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## EXISTENCE FOR A STATIONARY MODEL OF BINARY ALLOY SOLIDIFICATION (\*)

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*Abstract.* — A proof of existence is given for a stationary model of alloy solidification. The system is composed of heat equation, solute equation and Navier-Stokes equations. In the latter, Carman-Kozeny penalization of porous medium models the mushy zone. The problem is first regularized and a sequence of regularized solutions is built thanks to Leray-Schauder's fixed point Theorem. A solution is then extracted by compactness argument.

*Résumé.* — Une preuve d'existence pour un modèle de solidification d'alliage est donnée. Le système est composé de l'équation de la chaleur, de celle du soluté et de celles de Navier-Stokes. Dans ces dernières, la pénalisation des milieux poreux de Carman-Kozeny modélise la zone pâteuse. Le problème est tout d'abord régularisé et une suite de solutions régularisées est construite grâce au théorème de point fixe de Leray-Schauder. Une solution est ensuite extraite par un argument de compacité.

In a macroscopic approach, the solidification of a binary alloy is governed by the heat and solute equations coupled with Navier-Stokes equations. The latter ones contain a penalization term so that they apply not only to the zone where the alloy is liquid but also to the zone where both liquid and solid states coexist. This is the so-called mushy zone.

Since the liquid and mushy zones are *a priori* unknown the evolutive problem is very hard to handle mathematically, even for a simpler case [5]. We consider here the stationary problem and we prove an existence result using Leray-Schauder's homotopy Theorem (see Theorem 10.3, p. 222 in [7]). This work extends [4] and [2] to the alloy case.

The outline of this paper is as follows : Section 1 is devoted to a description of the physical model. In Section 2, the assumptions on the data are stated and

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the definition of a weak solution is given. A regularized problem is studied in Section 3 and our main existence result is established in Section 4.

To complete this introduction, we give some notations. We denote by  $W^{m,p}(\Omega)$  the space of functions in  $L^p(\Omega)$  whose derivatives up to the  $m$ -th order are  $L^p$ -integrable. For  $p = 2$  we shall write  $H^m(\Omega)$ . The space  $H_0^1(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  with respect to the norm  $\| \cdot \|_{H^1}$ . For conciseness of writing, we note  $(f|g)_\Omega = \int_\Omega fg \, d\Omega$  for  $f \in L^p(\Omega)$ ,  $g \in L^q(\Omega)$  and  $(\mathbf{f}|\mathbf{g})_\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{g} \, d\Omega = \int_\Omega \sum_{i=1}^n f_i g_i \, d\Omega$  for  $\mathbf{f} \in (L^p(\Omega))^n$ ,  $\mathbf{g} \in (L^q(\Omega))^n$  where  $p$  and  $q$  are conjugate exponents ( $p^{-1} + q^{-1} = 1$ ). Moreover, the study of Navier-Stokes equations requires the space  $\mathcal{V}(\Omega) = \{ \mathbf{u} \in (H_0^1(\Omega))^n \mid \operatorname{div}(\mathbf{u}) = 0 \}$  equipped with the norm  $\| \mathbf{u} \|_{\mathcal{V}(\Omega)} = ((\mathbf{u}|\mathbf{u})_\Omega)^{1/2}$  where, for  $\mathbf{f}, \mathbf{g} \in (H^1(\Omega))^n$ ,

$$((\mathbf{f}|\mathbf{g}))_\Omega = \int_\Omega \sum_{i=1}^n \nabla f_i \cdot \nabla g_i \, d\Omega .$$

**1. MODELLING**

Let us consider the solidification in a domain  $\Omega$  of a binary alloy composed of a principal compound (say Aluminium) in which there is a solute (say Silicon at 7 %). As time goes on, three domains  $\Omega_\ell$ ,  $\Omega_m$  and  $\Omega_s$  gradually appear, evolve, or disappear within  $\Omega$ . The first domain is fully liquid, the second is a fine mixing of both solid and liquid, and the third is fully solid (see *fig. 1.1*).

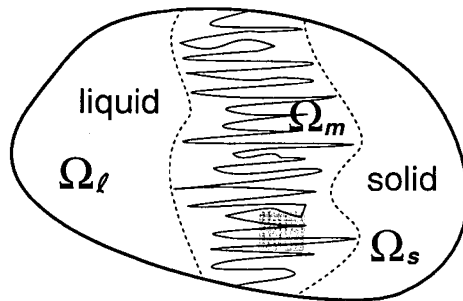


Figure 1.1. — Domains with a macroscopic box.

The mixing is due to the dendritic growth of the solid into the liquid, whose proportions are given by the solid fraction  $f_s$  and the liquid fraction  $f_l = 1 - f_s$ . In the following,  $\Omega_{ml} = \Omega \setminus \bar{\Omega}_s$  will refer to the domain not fully solid (*i.e.* fully liquid or with a mixing).

An averaged variable is introduced as a convenient treatment of the mixing. Take a macroscopic isothermal box containing a part of the dendritic zone (see *fig. 1.1*). We denote by  $c_l$  and  $c_s$  the concentrations of the solute in the liquid and solid, and define the mean concentration  $\bar{c}$  by the relation

$$\bar{c} = c_l f_l + c_s f_s = c_l (1 - f_s) + c_s f_s .$$

Clearly in  $\Omega_l$  we have  $f_s = 0$ , in  $\Omega_s$  we have  $f_s = 1$  and in  $\Omega_m$ , the solid fraction  $f_s$  will take a value given by a phase diagram describing the phase changes (see *fig. 1.2*). The temperature of the alloy is  $\theta$  with  $\theta_F$  and  $\theta_E$  being the fusion and eutectic temperature respectively.

