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A NONCONFORMING FINITE ELEMENT METHOD OF UPSTREAM TYPE
APPLIED TO THE STATIONARY NAVIER-STOKES EQUATION (*)

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Abstract. — We present a nonconforming finite element method with an upstream discretization of the convective term for solving the stationary Navier-Stokes equations. The existence of at least one solution of the discrete problem and the convergence of subsequences of such solutions to a solution of the Navier-Stokes equations are established. In addition, under certain assumptions on the data, uniqueness of the solution can be guaranteed and error estimates of the approximate solution are given. Moreover, some favourable properties of the discrete algebraic system are discussed.

Resumé — Nous présentons une méthode non conforme d'éléments finis avec une discrétisation décentrée amont du terme de convection pour la résolution des équations de Navier-Stokes stationnaires. On prouve l'existence d'une solution au moins du problème discret et la convergence des sous-suites de telles solutions vers une solution des équations de Navier-Stokes stationnaires. En outre on peut sous certaines hypothèses sur les données garantir l'unicité et on donne alors des estimations d'erreur de la solution approximative. En outre on discute quelques propriétés importantes du système algébrique discret.

1. INTRODUCTION

The Navier-Stokes equations for viscous, incompressible flow problems have been the object of considerable research efforts. Because of its great flexibility finite element methods have received considerable attention, both from a theoretical and computational point of view. In general one uses finite elements of higher-order shape functions in order to get better approximations of velocity and pressure fields. However, this can be guaranteed, at least theoretically, only for sufficiently smooth solutions of

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the considered problem. Moreover, the use of higher-order shape functions causes computational costs which can be too expensive for the problem under consideration. Therefore we propose a finite element method with lower-order shape functions. Taking into consideration the dominate influence of the convective term in the case of a higher Reynolds number, we shall use a special upstream discretization of this term.

In this paper we propose a method combining a $P_1-P_0$ nonconforming finite element method due to Crouzeix and Raviart [2] with an upstream discretization of the convective term which has been applied by Ohmori and Ushijima [9] in case of a scalar convection diffusion problem. The method in [2] proposed for the Stokes problem was extended to stationary Navier-Stokes equations in [7]. But the results concerning the nonconforming elements are stated without proof. An extension to time-dependent Navier-Stokes equations was done in [6].

A similar upwinding technique was first introduced in [8] to solve the Neutron transport equation. For solving the Navier-Stokes equations in terms of stream function and vorticity, this technique was applied in [3] and analyzed in [5].

The plan of the paper is the following. In Section 2 we introduce the notations used in the subsequent sections. The finite element method for the approximate solution is presented in Section 3. Section 4 contains a discussion of the properties of the algorithm and in Section 5 we give existence and convergence results for the discrete solutions.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, $\Omega$ is supposed to be a convex polygon in $R^2$ with boundary $\Gamma$. Let $n$ be the unit outer normal to $\Omega$. $D_j$, $j = 1, 2$ denotes the differential operator $\frac{\partial}{\partial x_j}$ and often we will use the summation convention, that one has to take the sum over an index occurring twice in some term. For a scalar function $s$ on a measurable subset $G \subset \Omega$, let $\|s\|_{k,p,G}$ and $|s|_{k,p,G}$ be the usual norm and seminorm on the Sobolev space $W^{k,p}(G)$ [1], respectively. Then for a vectorvalued function $v = (v_1, v_2)$ belonging to $(W^{k,p}(G))^2$ we will use the norm

$$\|v\|_{k,p,G} = \sum_{i=1}^{2} \|v_i\|_{k,p,G}$$

and the semi-norm

$$|v|_{k,p,G} = \sum_{i=1}^{2} |v_i|_{k,p,G}.$$
In this paper we consider the stationary Navier-Stokes problem for incompressible flows, i.e. we have to find the velocity field \( u = (u_1, u_2) \) and the pressure \( p \) such that

\[
-\nu \Delta u + u \cdot \nabla u + \text{grad} \ p = f \quad \text{in } \Omega \\
\text{div } u = 0 \quad \text{in } \Omega \\
u u = 0 \quad \text{on } \Gamma
\]

where \( \nu \) denotes the constant inverse Reynolds number and \( f \) a given body force. In order to write (2.1) in a weak form we introduce the notations

\[
V = (H^1_0(\Omega))^2
\]

\[
Q = L^2_0(\Omega) = \left\{ v \in L^2(\Omega): \int_\Omega v \, dx = 0 \right\}
\]

\((.,.)\) inner product in \( L^2(\Omega) \) and \((L^2(\Omega))^2\), respectively (the meaning becomes clear from the context)

\[
a(u, v) = \int_\Omega D_j u_i D_j v_i \, dx \quad u, v \in V
\]

\[
b(u, v, w) = \int_\Omega u_i D_j v_j w_j \, dx \quad u, v, w \in V.
\]

Then the variational form of (2.1) reads:

Find \( (u, p) \in V \times Q \) such that

\[
va(u, v) + b(u, u, v) - (p, \text{div } v) = (f, v) \quad \forall v \in V
\]

\[
(q, \text{div } u) = 0 \quad \forall q \in Q.
\]

It is well known that (2.6) admits at least one solution which is unique provided that \( \nu^{-2} \| f \| \) is sufficiently small [4].

3. FINITE ELEMENT APPROXIMATION OF U PSTREAM TYPE

For solving the continuous problem (2.6) approximately, we will combine a nonconforming finite element method due to Crouzeix/Raviart, Temam [2, 11] with an upstream discretization of the convective term which has been applied by Ohmori, Ushijima in case of a scalar convection-diffusion problem [9].

Let \( \{ \tau_h \} \) be a family of triangulations of \( \Omega \) into triangles \( K \) with

\[
\Omega = \bigcup_{K \in \tau_h} K,
\]

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which is assumed to be regular in the usual sense, and let $h_K$ be the diameter of the triangle $K$. We also assume that the inverse assumption on the mesh

$$\frac{h}{h_K} \leq C \quad \forall K \in \bigcup_h \tau_h$$

is fulfilled $\left( h = \max_{K \in \tau_h} h_K \right)$.

