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PARAMETRIC REPRESENTATION OF QUADRIC SURFACES

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Abstract. — This paper briefly considers some cases where triangular and quadrangular quadratic Bézier patches represent a quadric, which are used in geometric modeling. A detailed discussion may be found in [3] and [8].


INTRODUCTION

Quadric surfaces, such as circular cylinders, cones, and spheres, are frequently used in geometric modeling. On the other hand, in free-form design with predominantly parametric Bernstein-Bézier representations, implicitly defined quadrics are not used frequently. Therefore, as a step toward integrating quadrics surfaces into free-form design, this paper examines the parametric representation of quadrics with quadratic and biquadratic patches from a geometric point of view.

QUADRATIC BÉZIER PATCHES AND QUADRICS

An integral triangular quadratic $B$-patch, $b(u)$ where $u = (u, v, w)$ denotes barycentric coordinates in the domain, is controlled by six $B$-points, $b_{0,0}$, $b_{0,1}$, $b_{0,2}$, $b_{1,1}$, $b_{2,0}$, $b_{1,0}$. An integral rectangular biquadratic $B$-patch, $b(s)$, where $s = (s, t)$ denotes affine coordinates in the domain, is controlled by nine $B$-points, $b_{0,0}$, ..., $b_{2,2}$ [2, 4], as illustrated in figure 1.

In the case of a rational patch, each $B$-point $b_{i,k}$ has associated with it a weight $\beta_{i,k}$. The boundaries of the quadratic and biquadratic patches are conic sections, thus each is defined by three $B$-points. Note that the degree

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of a triangular quadratic $B$-patch is four or less, and the degree of a biquadratic patch is eight or less \[9\], therefore the $B$-points of quadratic and biquadratic patches on a quadric are not independent.

A quadric $Q$ can be defined as a surface which carries two families of straight lines, the so-called generatrices. If both families are real and different, the quadric is said to be doubly ruled or annular; it is a hyperboloid of one sheet or a hyperbolic paraboloid. If both families are non-real but different, the quadric is said to be non-ruled or oval; it is an ellipsoid, a hyperboloid of two sheets, or an elliptic paraboloid. In the special case, where both families coincide to one family, the quadric is singly ruled and degenerates to a quadratic cone or a cylinder.

Any plane section of a quadric is a conic section. Note that this conic section is not necessarily real, or if the plane is a tangent plane it decomposes into a pair of coinciding or different lines. Any conic section which contacts the infinite plane is a parabola. Any quadric which contacts the infinite plane is a paraboloid. Let $a$ denote such a point of contact at infinity. Viewed as a vector, it is called an axis direction of the parabola or the paraboloid under consideration [1].

A conic section is determined by five points or five tangents in the plane which may coincide in pairs, thus being a point with a tangent. Therefore six points or tangents are dependent, they are related by the configuration of Pascal and Brianchon [11].

A quadric is defined by nine points in space, where two or three may coincide to a point with tangent or to a point with tangent plane, respectively. However, three tangent planes at three points of $Q$ are related by Brianchon's configuration in the plane spanned by the three points.

Finally, any irreducible quadric as a whole can be viewed as the real projection of one of the three paraboloids shown in figure 2, below. More interesting properties of quadrics may be found in [1, 3].
INTEGRAL TRIANGULAR PATCHES ON A QUADRIC

All integral quadratic curves represent parabolas. Let $b_0$, $b_1$, $b_2$ denote the $B$-points of such a parabola, and its axis direction is $a = b_2 - 2b_1 + b_0$ [1, 3, 7]. Now consider an integral triangular quadratic $B$-patch, $b(u)$. Its three boundary curves are parabolas, and thus the quadric $\mathcal{Q}$ under consideration must be a paraboloid. Assuming that $\mathcal{Q}$ is an elliptic or a hyperbolic paraboloid, any parabola lies in a plane parallel to the axis. Consequently, the axes of the three boundary parabolas must be parallel to each other. In effect, this gives conditions on the $B$-points, as illustrated in figure 3.

Figure 3. — The general integral quadratic patch.
There is a simple method for classifying the paraboloid. As in figure 3, let $\tau$ denote the tangent plane at $b_0$, and let $C$ denote the opposite boundary parabola. If $C$ intersects $\tau$ in two different real points, in two non-real points, or $\tau$ is tangent to $C$, the quadric is a hyperbolic paraboloid, an elliptic paraboloid, or a parabolic cylinder, respectively.

Now assuming the quadric above is degenerate, it is a parabolic cylinder. Let $d$ denote the direction from $b_0$ to the point of contact of $C$ and $\tau$. Any translation of the $B$-points in the direction of $d$ will not effect the parabolic cylinder. Hence, the axes of all parabolas are parallel to a plane spanned by $a$ and $d$.

**RATIONAL QUADRATIC TRIANGULAR PATCHES**

Any rational $B$-patch in $\mathbb{R}^3$ can be defined as the projection of an integral $B$-patch in $\mathbb{R}^4$ [1, 4]. This projection can easily be realized by the simple procedure of inhomogeneizing.

Moreover, any rational triangular quadratic patch on a non-degenerate quadric $Q$ can be viewed as the projection of an integral triangular patch on a non-degenerate paraboloid.

