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J. BARANGER
D. SANDRI

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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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A FORMULATION OF STOKES'S PROBLEM AND THE LINEAR ELASTICITY EQUATIONS SUGGESTED BY THE OLDROYD MODEL FOR VISCOELASTIC FLOW (*)

J. BARANGER [1], D. SANDRI [1]

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Abstract. — *We propose a three fields formulation of Stokes's problem and the equations of linear elasticity, allowing conforming finite element approximation and using only the classical inf-sup condition relating velocity and pressure. No condition of this type is needed on the « non Newtonian » extra stress tensor. For the linear elasticity equations this method gives uniform results with respect to the compressibility.*

Résumé. — *On propose une formulation à trois champs du problème de Stokes et des équations de l'élasticité linéaire, permettant des approximations par éléments finis conformes et ne nécessitant que la classique condition inf-sup en vitesse pression à l'exclusion de toute condition sur le tenseur « non Newtonien » des extra contraintes. Sur les équations de l'élasticité linéaire la méthode est uniforme par rapport à la compressibilité.*

0. INTRODUCTION

A version of Stokes's problem with three unknown fields : σ extra stress tensor, u velocity and p pressure has been used in numerical finite element simulation, partly motivated by the study of viscoelastic fluids obeying Maxwell constitutive equation. Finite element approximation of this problem are known (see [8]) to converge if two Babuska-Brezzi (BB) conditions are satisfied : the classical one on (u, p) and an other one on (σ, u).

Regarding the equation of linear isotropic elasticity the ability of the method to perform independently of compressibility (particularly near the incompressible limit) is a major concern. Recently a method has been

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[1] UA CNRS 740 LAN, Bât. 101, Université de Lyon 1, 69622 Villeurbanne, France.

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proposed [12] to solve a three unknown fields version of the problem (σ extra stress, u displacement, p pressure) without the two BB conditions ; but this necessitate the addition of least squares terms. In a sense the (σ, u, p) equations studied are of Maxwell type.

The purpose of this paper is to show that the use of a modified version of the problem with three fields (σ, u, p), suggested by the use of Oldroyd model for viscoelastic fluids, allows to suppress the BB condition on (σ, u). This result applies to Stokes's problem and the linear elasticity equation uniformly with respect to the compressibility.

1. AN « OLDROYD VERSION » OF STOKES'S PROBLEM

We use the following notations : u velocity vector, p pressure, ∇u gradient velocity tensor $((\nabla u)_{ij} = u_{i,j})$, $d(u) = (1/2)(\nabla u + \nabla u^t)$ rate of strain tensor, $w(u) = (1/2)(\nabla u - \nabla u^t)$ vorticity tensor, σ_{tot} stress tensor, f body force, $(\nabla \cdot \sigma)_i = \sigma_{ij,j}$ divergence of a tensor, σ_t time derivative of σ.

The viscoelastic fluid is flowing in Ω, bounded open domain in $\mathbb{R}^N$ with Lipschitzian boundary Γ ; Γ is partitioned in Γ_1 and Γ_2 with meas $(\Gamma_1) \neq 0$; n is the outward unit normal to Γ.

For $a \in [-1, 1]$ one defines an objective derivative of a tensor σ by :

$$\frac{\partial_a \sigma}{\partial t} = \sigma_t + (u \cdot \nabla)\sigma + g_a(\sigma, \nabla u)$$

$$g_a(\sigma, \nabla u) = \sigma w(u) - w(u)\sigma - a(d(u)\sigma + \sigma d(u)) .$$

We use the dimensionless Reynolds number Re, Weissenberg number We and α (α may be considered as the quotient of the retardation time by the relaxation time or the part of viscoelastic viscosity in the total viscosity).

The equations of the Oldroyd model under consideration are obtained from the momentum equation :

$$\text{Re}\ (u_t + (u \cdot \nabla)u) - \nabla \cdot \sigma_{\text{tot}} = f .$$

Writing $\sigma_{\text{tot}} = -pI + \sigma_N + \sigma$ where σ and σ_N are respectively the non Newtonian and Newtonian part of the extra stress tensor $\sigma_{\text{tot}} + pI$, σ_N is defined by : $\sigma_N = 2(1-\alpha)\,d(u)$. Substituting σ_N in the momentum equation one gets :

$$\text{Re}\ (u_t + (u \cdot \nabla)u) - 2(1-\alpha)\nabla \cdot d(u) + \nabla p - \nabla \cdot \sigma = f .$$

The non Newtonian extra stress tensor σ satisfies the constitutive equation :

$$\sigma + \text{We}\frac{\partial_a \sigma}{\partial t} - 2\,\alpha d(u) = 0 . \tag{1.0}$$

In the following we consider the case of a stationary creeping flow (everything is independent of t and $(u \,.\, \nabla)\, u$ is neglected). A numerical method for a truly viscoelastic fluid (We > 0) must perform equally well in the limit We $= 0$ in (1.0).

