MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

W. BOEHM
D. HANSFORD

Parametric representation of quadric surfaces

Modélisation mathématique et analyse numérique, tome 26, nº 1 (1992), p. 191-200

<http://www.numdam.org/item?id=M2AN_1992__26_1_191_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 26, n° 1, 1992, p. 191 à 200)

PARAMETRIC REPRESENTATION OF QUADRIC SURFACES

by W. BOEHM [1] and D. HANSFORD [2]

Abstract. — *This paper briefly considers some cases where triangular and quadrangular quadratic Bézier patches represent a quadric, which are used in geometric modeling. A detailed discussion may be found in* [3] *and* [8].

Résumé. — Représentation paramétrique de surfaces quadriques. *Cet article considère brièvement quelques cas utilisés en modélisation géométrique où des carreaux de Bézier triangulaires ou quadrilataires représentent une quadrique. Les détails sont donnés en* [3] *et* [8].

INTRODUCTION

Quadric surfaces, such as circular cylinders, cones, and spheres, are frequently used in geometric modeling. On the other hand, in free-form design with predominantly parametric Bernstein-Bézier representations, implicitly defined quadrics are not used frequently. Therefore, as a step toward integrating quadrics surfaces into free-form design, this paper examines the parametric representation of quadrics with quadratic and biquadratic patches from a geometric point of view.

QUADRATIC BÉZIER PATCHES AND QUADRICS

An integral triangular quadratic B-patch, $\mathbf{b}(\mathbf{u})$ where $u = (u, v, w)$ denotes barycentric coordinates in the domain, is controlled by six B-points, $\mathbf{b}_{0,0}$, $\mathbf{b}_{0,1}$, $\mathbf{b}_{0,2}$, $\mathbf{b}_{1,1}$, $\mathbf{b}_{2,0}$, $\mathbf{b}_{1,0}$. An integral rectangular biquadratic B-patch, $\mathbf{b}(s)$, where $\mathbf{s} = (s, t)$ denotes affine coordinates in the domain, is controlled by nine B-points, $\mathbf{b}_{0,0}, \ldots, \mathbf{b}_{2,2}$ [2, 4], as illustrated in figure 1.

In the case of a rational patch, each B-point $\mathbf{b}_{i,k}$ has associated with it a weight $\beta_{i,k}$. The boundaries of the quadratic and biquadratic patches are conic sections, thus each is defined by three B-points. Note that the degree

(1) Technical University of Braunschweig.
(2) Arizona State University, Tempe, AZ 85287, U.S.A.

of a triangular quadratic B-patch is four or less, and the degree of a biquadratic patch is eight or less [9], therefore the B-points of quadratic and biquadratic patches on a quadric are not independent.

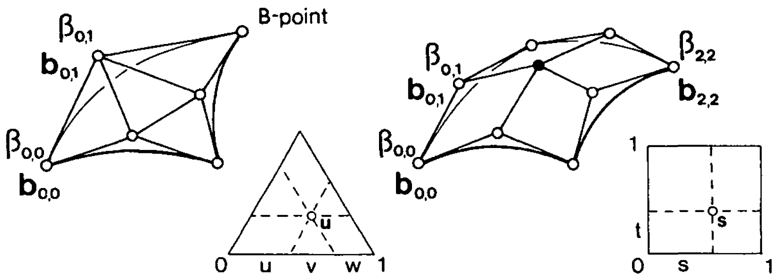

Figure 1. — Quadratic and biquadratic patches.

A quadric $\mathcal{Q}$ can be defined as a surface which carries two families of straight lines, the so-called generatrices. If both families are real and different, the quadric is said to be doubly ruled or annular ; it is a hyperboloid of one sheet or a hyperbolic paraboloid. If both families are non-real but different, the quadric is said to be non-ruled or oval ; it is an ellipsoid, a hyperboloid of two sheets, or an elliptic paraboloid. In the special case, where both families coincide to one family, the quadric is singly ruled and degenerates to a quadratic cone or a cylinder.

Any plane section of a quadric is a conic section. Note that this conic section is not necessarily real, or if the plane is a tangent plane it decomposes into a pair of coinciding or different lines. Any conic section which contacts the infinite plane is a parabola. Any quadric which contacts the infinite plane is a paraboloid. Let **a** denote such a point of contact at infinity. Viewed as a vector, it is called an axis direction of the parabola or the paraboloid under consideration [1].