We denote by $B_i$, $1 \leq i \leq N$, the midpoints of inner edges and by $B_i$, $N + 1 \leq i \leq N + M$, the midpoints of edges lying on the boundary $\Gamma$. Now we define the finite dimensional spaces $V_h$ and $Q_h$ for $V$ and $Q$, respectively, by

\begin{align}
V_h &= \left\{ v \in (L^2(\Omega))^2 : v|_K \in (P_1(K))^2 \forall K \in \tau_h, v \text{ is continuous at } B_i, 1 \leq i \leq N \right\} \\
& \quad \forall \; N + 1 \leq i \leq N + M \} \\
Q_h &= \left\{ q \in L^2_0(\Omega) : q|_K \in P_0(K) \forall K \in \tau_h \right\}
\end{align}

where $P_m(K)$, $m = 0, 1$, denotes the set of all polynomials on $K$ with degree not greater than $m$.

Because of $V_h \subseteq V$, we have to extend the divergence operator, the bilinear form $a$ and the trilinear form $b$, respectively.

For $u, v, w \in V + V_h$ and $q \in L^2(\Omega)$ we define these extensions by an elementwise calculation of the corresponding integrals such that

\begin{align}
(q, \text{div}_h u) &= \sum_K \int_K q \text{ div } u \, dx \\
a_h(u, v) &= \sum_K \int_K D_j u, D_j v, \, dx \\
b_h(u, v, w) &= \sum_K \int_K u, D_i v, w, \, dx.
\end{align}

It is well known [2] that $\| \cdot \|_h$ with

$$\| u \|_h = (a_h(u, u))^{1/2}$$

is a norm on $V_h$.

In [11] instead of (3.5) the trilinear form

$$\bar{b}_h(u, v, w) = \frac{1}{2} \sum_K \int_K (u_i D_j v_j w, - u, v, D_i w, ) \, dx$$

is used.
was used which can be regarded as an extension of \( b(u, v, w) \) too, because of

\[
b(u, v, w) = \frac{1}{2} \int_{\Omega} (u_i D_i v_j w_j - u_j D_j v_i w_i) \, dx
\]

(3.8)

\[ \forall u, v, w \in V \text{ with } \text{div} \, u = 0. \]

Moreover, \( \bar{b}_h \) satisfies the skew-symmetric property

\[
\bar{b}_h(u, v, w) = -\bar{b}_h(u, w, v) \quad \forall u, v, w \in V_h,
\]

which is useful in the analysis of existence and convergence. In the case of small value of \( v \), one needs a suitable discretization of the convective part \( b(u, u, v) \) of (2.6) in order to avoid instabilities and numerical oscillations, respectively. Therefore we will define a modified discretization of upstream type \( \bar{b}_h \) of \( b \) following the lines of [9].

![Diagram](image)

**Figure 1.**

Let each triangle \( K \) be divided into six barycentric fragments \( S_{ij} \), \( i, j \in \{k, l, m\}, i \neq j \), as it is indicated in figure 1. Then, for each node \( B_l, l = 1, \ldots, N + M \), we define a lumped region \( R_l \) by

\[
R_l = \bigcup_{k \in \Lambda_l} S_{lk},
\]

(3.9)

where \( \Lambda_l \) denotes the set of all indices \( k \), for which \( B_l \) and \( B_k \) are neighbour nodes. Furthermore, let \( \Lambda_{lk} \) be defined by

\[
\Lambda_{lk} = \partial S_{lk} \cap \partial S_{kl}
\]

(3.10)
and let $n_{lk}$ be the unit outer normal to $R_i$, which is associated with the part $\Gamma_{lk}$ or $\partial R_i$. In a similar way as in [9] we can derive the following upstream discretization $\tilde{b}_h$ of the trilinear form $b$

\begin{equation}
\tilde{b}_h(u, v, w) = \sum_{l=1}^{N+M} \sum_{k \in \Lambda_l} \int_{\Gamma_{lk}} u_{l} n_{lk} \, d\gamma (1 - \lambda_{lk}(u)) (v_j(B_k) - v_j(B_l)) w_j(B_l)
\end{equation}

with

\begin{equation}
\lambda_{lk}(u) = \begin{cases} 
1 & \text{if } \int_{\Gamma_{lk}} u_{l} n_{lk} \, d\gamma \geq 0 \\
0 & \text{otherwise}
\end{cases}
\end{equation}

Now our discretization of (2.6) reads

Find $(u_h, p_h) \in V_h \times Q_h$ such that

\begin{equation}
va_h(u_h, v) + \tilde{b}_h(u_h, u_h, v) - (p_h, \text{div}_h v) = (f, v) \quad \forall v \in V_h \\
(q, \text{div}_h u) = 0 \quad \forall q \in Q_h
\end{equation}

Remark 3.1 Contrary to $\tilde{b}_h$ of (3.7), in our discretization (3.13), $\tilde{b}_h$ is not a trilinear form on $V_h^3$. Actually, the mapping $(u, v, w) \rightarrow \tilde{b}_h(u, v, w)$ is linear in $v$ and $w$ only

4. SOME PROPERTIES OF THE PROPOSED METHOD

In order to establish results concerning existence and convergence of solutions of (3.13) we derive some properties of the mapping $\tilde{b}_h : V_h^3 \rightarrow R$

First of all let us define the lumping operator $L_h$ and the space $W_h$

For a given $v \in V_h$, the lumping operator $L_h$ is defined by

\begin{equation}
(L_h v)(x) = v(B_l) \quad \forall x \in R_l, \quad l = 1, \ldots, N + M
\end{equation}

Furthermore, let us define the space

\begin{equation}
W_h = \{ v \in V_h \mid (q, \text{div}_h v) = 0 \quad \forall q \in Q_h \}
\end{equation}