The equations for this case are :

$$\begin{array}{lll} \sigma - 2\,\alpha\, d(u) = 0 & \text{in} & \Omega\,, \\ -\nabla \,.\, \sigma - 2(1-\alpha)\, \nabla \,.\, d(u) + \nabla p = f & \text{in} & \Omega\,, \\ \nabla \,.\, u = 0 & \text{in} & \Omega\,, \\ u = 0 & \text{on} & \Gamma_1\,, \\ (\sigma + 2(1-\alpha)\, d(u) - pI) \,.\, n = g & \text{on} & \Gamma_2\,. \end{array}$$

The third equation is the incompressibility condition and the last two are boundary conditions.

We denote by $(\,.,\,.\,)$ the $L^2(\Omega)$ scalar product of functions, vectors and tensors and by $(\,.,\,.\,)_{\Gamma_2}$ the $L^2(\Gamma_2)$ scalar product ; we also define :

$$T = \left\{ \tau = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji}\,;\ \tau_{ij} \in L^2(\Omega)\,;\ 1 \leq i, j \leq N \right\}\,,$$

$$V = \left\{ v = (v_i)\,;\ v_i \in H^1(\Omega)\,;\ v_i|_{\Gamma_1} = 0\,;\ 1 \leq i \leq N \right\}\,,$$

$$Q = \left\{ q \in L^2(\Omega)\,;\ \int_\Omega q = 0 \right\} \text{ if meas } (\Gamma_2) = 0\,;\ Q = L^2(\Omega) \text{ else}\,.$$

$$\langle \ell, v \rangle = (f, v) + (g, v)_{\Gamma_2}\,.$$

Then the five equations above have the following weak formulation (Oldroyd version of Stokes's problem) :

Problem (SO) :

Find $(\sigma, u, p) \in T \times V \times Q$ such that :

$$(\sigma, \tau) - 2\,\alpha (d(u), \tau) = 0 \qquad \forall \tau \in T\,, \tag{1.1}$$

$$(\sigma, d(v)) + 2(1-\alpha)(d(u), d(v)) - (p, \nabla \,.\, v) = \langle \ell, v \rangle \qquad \forall v \in V\,, \tag{1.2}$$

$$(\nabla \,.\, u, q) = 0 \qquad \forall q \in Q\,. \tag{1.3}$$

For some years the numerical solution of the viscoelastic problem (1.0) (1.2) (1.3) with convenient boundary conditions on σ was limited to relatively small Weissenberg number because the hyperbolic character of (1.0) was not taken into account. In the pioneering works [7] and [14] this character was considered, suppressing the high Weissenberg number problem. In [7] (1.0) is solved by a discontinuous FEM of Lesaint Raviart, so σ is approximated in a space T_h of tensors with discontinuous components ; in [14] it is solved by a continuous FEM, so σ is approximated in a space T_h of tensors with continuous components.

In both cases the study is made on the Maxwell model corresponding to $\alpha = 1$. The numerical analysis of the corresponding Stokes model ((SO) with $\alpha = 1$ better called SM !) has been made in [8].

Given finite element spaces $T_h \subset T$, $V_h \subset V$, $Q_h \subset Q$, a finite element approximation of problem (SO) is :

Problem $(\mathrm{SO})_h$:

Find $(\sigma_h, u_h, p_h) \in T_h \times V_h \times Q_h$ such that :

$$(\sigma_h, \tau_h) - 2\,\alpha\,(d(u_h), \tau_h) = 0 \qquad \forall \tau_h \in T_h\,, \qquad (1.1)_h$$

$$(\sigma_h, d(v_h)) + 2(1-\alpha)(d(u_h), d(v_h)) - (p_h, \nabla \cdot v_h) = \langle \ell, v_h \rangle \qquad \forall v_h \in V_h\,, \qquad (1.2)_h$$

$$(\nabla \cdot u_h, q_h) = 0 \qquad \forall q_h \in Q_h\,. \qquad (1.3)_h$$

It is proved in [8] that when $\alpha = 1$ problem $(\mathrm{SO})_h$ is well posed and that its solution approximate the solution of problem (SO) if the following conditions are satisfied :

$C1$: Inf-sup condition on (u, p) :

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(\nabla \cdot v_h, q_h)}{|v_h|_1\,|q_h|_0} \geqslant \beta > 0\,.$$

$C2$: Inf-sup condition on (σ, u) : either $d(V_h) \subset T_h$ (case of discontinuous τ_h) or « the number of interiors degrees of freedom for τ_h in each K is greater or equal to the number of all the degrees of freedom of v_h in each K » (case of continuous τ_h).