A conic section is determined by five points or five tangents in the plane which may coincide in pairs, thus being a point with a tangent. Therefore six points or tangents are dependent, they are related by the configuration of Pascal and Brianchon [11].

A quadric is defined by nine points in space, where two or three may coincide to a point with tangent or to a point with tangent plane, respectively. However, three tangent planes at three points of $\mathcal{Q}$ are related by Brianchon's configuration in the plane spanned by the three points.

Finally, any irreducible quadric as a whole can be viewed as the real projection of one of the three paraboloids shown in figure 2, below. More interesting properties of quadrics may be found in [1, 3].

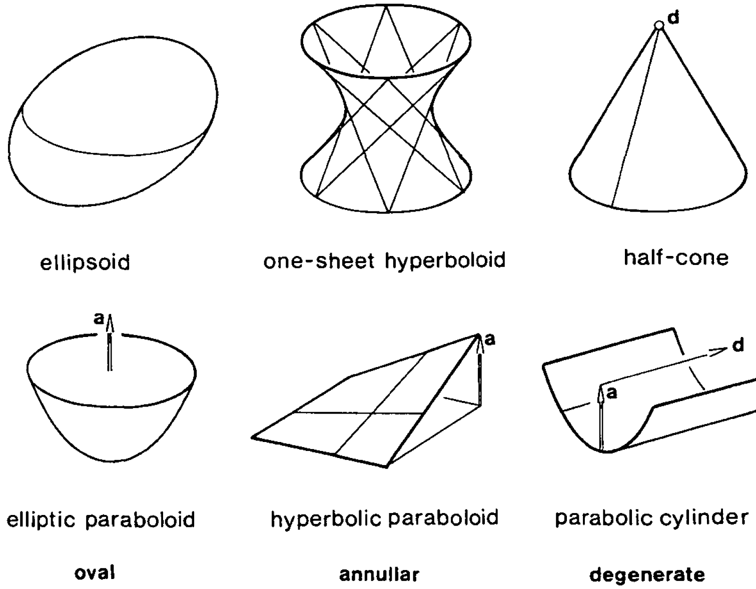

Figure 2. — Some quadrics in $\mathbb{R}^3$.

INTEGRAL TRIANGULAR PATCHES ON A QUADRIC

All integral quadratic curves represent parabolas. Let $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ denote the B-points of such a parabola, and its axis direction is $\mathbf{a} = \mathbf{b}_2 - 2\,\mathbf{b}_1 + \mathbf{b}_0$ [1, 3, 7]. Now consider an integral triangular quadratic B-patch, $\mathbf{b}(\mathbf{u})$. Its three boundary curves are parabolas, and thus the quadric $\mathcal{Q}$ under consideration must be a paraboloid. Assuming that $\mathcal{Q}$ is an elliptic or a hyperbolic paraboloid, any parabola lies in a plane parallel to the axis. Consequently, the axes of the three boundary parabolas must be parallel to each other. In effect, this gives conditions on the B-points, as illustrated in figure 3.

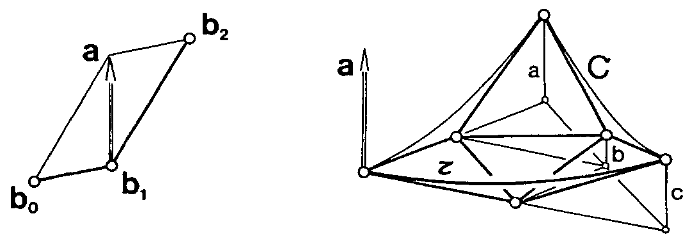

Figure 3. — The general integral quadratic patch.