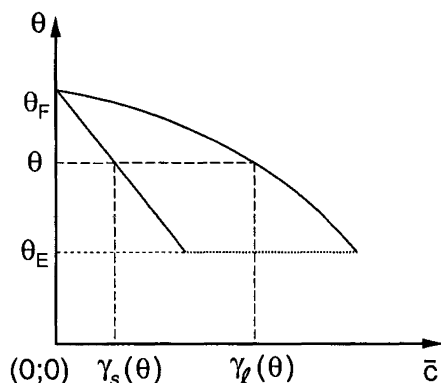


Figure 1.2. — Phase diagram.

On one hand, this diagram gives, for a given temperature  $\theta \in (\theta_E, \theta_F)$ , the concentration  $\gamma_s(\theta)$  of the solute for which the solid alloy begins to melt and the concentration  $\gamma_l(\theta)$  of the solute for which the liquid alloy begins to solidify. On the other hand it indicates the concentrations of solute in solid and liquid phases which can coexist at the thermodynamical equilibrium. The mushy zone corresponds in the phase diagram to points inbetween  $\gamma_s(\theta)$  and  $\gamma_l(\theta)$ .

We assume that the process of solidification holds at the thermodynamical equilibrium. In the mushy zone, where  $\gamma_l(\theta) < \bar{c} < \gamma_s(\theta)$  and  $\theta_E < \theta < \theta_F$ , we have

$$\bar{c} = \gamma_l(\theta) (1 - f_s) + \gamma_s(\theta) f_s$$

and therefore

$$f_s(\bar{c}, \theta) = \frac{\bar{c} - \gamma_\ell(\theta)}{\gamma_\ell(\theta) - \gamma_\ell(\theta)}$$

We extend  $f_s$  by 1 and 0 in the solid and liquid zone respectively. We have seen that in the mushy zone we had  $c_\ell = \gamma_\ell(\theta)$ ; clearly  $c_\ell = \bar{c}$  in the liquid zone. Denoting by  $\mathbf{v}$  the velocity field and by  $p$  the pressure our model ( $P$ ) reads

$$- \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p + \frac{C_0 f_s^2(\bar{c}, \theta)}{(1 - f_s(\bar{c}, \theta))^3} \mathbf{v} = \mathbf{F}(\bar{c}, \theta) \text{ in } \Omega_{ml}, \tag{1.1}$$

$$\operatorname{div}(\mathbf{v}) = 0 \text{ in } \Omega_{ml}, \tag{1.2}$$

$$\mathbf{v} = 0 \text{ in } \Omega_s, \tag{1.3}$$

$$- \Delta \theta + \mathbf{v} \cdot \nabla \theta = 0 \text{ in } \Omega, \tag{1.4}$$

$$- \Delta \bar{c} + \mathbf{v} \cdot \nabla c_\ell = 0 \text{ in } \Omega, \tag{1.5}$$

The term  $\mathbf{F}(\bar{c}, \theta)$  is here the external force acting upon the liquid. Using Boussinesq's approximation, it is given by  $\mathbf{F}(\bar{c}, \theta) = \rho \mathbf{g} (C_1(\theta - \theta_r) + C_2(c_\ell - c_r))$  where  $\rho$  is the mean value of the density,  $\mathbf{g}$  is the gravity,  $C_1$  and  $C_2$  are two real constants, and  $\theta_r, c_r$  are reference temperature and concentration. This system is completed by Dirichlet's boundary conditions for  $\mathbf{v}$  and  $\theta$  and Neumann's boundary conditions for  $\bar{c}$ . We set  $\mathbf{v} = \mathbf{0}$  on  $\partial\Omega_{ml}$ ,  $\theta = \theta_\delta$  and  $\partial_n \bar{c} = 0$  on  $\partial\Omega$ . We finally mention that the penalization term  $K(f_s) \mathbf{v}$  with  $K(x) = C_0 x^2 / (1 - x)^3$  which appears in Navier-Stokes equations and which accounts for the mushy effect is due to Carman and Kozeny. We refer to [1] for a complete description of this model.

2. ASSUMPTIONS AND WEAK FORMULATION

Throughout this paper we assume that  $\Omega, \gamma_s, \gamma_\ell, \mathbf{F}, K$  and  $\theta_\delta$  satisfy the assumptions mentioned below. The first concerns the domain  $\Omega$ :

$$(H_\Omega) \left\{ \begin{array}{l} \Omega \subset \mathbb{R}^n \text{ where } n = 2 \text{ or } 3 \text{ is an open, bounded,} \\ \text{connected set with a boundary } \partial\Omega \text{ of class } C^2. \end{array} \right.$$

Concerning the solidus and liquidus curves, we assume:

$$(H_\gamma) \left\{ \begin{array}{l} \gamma_s, \gamma_\ell \text{ are of class } C^1([\theta_E, \theta_F]), \\ \text{they are strictly decreasing, } \gamma_s < \gamma_\ell \text{ on } [\theta_E, \theta_F], \\ \gamma_s(\theta_F) = \gamma_\ell(\theta_F) = 0. \end{array} \right.$$