We show in § 4 that the use of problem $(\mathrm{SO})_h$ with $0 < \alpha < 1$ (excluding Maxwell case) allows to suppress condition $C2$ greatly enlarging the possible choices of approximations for the viscoelastic non Newtonian extra stress tensor σ.

2. LINEAR COMPRESSIBLE AND INCOMPRESSIBLE ELASTICITY

We denote by σ_{tot} the stress tensor, u the displacement and $\varepsilon(u) = (u_{i,j} + u_{j,i})/2$ the strain tensor. The elastic isotropic solid has a reference configuration Ω, open bounded domain in $\mathbb{R}^N$, ($N = 2$ or 3), with Lipschitzian boundary Γ partitioned as in the preceding paragraph. ν denotes the Poisson ratio and μ is the shear modulus. The constitutive equation of linear isotropic elasticity is then :

$$\sigma_{\text{tot}} = 2\,\mu\,\{\varepsilon(u) + (\nu/(1-2\,\nu))\,\nabla \cdot uI\}\,.$$

We introduce a parameter $\varepsilon = (1-2\,\nu)/2\,\nu \geqslant 0$ and for $\varepsilon > 0$ the pressure p by :

$$\varepsilon p + \nabla \cdot u = 0\,.$$

As suggested by problem (SO) we introduce a scaled « Newtonian » extra stress tensor $\sigma_N = 2(1-\alpha)\,\varepsilon(u)$ and a scaled « non Newtonian » extra stress tensor $\sigma = 2\,\alpha\,\varepsilon(u)$.

The momentum equation is then written :

$$-\nabla\,.\,\sigma - 2(1-\alpha)\,\nabla\,.\,\varepsilon(u) + \nabla p = f\,,$$

with $f = \mu^{-1} f'$ where f' is the body force.

The equations are :

$$\begin{array}{lll}
\sigma - 2\,\alpha\,\varepsilon(u) = 0 & \text{in} & \Omega\,, \\
-\nabla\,.\,\sigma - 2(1-\alpha)\,\nabla\,.\,\varepsilon(u) + \nabla p = f & \text{in} & \Omega\,, \\
\nabla\,.\,u + \varepsilon p = 0 & \text{in} & \Omega\,, \\
u = 0 & \text{on} & \Gamma_1\,, \\
(\sigma + 2(1-\alpha)\,\varepsilon(u) - pI)\,.\,n = g & \text{on} & \Gamma_2\,.
\end{array}$$

Using the functional spaces T, V, Q previously defined and with the obvious formal change $\varepsilon := d$, the five equations above have the following weak formulation (Oldroyd version of linear elasticity) :

Problem (EO) :

Find $(\sigma, u, p) \in T \times V \times Q$ such that :

$$\begin{array}{lll}
(\sigma, \tau) - 2\,\alpha\,(d(u), \tau) = 0 & \forall \tau \in T\,, & (2.1) \\
(\sigma, d(v)) + 2(1-\alpha)(d(u), d(v)) - (p, \nabla\,.\,v) = \langle \ell, v\rangle & \forall v \in V\,, & (2.2) \\
(\nabla\,.\,u, q) + \varepsilon(p, q) = 0 & \forall q \in Q\,. & (2.3)
\end{array}$$

Problem (EO) is a generalisation of problem (SO), the last one being the incompressible limit of the first one ($\varepsilon = 0$ corresponding to $\nu = 1/2$). A version of this problem with $\alpha = 1$ has been introduced in [9, 11].

Given finite element spaces $T_h \subset T$, $V_h \subset V$, $Q_h \subset Q$ we define an approximate problem :

Problem $(\text{EO})_h$:

Find $(\sigma_h, u_h, p_h) \in T_h \times V_h \times Q_h$ such that :

$$\begin{array}{lll}
(\sigma_h, \tau_h) - 2\,\alpha\,(d(u_h), \tau_h) = 0 & \forall \tau_h \in T_h\,, & (2.1)_h \\
(\sigma_h, d(v_h)) + 2(1-\alpha)(d(u_h), d(v_h)) & & (2.2)_h \\
\qquad - (p_h, \nabla\,.\,v_h) = \langle \ell, v_h\rangle & \forall v_h \in V_h\,, & \\
(\nabla\,.\,u_h, q_h) + \varepsilon(p_h, q_h) = 0 & \forall q_h \in Q_h\,. & (2.3)_h
\end{array}$$

We show in § 4 that when $0 < \alpha < 1$ under the inf-sup condition $C1$ on (u, p) only, problem $(\text{EO})_h$ is well posed and approximate the solution of problem (EO) uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$ (note that the incompressible limit $\varepsilon = 0$ is included).

Recent works have been dedicated to the development of FEM performing independently of compressibility for linear elasticity. Let us quote here [6, 10, 12] where Galerkin least squares methods are used on (u, p), (σ_{tot}, u) and $(T = \sigma_{tot} + pI, p, u)$ models, [9, 11] where the same method is applied to a four field (d, T, p, u) model $(d = d(u))$ is considered as an independent variable) and [1, 16] where a non conforming approximation of u is used, possibly with a post processing technique.