One can easily see that in our case $v \in V_h$ belongs to $W_h$ if and only if $\text{div}_h v \mid_K = 0 \quad \forall K \in \tau_h$, i.e. $W_h$ is the space of discrete-divergence-free functions in $V_h$

Now we have the following
LEMMA 1: It holds the estimate

\[ \tilde{b}_h(u, v, v) \geq 0 \quad \forall u \in W_h \quad \forall v \in V_h. \]

Proof: Writing \( \tilde{b}_h \) for \( u, v, w \in V_h \) in the form

\[ \tilde{b}_h(u, v, w) = \tilde{b}_h^1(u, v, w) + \tilde{b}_h^2(u, v, w) \]

with

\[ \tilde{b}_h^1(u, v, w) = \sum_{l=1}^{N+M} \sum_{k \in A} \int_{\Gamma_{ik}} u_i n_i^{lk} d\gamma v_j(B_l) + (1 - \lambda_{ik}(u)) v_j(B_k) w_j(B_l) \]

\[ \tilde{b}_h^2(u, v, w) = -\sum_{l=1}^{N+M} \sum_{k \in A} \int_{\Gamma_{ik}} u_i n_i^{lk} d\gamma v_j(B_l) w_j(B_l), \]

we obtain in an analogous way as in [9, Lemma 3]

\[ \tilde{b}_h(u, v, v) + \frac{1}{2} \tilde{b}_h^2(u, v, v) = \]

\[ = \sum_{l=1}^{N+M} \sum_{k \in A} \int_{\Gamma_{ik}} \frac{1}{2} u_i n_i^{lk} d\gamma (v_j(B_l) - v_j(B_k))^2 \left( \lambda_{ik} - \frac{1}{2} \right) \geq 0. \]

Using the fact that

\[ \sum_{k \in A} \int_{\Gamma_{ik}} u_i n_i^{lk} d\gamma = 0 \quad \forall u \in W_h, \quad l = 1, \ldots, N \]

one can easily verify that \( \tilde{b}_h^2(u, v, v) = 0 \) for \( u \in W_h, v \in V_h \). Together with (4.7) this proves (4.3).

The next statement implies the continuity of \( \tilde{b}_h \) on \( V_h^3 \).

LEMMA 2: There exists a constant \( c > 0 \) independent of \( h \), such that

\[ |\tilde{b}_h(u, v, w) - \tilde{b}_h(u^0, v, w)| \leq C \|u - u^0\|_h \|v\|_h \|w\|_h \]

holds for all \( u, u^0, v, w \in V_h \).

Proof: Let us define the set of indices

\[ I = \{(l, k): l = 1, \ldots, N + M, k \in A\} \].

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Then, we can write

\[(4.10) \quad |\tilde{b}_h(u, v, w) - \tilde{b}_h(u^0, v, w)| \leq \left| \sum_{(l,k) \in I} p_{lk} \right| + \left| \sum_{(l,k) \in I} q_{lk} \right|\]

with

\[(4.11) \quad p_{lk} = \int_{\Gamma_{lk}} (u_i - u_i^0) n_i^{lk} d\gamma (1 - \lambda_{lk}(u))(v_j(B_k) - v_j(B_i)) w_j(B_i)\]

and

\[(4.12) \quad q_{lk} = \int_{\Gamma_{lk}} u_i^0 n_i^{lk} d\gamma (\lambda_{lk}(u^0) - \lambda_{lk}(u))(v_j(B_k) - v_j(B_i)) w_j(B_i).\]

Using the fact, that \(D_i v_j, i, j = 1, 2\) is a constant on \(S_{lk} \cup S_{kl}\) for all \((l, k) \in I\), we can estimate for \(u, u^0, v, w \in V_h\)

\[|p_{lk}| \leq \text{mes} \Gamma_{lk} \|u - u^0\|_{0, \infty, S_{lk}} h|v|_{1, \infty, S_{lk}} \|w\|_{0, \infty, S_{lk}},\]

\[\leq h C_1 h^{-1/2} \| u - u^0 \|_{0, 4, S_{lk}} C_2 \| v \|_{1, 2, S_{lk}} C_1 h^{-1/2} \| w \|_{0, 4, S_{lk}}\]

where \(C_1\) and \(C_2\) are the constants of inverse inequalities which are independent of \(h, l\) and \(k\). This implies

\[\left| \sum_{(l,k) \in I} p_{lk} \right| \leq C \left( \sum_{(l,k) \in I} \|u - u^0\|_{0, 4, S_{lk}} \|v\|_{1, 2, S_{lk}} \|w\|_{0, 4, S_{lk}} \right)^{1/4} \left( \sum_{(l,k) \in I} \|v\|_{1, 2, S_{lk}} \right)^{1/4} \left( \sum_{(l,k) \in I} \|w\|_{0, 4, S_{lk}} \right)^{1/4}\]

\[= C \|u - u^0\|_{0, 4, \Omega} \|v\|_h \|w\|_{0, 4, \Omega}.\]

If we apply the estimate

\[(4.13) \quad \|z\|_{0, p, \Omega} \leq C(p, \Omega) \|z\|_h \quad \forall z \in V_h,\]

which can be proven for \(1 \leq p < \infty\) in the two-dimensional case along the lines of Rannacher and Heywood ([6, Proof of (4.36)]), we obtain

\[(4.14) \quad \left| \sum_{(l,k) \in I} p_{lk} \right| \leq C \|u - u^0\|_h \|v\|_h \|w\|_h.\]

