, *Regularity of solutions of non-linear variational inequalities* Arch Rat Mech and Anal 52 (1973) pp 389-393
- [12] D GILBARG and N TRUDINGER, *Elliptic Partial Differential Equations of second order* Springer, New York (1977)
- [13] R B GONZALEZ DE PAZ, *Sur un probleme d'optimisation de domaine* Numer Funct Anal and Optimiz 5 (1982) pp 173-197
- [14] B KAWOHL, *Geometrical properties of level sets of solutions to elliptic problems* Proc Symp in Pure Math A M S (1986), part 2, pp 25-36
- [15] D KINDERLEHRER and G STAMPACCHIA, *An introduction to variational inequalities and their applications* Academic Press, New York (1980)
- [16] H LANCHON, *Torsion elastoplastique d'un arbre* J de Mech 13 (1974) pp 267-318
- [17] J J MOREAU, *Fonctionnelles convexes Seminaire sur les equations aux derivees partielles* College de France (1966-67)
- [18] J NEÇAS, *Les methodes directes en theorie des equations elliptiques* Masson, Paris (1967)
- [19] O PIRONNEAU, *Optimal shape design for elliptic systems* Springer-Verlag, New York (1984)
- [20] G POLYA and A WEINSTEIN, *On the torsional rigidity of multiply-connected cross sections* Ann Math 52 (1950) pp 154-163
- [21] M VALADIER, *Sous-differentiels d'une borne superieure et d'une somme continue de fonctions convexes* C R Acad Sc Paris, Serie A, 268, pp 39-42 (janvier 1969)
- [22] J P ZOLESIO, *Domain variational formulation for free boundary problems, in « Optimization of distributed parameter structures NATO-Proceedings »* (Eds Haug, E J and Cea, J) Sijthoff and Noordhoff (1981)
- [23] J P ZOLESIO, *Identification de domaines par deformation* Doctoral Thesis, Universite de Nice (1979)
- [24] R JENSEN, *Boundary regularity for variational inequalities* Indiana Univ Math J 29 (1980) pp 495-511