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J.-F. CIAVALDINI
F. FOUCHER

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OPTIMAL CONTROL AND REGULARIZATION TO MODEL THE ATMOSPHERIC ELECTRON CONTENT FROM SATELLITE MEASUREMENTS (*)

by J.-F. CIAVALDINI (1) and F. FOUCHER (2)

Abstract — *We are concerned with the computing of the electron content of the ionosphere There already exists a statistical model, the model of Bent [1], which has been derived from physical data. The accuracy of this model appears to be insufficient when we deal with the real data that we get from satellite measurements Our aim is to derive a mathematical model from the equations of the physical model. The problem is ill-posed since it has an infinity of solutions. It is penalized and regularized by taking into account the values predicted by the model of Bent Hence, the problem is turned into an optimal control problem, the state of which is the electron content This problem is discretized in spaces of splines and error estimates are derived We give the numerical results we have computed from the data of the satellite SPOT-2 © Elsevier, Paris*

Résumé. — *Nous nous intéressons au calcul du contenu électronique de l'ionosphère Il existe déjà un modèle statistique, le modèle de Bent [1], déduit de mesures physiques La précision de ce modèle s'avère insuffisante quand on travaille avec des mesures réelles obtenues par satellite Notre objectif est de construire un modèle mathématique à partir des équations du modèle physique C'est un problème mal posé car il admet une infinité de solutions Il est pénalisé et régularisé en prenant en compte les valeurs du modèle de Bent Le problème est alors écrit sous la forme d'un problème de contrôle optimal ayant pour état le contenu électronique Ce problème est discrétisé dans des espaces de splines et des estimations d'erreurs sont établies Nous donnons des résultats numériques obtenus avec des mesures du satellite SPOT-2 © Elsevier, Paris*

1. INTRODUCTION

1.1. Physical problem

This problem is concerned with ionospheric modelling. The ionosphere is the upper part of the atmosphere (*fig.* 1), at a height of about 50 to 1 200 km, that contains free electrons and ionized particles [15]. The electron density varies with the local time, the location, the season and the solar activity. Moreover, it is affected by local and unpredictable variations due in particular to solar eruptions and associated with magnetic storms. The propagation of radio waves to or from a satellite is perturbed by the ionosphere and must be corrected for accurate measurements. Doppler frequency shifts brought about by the ionosphere are measured by the two-frequency positioning system DORIS between ground stations and both the satellite SPOT-2 (launched in 1990, at the altitude of 830 km) and the satellite Topex-Poseidon (launched in 1992, at the altitude of 1 300 km). Our purpose is to use these data to derive the total electron content which is the number of free electrons found in a vertical column of unit cross section. It is a key parameter both for ionospheric modelling and for correcting ionospheric effects on space systems. The problem of deriving the electron content from Doppler data was addressed in preliminary studies with simulated data (see [7], [8], [10]) and real data (see [10], [9]). New difficulties appear when we deal with real data of SPOT-2, which lead us to propose a more appropriate model.

Let us look to the problem along the satellite track (*fig.* 2), with the latitude θ as variable. Ionospheric Doppler shifts f_i are measured by DORIS stations S_i, $1 \leqslant i \leqslant p$. Let:

$$\Omega =]A_i, B_i[, 1 \leqslant i \leqslant p \tag{1.1}$$

(*) Manuscript received July 30, 1995, Revised October 25, 1996
(1) Département de Mathématiques, Faculté des Sciences, Université de Nantes, BP 92208, 44322 Nantes cedex 3
(2) Département d'Informatique et de Mathématiques, École Centrale de Nantes, BP 92101, 44321 Nantes cedex 3

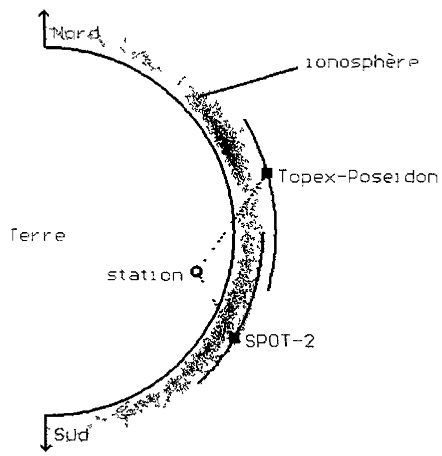

Figure 1. — Ionosphere and satellites.

