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ON NUMERICAL SOLUTION OF A MILDLY NONLINEAR TURNING POINT PROBLEM (*)

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Abstract — *A numerical method for a mildly nonlinear singularly perturbed boundary value problem with a turning point is considered The method is based on a careful analysis of the problem and uses finite differences on a special non-equidistant discretization mesh The uniformity of the method with respect to the perturbation parameter* ε *is investigated The convergence of the numerical solution towards the discretization of the continuous solution is proved in the discrete* L^1 *norm Numerical results show that the discrete* L^1 *error decreases when* ε *does, as well as that the pointwise convergence uniform in* ε *is present*

Résumé — *Nous considérons une méthode numérique pour un problème aux limites singulièrement perturbé et faiblement non linéaire ayant un point de retournement La méthode est basée sur une analyse minutieuse du problème et utilise les différences finies sur un réseau de discrétisation non équidistant et spécial Nous examinons l'uniformité de la méthode par rapport au paramètre de perturbation* ε *La convergence de la solution numérique vers la discretisation de la solution continue est prouvée dans la norme* L^1 *discrète Les résultats numériques montrent que l'erreur* L^1 *discrete décroît en même temps que* ε *et que la convergence aux nœuds uniforme en* ε *est aussi présente*

1. INTRODUCTION

Recently some efficient methods for solving general boundary value problems for stiff linear systems of ordinary differential equations were introduced in [8] and [5]. In [8] a procedure for constructing an appropriate discretization mesh was given and finite-difference schemes were used. The paper [5] presents a combination of essentially the same mesh construction procedure and high order collocation methods. In both papers singularly

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perturbed boundary value problems with turning points were tested numerically. The results from [5] show that the methods are not uniform in the perturbation parameter ε (as ε decreases, the accuracy remains unchanged only by increasing the number of mesh points). This is quite normal for such general mesh construction procedures (and it was not the intention of these two papers to investigate the uniformity in ε).

In this paper we shall consider a special singularly perturbed boundary value problem which models problems having solutions with single turning points. We shall allow a mild nonlinearity of the problem and show that properties of the continuous solution may be used in order to obtain numerical methods uniform in ε. We shall follow the technique from [18] (*cf.* [15], [16], [17], [20] and [9], [10] as well) which uses finite-difference schemes on special non-equidistant meshes. In this paper we shall be interested in the uniformity in ε, rather than in high accuracy. However, we believe that the high order collocation methods could be used on our meshes, so that the method would be both uniform in ε and highly accurate. In other words, we expect that a step analogous to the step from [8] to [5] could be made. Another numerical method for linear turning point problems, which uses analytical information is given in [12].

Our model problem is :

$$T_\varepsilon u := -\varepsilon u'' - xb(x)u' + c(x,u) = 0, x \in I = [-1, 1], \qquad (1a)$$

$$Ru := (u(-1), u(1)) = (U_-, U_+). \qquad (1b)$$

By ε we denote the perturbation parameter : $0 < \varepsilon \leqslant \varepsilon^* \ll 1$. The functions b, c and numbers U_-, U_+ are given. Our basic assumptions are :

$$c(x,u) = xc_1(x,u) + \varepsilon c_2(x,u), \qquad (2a)$$

$$b \in C^2(I), \quad c_k(x,u) \in C^2(I \times \mathbb{R}), \quad k = 1, 2, \qquad (2b)$$

$$b(x) \geqslant b_* > 0, \quad x \in I, \qquad (2c)$$

$$|c_{k,u}(x,u)| \leqslant c^*, \quad k = 1, 2, \quad x \in I, \quad u \in \mathbb{R}, \qquad (2d)$$

$$f_u(x,u) \geqslant f_* > 0, \quad x \in I, \quad u \in \mathbb{R}, \qquad (2e)$$

where :

$$f(x,u) := (xb(x))'u + c(x,u).$$

Moreover, throughout the paper we shall assume that ε^* is sufficiently small.

Under the given conditions, the operator (T_ε, R) is inverse monotone and there exists a unique solution to the problem (1) (see [11, Theorem 9]), which will be denoted by $u_\varepsilon : u_\varepsilon \in C^4(I)$. In section 2 we give estimates of the derivatives of u_ε. In particular, we estimate the quantities :

$$\begin{aligned} &|(u_\varepsilon - u_-)^{(k)}(x)|\,, \quad k = 0, 1, 2\,, \quad x \in [-1, 0]\,, \\ &|(u_\varepsilon - u_+)^{(k)}(x)|\,, \quad k = 0, 1, 2\,, \quad x \in [0, 1]\,, \end{aligned}$$

where u_-, $u_+ \in C^3(I)$ are the solutions to the reduced problems

$$-b(x)\, u'_\pm + c_1(x, u_\pm) = 0\,, \quad u_\pm(\pm 1) = U_\pm.$$

For instance, from Lemmas 2-4 we get :

$$|(u_\varepsilon - u_+)^{(k)}(x)| = O(\mu^j + \mu^{-k} \exp(-mx/\mu)),$$

$$k = 0,\ 1,\ 2,\ x \in [0, 1],\ j = 1 \text{ for } k = 0, 1 \text{ and } j = 0 \text{ for } k = 2\,.$$

Here m is a positive constant independent of ε, and throughout the paper $\mu := \varepsilon^{1/2}$. Analogous estimates hold for u_-. Thus, the estimates show an exponential interior layer at $x = 0$. The width of the layer is $O(\mu)$ and the steepness of u_ε in the layer is of order $O(\mu^{-1})$.