There is a simple method for classifying the paraboloid As in figure 3, let τ denote the tangent plane at $\mathbf{b}_{0\,0}$, and let C denote the opposite boundary parabola If C intersects τ in two different real points, in two non-real points, or τ is tangent to C, the quadric is a hyperbolic paraboloid, an elliptic paraboloid, or a parabolic cylinder, respectively

Now assuming the quadric above is degenerate, it is a parabolic cylinder Let $\mathbf{d}$ denote the direction from $\mathbf{b}_{0\,0}$ to the point of contact of C and τ Any translation of the B-points in the direction of $\mathbf{d}$ will not effect the parabolic cylinder Hence, the axes of all parabolas are parallel to a plane spanned by $\mathbf{a}$ and $\mathbf{d}$

RATIONAL QUADRATIC TRIANGULAR PATCHES

Any rational B-patch in $\mathbb{R}^3$ can be defined as the projection of an integral B-patch in $\mathbb{R}^4$ [1, 4] This projection can easily be realized by the simple procedure of inhomogeneizing

Moreover, any rational triangular quadratic patch on a non-degenerate quadric $\mathcal{Q}$ can be viewed as the projection of an integral triangular patch on a non-degenerate paraboloid

As a corollary, a rational triangular quadratic patch lies on a quadric if the three boundaries meet in one point, $\mathbf{q}$, where their three tangents, U, V, W, are coplanar, and $\mathbf{q}$ corresponds three times to the parameter value ∞, as illustrated in figure 4 [10] An example where these conditions are violated and a quadratic patch is not defined is discussed in [5], a rational quartic patch is needed to represent an octant of the sphere

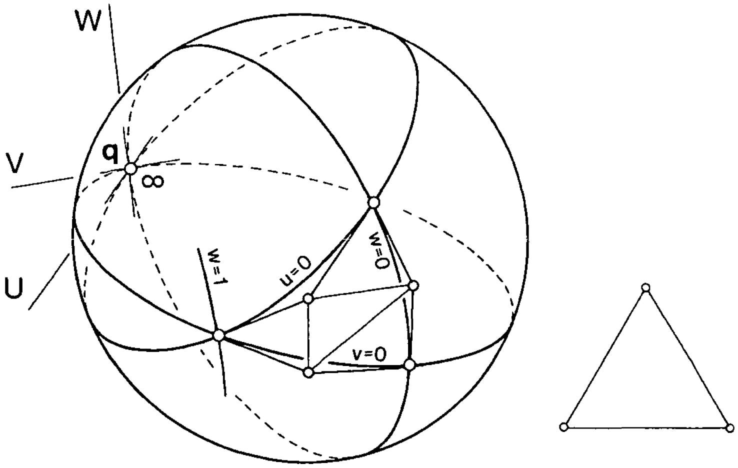

Figure 4. — The rational triangular patch on a quadric

Analogously one has that any rational triangular quadratic patch on a cone or cylinder, can be viewed as the projection of an integral patch on a parabolic cylinder, as illustrated in figure 5. Note that only in a special case do all three boundaries meet in a point $\mathbf{q}$. Note also that each of the three boundaries have to contact the opposite tangent plane. In effect this gives conditions on the B-points of the net and their weights.

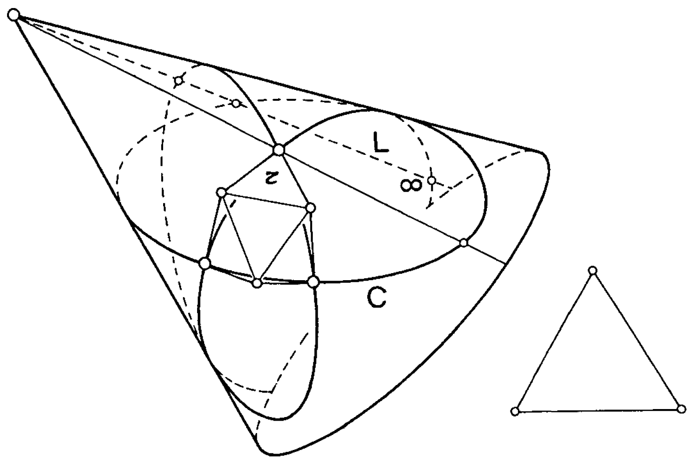

Figure 5. — The rational triangular patch on a cone.

A ONE TO ONE CORRESPONDENCE

There is a simple one to one correspondence between finite points $\mathbf{u}$ in the domain and the finite points $\mathbf{b}(\mathbf{u})$ of the nondegenerate paraboloids, as illustrated in figure 3. This correspondence can be generalized by perspectivity, and is also known as a stereographic projection.