With respect to  $\mathbf{F}$  we assume :

$$(H_F) \left\{ \begin{array}{l} \mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^n \text{ is uniformly Lipschitz} \\ \text{continuous and } \mathbf{F}(0, 0) = \mathbf{0} . \end{array} \right.$$

The penalization will not be restricted to Carman-Kozeny's term. We assume here :

$$(H_K) \left\{ \begin{array}{l} K \text{ is of class } C^1(-\infty, 1), \\ K \text{ is nondecreasing, } K \equiv 0 \text{ on } \mathbb{R}_- \text{ and} \\ \lim_{x \uparrow 1} K(x) = +\infty . \end{array} \right.$$

Observe that Carman-Kozeny's penalization

$$K(x) = \frac{C_0 x^2}{(1-x)^3},$$

for  $x \in [0, 1)$ , can be extended to  $(-\infty, 0)$  by 0 and so satisfies the hypothesis. We finally assume that the boundary condition  $\theta_\delta$  satisfies

$$(H_{\theta_\delta}) \left\{ \begin{array}{l} \theta_\delta \in W^{4/3, 3/2}(\partial\Omega) \cap C^\beta(\partial\Omega) \\ \text{for some } \beta \in (0, 1) \\ \text{and } \theta_\delta \geq 0 \text{ on } \partial\Omega . \end{array} \right.$$

The same symbol  $\theta_\delta$  denotes a function in  $W^{2, 3/2}(\Omega)$  whose trace on  $\partial\Omega$  is precisely  $\theta_\delta$ .

To define the weak formulation of Problem (P) we need a precise definition of the solid fraction  $f_s$ . Let  $\mathcal{O}_s$ ,  $\mathcal{O}_m$  and  $\mathcal{O}_\ell$  be the open sets given by figure 2.1. We recall that

$$f_s(\bar{c}, \theta) = \begin{cases} 0 & \text{in } \mathcal{O}_\ell, \\ \frac{\bar{c} - \gamma_\ell(\theta)}{\gamma_s(\theta) - \gamma_\ell(\theta)} & \text{in } \mathcal{O}_m, \\ 1 & \text{in } \mathcal{O}_s, \end{cases}$$

and we extend  $f_s$  to the boundaries of  $\mathcal{O}_i$  by

$$f_s(0, \theta_F) = 1, \quad f_s(\bar{c}, \theta_E) = 1,$$

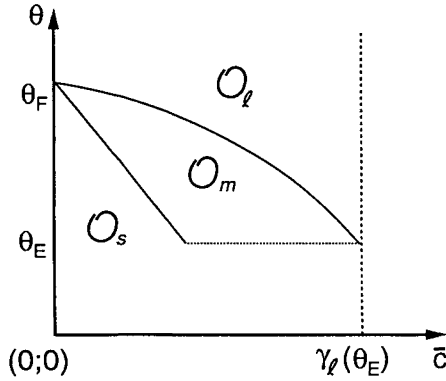


Figure 2.1. — Sets of phase diagram.

for all  $\bar{c} \in (\gamma_s(\theta_E); \gamma_l(\theta_E)]$ , and by continuity otherwise. For technical reasons we set  $f_s(\bar{c}, \theta) = 1$  for  $(\bar{c}; \theta) \in \mathbb{R}^2 \setminus \overline{\mathcal{O}_s \cup \mathcal{O}_m \cup \mathcal{O}_l}$ . Moreover it follows from the phase diagram that the solute concentration  $c_l$  is given by  $\alpha(\bar{c}, \theta)$  where

$$\alpha(\bar{c}, \theta) = \begin{cases} \gamma_l(\theta) & \text{in } \mathcal{O}_m, \\ \bar{c} & \text{in } \mathcal{O}_l. \end{cases}$$

We extend  $\alpha$  to  $\overline{\mathcal{O}_m \cup \mathcal{O}_l}$  by continuity. For the comparison principles to hold, the extension of  $\alpha$  to  $\mathbb{R}^2$  is not fully arbitrary. Level sets of extended  $\alpha$  are shown in figure 2.2 (where  $0 < a < \gamma_l(\theta_E)$ ).

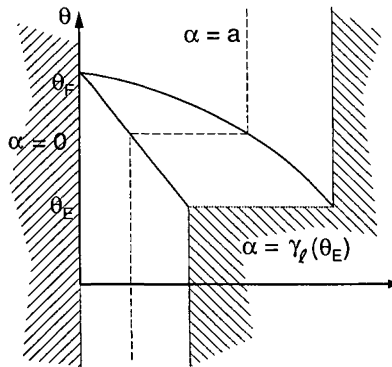


Figure 2.2. — Level sets of extended  $\alpha$ .