Our finite-difference scheme will use u_- and u_+, and the analysis of its consistency error requires the above estimates. The numerical error estimate has a similar form : the error is estimated in a discrete L^1 norm (*cf.* [1], [14]) by the following quantity :

$$M(\mu + \exp(-n))/n\,,$$

where n is the number of mesh steps and throughout the paper M denotes any (in the sense of $O(1)$) positive constant independent of ε and of n. This result is obtained by using the upwind difference scheme, stable in the discrete L^1 norm, on a discretization mesh which is dense in the layer. The mesh is generated by a suitable function λ which redistributes equidistant points and it depends on ε in such a way that the smaller ε becomes, the more the mesh is condensed in the layer.

In section 3 we give the discrete problem corresponding to the problem (1) and prove the L^1 stability result, uniform in ε. We use the technique of M-matrices [4].

In section 4 we prove the convergence result in the discrete L^1 norm. We have in mind that it is easy to prove the linear discrete L^1 convergence uniform in ε, of the numerical solution towards the restriction of u_ε on the mesh (see [18], [19]). Thus, our aim is to improve this result by using the functions u_- and u_+ in the numerical method. Such an approach was applied in [20] to the special nonlinear turning point problem having constant reduced solutions u_- and u_+. In the same time, numerical results presented in section 5, show the pointwise convergence uniform in ε, as well. Note that it was reported in [18] that the special non-equidistant mesh alone does not guarantee the uniform pointwise convergence. Because of that u_- and u_+ are introduced to improve the weakest part of the consistency error.

The problem (1), in the special case :

$$c_2 \equiv 0, \quad c_{1,u}(x, u) \geqslant c_* > 0, \quad x \in [0, 1], \quad u \in \mathbb{R},$$

was considered in [10], where the pointwise convergence uniform in ε was shown. Note that our condition (2 e) allows for c_u to be negative and because of that we have to use the discrete L^1 norm to prove our stability result, *cf.* [14].

Let us mention some other papers where linear $(c(x, u) = d_1(x)\, u + d_2(x))$ turning point problems have been treated numerically. In [21], the authors deal with problems of type (1), investigating the ill conditioning of the corresponding exponentially fitted discretization on the equidistant mesh. Other papers usually have the assumption $d_1(x) > 0$, $x \in I$, or at least $d_1(0) > 0$, [2], [3], [6], [9], [16]. In the first three of these papers, equidistant discretizations only are considered and upwind or (exponentially) fitted schemes are used.

2. ANALYSIS OF THE CONTINUOUS PROBLEM

Throughout this section we shall assume (2) and that ε^* is sufficiently small. Some positive constants independent of ε will be denoted by m, m_1, M_0, M_1 etc. Recall that $\mu = \varepsilon^{1/2}$ and let

$$B(x) = \int_0^x sb(s)\, ds .$$

LEMMA 1 : $|u_\varepsilon(x)| \leqslant M, \quad x \in I.$

Proof : Define the linear operator :

$$L_\varepsilon u := -\varepsilon u'' - xb(x)\, u' + q(x)\, u ,$$

where

$$q(x) = \int_0^1 c_u(x, su_\varepsilon(x))\, ds .$$

The operator (L_ε, R) is inverse monotone, see [11], since from (2 e) it follows :

$$r(x) := (xb(x))' + q(x) \geqslant f_* > 0 , \quad x \in I .$$

Let $p(x)$ be a $C^2(I)$-function, such that

$$p(x) = |x| , \quad x \in I \backslash [-\mu, \mu] ,$$
$$|p(x)|, \ |p'(x)|, \ \varepsilon |p''(x)| \leq M, \quad x \in [-\mu, \mu] , \tag{3}$$

(for instance :

$$p(x) = -x^4/(8\mu^3) + 3x^2/(4\mu) + 3\mu/8 , \quad x \in [-\mu, \mu]) .$$

Furthermore, let σ denote a positive number, such that

$$\sigma b_* - c^* \geq m_1 > 0 .$$

Let

$$y_\varepsilon(x) = M_0 \exp(-\sigma p(x)) + M_1 \exp(-B(x)/\varepsilon) ,$$

where M_0, M_1 are such constants that

$$y_\varepsilon(\pm 1) \geq |U_\pm| , \tag{4a}$$

and for $x \in I$:

$$L_\varepsilon y_\varepsilon(x) \geq M_2(|x| + \varepsilon) , \tag{4b}$$

where

$$L_\varepsilon(\pm u_\varepsilon(x)) = \mp c(x, 0) \leq M_2(|x| + \varepsilon) ,$$
$$|c_1(x, 0)| + |c_2(x, 0)| \leq M_2 , \quad x \in I .$$