Consider a triangular patch on a quadric $\mathcal{Q}$ and let $\mathbf{q}$ denote the point where the three boundary curves meet. Let ρ denote the tangent plane of $\mathcal{Q}$ at $\mathbf{q}$, and let σ be a plane parallel to ρ, as in figure 6. Consider the projection which projects the points of $\mathcal{Q}$ not lying in ρ from $\mathbf{q}$ onto σ. The ∞^2 lines of σ correspond to the ∞^2 conic sections through $\mathbf{q}$ and vice versa. Note that such a one to one correspondence is a necessary and sufficient condition for a quadratic triangular patch to lie on a non-degenerate quadric [9].

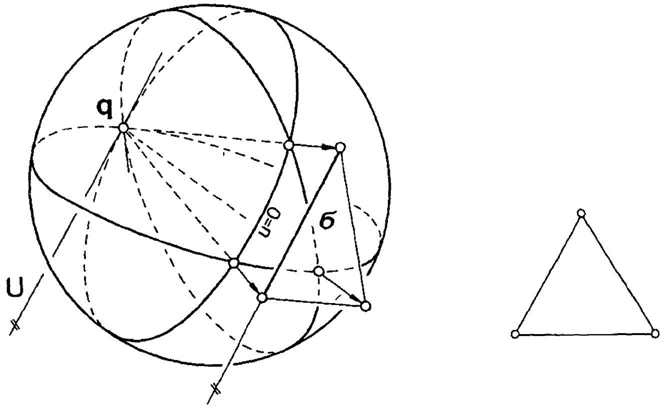

Figure 6. — The one to one correspondence.

QUADRANGULAR FROM TRIANGULAR PATCHES

Any integral quadratic triangular patch can easily be extended to a biquadratic patch by the construction of the B-points of the isolines $u = 1$ and $v = 1$ via the de Casteljau algorithm [2, 4]. Note that all isolines are parabolas too. Moreover, all isolines of the same family are parallel and congruent in space. The interior B-point must be adjusted so that the twist of the rectangular patch is identical to the twist of the triangular patch.

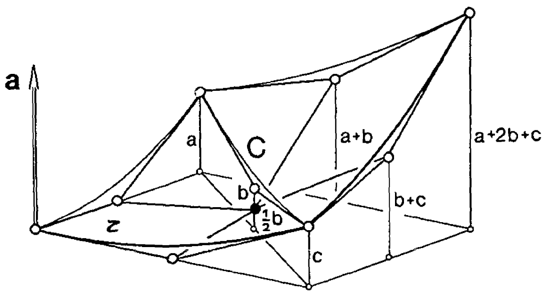

Figure 7. — Extension of an integral triangular patch.

The extension to a rational patch is shown in figure 8. The boundaries $u = 0$ and $u = 1$ have a common tangent U at $\mathbf{q}$, and similarly the boundaries $v = 0$ and $v = 1$ have a common tangent V at $\mathbf{q}$. Note that a final linear rational change of the parametrization will not change the surface.

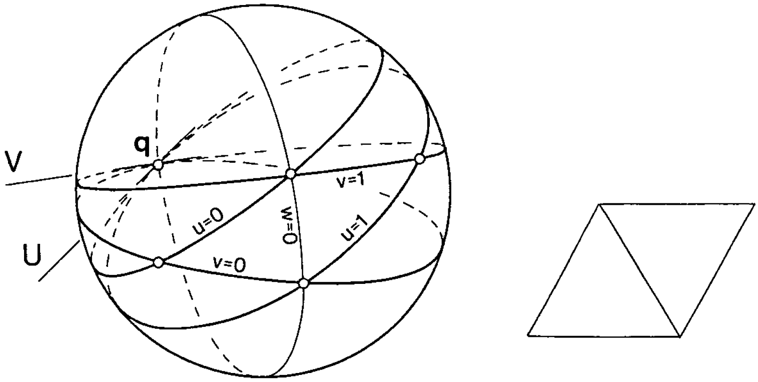

Figure 8. — Extension of rational triangular patch.

This can be generalized. Let the barycentric coordinates **u** be expressed as the bilinear interpolant, s, of four points, $\mathbf{u}_1, \ldots, \mathbf{u}_4$, in the **u**-plane, as in figure 8. This is done by substituting into s, the barycentric definitions of $\mathbf{u}_1, \ldots, \mathbf{u}_4$ and rearranging. Now **u** has been expressed bilinearly in terms of s. Substituting this expression of **u** into the representation of the quadratic patch $\mathbf{b}(\mathbf{u})$ one gets a biquadratic patch in s.