DEFINITION 2.1 : Under the assumptions  $(H_\Omega)$ ,  $(H_\gamma)$ ,  $(H_F)$ ,  $(H_K)$  and  $(H_{\theta_s})$ , the triple  $(\mathbf{v}; \theta; \bar{c}) \in \mathcal{V}(\Omega) \times (H^1(\Omega) \cap C^0(\Omega))^2$  is a weak solution to Problem (P) if the relations

$$\begin{aligned} \nu((\mathbf{v}|\mathbf{w}))_{\Omega_{m\ell}} + ((\mathbf{v} \cdot \nabla) \mathbf{v}|\mathbf{w})_{\Omega_{m\ell}} + (K(f_s(\bar{c}, \theta)) \mathbf{v}|\mathbf{w})_{\Omega_{m\ell}} &= (\mathbf{F}(\bar{c}, \theta)|\mathbf{w})_{\Omega_{m\ell}}, \\ (\nabla\theta|\nabla\varphi)_\Omega + (\mathbf{v} \cdot \nabla\theta|\varphi)_\Omega &= 0, \\ (\nabla\bar{c}|\nabla\eta)_\Omega + (\mathbf{v} \cdot \nabla\alpha(\bar{c}, \theta)|\eta)_\Omega &= 0, \end{aligned}$$

hold for all  $\mathbf{w} \in \mathcal{V}(\Omega)$  with  $\text{supp}(\mathbf{w}) \subset \Omega_{m\ell}$ ,  $\varphi \in H_0^1(\Omega)$  and  $\eta \in H^1(\Omega)$  and moreover if  $\mathbf{v} = 0$  in  $\Omega_s$ ,  $\theta \geq 0$ ,  $0 \leq \bar{c} \leq \gamma_\ell(\theta_E)$  in  $\Omega$ ,  $\theta - \theta_\delta \in H_0^1(\Omega)$ ,  $\int_\Omega \bar{c} \, d\Omega = \bar{c}_g$  where  $\bar{c}_g$  is a data satisfying  $0 \leq \bar{c}_g \leq \text{meas}(\Omega) \gamma_\ell(\theta_E)$ . The two sets  $\Omega_s$  and  $\Omega_{m\ell}$  are defined by

$$\begin{aligned} \Omega_s &= \{ \mathbf{x} \in \Omega \mid (\bar{c}(\mathbf{x}); \theta(\mathbf{x})) \in \overline{\mathcal{O}_s} \setminus \partial\mathcal{O}_m \}, \\ \Omega_{m\ell} &= \{ \mathbf{x} \in \Omega \mid (\bar{c}(\mathbf{x}); \theta(\mathbf{x})) \in \overline{\mathcal{O}_m \cup \mathcal{O}_\ell} \setminus \partial\mathcal{O}_s \}. \end{aligned}$$

The remaining part of this work establishes the theorem.

THEOREM 2.2 : There exists a weak solution to Problem (P).

### 3. A REGULARIZED PROBLEM

In this section we introduce a regularized problem and we prove an existence result using Leray-Schauder's homotopy Theorem. For  $\varepsilon \in (0, 1]$ , let  $\mathcal{F}_\varepsilon \in C_0^\infty(\mathbb{R}^2)$  be a family of mollifiers. Functions  $f_s^\varepsilon$  and  $\alpha^\varepsilon$  are regularizations of  $f_s$  and  $\alpha$  given by  $f_s^\varepsilon = \mathcal{F}_\varepsilon * f_s$  and  $\alpha^\varepsilon = \mathcal{F}_\varepsilon * \alpha$ .

THEOREM 3.1 : For each  $\varepsilon \in (0, 1]$  there exists a triple  $(\mathbf{v}_\varepsilon; \theta_\varepsilon; \bar{c}_\varepsilon) \in \mathcal{V}(\Omega) \times (H^1(\Omega))^2$ , where  $\theta_\varepsilon - \theta_\delta \in H_0^1(\Omega)$  and  $\int_\Omega \bar{c}_\varepsilon \, d\Omega = \bar{c}_g$ , verifying Problem  $(P_\varepsilon)$ :

$$\begin{aligned} \nu((\mathbf{v}_\varepsilon|\mathbf{w}))_\Omega + ((\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon|\mathbf{w})_\Omega + \\ + (K(f_s^\varepsilon(\bar{c}_\varepsilon, \theta_\varepsilon) - \varepsilon) \mathbf{v}_\varepsilon|\mathbf{w})_\Omega &= (\mathbf{F}(\bar{c}_\varepsilon, \theta_\varepsilon)|\mathbf{w})_\Omega, \end{aligned} \tag{3.1}$$

$$(\nabla\theta_\varepsilon|\nabla\varphi)_\Omega + (\mathbf{v}_\varepsilon \cdot \nabla\theta_\varepsilon|\varphi)_\Omega = 0, \tag{3.2}$$

$$(\nabla\bar{c}_\varepsilon|\nabla\eta)_\Omega + (\mathbf{v}_\varepsilon \cdot \nabla\alpha^\varepsilon(\bar{c}_\varepsilon, \theta_\varepsilon)|\eta)_\Omega = 0, \tag{3.3}$$



for all  $\mathbf{w} \in \mathcal{V}(\Omega)$ ,  $\varphi \in H_0^1(\Omega)$  and  $\eta \in H^1(\Omega)$ . Moreover any solution  $(\mathbf{v}_\varepsilon; \theta_\varepsilon; \bar{c}_\varepsilon) \in \mathcal{V}(\Omega) \times (H^1(\Omega))^2$  of the above problem is bounded by a constant independent of  $\varepsilon$ .

*Remark 3.2 :* It is interesting to note the following non-trivial result (see [3]) : let  $\mathbf{v}_\varepsilon \in (H_0^1(\Omega))^2$  with  $\text{div}(\mathbf{v}_\varepsilon) = 0$  and  $\theta_\varepsilon \in L^2(\Omega)$  be given, then equation (3.3) has a *unique* solution  $\bar{c}_\varepsilon \in H^1(\Omega)$  such that  $\int_\Omega \bar{c}_\varepsilon d\Omega = \bar{c}_g$ .