We choose the constants M_0 and M_1 in the following way. First let us determine M_0 so that :

$$M_0 \exp(-\sigma) \geq \max \{|U_\pm|, 4M_2/m_1\} .$$

Then, (4a) is obvious and (4b) follows for $x \in I \backslash [-\mu, \mu]$:

$$\begin{aligned} L_\varepsilon y_\varepsilon &= M_0 L_\varepsilon \exp(-\sigma p(x)) + M_1 r(x) \exp(-B(x)/\varepsilon) \\ &\geq M_0 L_\varepsilon \exp(-\sigma |x|) \\ &= M_0(-\varepsilon\sigma^2 + \sigma |x| \, b(x) + q(x)) \exp(-\sigma |x|) \\ &\geq M_0(-\varepsilon\sigma^2 + \sigma b_* |x| - (\varepsilon + |x|) c^*) \exp(-\sigma |x|) \\ &\geq M_0(m_1 |x| - \varepsilon(\sigma^2 + c^*)) \exp(-\sigma |x|) \\ &\geq M_0 m_1 |x| \exp(-\sigma)/2 \geq 2 M_2 |x| \geq M_2(|x| + \varepsilon) . \end{aligned}$$

Then choose M_1 so that :

$$M_1 r(x) \exp(-B(x)/\varepsilon) \geq M_2(|x| + \varepsilon) - M_0 L_\varepsilon \exp(-\sigma p(x)) ,$$
$$x \in [-\mu, \mu] .$$

This is possible because of (3) and

$$r(x)\exp(-B(x)/\varepsilon) \geqslant f_* \exp(-b^* x^2/(2\varepsilon)) \geqslant f_* \exp(-b^*/2)\,, \quad x \in [-\mu, \mu]\,,$$

where

$$b(x) \leqslant b^*\,, \quad x \in I\,.$$

Thus, (4) is proved and because of the inverse monotonicity of (L_ε, R), we have :

$$|u_\varepsilon(x)| \leqslant y_\varepsilon(x) \leqslant M\,, \quad x \in I\,. \quad \square$$

Let

$$I^- = [-1, 0]\,, \quad I^+ = [0, 1]\,,$$

$$v_\pm = u_\varepsilon - u_\pm\,.$$

LEMMA 2 : $|v_\pm(x)| \leqslant M(\mu + \exp(-B(x)/\varepsilon))\,, \quad x \in I^\pm.$

Proof : The proof is similar to the proof of Lemma 1. Let

$$L_\varepsilon^\pm u := -\varepsilon u'' - xb(x)u' + q_\pm(x)u\,,$$

where

$$q_\pm(x) = \int_0^1 c_u(x, u_\pm(x) + s(u_\varepsilon - u_\pm)(x))\,ds\,.$$

We have

$$L_\varepsilon^+ v_+ = \varepsilon u_+'' - \varepsilon c_2(x, u_+)\,, \quad x \in I^+\,. \tag{5}$$

Let

$$z_\varepsilon(x) = \mu M_3 \exp(-\sigma x) + M_4 \exp(-B(x)/\varepsilon)\,,$$

where σ is the same as in the proof of Lemma 1. Again, M_3 and M_4 can be chosen in such a way that

$$L_\varepsilon^+ z_\varepsilon(x) \geqslant M_5\,\varepsilon \geqslant L_\varepsilon^+(\pm v_+)\,, \quad x \in I^+\,,$$

$$z_\varepsilon(0) \geqslant |v_+(0)|\,, \quad z_\varepsilon(1) \geqslant 0\,.$$

We can take

$$M_3 \geqslant 2\,M_5 \exp(\sigma)/m_1\,,$$

so that (see the proof of Lemma 1) :

$$\begin{aligned} L_\varepsilon^+ z_\varepsilon(x) &\geqslant \mu M_3\, m_1\, x \exp(-\sigma)/2 \\ &\geqslant M_3\, m_1\, \varepsilon \exp(-\sigma)/2 \geqslant M_5\,\varepsilon\,, \quad x \in [\mu, 1]\,. \end{aligned}$$

Then it is possible to choose M_4 in such a way that

$$M_4 \geq |v_+(0)| - M_3 \mu$$

(note that $|v_+(0)| \leq M$ because of Lemma 1), and that

$$M_4 r(x) \exp(-B(x)/\varepsilon) \geq M_5 \varepsilon - \mu M_3 L_\varepsilon^+ \exp(-\sigma x), \quad x \in [0, \mu].$$

Then because of inverse monotonicity, we get

$$|v_+(x)| \leq z_\varepsilon(x) \leq M(\mu + \exp(-B(x)/\varepsilon)), \quad x \in I^+.$$

Analogously, we consider L_ε^- on the interval I^- and prove the rest of the lemma. □

LEMMA 3:

$$|v'_\pm(x)| \leq M(\mu + (x^2/\varepsilon + 1/\mu) \exp(-B(x)/\varepsilon)), \quad x \in I^\pm.$$

Proof: Again, we shall give the proof for $v_+(x)$, $x \in I^+$, only, since the rest can be proved analogously. For the technique *cf.* [7].