3. EXISTENCE AND UNIFORM CONTINUITY OF SOLUTIONS OF PROBLEM (EO)

We prove in this paragraph that problem (EO) admits a unique solution $x = (\sigma, u, p)$ and that x is an uniformly continuous function of ℓ with respect to ε. This prepare the uniform FE approximation result of § 4.

The space T of symmetric tensors with $L^2(\Omega)$ components is equiped with the scalar product $(\sigma, \tau) = \int_\Omega \sigma : \tau = \int_\Omega \sigma_{ij}\, \tau_{ij}$ with associated norm $|\tau|_0$; V is equiped with the scalar product $(u, v)_V = (d(u), d(v))$ with associated norm $|v|_1 = (d(u), d(u))^{1/2}$ which is a norm by Korn's inequality ; $Q = L^2(\Omega)$ if meas $(\Gamma_2) \neq 0$ is equiped with the usual scalar product and $Q = L_0^2(\Omega)$ if $\Gamma_2 = \varnothing$ is equiped with the quotient scalar product, both denoted by (p, q) with associated norm $|q|_0$.

$H = T \times V \times Q$ is equiped with the scalar product given by :

$$x = (\sigma, u, p), \quad y = (\tau, v, q),$$

$$(x, y) = (\sigma, \tau) + (u, v)_V + (p, q)$$

with corresponding norm $\|x\|$.

The variational formulation of problem (EO) can be written in the following abstract form :

Problem (EO)' :

Find $x \in H$ such that :

$$B(x, y) = \langle \ell', y \rangle \quad \forall y \in H, \tag{3.1}$$

where B is the bilinear symmetric form :

$$\begin{aligned} B(x, y) = &(\sigma, \tau) - 2\,\alpha(d(u), \tau) - 2\,\alpha(d(v), \sigma) \\ &- 4\,\alpha(1 - \alpha)(d(u), d(v)) \\ &+ 2\,\alpha(\nabla . v, p) + 2\,\alpha(\nabla . u, q) + 2\,\alpha\,\varepsilon(p, q) \end{aligned} \tag{3.2}$$

and where $\langle \ell', y \rangle = -2\,\alpha \langle \ell, v \rangle \quad \forall y \in H$.

This formulation is obtained by multiplying equation (2.2) by $-2\,\alpha$, equation (2.3) by $2\,\alpha$ and by adding the three equations obtained.

For the study of this variational formulation we use the following abstract result [2] :

THEOREM 3.1 : *Let H be a real Hilbert space and $\ell' \in H'$, topological dual space of H, and let B be a bilinear form on H satisfying the following three hypotheses :*

(H1) *There exists a constant $\eta > 0$ such that :*

$$B(x, y) \leq \eta \|x\| \|y\| \quad \forall x, y \in H .$$

(H2) *There exists a constant $\gamma > 0$ such that :*

$$\sup_{x \in H} \frac{B(x, y)}{\|x\|} \geq \gamma \|y\| \quad \forall y \in H .$$

(H3) *There exists a constant $\gamma' > 0$ such that :*

$$\sup_{y \in H} \frac{B(x, y)}{\|y\|} \geq \gamma' \|x\| \quad \forall x \in H .$$

Then problem (EO)$'$ *has a unique solution $x \in H$ such that* $\|x\| \leq (1/\gamma')\|\ell'\|_{H'}$.

We remark that hypotheses (H2) and (H3) are equivalent when B is symmetric.

We now show that these hypotheses are satisfied for the form B, with constants independent of ε when $\varepsilon \in [0, \varepsilon_0]$.

THEOREM 3.2 : *The bilinear symmetric form B given by* (3.2) *satisfies the hypothesis* (H1) *of Theorem* 3.1 *with constant independent of ε for $\varepsilon \in [0, \varepsilon_0]$ and the hypotheses* (H2) *and* (H3) *with constants independent of ε.*

Proof :

$$\begin{aligned}(\mathrm{H1})\ B(x, y) \leq\ & |\sigma|_0 |\tau|_0 + 2\alpha |\tau|_0 |u|_1 + 2\alpha|\sigma|_0 |v|_1 \\ & + 4\alpha(1-\alpha)|u|_1 |v|_1 + 2N^{1/2}\alpha |p|_0 |v|_1 \\ & + 2N^{1/2}\alpha |q|_0 |u|_1 + 2\alpha\varepsilon|p|_0 |q|_0 \\ \leq\ & C_0(1+\varepsilon_0)\|x\| \|y\| ,\end{aligned}$$

with C_0 independent of ε_0 and then (H1) is satisfied with $\eta = C_0(1+\varepsilon_0)$.

Before proving (H2) (H3) we recall the following result.