Theorem 3.1 will be a consequence of the following lemmata. We first introduce four operators and prove that they are well-defined and satisfy continuity and compactness properties. Let  $\chi$  be the product space  $\mathcal{V}(\Omega) \times (H^1(\Omega))^2$  and let  $G_1 : \chi \rightarrow (L^{3/2}(\Omega))^n$ ,  $G_2 : \chi \rightarrow L^{3/2}(\Omega)$ , and  $G_3 : \chi \rightarrow L^{3/2}(\Omega)$  be three operators defined by

$$G_1(\mathbf{v}, \theta, c) = -(\mathbf{v} \cdot \nabla) \mathbf{v} - K(f_s^\varepsilon(c, \theta) - \varepsilon) \mathbf{v} + \mathbf{F}(c, \theta), \tag{3.4}$$

$$G_2(\mathbf{v}, \theta, c) = -\mathbf{v} \cdot \nabla \theta, \tag{3.5}$$

$$G_3(\mathbf{v}, \theta, c) = -\mathbf{v} \cdot \nabla \alpha^\varepsilon(c, \theta). \tag{3.6}$$

We further introduce the operator

$$\begin{aligned} \mathbf{G} : \chi &\rightarrow \chi \\ (\mathbf{v} ; \theta ; c) &\mapsto (\mathbf{u} ; \hat{\theta} ; \hat{c}) \end{aligned}$$

where  $(\mathbf{u} ; \hat{\theta} ; \hat{c}) \in \chi$  is the unique solution to the system

$$\nu((\mathbf{u} | \mathbf{w}))_\Omega = (G_1(\mathbf{v}, \theta, c) | \mathbf{w})_\Omega, \tag{3.7}$$

$$(\nabla \hat{\theta} | \nabla \varphi)_\Omega = (G_2(\mathbf{v}, \theta, c) | \varphi)_\Omega, \tag{3.8}$$

$$(\nabla \hat{c} | \nabla \eta)_\Omega = (G_3(\mathbf{v}, \theta, c) | \eta)_\Omega, \tag{3.9}$$

for all  $\mathbf{w} \in \mathcal{V}(\Omega)$ ,  $\varphi \in H_0^1(\Omega)$  and  $\eta \in H^1(\Omega)$ , which also satisfies the conditions  $\hat{\theta} - \theta_\delta \in H_0^1(\Omega)$  and  $\int_\Omega \hat{c} d\Omega = \bar{c}_g$ .

Clearly  $(\mathbf{v}_\varepsilon ; \theta_\varepsilon ; \bar{c}_\varepsilon)$  is a solution of (3.1)-(3.3) if and only if it is a fixed point of the operator  $\mathbf{G}$ . In the following, we prove that  $\mathbf{G}$  has at least one fixed point and consequently the first part of Theorem 3.1 will be established.

**LEMMA 3.3 :** *The operators  $G_1, G_2$  and  $G_3$  are well-defined and continuous.*

*Proof :* Since  $\mathbf{v} \in \mathcal{V}(\Omega)$  and  $n = 2$  or  $3$ , then  $\mathbf{v} \in (L^6(\Omega))^n$  and  $(\mathbf{v} \cdot \nabla) \mathbf{v} \in (L^{3/2}(\Omega))^n$ . The continuity of  $G_1$  is a consequence of the Lipschitz-continuity of the functions  $K(f_s^\varepsilon(\dots) - \varepsilon)$  and  $\mathbf{F}(\dots)$  seen as

functions from  $\mathbb{R}^2$  into  $\mathbb{R}$  and  $\mathbb{R}^n$  respectively. To check the continuity of  $G_2$  and  $G_3$  in  $\chi$  with respect to the product norm, we take a sequence  $(\mathbf{v}_m; \theta_m; c_m) \in \chi$  converging to  $(\mathbf{v}; \theta; c) \in \chi$ . Using Hölder's inequality we get

$$\begin{aligned} \|G_2(\mathbf{v}; \theta; c) - G_2(\mathbf{v}_m; \theta_m; c_m)\|_{L^{3/2}(\Omega)} &\leq \| \mathbf{v} - \mathbf{v}_m \|_{(L^6(\Omega))^n} \| \nabla \theta \|_{(L^2(\Omega))^n} + \\ &+ \| \mathbf{v}_m \|_{(L^6(\Omega))^n} \| \nabla \theta - \nabla \theta_m \|_{(L^2(\Omega))^n} \end{aligned}$$

and therefore

$$\lim_{m \rightarrow \infty} \|G_2(\mathbf{v}; \theta; c) - G_2(\mathbf{v}_m; \theta_m; c_m)\|_{L^{3/2}(\Omega)} = 0$$

which proves the continuity of  $G_2$ .