be the interiors of the measurement intervals (*fig.* 3). We put:

$$\Omega_i =]A_0, B_0[\text{ where } A_0 = \min_{1 \leqslant i \leqslant p} A_i, B_0 = \max_{1 \leqslant i \leqslant p} B_i . \tag{1.2}$$

Then, let u denote the electron content on Ω. Each function f_i, $1 \leqslant i \leqslant p$, is proportional on Ω_i to the derivative of the slant electron content, which is the electron content along the line of sight from the satellite to the station S_i. Making use of assumptions on the ionosphere space variations (see [10]), the slant content is converted into the product of the vertical content beneath the satellite by a geometric factor α_i. Hence, u satisfies the equations:

$$(\alpha_i \cdot u)' + y_i = f_i \text{ on } \Omega_i \,;\ 1 \leqslant i \leqslant p \tag{1.3}$$

where y_i are unknown corrective terms we add to take into account all the approximations of the model. These terms depend on longitudinal and latitudinal gradients of the electron content and come from local and instantaneous variations we cannot explicite further more. For $1 \leqslant i \leqslant p, f_i$, respectively α_i, are assumed to be continuously derivable, respectively twice continuously derivable, on Ω_i ; moreover, α_i are non negative and there exists a real constant c such that for every $1 \leqslant i \leqslant p$, $0 < c \leqslant \alpha_i$ on Ω_i.

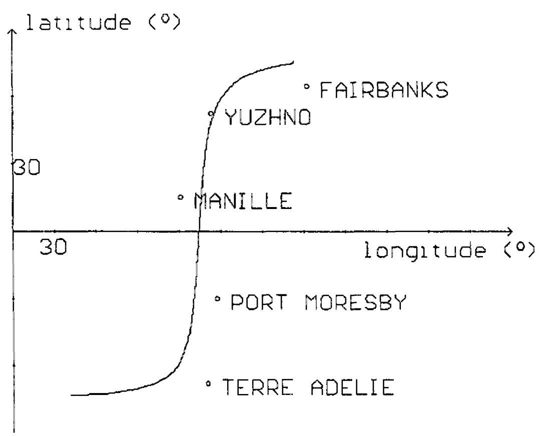

Figure 2. — Satellite track and stations.

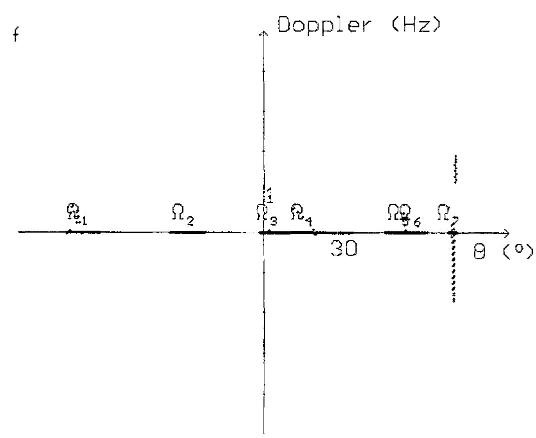

Figure 3. — Ionospheric Doppler shifts

1.2. Mathematical model

Let L^2, H^1, H^2 be the usual Sobolev spaces (see for example [11]) provided with the following semi-norms and norms and the scalar products $\langle \cdot, \cdot \rangle_m$ associated to the norms $\| \cdot \|_m$:

$$\left\{ \begin{array}{ll} |u|_m = \left(\int_\Omega (u^{(m)})^2 \, d\theta \right)^{1/2} \text{ and } \|u\|_m = \left(\sum_{j=0}^{m} |u|_j^2 \right)^{1/2} \quad \text{for } u \in H^m(\Omega) & \\ & , m = 0, 1, 2 \\ |y|_m = \left(\sum_i \int_{\Omega_i} (y_i^{(m)})^2 \, d\theta \right)^{1/2} \text{ and } \|y\|_m = \left(\sum_{j=0}^{m} |y|_j^2 \right)^{1/2} \quad \text{for } y \in \prod_i H^m(\Omega_i) & \end{array} \right. \tag{1.4}$$