First let us rewrite (5) in the form:

$$-\varepsilon v''_+ - (xb(x) v_+)' + (xb(x) + q_+(x)) v_+ = \varepsilon(u''_+(x) - c_2(x, u_+(x)))$$

and integrate this equality from 0 to x_*, where $x_* \in (0, \mu)$ is such a point that

$$v'_+(x_*) = (v_+(\mu) - v_+(0))/\mu,$$

hence

$$|v'_+(x_*)| \leq M/\mu.$$

Then it follows:

$$|v'_+(0)| \leq M/\mu.$$

Now from (5) we get

$$v'_+(x) = \{\varepsilon^{-1} \int_0^x [(q_+ v_+)(t) + \varepsilon(c_2(t, u_+(t)) - u''_+(t))] \exp(B(t)/\varepsilon)\, dt + v'_+(0)\} \exp(-B(x)/\varepsilon).$$

Then from Lemma 2 and $|q_+(t)| \leq M(\varepsilon + t)$, it follows:

$$|v'_+(x)| \leq M(S_1 + S_2 + \mu^{-1} \exp(-B(x)/\varepsilon)),$$

where

$$\begin{aligned} S_1 &= \int_0^x (1 + t/\mu) \exp[(B(t) - B(x))/\varepsilon]\, dt \\ &\leq \int_0^x (1 + t/\mu) \exp[b_*(t^2 - x^2)/(2\,\varepsilon)]\, dt \\ &\leq \int_0^x \exp[b_*\, x(t - x)/(2\,\varepsilon)]\, dt \\ &\quad + \mu^{-1} \int_0^x t \exp[b_*(t^2 - x^2)/(2\,\varepsilon)]\, dt \\ &\leq M\mu , \end{aligned}$$

and

$$\begin{aligned} S_2 &= \exp(- B(x)/\varepsilon) \int_0^x (1 + t/\varepsilon)\, dt \\ &\leq M(x + x^2/\varepsilon) \exp(- B(x)/\varepsilon) \\ &\leq M(\mu + (x^2/\varepsilon) \exp(- B(x)/\varepsilon)) . \quad \square \end{aligned}$$

LEMMA 4 :

$$|v''_\pm(x)| \leq M(1 + (|x|/\varepsilon)(x^2/\varepsilon + 1/\mu) \exp(- B(x)/\varepsilon)) ,$$
$$|xv''_\pm(x)| \leq M(\mu + (x^2/\varepsilon)(x^2/\varepsilon + 1/\mu) \exp(- B(x)/\varepsilon)) , \quad x \in I^\pm .$$

Proof : Let $x \in I^+$ (the case $x \in I^-$ is similar). Differentiate (5) and obtain :

$$\varepsilon v_+^{(3)} + xb(x)\, v''_+ = s(x) , \tag{6}$$

where by previous lemmas we have

$$|s(x)| \leq M(\mu + (x^2/\varepsilon + 1/\mu) \exp\,(- B(x)/\varepsilon)) . \tag{7}$$

Then from (6) it follows :

$$|v''_+(x)| \leq M\left[1 + \varepsilon^{-1} \int_0^x |s(t)| \exp(B(t)/\varepsilon)\, dt\right] \exp(- B(x)/\varepsilon) ,$$

(note that from (5) we have $|v''_+(0)| \leq M$). Now, using (7) and the technique from the proof of Lemma 3, we can complete the proof. $\square$

LEMMA 5 :

$$|u_\varepsilon^{(3)}(x)| \leq M[1/\mu + (|x|/\varepsilon)(|x|^3/\varepsilon^2 + x^2/\varepsilon + $$
$$+ |x|/\mu^3 + 1/\mu) \exp(- B(x)/\varepsilon)] , \quad x \in I .$$

Proof: Differentiate (1*a*) twice and express $u_\varepsilon^{(3)}$. Then use the estimates of u_ε' and u_ε'', which follow from Lemmas 3 and 4, and the same technique, to prove the assertion. □

From Lemmas 2-5 we derive simpler estimates, which will be sharp enough to use them in the consistency error analysis, in section 4. Let

$$V_\varepsilon(x) = \exp(-m|x|/\mu),$$

where $m > 0$ is an arbitrary constant independent of ε. We have:

THEOREM:

$$|(xv_\pm(x))'| \leq M(\mu + V_\varepsilon(x)), \qquad x \in I^\pm; \tag{8a}$$
$$|(xv_\pm(x))''| \leq M(\mu + \mu^{-1} V_\varepsilon(x)), \quad x \in I^\pm; \tag{8b}$$
$$\varepsilon|u_\varepsilon''(x)| \leq M(\varepsilon + V_\varepsilon(x)), \qquad x \in I; \tag{8c}$$
$$\varepsilon|u_\varepsilon^{(3)}(x)| \leq M(\mu + \mu^{-1} V_\varepsilon(x)), \quad x \in I. \tag{8d}$$

Proof: Let us illustrate the proof by showing the last inequality. From Lemma 5 it follows:

$$\begin{aligned}\varepsilon|u_\varepsilon^{(3)}(x)| &\leq M[\mu + |x|(|x|^3/\varepsilon^2 + x^2/\varepsilon \\ &\quad + |x|/\mu^3 + 1/\mu)\exp(-b_* x^2/(2\varepsilon))] \\ &\leq M[\mu + \mu^{-1}\exp(-b_* x^2/(4\varepsilon))] \\ &\leq M[\mu + \mu^{-1} V_\varepsilon(x)]. \quad \square\end{aligned}$$

3. THE DISCRETIZATION AND ITS STABILITY

Let I^h be the discretization mesh with the mesh points:

$$x_i = \lambda(t_i), \quad t_i = -1 + 2i/n, \quad i = 0, 1, \ldots, n,$$
$$n = 2n_0, \quad n_0 \in \mathbb{N},$$

where

$$\lambda(t) = \begin{cases} \omega(t) := \beta\mu t/(\gamma - t), & t \in [0, \alpha] \\ \pi(t) := \delta(t-\alpha)^3 + \omega''(\alpha)(t-\alpha)^2/2 + \\ \qquad + \omega'(\alpha)(t-\alpha) + \omega(\alpha), & t \in [\alpha, 1] \\ -\lambda(-t), & t \in [-1, 0]. \end{cases}$$

Here $\alpha \in (0, 1)$ is an arbitrary parameter (independent of ε),

$$\gamma = \alpha + \mu^{1/3},$$

and δ is determined from:

$$\pi(1) = 1 .$$

We have:

$$\lambda \in C^2(I^{\pm}), \quad \lambda \in C^1(I),$$

and

$$\omega(\alpha) = \alpha\beta\mu^{2/3}, \quad \omega'(\alpha) = \beta\gamma\mu^{1/3}, \quad \omega''(\alpha) = 2\beta\gamma . \tag{9}$$

The parameter β should satisfy:

$$0 < \beta \leqslant \gamma^{-1}(1-\alpha)^{-2},$$

which implies

$$\delta \geqslant 0, \quad \text{i.e.} \quad \pi^{(3)} \geqslant 0,$$

provided ε^* be sufficiently small (see (9)). Then it follows:

$$\pi^{(k)}(t) \geqslant \pi^{(k)}(\alpha) = \omega^{(k)}(\alpha) > 0, \quad t \in [\alpha, 1],$$

first for $k = 2$ and then for $k = 1$. Obviously:

$$\omega^{(k)}(t) > 0, \quad k = 0, 1, \ldots, t \in [0, \alpha],$$

and taking (9) into account we get:

$$0 < \lambda^{(k)}(t) \leqslant M, \quad k = 1, 2, \quad t \in I^{+} . \tag{10a}$$

Furthermore, note the inequality:

$$\exp(-\omega(t)/\mu) \leqslant M \exp(-M/(q-t)), \quad t \in [0, q], \tag{10b}$$

which will be used in section 4.

It is easy to derive analogous properties of the function λ in I^-. The same function λ was used in [18] and a very similar in [17] (see mesh generating functions in [15], [16] and [20] as well). Essentially, the part ω is a certain modification of the inverse of the interior layer function $V_\varepsilon(x)$ for $x \geqslant 0$ (*cf.* [15]), and π is merely its continuous extension.

Let

$$h_i = x_i - x_{i-1}, \qquad i = 1, 2, \ldots, n;$$
$$\bar{h}_i = (h_i + h_{i+1})/2, \quad i = 1, 2, \ldots, n-1 .$$

We shall discretize the following forms of the equation (1*a*):

$$-\varepsilon u'' - g_{\pm}(x, u)' + s_{\pm}(x, u) = 0, \tag{11}$$

where

$$g_{\pm}(x, u) = xb(x)(u - u_{\pm}(x)),$$
$$s_{\pm}(x, u) = f(x, u) - (xb(x) u_{\pm}(x))'$$
$$= (xb(x))' (u - u_{\pm}(x)) + c(x, u) - xc_1(x, u_{\pm}(x)).$$