The continuity of  $G_3$  follows from the relations

$$\lim_{m \rightarrow \infty} \|(\partial_1 \alpha^\varepsilon(c, \theta) - \partial_1 \alpha^\varepsilon(c_m, \theta_m)) \mathbf{v} \cdot \nabla c\|_{L^{3/2}(\Omega)} = 0, \tag{3.10}$$

$$\lim_{m \rightarrow \infty} \|(\partial_2 \alpha^\varepsilon(c, \theta) - \partial_2 \alpha^\varepsilon(c_m, \theta_m)) \mathbf{v} \cdot \nabla \theta\|_{L^{3/2}(\Omega)} = 0. \tag{3.11}$$

To ease the reading, we set

$$g_m = |(\partial_1 \alpha^\varepsilon(c, \theta) - \partial_1 \alpha^\varepsilon(c_m, \theta_m)) \mathbf{v} \cdot \nabla c|^{3/2}.$$

Since  $\partial_1 \alpha^\varepsilon(\dots)$  is continuous, there exists a subsequence  $g_k$  of  $g_m$  such that  $g_k$  converges to 0 almost everywhere in  $\Omega$ . Further  $|g_k| \leq (2 \|\partial_1 \alpha^\varepsilon\|_{L^\infty(\mathbb{R}^2)} |\mathbf{v} \cdot \nabla c|)^{3/2}$  almost everywhere in  $\Omega$ . So  $\lim_{k \rightarrow \infty} \|g_k\|_{L^1(\Omega)} = 0$  by Lebesgue's Dominated Convergence Theorem. We complete the proof of (3.10) by observing that the sequence  $g_m$  itself converges to 0 in  $L^1(\Omega)$ . The proof of (3.11) is very similar. ■

LEMMA 3.4 : *The operator  $\mathbf{G}$  is well-defined continuous and compact.*

*Proof :* It follows from Proposition 2.3, p. 30 in [9] for Stokes problem and from standard regularity results for the inverse of the operator  $-\Delta$  that the operator  $\mathbf{G}$  is continuous from  $\chi$  into  $(W^{2,3/2}(\Omega))^n \times (W^{2,3/2}(\Omega))^2$ . For  $n \leq 3$  the embedding  $W^{2,3/2}(\Omega) \subset H^1(\Omega)$  is compact and  $\mathbf{G}$  is compact.

The *a priori* bounds for fixed points of operator  $\lambda \mathbf{G}$ ,  $\lambda \in [0, 1]$  are direct consequences of the next lemma. The case  $\lambda = 0$  being trivial, we only consider the cases  $\lambda \in (0, 1]$ .

LEMMA 3.5 : Let  $(\mathbf{v}_\lambda; \theta_\lambda; c_\lambda) \in \chi$  be a fixed point of  $\lambda G$ ,  $\lambda \in (0, 1]$ . Then the following estimates hold almost everywhere in  $\Omega$  :

$$\lambda \operatorname{ess\,inf}_{\partial\Omega} \theta_\delta \leq \theta_\lambda \leq \lambda \operatorname{ess\,sup}_{\partial\Omega} \theta_\delta, \tag{3.12}$$

$$-\varepsilon \leq c_\lambda \leq \gamma_\ell(\theta_E) + \varepsilon. \tag{3.13}$$

*Proof:* First we notice that  $\theta_\lambda = \lambda\theta_\delta$  on the boundary  $\partial\Omega$ . If we set  $w = \lambda \operatorname{ess\,sup}_{\partial\Omega} \theta_\delta$  (considered as a constant function on  $\partial\Omega$ ) and if we consider  $[\theta_\lambda - w]_+$  as the positive part of  $\theta_\lambda - w$ , we have  $[\theta_\lambda - w]_+ \in H^1_0(\Omega)$ . By choosing  $\varphi = [\theta_\lambda - w]_+$  in the equation corresponding to (3.2) in which we have replaced  $\mathbf{v}_\varepsilon \cdot \nabla \theta_\varepsilon$  by  $\lambda \mathbf{v}_\lambda \cdot \nabla \theta_\lambda$ , we obtain

$$(\nabla \theta_\lambda | \nabla [\theta_\lambda - w]_+)_\Omega + \lambda (\mathbf{v}_\lambda \cdot \nabla \theta_\lambda | [\theta_\lambda - w]_+)_\Omega = 0.$$

Consequently

$$(\nabla [\theta_\lambda - w]_+ | \nabla [\theta_\lambda - w]_+)_\Omega + \lambda (\mathbf{v}_\lambda \cdot \nabla [\theta_\lambda - w]_+ | [\theta_\lambda - w]_+)_\Omega = 0$$

and we use the fact that for  $\mathbf{v} \in \mathcal{V}(\Omega)$  and  $\psi \in H^1(\Omega)$  the relation  $(\mathbf{v} \cdot \nabla \psi | \psi)_\Omega = 0$  holds to conclude that

$$(\nabla [\theta_\lambda - w]_+ | \nabla [\theta_\lambda - w]_+)_\Omega = 0.$$

Hence  $[\theta_\lambda - w]_+ = 0$  and  $\theta_\lambda \leq \lambda \operatorname{ess\,sup}_{\partial\Omega} \theta_\delta$  almost everywhere in  $\Omega$ . The lower bound is proved similarly using the test function  $\varphi = [\theta_\lambda - \lambda \operatorname{ess\,inf}_{\partial\Omega} \theta_\delta]_-$ .