To estimate the second sum in (4.10), we split the set \(I\) of indices into

\[I^+ = \{(l, k) \in I : |(u_i^0 n_i^{lk})(P_{lk})| > \|u - u^0\|_{0, \infty, S_{lk}}\} \]
and $I^- = I \setminus I^+$, where $P_{lk}$ denotes the midpoint of $\Gamma_{lk}$. For $(l, k) \in I^+$ we have

$$\left| \left( (u_t - u_t^0) n_t^l k(P_{lk}) \right) \right| \leq \| u - u^0 \|_{0, \infty, s_{ik}} < \left| (u_t^n n_t^l k)(P_{lk}) \right|,$$

which implies

$$\text{sign} \left( (u_t n_t^l k)(P_{lk}) \right) = \text{sign} \left( (u_t^n n_t^l k)(P_{lk}) \right)$$

and consequently, since $u$ is linear on $\Gamma_{lk}$,

$$(4.15) \quad \lambda_{lk}(u) = \lambda_{lk}(u^0) \quad \text{for} \quad (l, k) \in I^+. \quad \text{\}$$

$$(4.15) \quad \text{yields} \quad \sum_{(l, k) \in I^+} q_{lk} = 0. \quad \text{For} \quad (l, k) \in I^- \quad \text{we have}$$

$$\left| q_{lk} \right| \leq \text{mes} \Gamma_{lk} \| u - u^0 \|_{0, \infty, s_{ik}} h \| v \|_{1, \infty, s_{ik}} \| w \|_{0, \infty, s_{ik}}.$$

Thus, we obtain in an analogous way as for $P_{lk}$ the estimate (4.14) also for $\sum_{(l, k) \in I^-} q_{lk}$, which completes the proof of (4.8).

To prove our convergence result in Section 5, we need for arbitrary $\kappa \in (0, 1)$ the inequality

$$(4.16) \quad \| v \|_{0, \infty, \Omega} \leq C_{\kappa} h^{-\kappa} \| v \|_h \quad \forall v \in V_h$$

which is a consequence of (4.13) and the inverse inequality

$$\| v \|_{0, \infty, \Omega} \leq C_{\kappa} h^{-\kappa} \| v \|_{0, p, \Omega} \quad \forall v \in V_h$$

with $\kappa = 2/p$.

Now we will estimate the difference between the two different discretizations of the convective term $b$.

**Lemma 3:** There exists a constant $C$ independent of $h$, such that the estimate

$$(4.17) \quad \left| b_h(u, v, w) - \tilde{b}_h(u, v, w) \right| \leq C_{\kappa} h^{1-\kappa} \| u \|_h \| v \|_h \| w \|_h$$

holds for all $u, v, w \in V_h$ and $\kappa \in (0, 1)$.

**Proof:** We decompose $b_h$ into

$$b_h(u, v, w) = b_h^1(u, v, w) + b_h^2(u, v, w)$$

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with
\begin{equation}
(4.19) \quad b_h^1(u, v, w) = \sum_K \int_K D_i(u, v_j) w_j \, dx
\end{equation}
and
\begin{equation}
(4.20) \quad b_h^2(u, v, w) = -\sum_K \int_K D_i u_i v_j w_j \, dx .
\end{equation}

Using the decomposition (4.4)-(4.6) of $\tilde{b}_h$ we may write
\begin{equation}
(4.21) \quad b_h(u, v, w) - \tilde{b}_h(u, v, w) = Y_1 + Y_2 + Y_3
\end{equation}
with
\begin{equation}
(4.22) \quad Y_1 = b_h^1(u, v, w - L_h w) ,
\end{equation}
\begin{equation}
(4.23) \quad Y_2 = b_h^1(u, v, L_h w) - \tilde{b}_h^1(u, v, w) ,
\end{equation}
\begin{equation}
(4.24) \quad Y_3 = b_h^2(u, v, w) - \tilde{b}_h^2(u, v, w) .
\end{equation}

At first let us estimate $Y_1$ by
\[
|Y_1| \leq \sum_K \left| \int_K (D_i u_i v_j + u_i D_i v_j) (w_j - L_h w_j) \, dx \right| \\
\leq \sum_K \left( |u|_{1, 2, K} \|v\|_{0, \infty, K} + \|u\|_{0, \infty, K} |v|_{1, 2, K} \right) \|w - L_h w\|_{0, 2, K} \\
\leq (\|u\|_h + \|v\|_{0, \infty, \Omega} + \|u\|_{0, \infty, \Omega} \|v\|_h) h \|w\|_h.
\]

Using (4.16) we obtain
\begin{equation}
(4.25) \quad |Y_1| \leq C_\kappa h^{1-\kappa} \|u\|_h \|v\|_h \|w\|_h .
\end{equation}

To estimate $Y_2$ we start with the first sum $Y_{21}$ in
\[
Y_2 = \sum_{l=1}^N \int_{\Gamma_l} \left[ u_i n_l^i v_j \right]_{\Gamma_l} d\gamma w_j(B_l) + \sum_{(i,k) \in I} \int_{\Gamma_{lk}} u_i n_l^{ik} (\lambda_{lk}(u) (v_j - v_j(B_l)) \\
+ (1 - \lambda_{lk}(u)) (v_j - v_j(B_k))) w_j(B_l) d\gamma
\]
where $\Gamma_l$ denotes the edge containing the node $B_l$, $[z]_{\Gamma_l}$ the jump of $z$ along $\Gamma_l$, $n_l^i$ the unit normal vector on $\Gamma_l$ and $I$ the index set defined by (4.9). To be more specific, let $K_1$, $K_2$ be the two triangles with the common edge
Γ_l and let n' be directed outward with respect to K_l. Then, the jump is defined by \( [[z]]_{Γ_l} = z|_{K_1} - z|_{K_2} \). Obviously it holds

\[
\int_{Γ_l} [[u, n_i v_j]]_{Γ_l} dγ w_j(B_l) =
\int_{Γ_l} \left( (u_i - L_h u_i) n_i'(v_j - L_h v_j) \right)_{Γ_l} dγ w_j(B_l)
\leq \text{mes} (Γ_l) \sum_{m=1}^2 h |u|_{1, ∞, K_m} h |v|_{1, ∞, K_m} \|w\|_{0, ∞, Ω}.
\]

By means of inverse inequalities we obtain

\[
\int_{Γ_l} [[u, n_i v_j]]_{Γ_l} dγ w_j(B_l) \leq C h \|w\|_{0, ∞, Ω} \sum_{m=1}^2 |u|_{1, 2, K_m} |v|_{1, 2, K_m}.
\]