THEOREM 3.3 : *For each $p \in Q$ there exists a $v \in V$ such that $\nabla . v = p$ and*

$$|v|_1 \leq C |p|_0 \quad \textit{with } C \textit{ independent of } p . \tag{3.3}$$

When $\Gamma_2 = \varnothing$ this result is a consequence of ([13], Corollary 2.4, p. 24). We give below a sketch of the proof when meas $(\Gamma_2) \neq 0$.

Let $v_1 \in V$ such that $\int_\Omega \nabla . v_1 > 0$ (v_1 exists because meas $(\Gamma_2) \neq 0$) and $C_1 = \int_\Omega \nabla . v_1$.

Let $p \in Q$, we have $p = p_1 + p_2$ where $p_1 = \text{meas}\,(\Omega)^{-1} \int_\Omega p$. Then there exists a $v_2 \in V$ such that $v_2|_\Gamma = 0$ satisfying $\nabla . v_2 = p_2$ and $|v_2|_1 \leqslant C_2 |p_2|_0$ with C_2 independent of p. Then it is easy to check that :

$$v = 2\, p_1\, \text{meas}\,(\Omega)\, C_1^{-1} v_1 + (1 + \text{meas}\,(\Omega)\, N C_1^{-2} |v_1|_1^2)\, v_2$$

satisfies :

$$\int_\Omega p \nabla . v \geqslant |p|_0^2$$

and

$$|v|_1 \leqslant C\, |p|_0 \quad \text{with} \quad C \text{ independent of } p\,.$$

The desired result is then a consequence of ([3], Theorem 0.1). □

Proof of (H3) : For each $x \in H$ select $y \in H$ such that :

$$\begin{aligned} \tau &= \sigma\,, \\ v &= -u + (1/2)\, C^{-2} \hat{u}\,, \\ q &= p\,, \end{aligned} \tag{3.4}$$

where $\hat{u}$ satisfies $\nabla . \hat{u} = p$, $|\hat{u}|_1 \leqslant C\, |p|_0$ with C independent of p.

Then

$$\begin{aligned} B(x, y) = {} & (\sigma, \sigma) - 2\,\alpha\,(d(u), \sigma) + 2\,\alpha\,(d(u), \sigma) - C^{-2} \alpha\,(d(\hat{u}), \sigma) \\ & + 4\,\alpha\,(1 - \alpha)(d(u), d(u)) - 2\,C^{-2} \alpha\,(1 - \alpha)(d(u), d(\hat{u})) \\ & - 2\,\alpha\,(\nabla . u, p) + C^{-2} \alpha\,(\nabla . \hat{u}, p) + 2\,\alpha\,(\nabla . u, p) + 2\,\alpha\,\varepsilon(p, p) \\ \geqslant {} & |\sigma|_0^2 + 4\,\alpha\,(1 - \alpha)|u|_1^2 + (2\,\alpha\,\varepsilon + \alpha C^{-2})|p|_0^2 - C^{-1} \alpha\,|\sigma|_0\, |p|_0 \\ & - 2\,C^{-1} \alpha\,(1 - \alpha)|u|_1\, |p|_0 \\ \geqslant {} & \frac{1}{2}\,|\sigma|_0^2 + 3\,\alpha\,(1 - \alpha)|u|_1^2 \\ & + \left(2\,\alpha\,\varepsilon + \alpha C^{-2} - \frac{1}{2}\,\alpha^2 C^{-2} - \alpha\,(1 - \alpha)\, C^{-2}\right) |p|_0^2 \\ \geqslant {} & \frac{1}{2}\,|\sigma|_0^2 + 3\,\alpha\,(1 - \alpha)|u|_1^2 + \left(\frac{1}{2} C^{-2} \alpha^2 + 2\,\alpha\,\varepsilon\right) |p|_0^2 \\ \geqslant {} & \alpha_0 \|(\sigma, u, p)\|^2\,, \end{aligned} \tag{3.5}$$

with $\alpha_0 = \min\left\{\frac{1}{2}, 3\,\alpha(1-\alpha), \frac{1}{2}C^{-2}\alpha^2\right\}$ independent of ε.

On the other hand :

$$\|(\tau, v, q)\|^2 \leqslant |\sigma|_0^2 + \left(|u|_1 + \frac{C^{-2}}{2}|\hat{u}|_1\right)^2 + |p|_0^2$$

$$\leqslant |\sigma|_0^2 + 2|u|_1^2 + 2\left(\frac{C^{-1}}{2}|p|_0\right)^2 + |p|_0^2$$

$$\leqslant |\sigma|_0^2 + 2|u|_1^2 + \left(1 + \frac{N}{2}\right)|p|_0^2 \quad \text{(because from (3.3) } C^{-1} \leqslant \sqrt{N}\text{)}$$