Then, for u, v in $H^1(\Omega)$ and $y = (y_i)$ in $\prod_{i=1}^{p} L^2(\Omega_i)$, we define the functions:

$$\left\{ \begin{array}{l} J(y, u) = \sum_{i=1}^{p} \int_{\Omega_i} ((\alpha_i \cdot u)' + y_i - f_i)^2 \, d\theta \\ a(u, v) = \sum_{i=1}^{p} \int_{\Omega_i} (\alpha_i \cdot u)' (\alpha_i \cdot v)' \, d\theta \\ b(y, v) = \sum_{i=1}^{p} \int_{\Omega_i} y_i (\alpha_i \cdot v)' \, d\theta \end{array} \right. \tag{1.5}$$

We are looking for couples (y, u) minimizing $J(y, u)$. Such couples exist, for example the trivial solution $(f, 0)$. The original problem:

$$\text{for } y \text{ given in } Y = \prod_{i=1}^{p} H_1(\Omega_i), \text{ find } u \text{ that minimizes } J(y, u) \tag{1.6}$$

is penalized. In a preliminary study (see [7]), we assumed that $y = 0$ and we were looking for u minimizing $J(0, u) + \varepsilon |u|_2^2$ in $H^2(\Omega)$ where ε is strictly positive and intends to vanish. This model gave interesting results as long as we were concerned with simulated data but when we have dealt with real data, new difficulties appeared, actually due to variations of the satellite measurements. The main effect was that the computed u was larger than the admissible u. We have therefore improved our model as follows. We take into account the solution $\bar{u}$ of the model of Bent, which gives an average of the electron content for a given month. We assume that the data are regular in such a way that $\bar{u} \in H^4(\Omega)$ and we look for $u(y)$ minimizing in $U = H^2(\Omega)$ the regularized function:

$$J(y, u) + \varepsilon \|u - \bar{u}\|_2^2 .$$

As ε vanishes, $u(y)$ converges to the closest function to $\bar{u}$ (for the H^2-norm) among the solutions of (1.6) (see Lions and Stampacchia, [14]). Then, since y is unknown, we assume that it minimizes in Y the penalized function:

$$J(y, u(y)) + \tau \|y\|_1^2$$

where τ is a suitable positive weight. To sum up, we model the electron content with the state u of the following optimal control problem (according to the definition of Lions, [12]):

$$\begin{aligned} &\text{Find } y \text{ in } Y = \prod_{i=1}^{p} H^1(\Omega_i) \text{ that minimizes : } H(y) = J(y, u(y)) + \tau \|y\|_1^2 \\ &\text{where the state } u(y) \text{ is in } U = H^2(\Omega) \text{ and minimizes : } J(y, u(y)) + \varepsilon \|u(y) - \bar{u}\|_2^2 \end{aligned} \tag{1.7}$$

An outline of the paper is as follows. In section 2, we prove that the problem (1.7) admits a unique solution. It is written as three variational equalities which define the state, the adjoint state and the control. In section 3, we deduce an approximate problem with spaces of splines and numerical quadrature. We study the convergence of approximate solutions and establish error estimates. In section 4, we discuss our model with numerical results we got using ionospheric Doppler shifts measured by DORIS on the satellite SPOT-2. Throughout the paper, c will denote any real constant which does not depend on the weights ε, τ and on the mesh size h but only on the data α_i and f_i.