Thus, using (4.16) we can estimate

\[
Y_{21} \leq 2 C h \|w\|_{0, ∞, Ω} \sum_{K \in τ_h} |u|_{1, 2, K} |v|_{1, 2, K} \\
\leq 2 C h \|w\|_{0, ∞, Ω} \|u\|_h \|v\|_h \\
\leq C_2 h^{1-κ} \|u\|_h \|v\|_h \|w\|_h.
\]

Now, let us consider the second sum in Y_2 denotes by Y_{22}. If we take into consideration that Γ_{lk} = Γ_{kl}, \( λ_{lk}(u) = 1 - λ_{kl}(u) \) and \( n'^{lk} = -n'^{kl} \), we get

\[
Y_{22} = \frac{1}{2} \sum_{(l, k) \in I} \int_{Γ_{lk}} u, n_i^{lk}(λ_{lk}(u))(v_j - v_j(B_l)) + \\
+ (1 - λ_{lk}(u))(v_j - v_j(B_l))(w_j(B_l) - w_j(B_k)) dγ.
\]

Since u, v and w are linear on Γ_{lk}, we can estimate

\[
\left|Y_{22}\right| \leq \frac{1}{2} \sum_{(l, k) \in I} \text{mes} Γ_{lk} \|u\|_{0, ∞, s_{lk}} h |v|_{1, ∞, s_{lk}} h |w|_{1, ∞, s_{lk}} \\
\leq \frac{1}{2} \sum_{(l, k) \in I} h \|u\|_{0, ∞, s_{lk}} C_2 |v|_{1, 2, s_{lk}} C_2 |w|_{1, 2, s_{lk}} \\
\leq C h \|u\|_{0, ∞, Ω} \|v\|_h \|w\|_h,
\]

where \( C_1 \) and \( C_2 \) are again the constants of inverse inequalities which are independent of h, l and k. Together with (4.16) and the estimate for Y_{21} we receive

\[
(4.26) \quad \left|Y_2\right| \leq C_2 h^{1-κ} \|u\|_h \|v\|_h \|w\|_h.
\]

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Finally, we have to estimate $Y_3$. Using the identity
\[ b_h^2(u, v, w) = b_h^2(u, L_h v, L_h w) \quad \forall u, v, w \in V_h \]
we get
\[ Y_3 = b_h^2(u, v - L_h v, w) + b_h^2(u, L_h v, w - L_h w) \]
It is easy to check that the lumping operator $L_h$ satisfies
\[ \| L_h v \|_0^\infty_K = \| v \|_0^\infty_K \quad \forall v \in V_h, \quad K \in \tau_h \]
and
\[ \| v - L_h v \|_0^2_K \leq h \| v \|_1^2_K \quad \forall v \in V_h, \quad K \in \tau_h \]
Thus, from (4.27), the definition (4.20) of $b_h^2$ and (4.16) we obtain
\[ |Y_3| \leq \sum_K \| u \|_1^2_K h \| v \|_1^2_K \| w \|_0^\infty_K + \| u \|_1^2_K \| v \|_0^\infty_K h \| w \|_1^2_K \leq h \| u \|_h \| v \|_h \| w \|_0^\infty \Omega + h \| u \|_h \| v \|_0^\infty \Omega \| w \|_h \leq C_\kappa h^{1-\kappa} \| u \|_h \| v \|_h \| w \|_h, \]
which completes together with (4.25), (4.26) the proof of (4.17) \qed

5. EXISTENCE AND CONVERGENCE OF THE DISCRETE SOLUTIONS

In this section we study solvability of the discrete problem (3.13) and convergence properties of its solutions to a solution of the continuous problem (2.6).

It can be shown that our nonconforming finite element discretization fulfills the discrete LBB-condition, i.e. there is a constant $\alpha > 0$, independent of $h$, such that
\[ \sup_{v \in V_h} \frac{(p, \text{div} v_h)}{\| v_h \|} \leq \alpha \| p \|_0^2 \Omega \quad \forall p \in Q_h \]
Therefore, it is possible to separate the problem of finding a solution $(u_h, p_h)$ of (3.13) into one for determining $u_h$ and another one for determining $p_h$ with a known $u_h$ [4]. The discrete velocity field $u_h$ solves the problem
Find $u_h \in W_h$ such that
\[ v_a(u_h, v) + \bar{b}_h(u_h, u_h, v) = (f, v) \quad \forall v \in W_h \]
where $W_h$ denotes the space of discrete-divergence-free functions defined in (4.2).

**Theorem 1:** Assume that $f \in (L^2(\Omega))^2$. Then there exists at least one solution $(u_h, p_h) \in V_h \times Q_h$ of (3.13).

**Proof:** Let $P : W_h \to W_h$ be the mapping defined by

$$ a_h(Pv, w) = va_h(v, w) - \tilde{b}_h(v, v, w) - (f, w) $$

for all $v, w \in W_h$. Then, if $k$ is sufficiently large, from Lemma 1 we conclude for $\|v\|_h = k$

$$ a_h(Pv, v) = va_h(v, v) - (f, v) \geq \|v\|_h (\|v\|_h - C(2, \Omega) \|f\|_{0,2,\Omega}) > 0 $$

where $C(2, \Omega)$ is the constant from (4.13). In order to show the continuity of $P$ we apply Lemma 2

$$ \|Pv - Pw\|_h^2 = va_h(v - w, Pv - Pw) + \tilde{b}_h(v, v, Pv - Pw) - \tilde{b}_h(w, w, Pv - Pw) \leq (v + C(\|v\|_h + \|w\|_h)) \|v - w\|_h \|Pv - Pw\|_h $$

and obtain for bounded $v$ and $w$

$$ \|Pv - Pw\|_h \leq C \|v - w\|_h. $$

Then, by means of [11, II Lemma 1.4] we obtain the existence of at least one solution $u_h \in W_h$ of (5.2). The existence of a unique $p_h \in Q_h$ such that the pair $(u_h, p_h)$ fulfills (3.13) follows in the usual way from (5.1) [4]. $\square$