$$\leqslant \left(2 + \frac{N}{2}\right)\|(\sigma, u, p)\|^2 . \tag{3.6}$$

From (3.5) and (3.6) we deduce that (H2) and (H3) are satisfied with :

$$\gamma = \gamma' = \alpha_0(N/2 + 2)^{-1/2} .$$

Remark 3.1 : We can also consider for $\alpha \in [0, 1[$ (including $\alpha = 0$) the non symmetric bilinear form $\bar{B}$ defined by :

$$\begin{aligned}\bar{B}(x, y) = {} & (\sigma, \tau) - 2\,\alpha(d(u), \tau) + (d(v), \sigma) + 2(1-\alpha)(d(u), d(v)) \\ & - (\nabla . v, p) + (\nabla . u, q) + \varepsilon(p, q) ,\end{aligned}$$

then $\bar{B}$ satisfies also the continuity condition (H1) and the inf-sup conditions (H2) (H3). We give the beginning of a proof, which can be adapted to the discrete case, for (H2) and (H3) :

(H3) Let $x \in H$, select $y \in H$ such that :

$$\begin{aligned} \tau &= \sigma + 2(\alpha - 1)\,\phi , \\ v &= 2\,u - (1/3)\,C^{-2}\,\hat{u} , \\ q &= 2\,p , \end{aligned} \tag{3.7}$$

where $\hat{u}$ is choosen as in (3.4) and where $\phi \in T$ satisfies $(\phi, \tau) = (d(u), \tau)$ $\forall \tau \in T$ (in the continuous case $\phi = d(u)$ is the unique solution because $d(V) \subset T$). Then a straightforward computation gives (H3) with :

$$\gamma' = C_3(1 - \alpha) ,$$

with C_3 independent of ε and α.

(H2) In the same way, let $y \in H$; we take $x \in H$ such that :

$$\begin{aligned} \sigma &= 4\,\tau + 4(5\,\alpha - 1)\,\phi' , \\ u &= 10\,v + C^{-2}v' , \\ p &= 10\,q , \end{aligned} \tag{3.8}$$

where v' and ϕ' are choosen in the same manner (relatively to q, τ) as in (3.7), then we obtain (H2) with :

$$\gamma = C_4(1 - \alpha),$$

with C_4 independent of α and ε.

Then the inf-sup conditions are uniformly satisfied for $\alpha \in [0, \alpha_0]$, $\alpha_0 < 1$, including $\alpha = 0$ corresponding to the classical Stokes formulation.

In the same way it is possible to verify that inf-sup conditions are satisfied with constants independent of $\alpha \in [0, 1]$ and ε if we use the fact that $d(V) \subset T$.

Remark 3.2 : The idea that there is no need of an inf-sup condition on (σ, u) provided $0 < \alpha < 1$ can be seen on (SO) problem by using a non symmetric global formulation without the pressure variable.

Let $K = \{v \in V ; \nabla . v = 0\}$ and consider the product space $T \times K$. Then problem (SO) is equivalent to :

Problem (SO)' :

Find $(\sigma, u) \in K$ such that :

$$A((\sigma, u), (\tau, v)) = 2\alpha \langle \ell, v \rangle \ \forall (\tau, v) \in K, \tag{3.9}$$

where A is the bilinear form :

$$A((\sigma, u), (\tau, v)) = (\sigma, \tau) - 2\alpha(d(u), \tau) + 2\alpha(d(v), \sigma) + 4\alpha(1 - \alpha)(d(u), d(v)).$$

Then it obvious that A is K-elliptic and then from Lax & Milgram Theorem, (3.9) admits a unique solution $(\sigma, u) \in K$. Besides from theory of saddle-point problem [3] it is easy to show that there exists a unique $p \in Q$ such that (σ, u, p) is the unique solution of (SO).

4. FINITE ELEMENT APPROXIMATION

Given a closed subspace $H_h \subset H$, with equation (3.1) we associate the discrete problem :

Find $x_h \in H_h$ such that :

$$B(x_h, y_h) = \langle \ell', y_h \rangle \quad \forall y_h \in H_h. \tag{4.1}$$

Then the following result holds [2] :

THEOREM 4.1 : *Assume that hypotheses of Theorem* 3.1 *are satisfied. Let* $x \in H$ *be the solution of* (3.1). *Assume also the following :*

$(\mathrm{H2})_h$ *There exists a constant* $\gamma_h > 0$ *such that :*

$$\sup_{x_h \in H_h} \frac{B(x_h, y_h)}{\|x_h\|} \geq \gamma_h \|y_h\| \quad \forall y_h \in H_h .$$

$(\mathrm{H3})_h$ *There exists a constant* $\gamma_h' > 0$ *such that :*

$$\sup_{y_h \in H_h} \frac{B(x_h, y_h)}{\|y_h\|} \geq \gamma_h' \|x_h\| \quad \forall x_h \in H_h .$$