2. MATHEMATICAL MODEL

2.1. Existence and smoothness of the state $u(y)$ associated to a control y

Fixing y, the function $u \in U \to J(y, u) + \varepsilon \|u - \bar{u}\|_2^2$ is quadratic (up to some additive constant) and strictly convex. Thus, $u(y)$ exists and is unique, defined by the variational equality:

$$u(y) \in U \text{ and } a(u(y), v) + \varepsilon \langle u(y) - \bar{u}, v \rangle_2 = b(f - y, v),\ \forall v \in U \tag{2.1}$$

Let O be an open set of Ω. We write the equality (2.1) for v in the space $D(O)$ of functions indefinitely derivable on O and the support of which is compact in O. We do that when O is successively the interior of the following subsets:

$$\begin{cases} \Omega \backslash \bigcup_{i=1}^{p} \Omega_i \\ \Omega_i \backslash \bigcup_{\substack{j=1 \\ j \neq i}}^{p} (\Omega_i \cap \Omega_j),\ 1 \leqslant i \leqslant p \\ (\Omega_i \cap \Omega_j) \backslash \bigcup_{\substack{k=1 \\ k \neq i,j}}^{p} (\Omega_i \cap \Omega_j \cap \Omega_k),\ 1 \leqslant i < j \leqslant p \\ (\Omega_i \cap \Omega_j \cap \Omega_k) \backslash \bigcup_{\substack{l=1 \\ l \neq i,j,k}}^{p} (\Omega_i \cap \Omega_j \cap \Omega_k \cap \Omega_l),\ 1 \leqslant i < j < k \leqslant p \\ \dots \end{cases}$$

For the sake of simplicity, let $(K_k)_{1 \leqslant k \leqslant q}$ denote all these subsets O. They are such that:

$$\bar{\Omega} = \bigcup_{k=1}^{q} \bar{K}_k \text{ and } K_k \cap K_{k'} = \varnothing \text{ for } 1 \leqslant k < k' \leqslant q .$$

We obtain that $u(y)$ satisfies on K_k, $1 \leqslant k \leqslant q$:

$$u(y)^{(4)} = \bar{u}^{(4)} - (u(y) - \bar{u}) + (u(y) - \bar{u})'' + \sum_{\substack{i=1 \\ \Omega_i \supset K_k}}^{p} \frac{\alpha_i}{\varepsilon}((\alpha_i u(y))'' - (f_i - y_i)') . \tag{2.2}$$

Since $\bar{u} \in H^4(\Omega)$, we deduce that:

$$u(y) \in V \text{ where } V = \{ v \in H^2(\Omega) / v|_{K_k} \in H^4(K_k) \, ; 1 \leqslant k \leqslant q \} . \tag{2.3}$$

Let V be provided with the semi-norm of the product space $\prod_{k=1}^{q} H^4(K_k)$:

$$|v|_{4,V} = \left(\sum_{k=1}^{q} |v|_{K_k}|_4^2 \right)^{1/2} \quad \text{for} \quad v \in V . \tag{2.4}$$

We get from (2.2) the inequality:

$$|u(y) - \bar{u}|_{4,V} \leqslant c\left(\| u(y) - \bar{u} \|_2 + \frac{1}{\varepsilon} (\| u(y) \|_2 + \| f - y \|_1) \right)$$

Therefore, since the function $a(u,u)$ is positive and $u(y)$ satisfies (2.1), we show that:

$$\| u(y) - \bar{u} \|_2 \leqslant \frac{c}{\varepsilon} (\| f - y \|_0 + \| \bar{u} \|_1) , \tag{2.5}$$

hence:

$$|u(y) - \bar{u}|_{4,V} \leqslant c\left(\frac{1}{\varepsilon^2} \| f - y \|_1 + \frac{1}{\varepsilon} \| \bar{u} \|_2 \right) . \tag{2.6}$$

2.2. Existence and uniqueness of a solution to the optimal control problem

The function H is derivable on Y:

$$H'(y)(z) = 2\, a(u(y), u(z) - u(0)) + 2\, b(y - f, u(z) - u(0)) + 2\, b(z, u(y))$$
$$+ 2 \langle y - f, z \rangle_0 + 2\, \tau \langle y, z \rangle_1, \forall y, z \in Y \tag{2.7}$$

It is easy to see that:

$$\frac{1}{2} (H'(y) - H'(z)) (y - z) \geqslant \tau \| y - z \|_1^2, \forall y, z \in Y \tag{2.8}$$

(2.8) proves that H is strictly convex. Then, the optimal control problem (1.7) admits a unique solution y in Y, defined by the variational equality $H'(y)(z) = 0$ for every z in Y.