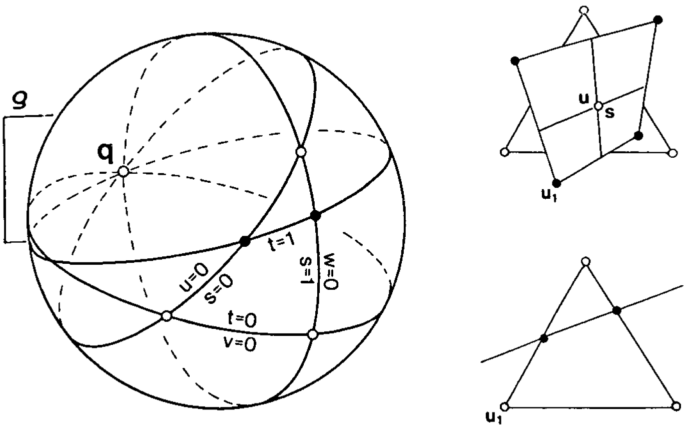

Figure 9. — Corner cutting of a triangular patch.

In a simple case, the fourth side of the rectangle can be constructed by a corner cutting construction of the triangle, as illustrated in figure 9. The fourth boundary also meets **q**.

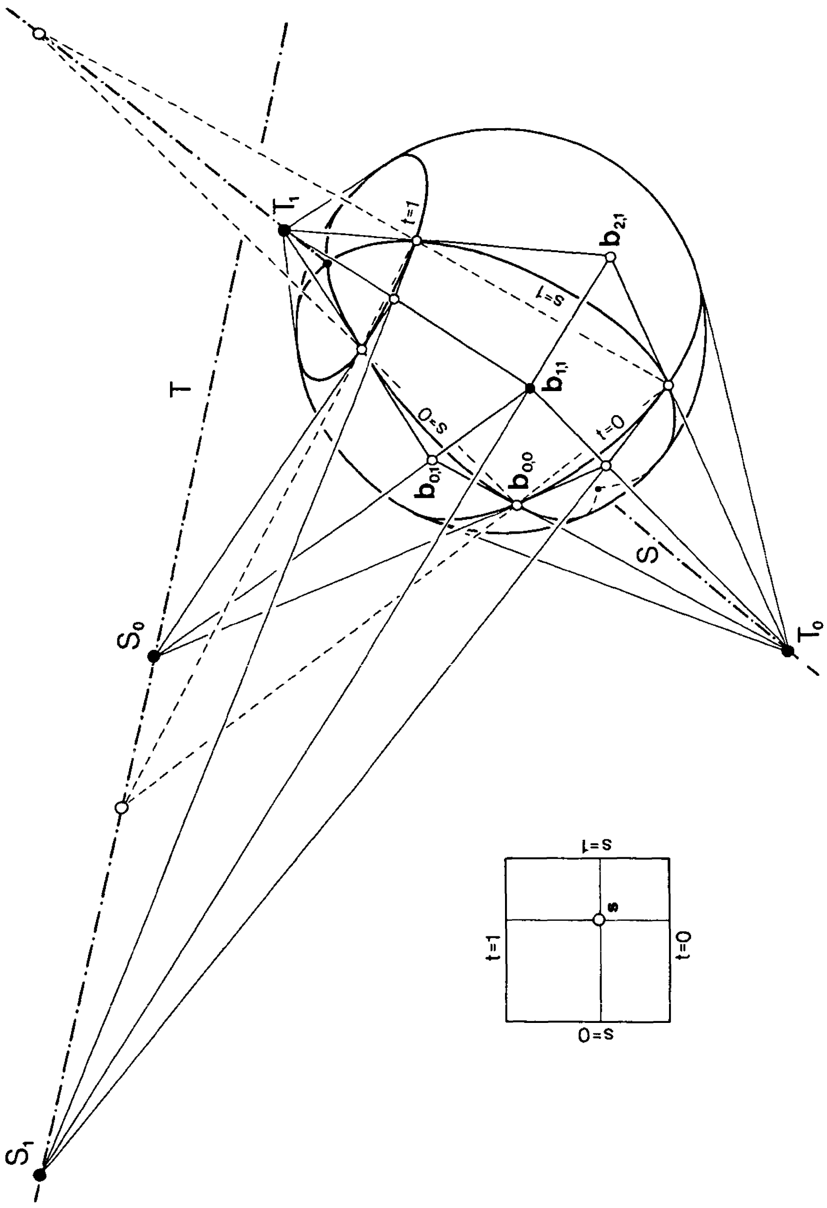

Figure 10. — The rational biquadratic patch.