Then equation (4.1) *admits a unique solution* $x_h \in H_h$ *such that :*

$$\|x_h\| \leq (\gamma_h')^{-1} \|\ell'\|_{H'}$$

and we have :

$$\|x - x_h\| \leq \left(1 + \frac{\eta}{\gamma_h'}\right) \inf_{y_h \in H_h} \|x - y_h\| .$$

Consider now three finite element subspaces $T_h \subset T$, $V_h \subset V$ and $Q_h \subset Q$. Then problem $(\mathrm{EO})'$ is approximated by :

Problem $(\mathrm{EO})_h'$:

Find $x_h = (\sigma_h, u_h, p_h) \in H_h = T_h \times V_h \times Q_h$ satisfying (4.1) with the bilinear form B associated with problem $(\mathrm{EO})'$ and defined by (3.2).

The purpose of this section is to establish the following : problem $(\mathrm{EO})_h'$ has a unique solution x_h which converges in H uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$ towards the unique solution x of problem $(\mathrm{EO})'$, provided the FEM satisfies a velocity-pressure inf-sup condition (note that no inf-sup condition relating the viscoelastic extra stress and the velocity is needed).

This result is a consequence of Theorem 4.1 and the following :

THEOREM 4.2 : *Assume that the following velocity-pressure inf-sup condition holds :*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(\nabla . v_h, q_h)}{|v_h|_1 \, |q_h|_0} \geq \beta > 0 , \tag{4.2}$$

with β *independent of h, then the hypotheses* $(\mathrm{H2})_h$ *and* $(\mathrm{H3})_h$ *are satisfied with constants independent of h and* ε.

Proof : The proof is analogous to the proof of Theorem 3.2 and it suffices to check $(\mathrm{H3})_h$; this can be done in the following way :

Let $x_h \in H_h$, select $y_h \in H_h$ such that :

$$\begin{aligned} \tau_h &= \sigma_h , \\ v_h &= -u_h + (1/2)\,\beta^2\,\hat{u}_h , \\ q_h &= p_h , \end{aligned}$$

with $\hat{u}_h \in V_h$ satisfying :

$$(\nabla \,.\, \hat{u}_h, q_h) = (p_h, q_h) \quad \forall q_h \in Q_h ,$$
$$|\hat{u}_h|_1 \leq \beta^{-1} |p|_0 .$$

The existence of $\hat{u}_h$ is given by (4.2) and ([3], Theorem 0.1). Proceeding as in Theorem 3.2 we obtain :

$$\gamma_h = \gamma_h' = (N/2 + 2)^{-1/2} \min \left\{ \frac{1}{2}, 3\,\alpha(1-\alpha), \frac{1}{2}\beta^2\alpha^2 \right\} . \quad \square$$

Remark 4.1 : A discrete version of Remark 3.1 is valid.

Remark 4.2 : It is possible to build up a Galerkin least squares formulation of problem (EO)′ following the ideas of [12]. Due to the fact that $\alpha < 1$ the least squares terms are different.

In order to give an example of finite element spaces for which convergence is obtained, we introduce the following notations : Ω is assumed to be polygonal in $\mathbb{R}^2$. Let $\Sigma_h = \{K\}$ be a regular triangulation of Ω by triangles. As usual h denotes the size of the mesh. Let $P_k(K)$ denote the space of polynomials of degree less than or equal to k on $K \in \Sigma_h$. We choose for H_h :

$$T_h = \left\{ \tau_h \in T \,;\, \tau_h|_K \in P_m(K)^4, \quad K \in \Sigma_h \right\} , \quad m \geq 0$$

or

$$\begin{aligned} T_h &= \left\{ \tau_h \in T \cap C^0(\bar{\Omega})^4 \,;\, \tau_h|_K \in P_m(K)^4 , \quad K \in \Sigma_h \right\} , \quad m \geq 1 , \\ V_h &= \left\{ v_h \in V \,;\, v_h|_K \in P_k(K)^2 , \quad K \in \Sigma_h \right\} , \quad k \geq 1 , \\ Q_h &= \left\{ q_h \in Q \,;\, q_h|_K \in P_\ell(K) , \quad K \in \Sigma_h \right\} , \quad \ell \geq 0 , \end{aligned}$$

or

$$Q_h = \left\{ q_h \in Q \cap C^0(\bar{\Omega}) \,;\, q_h|_K \in P_\ell(K) , \quad K \in \Sigma_h \right\} , \quad \ell \geq 1 .$$

Assume $k \geq 2$, $m = k - 1$, $\ell = k - 1$ and $Q_h \subset C^0(\bar{\Omega})$. If $\Gamma_2 = \emptyset$, it's a known fact that under suitable hypotheses on Σ_h the inf-sup condition (4.2) is satisfied (see [13, 15, 17]). If meas $(\Gamma_2) \neq 0$ it is possible, with an argument similar to the proof of Theorem 3.3, to show that this condition is also satisfied for a reasonable mesh. Then combining Theorem 4.1 with standard interpolation theory [4, 5] and assuming that $\sigma \in H^k(\Omega)^4$, $u \in H^{k+1}(\Omega)^2$ and $p \in H^k(\Omega)$, we get :

$$|\sigma - \sigma_h|_0 + |u - u_h|_1 + |p - p_h|_0 \leq C(|\sigma|_k + |u|_{k+1} + |p|_k)\, h^k ,$$

with C independent of ε when $\varepsilon \in [0, \varepsilon_0]$.