Concerning the concentration equation we note that for all  $c \geq \gamma_\ell(\theta_E) + \varepsilon$

$$\alpha^\varepsilon(c, \theta) = \alpha(\gamma_\ell(\theta_E), \theta_E).$$

Using the test function  $\eta = [c_\lambda - \gamma_\ell(\theta_E) - \varepsilon]_+ \in H^1(\Omega)$  in the equation corresponding to (3.3) in which we have replaced  $\mathbf{v}_\varepsilon \cdot \nabla \alpha^\varepsilon(\bar{c}_\varepsilon, \theta_\varepsilon)$  by  $\lambda \mathbf{v}_\lambda \cdot \nabla \alpha^\varepsilon(\bar{c}_\lambda, \theta_\lambda)$ , we get

$$(\nabla [c_\lambda - \gamma_\ell(\theta_E) - \varepsilon]_+ | \nabla [c_\lambda - \gamma_\ell(\theta_E) - \varepsilon]_+)_\Omega = 0$$

and hence  $[c_\lambda - \gamma_\ell(\theta_E) - \varepsilon]_+$  is constant. Since  $\int_\Omega c_\lambda \, d\Omega = \lambda \bar{c}_g \leq \operatorname{meas}(\Omega) \gamma_\ell(\theta_E)$ , we have  $[c_\lambda - \gamma_\ell(\theta_E) - \varepsilon]_+ = 0$  and the upper bound holds. Finally we use the value  $\alpha^\varepsilon(c, \theta) = 0$  for all  $c \leq -\varepsilon$  and the test function  $\eta [c_\lambda + \varepsilon]_-$  to check the lower bound. ■

LEMMA 3.6 : *The operator  $\mathbf{G}$  has a fixed point.*

*Proof:* We check that any fixed point of  $\lambda\mathbf{G}$ ,  $\lambda \in [0, 1]$  is bounded in  $\chi$ . We choose  $\mathbf{w} = \mathbf{v}_\lambda$  in equation

$$v((\nabla \mathbf{v}_\lambda | \nabla \mathbf{w}))_\Omega = \lambda(G_1(\mathbf{v}_\lambda, \theta_\lambda, c_\lambda) | \mathbf{w})_\Omega$$

(see (3.8)) and using the relation  $\int_\Omega (\mathbf{v}_\lambda \cdot \nabla) \mathbf{v}_\lambda \cdot \mathbf{v}_\lambda \, d\Omega = 0$  we conclude that

$$v \|\mathbf{v}_\lambda\|_{\mathcal{V}(\Omega)}^2 + \lambda(K(f_s^e(c_\lambda, \theta_\lambda) - \varepsilon) \mathbf{v}_\lambda | \mathbf{v}_\lambda)_\Omega = \lambda(\mathbf{F}(c_\lambda, \theta_\lambda) | \mathbf{v}_\lambda)_\Omega.$$

Since  $K(\cdot)$  is non-negative and  $\mathbf{F}$  has the Lipschitz property we get

$$v \|\mathbf{v}_\lambda\|_{\mathcal{V}(\Omega)}^2 \leq L_F(\|c_\lambda\|_{L^2(\Omega)} + \|\theta_\lambda\|_{L^2(\Omega)}) \|\mathbf{v}_\lambda\|_{(L^2(\Omega))^n}.$$

Poincaré's inequality and estimates (3.12) and (3.13) allow us to conclude that there exist a constant  $C$  independent of  $\lambda$  such that

$$\|\mathbf{v}_\lambda\|_{\mathcal{V}} \leq C.$$

We finally choose  $\varphi = \theta_\lambda - \lambda\theta_\delta$  and  $\eta = c_\lambda$  in equations

$$(\nabla \theta_\lambda | \nabla \varphi)_\Omega = \lambda(G_2(\mathbf{v}_\lambda, \theta_\lambda, c_\lambda) | \varphi)_\Omega,$$

$$(\nabla c_\lambda | \nabla \eta)_\Omega = \lambda(G_3(\mathbf{v}_\lambda, \theta_\lambda, c_\lambda) | \eta)_\Omega,$$

to prove that  $\theta_\lambda$  and  $c_\lambda$  are bounded in  $H^1(\Omega)$  uniformly with respect to  $\lambda$ . This lemma is now a consequence of Leray-Schauder's homotopy Theorem. ■

*Proof of Theorem 3.1 :* As said before, the first part of Theorem 3.1 is immediate since a fixed point of  $\mathbf{G}$  is a solution of Problem  $(P_\varepsilon)$ . Moreover, it is easy to see that any solution  $(\mathbf{v}_\lambda; \theta_\varepsilon; \bar{c}_\varepsilon) \in \chi$  of Problem  $(P_\varepsilon)$  is uniformly bounded with respect to  $\varepsilon$ . ■

#### 4. EXISTENCE OF A WEAK SOLUTION

We are now in position to prove our main result. As a consequence of Theorem 3.1, for  $\varepsilon \in (0, 1]$ , any solution  $(\mathbf{v}_\varepsilon, \theta_\varepsilon, \bar{c}_\varepsilon) \in \chi$  of Problem  $(P_\varepsilon)$  is uniformly bounded in  $\chi$  with respect to  $\varepsilon$ . By compact embeddings, there exists  $(\mathbf{v}; \theta; \bar{c}) \in \chi$  and a sequence, still indexed by  $\varepsilon$ , such that

$$\begin{aligned} \mathbf{v}_\varepsilon &\rightarrow \mathbf{v} && \text{strongly in } (L^4(\Omega))^n, \\ \theta_\varepsilon &\rightarrow \theta && \text{strongly in } L^2(\Omega), \\ \bar{c}_\varepsilon &\rightarrow \bar{c} && \text{strongly in } L^2(\Omega), \\ \nabla \mathbf{v}_\varepsilon &\rightharpoonup \nabla \mathbf{v} && \text{weakly in } (L^2(\Omega))^n, \\ \theta_\varepsilon &\rightharpoonup \theta && \text{weakly in } H^1(\Omega), \\ \bar{c}_\varepsilon &\rightharpoonup \bar{c} && \text{weakly in } H^1(\Omega). \end{aligned}$$