In order to study the convergence properties of the solutions $(u_h, p_h)$ of (3.13) we introduce the embedding operator $I_h : V + V_h \to (L^2(\Omega))^6$ defined on each element $K$ by

$$ (I_h v)(x) = (v(x), \text{grad } v(x)) \quad \forall x \in \hat{K}. $$

As a consequence of inequality (4.13) the embedding operator $I_h$ is continuous uniformly in $h$, i.e. there is a constant $C > 0$ such that

$$ \|I_h v\| \leq C \|v\|_h \quad \forall v \in V + V_h. $$
THEOREM 2 Let \( \{ (u_h, p_h) \} \) be a sequence of solutions of the discrete problem (3.13) where \( h \) tends to zero. Then there exists a subsequence \( \{ (u_h, p_h) \} \) and an element \((u, p)\) belonging to \( V \times Q \) such that \( I_h u_h \) converges to \((u, \text{grad} u)\) in \( (L^2(\Omega))^6 \), \( p_h \) converges to \( p \) weakly in \( L^2(\Omega) \) and the pair \((u, p)\) is a solution of the continuous problem (2.6). Moreover, if \((u, p)\) belongs to \( (H^2(\Omega))^2 \times H^1(\Omega) \) the pressure \( p_h \) converges to \( p \) also strongly in \( L^2(\Omega) \).

Proof Following the lines of Temam [11], we only have to modify some details, which result from replacing the discretization \( b_h \) defined in (3.7) by our upstream discretization \( \tilde{b}_h \) defined in (3.11). Therefore, we will only mention the important steps of the proof.

Setting \( v_h = u_h \) in (5.2) we obtain from Lemma 1 the a priori estimate
\[
\| u_h \|_h \lesssim v^{-1} C(2, \Omega) \| f \|_{0,2} \Omega
\]
By means of the discrete LBB-condition (5.1) we conclude from (3.13) and (5.4) for all \( h \)
\[
\| p_h \|_{0,2} \Omega \lesssim C
\]
such that the sequence \( \{ I_h u_h, p_h \} \) is uniformly bounded in \( (L^2(\Omega))^7 \). Consequently, we are able to select a subsequence being weakly convergent. For simplicity, we will denote this subsequence again by \( \{ (u_h, p_h) \} \). The weak limit \((u, p)\) of \( \{ (u_h, p_h) \} \) belongs to the space \( W \times Q \) (cf. [11]) where
\[
W = \{ v \in V . \text{div} v = 0 \}
\]
In order to show that \((u, p)\) is a solution of the continuous problem we introduce the restriction operator \( r_h . V \to V_h \) and \( r_h W \to W_h \), respectively, which is defined by
\[
(r_h v)(B_j) = \frac{1}{\text{mes } \Gamma_j} \int_{\Gamma_j} v \, ds
\]
and consider (3.13) for \( v \) replaced by \( r_h v \) with \( v \in (C_0^\infty(\Omega))^2 \). As in [11] it holds
\[
\begin{align*}
a_h(u_h, r_h v) &\to a(u, v) \\
(p_h, \text{div} r_h v) &\to (p, \text{div} v) \\
(f, r_h v) &\to (f, v)
\end{align*}
\]
for \( h \to 0 \) \( \forall v \in (C_0^\infty(\Omega))^2 \)
and we have to verify
\[
(5.6) \; \tilde{b}_h(u_h, u_h, r_h v) \to b(u, u, v) \quad \text{for} \; h \to 0 \; \forall v \in (C_0^\infty(\Omega))^2
\]
Analogously to the proof of Lemma 3.3 in [11, II.3) we can show that

\[(5.7) \quad b_h(u_h, u_h, r_h v) \to b(u, u, v) \quad \text{for} \quad h \to 0 \quad \forall v \in (C_0^\infty(\Omega))^2.\]

Using Lemma 3 we can estimate

\[
|b_h(u_h, u_h, r_h v) - b_h(u_h, u_h, r_h v)| \leq C \kappa \|u_h\|_h^2 \|r_h v\|_h.
\]

From (5.4) and the fact that \(I_h(r_h v - v)\) tends to zero in the norm of \((L^2(\Omega))^6\) (cf. [11]) we see that \(\|u_h\|_h\) and \(\|r_h v\|_h\) are uniformly bounded. Thus, (5.7) implies (5.6) and the weak limit \((u, p)\) fulfills

\[
va(u, v) + b(u, u, v) - (p, \text{div } v) = (f, v) \quad \forall v \in (C_0^\infty(\Omega))^2
\]

\[
(q, \text{div } u) = 0 \quad \forall q \in Q.
\]

Since \(C_0^\infty(\Omega)\) is a dense subset of \(H_0^1(\Omega)\), \((u, p)\) is a solution of the continuous problem (2.6).

Now we prove the strong convergence of \(I_h(u_h - u)\) in \((L^2(\Omega))^6\). For this we consider

\[
X_h = a_h(u_h - r_h u, u_h - r_h u) = \|u_h - r_h u\|_h^2 \geq 0.
\]

Since \(u_h\) fulfills (5.2), we obtain

\[
X_h = a_h(u_h, u_h) - 2 a_h(u_h, r_h u) + a_h(r_h u, r_h u)
\]

\[
= \frac{1}{\nu} \left\{ (f, u_h) - \tilde{b}_h(u_h, u_h, u_h) \right\} - 2 a_h(u_h, r_h u) + a_h(r_h u, r_h u)
\]

and with lemma 1

\[
X_h \leq \frac{1}{\nu} (f, u_h) - 2 a_h(u_h, r_h u) + a_h(r_h u, r_h u).
\]

The right hand side of (5.8) for \(h \to 0\) converges to

\[
\frac{1}{\nu} (f, u) - a(u, u) = \frac{1}{\nu} b(u, u, u) = 0,
\]

which implies \(\|u_h - r_h u\|_h \to 0\) for \(h \to 0\). The triangle inequality concludes the proof of the strong convergence of \(I_h(u_h - u)\) to zero in \((L^2(\Omega))^6\).