THE RATIONAL BIQUADRATIC PATCH

A simple example of a rational biquadratic patch can be constructed on the sphere, where the isoparametric lines are formed by parallels and meridians. As before, its projection gives a rational biquadratic patch, illustrated in figure 10.

Let the planes through $s = 0$ and $s = 1$ intersect in S, and let S_0 and S_1 denote the poles of these planes with respect to $\mathcal{Q}$, both determining an axis T. The planes through $t = 0$ and $t = 1$ intersect in T. The poles T_0 and T_1 of both planes lie on S. Moreover, the isolines of the patch lie in the two pencils of planes supported by S and T, while their poles vary on T and S, respectively. The chords of opposite boundaries meet on S and T, respectively. Consequently the four patch corners lie in a plane. Moreover, the tangents to $s = 0$ and $s = 1$ at $t = 0$ meet in T_0, and similarly for the other sets of tangents. Finally, the connection of $\mathbf{b}_{0,1}$, $\mathbf{b}_{1,1}$ meets T in S_0. These relations may be used to construct the weighted B-points of the patch.

It should be mentioned that two arbitrary plane intersections of a quadric intersect each other in two (not necessarily real) points. Consequently, by the correspondence of $\mathbf{s}$ to $\mathbf{b}(\mathbf{s})$ the quadric $\mathcal{Q}$ will be covered twice. It should be mentioned that special rational tensor product patches are considered in [6].

BILINEAR PATCHES

There is an interesting special case, where the boundary parabolas of a quadrangular patch degenerate to straight lines. In the case of an integral patch the resulting surface is well-known as a bilinear interpolant [2], representing a hyperbolic paraboloid. Its projection gives a ruled quadric. Let this ruled quadric be defined by two skew pairs of its generatrices and a point $\mathbf{p}$. There are two further generatrices through $\mathbf{p}$. Where they intersect the boundary lines, may serve as points corresponding to parameter values 1/2 for a rational parametrization of the four boundaries, *cf.* [4], followed by bilinear rational interpolation [3].

ACKNOWLEDGEMENTS

The work of the second author was supported by NSF grant DMC-8807747 and by DoE grant DE-FG02-87ER25041 to Arizona State University. The authors would like to thank Gerald Farin and Andreas Johannsen for helpful comments.

REFERENCES

[1] W BOEHM, On Cubics A Survey *Comput Graph Image Process* 19 (1982) pp 201-226

[2] W BOEHM, G FARIN and J KAHMANN, A survey of curve and surface methods in CAGD *Comput Aided Geom Design*, 1 (1984) pp 1-60

[3] W BOEHM and D HANSFORD, Bézier Patches on Quadrics, in G Farin, editor, NURBS for Curve and Surface Design, SIAM, Philadelphia (1991) pp 1-14

[4] G FARIN, Curves and Surfaces for Computer Aided Geometric Design Academic Press, 1988, 2nd Edition 1990

[5] G FARIN, B PIPER and A WORSEY, The octant of a sphere as a non-degenerate triangular Bezier patch *Comput Aided Geom Design,* 4 (1987) pp 329-332

[6] G GEISE and U LANGBECKER, Finite quadrics segments with four conic boundary curves *Comput Aided Geom Design*, 7 (1990) pp 141-150

[7] E LEE, The rational Bezier representation for conics, in G Farin, editor, Geometric Modeling Algorithms and New Trends, SIAM, Philadelphia (1987) pp 3-19

[8] J NIEBUHR, *B*-patches on Quadrics Dissertation TU Braunschweig, to be published in 1991 (in German)

[9] T SEDERBERG, Implicit curves and surfaces Ph D Thesis, Purdue, 1983

[10] T SEDERBERG and D ANDERSON, Steiner surface patches *IEEE CG&A*, May (1985)

[11] D STRUIK, Lectures on Analytic and Projective Geometry Addison-Wesley, 1953