Let w^h denote a mesh function on $I^h \setminus \{-1, 1\}$, which will be identified with the vector:

$$w^h = [w_1, w_2, \dots, w_{n-1}]^T \in \mathbb{R}^{n-1}, \quad (w_i := w_i^h),$$

and let $T^h = T^h_\varepsilon$ be the discrete operator corresponding to (11):

$$T^h : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1},$$

$$T^h w_i := (T^h w^h)_i = \begin{cases} T^h_- w_i, & i = 1, 2, \dots, n_0 \\ T^h_+ w_i, & i = n_0 + 1, \dots, n - 1, \end{cases}$$

$$T^h_{\pm} w_i = -\varepsilon D'' w_i - D'_{\pm} g_{\pm}(x_i, w_i) + s_{\pm}(x_i, w_i),$$

where

$$D'' w_i = [(w_{i-1} - w_i)/h_i + (w_{i+1} - w_i)/h_{i+1}]/\bar{h}_i,$$
$$D'_{\pm} w_i = \pm (w_{i \pm 1} - w_i)/\bar{h}_i,$$

cf. [1] for $D'_{\pm}$. Of course, the quantities w_0 and w_n should be replaced by U_- and U_+, respectively. (Instead of T^h at $x_{n_0} = 0$ it is possible to use T^h_+, as well as $(T^h_- + T^h_+)/2$.)

Thus, the discrete problem reads:

$$T^h w^h = 0. \tag{12}$$

Let $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote the usual vector (matrix) norms in $\mathbb{R}^{n-1}$ ($\mathbb{R}^{n-1, n-1}$). Furthermore, in $\mathbb{R}^{n-1}$ we shall use the following discrete L^1 norm (*cf.* [1]):

$$\|w^h\|_1^h = \sum_{i=1}^{n-1} \bar{h}_i |w_i|,$$

which can be written down in the form:

$$\|w^h\|_1^h = \|Hw^h\|_1, \quad H = \operatorname{diag}(\bar{h}_1, \bar{h}_2, \dots, \bar{h}_{n-1}).$$

The corresponding matrix norm is:

$$\|A\|_1^h = \|HAH^{-1}\|_1, \quad A \in \mathbb{R}^{n-1, n-1}.$$

Now we shall prove the stability inequality :

$$\| w^h - z^h \|_1^h \leq f_*^{-1} \| T^h w^h - T^h z^h \|_1^h , \tag{13}$$

which is valid for all w^h, $z^h \in \mathbb{R}^{n-1}$.

THEOREM 2 : *Let* (2*b*, *c*, *e*) *hold. Then we have* (13) *and there exists a unique solution* $w_\varepsilon^h \in \mathbb{R}^{n-1}$ *to the discrete problem* (12).

Proof : Let

$$A_h = (T^h)' (w^h) ,$$

where $(T^h)' (w^h)$ denotes the Frechet derivative of T^h at any w^h. It is easy to see that A_h is an L-matrix (the diagonal elements are positive and the off-diagonal elements are non-negative). Furthermore :

$$(HA_h H^{-1})^T e^h \geq f_* e^h ,$$

where $e^h = [1, 1, ..., 1]^T \in \mathbb{R}^{n-1}$. (This inequality should be understood componentwise.) Thus A_h is an M-matrix ($A_h^{-1} \geq 0$, see [4]) and we get :

$$\| A_h^{-1} \|_1^h = \| ((HA_h H^{-1})^T)^{-1} \|_\infty \leq f_*^{-1} .$$

This guarantees that w_ε^h exists uniquely (see the Hadamard's Theorem in [13]), and (13) follows from :

$$w^h - z^h = ((T^h)' (\theta^h))^{-1} (T^h w^h - T^h z^h) ,$$

which is valid with some $\theta^h \in \mathbb{R}^{n-1}$. □

4. THE CONVERGENCE RESULT

Let us consider the consistency error

$$r_i := T^h u_\varepsilon(x_i) - (T_\varepsilon u_\varepsilon)(x_i) , \quad i = 1, 2, ..., n-1 .$$

We have

$$\begin{aligned} r_i &= r_i'' + r_i' , \\ r_i'' &= \varepsilon [u_\varepsilon''(x_i) - D'' u_\varepsilon(x_i)] , \\ r_i' &= g_\varepsilon'(x_i) - D_-' g_\varepsilon(x_i) , \qquad i = 1, 2, ..., n_0 , \\ r_i' &= g_\varepsilon'(x_i) - D_+' g_\varepsilon(x_i) , \qquad i = n_0 + 1, ..., n-1 , \end{aligned}$$

where

$$g_\varepsilon(x) = g_\pm(x, u_\varepsilon(x)), \quad x \in I^\pm,$$

(at $x_{n_0} = 0$ the derivatives of g_ε should be taken from the left).