The strong convergence in \(L^2(\Omega)\) of the pressure \(p_h\) in the case \((u, p) \in (H^2(\Omega))^2 \times H^1(\Omega)\) follows from (5.1) in the following way. Multiplying the equation

\[-\nu \Delta u + u, D, u + \text{grad } p = f,
\]

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which now holds in \((L^2(\Omega))^2\), with \(v \in V_h\), integrating over \(K\), applying Green’s formula and summing-up over all finite elements \(K\) we get

\[
va_h(u, v) + b_h(u, u, v) - (p, \text{div}_h v) = (f, v) + l(v)
\]

for all \(v \in V_h\) where \(l\) is defined by

\[
l(v) = \sum_K \left( v \int_{\partial K} \frac{\partial u}{\partial n} v \, ds - \int_{\partial K} p(v \cdot n) \, ds \right).
\]

Together with (3.13) we have for each \(v \in V_h\)

\[
(p_h - p, \text{div}_h v) = va_h(u_h - u, v) + b_h(u_h, u_h, v) - b_h(u_h, u_h, v) + b_h(u_h, u_h - u, v) + l(v).
\]

Using the representation

\[
b_h(u_h, u_h, v) - b_h(u, u, v) = b_h(u_h - u, u, v) + b_h(u_h, u_h - u, v)
\]

and taking into consideration that \(u_h\) is uniformly bounded and \(b_h\) is a continuous trilinear form on \(V + V_h\) we obtain

\[
|b_h(u_h, u_h, v) - b_h(u, u, v)| \leq C \|u - u_h\|_h \|v\|_h
\]

for some positive constant \(C\) independent of \(h\). In [2] it was already shown that

\[
|l(v)| \leq Ch\|v\|_h \quad \text{for all} \quad v \in V_h.
\]

Together with lemma 3 and (5.4) it follows from (5.9)

\[
| (p - p_h, \text{div}_h v) | \leq (C \|u - u_h\|_h + C_\kappa h^{1-\kappa} + Ch) \|v\|_h.
\]

Let \(\tilde{p}_h\) be the orthogonal projection in \(L^2(\Omega)\) of \(p\) on \(Q_h\). Then by means of (5.1) we have

\[
\|\tilde{p}_h - p_h\|_{0,2,\Omega} \leq \frac{1}{\alpha} \sup_{v \in V_h} \frac{(\tilde{p}_h - p_h, \text{div}_h v)}{\|v\|_h} = \frac{1}{\alpha} \sup_{v \in V_h} \frac{(p - p_h, \text{div}_h v)}{\|v\|_h} \leq C \|u - u_h\|_h + C_\kappa h^{1-\kappa}.
\]

Thus, we get the estimate

\[
(5.10) \quad \|p - p_h\|_{0,2,\Omega} \leq \inf_{q \in Q_h} \|p - q\|_{0,2,\Omega} + C \|u - u_h\|_h + C_\kappa h^{1-\kappa}
\]
such that for the convergent subsequence \( \{u_h\} \) the associated sequence \( \{p_h\} \) converges to the solution \( p \) in \( L^2(\Omega) \).

Now, we will study the case of sufficiently large \( v \) in which the unique solvability of the problems (2.6) and (3.13) can be guaranteed and give a result concerning the rate of convergence.

**Theorem 3**: Let \( v \) be sufficiently large. Then both problems (2.6) and (3.13) have uniquely determined solutions. Moreover, if the solution \( (u, p) \) of (2.6) belongs to \( (H^2(\Omega))^2 \times H^1(\Omega) \) the error estimate

\[
\|p - p_h\|_{0,2,\Omega} + \|u - u_h\|_{h} \leq C_{\kappa} h^{1-\kappa}
\]

with an arbitrary \( \kappa \in (0,1) \) is satisfied.

**Proof**: Let \((u_1, p_1)\) and \((u_2, p_2)\) be two different solutions of (3.13). From (5.2) we have for \( v = u_1 - u_2 \in W_h \)

\[
v a_h(v, v) = \bar{b}_h(u_2, u_2, v) - \bar{b}_h(u_1, u_1, v).
\]

Applying Lemma 1 and Lemma 2 we can estimate

\[
v \|v\|_h^2 \leq C \|v\|_h^2 \|u_2\|_h.
\]

By means of the a priori estimate (5.4) it follows \( v = 0 \) if \( v^2 \|f\|_{0,2,\Omega}^{-1} \) is sufficiently large. The relation \( p_1 = p_2 \) can be easily concluded from the discrete LBB-condition (5.1). In a similar way we can also prove uniqueness of the solution of problem (2.6).

In order to prove the error estimate let us consider \( w = u_h - v \in W_h \) with an arbitrary \( v \in W_h \). Then we have

\[
v \|w\|_h^2 \leq v a_h(u_h - v, w) = v a_h(u - v, w) + v a_h(u_h - u, w)
\]

\[
\leq v \|u - v\|_h \|w\|_h + (f, w) - b_h(u, u, w) - v a_h(u, w)
\]

\[
+ b_h(u_h, u_h, w) - \bar{b}_h(u_h, u_h, w) + b_h(u, u, w) - b_h(u_h, u_h, w).
\]

We split the term \( R = b_h(u, u, w) - b_h(u_h, u_h, w) \) into

\[
R = b_h(u, u - u_h, w) + b_h(u - u_h, u_h, w)
\]

\[
\leq C (\|u\|_h + \|u_h\|_h) \|u - u_h\|_h \|w\|_h
\]

and take into consideration that \( u \) and \( u_h \) are uniformly bounded such that

\[
R \leq C v^{-2} \|u - u_h\|_h \|w\|_h.
\]
From the triangle inequality it follows
\[
\|u - u_h\|_h \leq v (\|u - v\|_h + \|w\|_h)
\]
\[
\leq 2v\|u - v\|_h + \frac{(f, w) - b_h(u, u, w) - va_h(u, w)}{\|w\|_h}
\]
\[
+ \frac{b_h(u_h, u_h, w) - \tilde{b}_h(u_h, u_h, w)}{\|w\|_h} + \frac{C}{v} \|u - u_h\|_h.
\]

Now, if \(v^2\) is greater than \(C\) we have the estimate
\[
\|u - u_h\|_h \leq C \inf_{v \in W_h} \|u - v\|_h + \sup_{w \in W_h} \frac{|(f, w) - b_h(u, u, w) - va_h(u, w)|}{\|w\|_h}
\]
\[
+ \sup_{w \in W_h} \frac{|b_h(u_h, u_h, w) - \tilde{b}_h(u_h, u_h, w)|}{\|w\|_h}
\]
and the error is decomposed into three parts, the approximation error, the discretization error caused by the nonconforming finite element method and the error due to the upstream discretization.