As a corollary, a rational triangular quadratic patch lies on a quadric if the three boundaries meet in one point, $q$, where their three tangents, $U$, $V$, $W$, are coplanar, and $q$ corresponds three times to the parameter value $\infty$, as illustrated in figure 4 [10]. An example where these conditions are violated and a quadratic patch is not defined is discussed in [5]. A rational quartic patch is needed to represent an octant of the sphere.

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**Figure 4. — The rational triangular patch on a quadric**
Analogously one has that any rational triangular quadratic patch on a cone or cylinder, can be viewed as the projection of an integral patch on a parabolic cylinder, as illustrated in figure 5. Note that only in a special case do all three boundaries meet in a point \( q \). Note also that each of the three boundaries have to contact the opposite tangent plane. In effect this gives conditions on the \( B \)-points of the net and their weights.

![Figure 5. — The rational triangular patch on a cone.](image)

**A ONE TO ONE CORRESPONDENCE**

There is a simple one to one correspondence between finite points \( u \) in the domain and the finite points \( b(u) \) of the nondegenerate paraboloids, as illustrated in figure 3. This correspondence can be generalized by perspectivity, and is also known as a stereographic projection.

Consider a triangular patch on a quadric \( \mathcal{Q} \) and let \( q \) denote the point where the three boundary curves meet. Let \( \rho \) denote the tangent plane of \( \mathcal{Q} \) at \( q \), and let \( \sigma \) be a plane parallel to \( \rho \), as in figure 6. Consider the projection which projects the points of \( \mathcal{Q} \) not lying in \( \rho \) from \( q \) onto \( \sigma \). The \( \infty^2 \) lines of \( \sigma \) correspond to the \( \infty^2 \) conic sections through \( q \) and vice versa. Note that such a one to one correspondence is a necessary and sufficient condition for a quadratic triangular patch to lie on a non-degenerate quadric [9].
QUADRANGULAR FROM TRIANGULAR PATCHES

Any integral quadratic triangular patch can easily be extended to a biquadratic patch by the construction of the $B$-points of the isolines $u = 1$ and $v = 1$ via the de Casteljau algorithm [2, 4]. Note that all isolines are parabolas too. Moreover, all isolines of the same family are parallel and congruent in space. The interior $B$-point must be adjusted so that the twist of the rectangular patch is identical to the twist of the triangular patch.

The extension to a rational patch is shown in figure 8. The boundaries $u = 0$ and $u = 1$ have a common tangent $U$ at $q$, and similarly the boundaries $v = 0$ and $v = 1$ have a common tangent $V$ at $q$. Note that a final linear rational change of the parametrization will not change the surface.
This can be generalized. Let the barycentric coordinates $u$ be expressed as the bilinear interpolant, $s$, of four points, $u_1, ..., u_4$, in the $u$-plane, as in figure 8. This is done by substituting into $s$, the barycentric definitions of $u_1, ..., u_4$ and rearranging. Now $u$ has been expressed bilinearly in terms of $s$. Substituting this expression of $u$ into the representation of the quadratic patch $b(u)$ one gets a biquadratic patch in $s$.

In a simple case, the fourth side of the rectangle can be constructed by a corner cutting construction of the triangle, as illustrated in figure 9. The fourth boundary also meets $q$. 

vol. 26, n° 1, 1992
Figure 10. — The rational bi-quadratic patch.

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THE RATIONAL BIQUADRATIC PATCH

A simple example of a rational biquadratic patch can be constructed on the sphere, where the isoparametric lines are formed by parallels and meridians. As before, its projection gives a rational biquadratic patch, illustrated in figure 10.

Let the planes through $s = 0$ and $s = 1$ intersect in $S$, and let $S_0$ and $S_1$ denote the poles of these planes with respect to $\mathcal{Q}$, both determining an axis $T$. The planes through $t = 0$ and $t = 1$ intersect in $T$. The poles $T_0$ and $T_1$ of both planes lie on $S$. Moreover, the isolines of the patch lie in the two pencils of planes supported by $S$ and $T$, while their poles vary on $T$ and $S$, respectively. The chords of opposite boundaries meet on $S$ and $T$, respectively. Consequently the four patch corners lie in a plane. Moreover, the tangents to $s = 0$ and $s = 1$ at $t = 0$ meet in $T_0$, and similarly for the other sets of tangents. Finally, the connection of $b_{0,1}$, $b_{1,1}$ meets $T$ in $S_0$. These relations may be used to construct the weighted $B$-points of the patch.

It should be mentioned that two arbitrary plane intersections of a quadric intersect each other in two (not necessarily real) points. Consequently, by the correspondence of $s$ to $b(s)$ the quadric $\mathcal{Q}$ will be covered twice. It should be mentioned that special rational tensor product patches are considered in [6].

BILINEAR PATCHES

There is an interesting special case, where the boundary parabolas of a quadrangular patch degenerate to straight lines. In the case of an integral patch the resulting surface is well-known as a bilinear interpolant [2], representing a hyperbolic paraboloid. Its projection gives a ruled quadric. Let this ruled quadric be defined by two skew pairs of its generatrices and a point $p$. There are two further generatrices through $p$. Where they intersect the boundary lines, may serve as points corresponding to parameter values $1/2$ for a rational parametrization of the four boundaries, cf. [4], followed by bilinear rational interpolation [3].

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vol. 26, n° 1, 1992
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