Let w_ε^h denote the solution to the discrete problem (12), let

$$u_\varepsilon^h = [u_\varepsilon(x_1), u_\varepsilon(x_2), \dots, u_\varepsilon(x_{n-1})]^T,$$

and

$$r^h = [r_1, r_2, \dots, r_{n-1}]^T.$$

We have :

THEOREM 3 : *Let* (2) *hold and let* ε^* *be sufficiently small. Then :*

$$\|w_\varepsilon^h - u_\varepsilon^h\|_1^h \leq Md,$$

where

$$d = (\mu + \exp(-n))/n.$$

Proof : Because of the stability inequality (13) it is sufficient to prove :

$$\|r^h\|_1^h \leq Md.$$

We shall consider r_i for $i = n_0 + 1, \dots, n-1$ only, since the analysis for $i = 1, 2, \dots, n_0$ is analogous. Thus, we shall prove :

$$\sum_{i=n_0+1}^{n-1} \bar{h}_i |r_i| \leq Md. \tag{14}$$

The following estimates hold :

$$|r_i''| \leq M\varepsilon h_{i+1} |u'''(\sigma_i'')|, \ \sigma_i'' \in (x_{i-1}, x_{i+1}), \tag{15a}$$

$$|r_i''| \leq 2\varepsilon \max_{x_{i-1} \leq x \leq x_{i+1}} |u''(x)|, \tag{15b}$$

$$|r_i'| \leq Q_i + G_i/h_{i+1}, \tag{16}$$

where

$$Q_i = ((h_{i+1} - h_i)/(h_i + h_{i+1})) |g_\varepsilon'(\sigma_i')|,$$

$$\sigma_i' \in (x_i, x_{i+1}),$$

and

$$G_i = \int_{x_i}^{x_{i+1}} (x_{i+1} - x) |g_\varepsilon''(x)| \, dx.$$

The proof of (14) is divided into three steps (*cf.* [15], [16], [17], [18], [20]) :

1° $t_{i-1} \geq \alpha - \mu^{1/3}$,

2° $t_{i-1} < \alpha - \mu^{1/3}$ and $t_{i-1} \leq \alpha - 6/n$,

3° $\alpha - 6/n < t_{i-1} < \alpha - \mu^{1/3}$.

In the steps 1° and 2° we shall prove :

$$h_{i+1}|r_i''| \leq M\mu/n^2, \tag{17a}$$

$$h_{i+1}|r_i'| \leq M\mu/n^2. \tag{17b}$$

1° From (15*a*), (10) and (8*d*) we have :

$$\begin{aligned} h_{i+1}|r_i''| &\leq Mn^{-2}[\mu + \mu^{-1} V_\varepsilon(x_{i-1})] \\ &\leq Mn^{-2}[\mu + \mu^{-1} V_\varepsilon(\omega(\alpha - \mu^{1/3}))] \leq M\mu/n^2, \end{aligned}$$

hence, (17*a*) holds in this case. Similarly, from (16), (10) and (8*a*, *b*) it follows :

$$\begin{aligned} h_{i+1} Q_i &\leq Mn^{-2}[\mu + V_\varepsilon(x_{i-1})] \leq M\mu/n^2, \\ h_{i+1} G_i &\leq Mh_{i+1}^2[\mu + \mu^{-1} V_\varepsilon(x_{i-1})] \leq M\mu/n^2, \end{aligned}$$

and (17*b*) holds as well.

2° Now we have

$$\alpha - t_{i+1} \geq (\alpha - t_{i-1})/3,$$

and thus

$$\gamma - t_{i+1} > (\gamma - t_{i-1})/3.$$

Using this and (15*a*), (10) and (8*d*) again, we get :

$$\begin{aligned} h_{i+1}|r_i''| &\leq Mn^{-2}\omega'(t_{i+1})^2[\mu + \mu^{-1} V_\varepsilon(x_{i-1})] \\ &\leq Mn^{-2}\mu[1 + (\gamma - t_{i+1})^{-4} V_\varepsilon(\omega(t_{i-1}))] \\ &\leq Mn^{-2}\mu[1 + (\gamma - t_{i-1})^{-4} V_\varepsilon(\omega(t_{i-1}))] \leq M\mu/n^2. \end{aligned}$$

In the same way :

$$\begin{aligned} h_{i+1} Q_i &\leq Mn^{-2}\omega''(t_{i+1})[\mu + V_\varepsilon(x_{i-1})] \\ &\leq Mn^{-2}\mu[1 + (\gamma - t_{i+1})^{-3} V_\varepsilon(\omega(t_{i-1}))] \leq M\mu/n^2, \\ h_{i+1} G_i &\leq Mn^{-2}[\mu + \mu^{-1} V_\varepsilon(x_{i-1})] \leq M\mu/n^2. \end{aligned}$$

3° This case occurs at most at two points. Because of that it is sufficient to show

$$h_{i+1}|r_i''| \leq Md, \tag{18a}$$
$$h_{i+1}|r_i'| \leq Md. \tag{18b}$$

Together with (17) this will imply (14).