The estimates of the first and second error are obtained as in [2, 11]. On the third term we apply Lemma 3 and (5.10) yields the estimation for the pressure. □

Finally, we shall give a result about the algebraic system corresponding to our discrete problem (3.13). Splitting the algebraic system by means of a pressure-velocity iteration and solving the nonlinear system by a simple iteration technique we get the linear system

\[
A(u^m) \ u^{m+1} = F
\]
where \(u^m\) denotes the \(m\)-th iterate of the vector of velocity components. We will show that under a certain assumption on the triangulation the matrix \(A(u)\) is an M-matrix. To verify that \(A = (a_{ij})\) is an M-matrix it is sufficient to show that

(i) \(a_{ij} \leq 0\) for \(i \neq j\) and

(ii) \(\exists e \geq 0\) such that \(Ae \geq 0\) and for all \(i \in \{1, \ldots, n\}\) with \((Ae)_i = 0\) there exists a chain \(i_0 = i, i_1, \ldots, i_p\) such that \((Ae)_{i_p} > 0\) and \(a_{i_{q-1}, i_q} < 0\) for \(q = 1, \ldots, p\).

Let the triangulation of \(\Omega\) be of weakly acute type, i.e. the interior angles of all triangles are not greater than \(\pi/2\). Moreover, let \(\psi_i = (\varphi_i, 0)\),
ψ_{i+N} = (0, φ_i), i = 1, ..., N be the basis of $V_h$ satisfying $φ_i(B_j) = 0$ for $i \neq j$ and $φ_i(B_i) = 1$. Then the matrix $A(φ)$ in (5.12) is given by

\begin{equation}
(5.13) \quad a_{ij} = \begin{cases} \nabla_h(φ_j, φ_i) + \bar{b}_h(u, φ_j, φ_i) & i, j = 1, ..., 2N, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

**Theorem 4**: Let the triangulation of $Ω$ be of weakly acute type. Then the matrix $A(φ)$ of (5.12) defined by (5.13) is an $M$-matrix.

**Proof**: Taking into account the representation (3.11) with (3.12) we get the nonpositivity of $\bar{b}_h(u, φ_j, φ_i)$ for $i \neq j$ and the nonnegativity for $i = j$. The direction of $∇φ_i$ on a triangle $K$ corresponds to the outer normal on the boundary $∂K$ in the node $B_i$ of $K$. Therefore, $a_{i}(φ_j, φ_i)$ is nonpositive for $i \neq j$ and negative only in the case where $i, j$ are neighbour nodes and the angle between both edges with midpoints $B_i, B_j$ is smaller than $π/2$. Consequently, the assumption (i) is fulfilled. We set $e = (1, ..., 1)$ such that $(Ae)$ corresponds to the $i$-th row sum. Obviously, it follows that

\[(Ae)_i \geq 0 \quad \text{for} \quad i = 1, ..., 2N.\]

If for some $i = i_0$ $(Ae)_i = 0$ we have to construct a chain $i_0, i_1, ..., i_p$ such that $a_{i_{q-1}, i_q} < 0$, $q = 1, ..., p$. For this aim it is sufficient to show that

\begin{equation}
(5.14) \quad \int_Ω \nabla φ_{i_{q-1}} \cdot \nabla φ_{i_q} \, dx < 0 \quad q = 1, ..., p.
\end{equation}

Let $B_i$ and $B_j$ are neighbour nodes, $K$ the triangle containing these nodes and $k$ the third node of $K$. Since the triangulation is of weakly acute type we have

\begin{equation}
(5.15) \quad \int_K \nabla φ_i \cdot \nabla φ_j \, dx = 0 \Rightarrow \quad \Rightarrow \left( \int_K \nabla φ_i \cdot \nabla φ_k \, dx < 0 \text{ and } \int_K \nabla φ_k \cdot \nabla φ_j \, dx < 0 \right).
\end{equation}

Therefore, starting with an index $i_0$ not belonging to a node of a boundary triangle we can find a chain of two or three indices satisfying (5.14) and connecting $B_i$ with any of the four neighbour nodes. Continuing this procedure we come to a node $B_i, i = i_p$, of a boundary triangle $K$. Now we have

\begin{equation}
(5.16) \quad (Ae)_i \geq -ν \sum_{j \in Δ_i \atop B_j \in Γ} \int_Ω \nabla φ_j \cdot \nabla φ_i \, dx.
\end{equation}
Every integral in the above sum is nonpositive. In order to prove\((Ae)_k > 0\) we have to exclude the case that every integral in (5.16) is equal to zero. In the following let us consider this case.

For the boundary triangle \(K\) containing the node \(B_i\), we denote by \(B_j\) the boundary node and by \(B_k\) the third node. Because of (5.15), \(B_k\) does not belong to \(\Gamma\) and (5.14) holds for \(q = p + 1\) and \(t_{p+1} = k\). Therefore, let us take the chain which consists of the above chain with \(t_p = i\) and \(t_{p+1} = k\) (5.15) also implies

\[\int_\Omega \nabla \varphi_i \nabla \varphi_k \, dx < 0\]

and therefore \((Ae)_k > 0\). Consequently, the assumption (ii) holds and \(A(u)\) is an \(M\)-matrix.

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