Furthermore, thanks to Theorem 14.1, p. 201 in [8] and Proposition 6, p. 683 in [6] there exists a real  $a \in (0, 1)$  and a non-negative real  $M$  such that

$$\|\theta_\varepsilon\|_{C^a(\bar{\Omega})} + \|c_\varepsilon\|_{C^a(\bar{\Omega})} \leq M.$$

Because of Arzela-Ascoli's Theorem, there is a sequence such that

$$\begin{aligned} \theta_\varepsilon &\rightarrow \theta \text{ in } C^0(\bar{\Omega}), \\ \bar{c}_\varepsilon &\rightarrow \bar{c} \text{ in } C^0(\bar{\Omega}). \end{aligned}$$

We check now that  $(\mathbf{v}; \theta; \bar{c}) \in \chi$  is a solution to Problem  $(P)$ . Recall that  $\Omega_{m\ell} = \{\mathbf{x} \in \Omega \mid (\bar{c}(\mathbf{x}); \theta(\mathbf{x})) \in \overline{\mathcal{O}_m \cup \mathcal{O}_\ell} \setminus \partial\mathcal{O}_s\}$  and let  $\mathbf{w} \in (C_0^\infty(\Omega))^n$  with its compact support  $\mathcal{K} \subset \Omega_{m\ell}$ . Note that for a certain  $\delta \in (0, \infty)$ , and for every  $\mathbf{x} \in \mathcal{K}$ ,

$$f_s(\bar{c}(\mathbf{x}), \theta(\mathbf{x})) < 1 - \delta.$$

Due to the uniform convergence of  $f_s^\varepsilon$  towards  $f_s$  on any compact subset of  $\Omega_{m\ell}$ , there is an  $\varepsilon_\delta$  such that for all  $\varepsilon \in (0, \varepsilon_\delta)$  and for all  $\mathbf{x} \in \mathcal{K}$ ,

$$f_s^\varepsilon(\bar{c}_\varepsilon(\mathbf{x}), \theta_\varepsilon(\mathbf{x})) < 1 - \frac{\delta}{2}.$$

Consequently  $K(f_s^\varepsilon(\bar{c}_\varepsilon, \theta_\varepsilon) - \varepsilon)$  converges to  $K(f_s(\bar{c}, \theta))$  in  $C^0(\mathcal{K})$ . Using the properties of the subsequences and passing to the limit in the velocity equation (3.1) we are led to

$$\begin{aligned} \nu((\mathbf{v}|\mathbf{w}))_{\Omega_{m\ell}} + ((\mathbf{v} \cdot \nabla) \mathbf{v}|\mathbf{w})_{\Omega_{m\ell}} + (K(f_s(\bar{c}, \theta)) \mathbf{v}|\mathbf{w})_{\Omega_{m\ell}} = \\ = (\mathbf{F}(\bar{c}, \theta)|\mathbf{w})_{\Omega_{m\ell}}. \end{aligned}$$

By density and definition of  $\mathcal{V}(\Omega_{m\ell})$ , this equation holds for any  $\mathbf{w} \in \mathcal{V}(\Omega_{m\ell})$  with  $\text{supp}(\mathbf{w}) \subset \Omega_{m\ell}$ .

To check that  $\mathbf{v} = \mathbf{0}$  in  $\Omega_s$ , take a compact set  $\mathcal{K} \subset \Omega_s$ . Since the solid domain is open, there is an  $\varepsilon_{\mathcal{K}} > 0$  such that

$$f_s^\varepsilon(\bar{c}_\varepsilon, \theta_\varepsilon) = 1 \quad \text{in } \mathcal{K}$$

whenever  $\varepsilon \in (0, \varepsilon_{\mathcal{K}})$ . Choosing  $\mathbf{w} = \mathbf{v}_\varepsilon$  into the regularized velocity equation (3.1) and using Hypothesis  $(H_K)$ , it comes

$$K(1 - \varepsilon) (\mathbf{v}_\varepsilon|\mathbf{v}_\varepsilon)_{\mathcal{K}} \leq C$$

where  $C$  is an independent of  $\varepsilon$ . As  $\varepsilon$  vanishes,  $K(1 - \varepsilon)$  blows up and compels  $\|\mathbf{v}_\varepsilon\|_{(L^2(\mathcal{X}))^n}$  to converge to 0. Consequently  $\mathbf{v} = \mathbf{0}$  in  $\mathcal{K}$  and the arbitrary choice of  $\mathcal{K}$  means that  $\mathbf{v} = \mathbf{0}$  in  $\Omega_s$ .

The convergence in the heat and solute equations are standard. The conditions  $\theta \geq 0$  and  $0 \leq \bar{c} \leq \gamma_\ell(\theta_E)$  follow at once from Lemma 3.4. The proof of our main result is now complete.

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