In this case it holds that

$$\mu^{1/3} < 6/n.$$

We use this, (10) and (8*c*) to get :

$$\begin{aligned} h_{i+1}|r_i''| &\leq Mn^{-1}[\varepsilon + V_\varepsilon(x_{i-1})] \\ &\leq Mn^{-1}[\varepsilon + V_\varepsilon(\omega(\alpha - 6/n))] \\ &\leq Mn^{-1}[\varepsilon + \exp(-m\beta\gamma/(\mu^{1/3} + 6/n))] \\ &\leq Mn^{-1}[\varepsilon + \exp(-m\beta\gamma n/12)]. \end{aligned}$$

Now choose an appropriate m to get (18*a*). On the other hand, from (10) and (8*a*) it follows :

$$\begin{aligned} h_{i+1} Q_i &\leq Mn^{-2}[\mu + V_\varepsilon(x_{i-1})] \\ &\leq Mn^{-2}[\mu + \exp(-n)], \end{aligned}$$

and

$$h_{i+1} G_i \leq M \int_{x_i}^{x_{i+1}} (x_{i+1} - x)[\mu + \mu^{-1} V_\varepsilon(x)]\, dx.$$

After integration we have

$$\begin{aligned} h_{i+1} G_i &\leq M[\mu/n^2 + n^{-1} V_\varepsilon(x_i)] \\ &\leq M[\mu/n^2 + n^{-1}\exp(-n)], \end{aligned}$$

thus (18*b*) is proved. □

Remark : The case 3° of the proof of the previous theorem seems to be technical and not essential. It fills the gap between the cases 1° and 2°. A more natural result, which is confirmed by numerical results in the next section, is :

$$\| w_\varepsilon^h - u_\varepsilon^h \|_1^h \leq M\mu/n,$$

but we are unable to prove it.

5. NUMERICAL RESULTS

We consider the test example from [5] :

$$-\varepsilon u'' - xu' - \pi x \sin(\pi x) - \varepsilon\pi^2 \cos(\pi x) = 0, \quad x \in I,$$
$$u(-1) = -2, \quad u(1) = 0.$$

The exact solution and the solutions to the reduced problems are given by

$$u_\varepsilon(x) = \cos(\pi x) + \operatorname{erf}[x/(2\varepsilon)^{1/2}]/\operatorname{erf}[(2\varepsilon)^{-1/2}],$$

(erf is the error function),

$$u_-(x) = \cos(\pi x) - 1,$$
$$u_+(x) = \cos(\pi x) + 1.$$

When it is not possible to find u_- and u_+ exactly, we have to solve the reduced problems numerically, and to use approximate values of $u_-(x_i)$, $i = 1, 2, \dots, n_0$, and $u_+(x_i)$, $i = n_0 + 1, \dots, n - 1$.

TABLE 1

$\alpha = 0.8$, $P = 20\,\%$.

ε	n	50	100	150
1.- 2	E_∞	2.81-2	1.25-2	8.02-3
	E_1	6.32-3	3.06-3	2.03-3
1.- 4	E_∞	2.80-2	1.25-2	7.98-3
	E_1	6.80-4	3.23-4	2.13-4
1.- 6	E_∞	2.80-2	1.24-2	7.98-3
	E_1	6.40-5	3.19-5	2.13-5
1.- 8	E_∞	2.79-2	1.24-2	7.99-3
	E_1	6.39-6	3.21-6	2.16-6
1.-10	E_∞	2.78-2	1.24-2	8.00-3
	E_1	6.43-7	3.24-7	2.18-7
1.-12	E_∞	2.78-2	1.24-2	8.00-3
	E_1	6.57-8	3.37-8	2.27-8

(As usual, 1.-2 means 10^{-2} etc.)

In the tables we present the errors :

$$E_\infty := \| w_\varepsilon^h - u_\varepsilon^h \|_\infty, \quad E_1 := \| w_\varepsilon^h - u_\varepsilon^h \|_1^h,$$

for different values of n and ε. The numerical results confirm the theoretical ones, and, moreover, the pointwise convergence, uniform in ε, can be observed.

Let P denote the percentage of the intervals $[x_i, x_{i+1}]$, which lie within $[-\mu, \mu]$. By changing the mesh generating function parameters α and β we can change P. However, for α and β given, P changes slightly when ε and n do. In order to avoid this, we use the following procedure : we choose P and α, and then we change β so that P remains fixed for all ε and n, *cf.* [19].

Comparing E_∞ with the results from [5] we can conclude that our method is better with respect to the uniformity in ε.

TABLE 2

$\alpha = 0.8$, $P = 40\ \%$.

ε	n	50	100	150
1.- 2	E_∞ E_1	1.24-2 4.69-3	5.96-3 2.03-3	3.92-3 1.30-3
1.- 4	E_∞ E_1	1.27-2 5.37-4	6.18-3 2.36-4	4.07-3 1.51-4
1.- 6	E_∞ E_1	1.36-2 4.71-5	6.51-3 2.33-5	4.28-3 1.55-5
1.- 8	E_∞ E_1	1.41-2 4.83-6	6.72-3 2.43-6	4.41-3 1.63-6
1.-10	E_∞ E_1	1.44-2 4.96-7	6.83-3 2.51-7	4.49-3 1.69-7
1.-12	E_∞ E_1	1.45-2 5.04-8	6.89-3 2.66-8	4.53-3 1.76-8

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