Remark 4.3 : In this example the approximation of p being continuous, if we choose $T_h \subset C^0(\bar{\Omega})^4$, then from

$$\sigma_{\text{tot}} = (\mu/\alpha)\,\sigma - \mu p I$$

and

$$\varepsilon(u) = (1/2\,\alpha)\,\sigma ,$$

we obtain a continuous approximation of σ_{tot} and $\varepsilon(u)$ by :

$$|\sigma_{\text{tot}} - ((\mu/\alpha)\,\sigma_h - \mu p_h I)|_0 + |\varepsilon(u) - (1/2\,\alpha)\,\sigma_h|_0 \leq$$
$$\leq C'(|\sigma|_k + |u|_{k+1} + |p|_k)\, h^k ,$$

with C' independent of ε when $\varepsilon \in [0, \varepsilon_0]$.

Remark 4.4 : Other families of elements satisfying (4.2) are possible ; for example the « Mini » Finite Element for the displacement with P_1 continuous approximation for the pressure [13] and P_1 continuous or P_0 discontinuous approximation of the tensors, Finite Element using discontinuous pressure [13], etc...

Remark 4.5 : When $T_h \subset C^0(\bar{\Omega})^4$, problem $(\mathrm{EO}_h)'$ can be solved by a fixed point method if $\alpha < 1/2$:

Given $\{(\sigma_n, u_n, p_n)\}_{n \geq 0} \in H_h$, $(\sigma_{n+1}, u_{n+1}, p_{n+1})$ is defined by :

$$(\sigma_n, d(v)) + 2(1-\alpha)(d(u_{n+1}), d(v)) - (p_{n+1}, \nabla \cdot v) = \langle \ell, v \rangle \quad \forall v \in V_h , \tag{4.3}$$

$$(\nabla \cdot u_{n+1}, q) + \varepsilon(p_{n+1}, q) = 0 \qquad \forall q \in Q_h , \tag{4.4}$$

and

$$(\sigma_{n+1}, \tau) - 2\,\alpha(d(u_{n+1}), \tau) = 0 \quad \forall \tau \in T_h . \tag{4.5}$$

Let (σ_h, u_h, p_h) be the solution of problem $(\mathrm{EO}_h)'$. We deduce from (4.4) and (4.5) that :

$$\varepsilon |p_{n+1} - p_h|_0^2 \leq -(\nabla \cdot (u_{n+1} - u_h), p_{n+1} - p_h) , \tag{4.6}$$

$$|\sigma_{n+1} - \sigma_h|_0 \leq 2\,\alpha\,|u_{n+1} - u_h|_1 . \tag{4.7}$$

Then from (4.3), (4.6) and (4.7) we obtain :

$$2(1-\alpha)|u_{n+1}-u_h|_1 \leq 2\,\alpha\,|u_n-u_h|_1 \tag{4.8}$$

and then :

$$\begin{aligned} |u_{n+1}-u_h|_1 &\leq (\alpha/(1-\alpha))^{n+1}\,|u_0-u_h|_1\,, \\ |\sigma_{n+1}-\sigma_h|_1 &\leq 2\,\alpha\,(\alpha/(1-\alpha))^{n+1}\,|u_0-u_h|_1\,. \end{aligned}$$

Otherwise (4.3) gives :

$$\begin{aligned} (p_{n+1}-p_h, \nabla \cdot v) = (\sigma_n - \sigma_h, d(v)) \\ + 2(1-\alpha)(d(u_{n+1}-u_h), d(v)) \quad \forall v \in V_h\,, \end{aligned}$$

and then if the discrete inf-sup condition (4.2) holds, we obtain :

$$\begin{aligned} |p_{n+1}-p_h|_0 &\leq \beta^{-1}(|\sigma_n-\sigma_h|_0 + 2(1-\alpha)|u_{n+1}-u_h|_1) \\ &\leq 4\,\beta^{-1}\,\alpha\,(\alpha/(1-\alpha))^n\,|u_0-u_h|_1\,. \end{aligned}$$

Then for $\alpha < 1/2$ the convergence of the method is obtained. □

If an iterative method is used to solve (4.3) (4.4), the cost of the global fixed point iteration will be approximately proportional to the cost of solving (4.3) (4.4).

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