Since $H(y) \leqslant H(z)$ for every z in Y, we establish some estimates of the remainder $J(y, u(y))$ and of $\| y \|_1$. First, let $\bar{y}$ be the following function of Y:

$$\bar{y} = (\bar{y}_i) \quad \text{where} \quad \bar{y}_i = f_i - (\alpha_i \, . \, \bar{u})'$$

This function satisfies $J(\bar{y}, \bar{u}) = 0$ so that $u(\bar{y}) = \bar{u}$ and $H(\bar{y}) = \tau \|\bar{y}\|_1^2$. Thus, writing $H(y) \leqslant H(\bar{y})$, we obtain that:

$$\|y\|_1 \leqslant c(\|f\|_1 + \|\bar{u}\|_2) \tag{2.9}$$

Second, we write $H(y) \leqslant H(f)$ that is:

$$H(y) \leqslant J(f, u(f)) + \tau \|f\|_1^2$$

Since $u(f)$ minimizes $J(f, v) + \varepsilon \|v - \bar{u}\|_2^2$ for all v in U, let us choose $v = 0$:

$$J(f, u(f)) \leqslant \varepsilon \|\bar{u}\|_2^2$$

Hence $H(y) \leqslant H(f)$ yields:

$$\sqrt{J(y, u(y))} \leqslant \sqrt{\varepsilon}\ \|\bar{u}\|_2 + \sqrt{\tau}\ \|f\|_1 \tag{2.10}$$

and

$$\|y\|_1 \leqslant \sqrt{\frac{\varepsilon}{\tau}}\ \|\bar{u}\|_2 + \|f\|_1 \tag{2.11}$$

2.3. Definition and smoothness of the adjoint state $u^*(y)$ associated to a control y

We introduce $u^*(y)$, which is the solution of the variational equality:

$$u^*(y) \in U \text{ and } a(v, u^*(y)) + \varepsilon\langle v, u^*(y)\rangle_2 = a(u(y), v) + b(y - f, v),\ \forall v \in U \tag{2.12}$$

We turn the equality (2.7) into:

$$\frac{1}{2} H'(y)(z) = a(u(z) - u(0), u^*(y)) + \varepsilon\langle u(z) - u(0), u^*(y)\rangle_2 + b(z, u(y)) + \langle y - f, z\rangle_0 + \tau\langle y, z\rangle_1$$

Then, let $u(z)$ and $u(0)$ be two solutions of (2.1). We get:

$$\frac{1}{2} H'(y)(z) = b(z, u(y) - u^*(y)) + \langle y - f, z\rangle_0 + \tau\langle y, z\rangle_1 \tag{2.13}$$

Consequently, putting together (2.1), (2.12) and $H'(y)(z) = 0$ for every z in Y, we write the optimal control problem (1.7) as three variational equalities:

Find $y \in Y$, $u(y) \in U$, $u^*(y) \in U$ such that:

$$(S)\begin{cases} \forall v \in U,\ a(u(y), v) + \varepsilon\langle u(y), v\rangle_2 + b(y, v) = b(f, v) + \varepsilon\langle \bar{u}, v\rangle_2 \\ \forall v \in U,\ \varepsilon\langle u(y), v\rangle_2 + a(u^*(y), v) + \varepsilon\langle u^*(y), v\rangle_2 = \varepsilon\langle \bar{u}, v\rangle_2 \\ \forall z \in Y,\ b(z, u(y) - u^*(y)) + \tau\langle y, z\rangle_1 + \langle y, z\rangle_0 = \langle f, z\rangle_0 \end{cases} \tag{2.14}$$

In the same way we have done with $u(y)$, we show that $u^*(y)$ satisfies on K_k, $1 \leqslant k \leqslant q$:

$$u^*(y)^{(4)} = - u^*(y) + u^*(y)'' + \sum_{\substack{i=1 \\ \Omega_i \supset K_k}}^{p} \frac{\alpha_i}{\varepsilon} (\alpha_i u^*(y))''$$

$$+ (\bar{u} - u(y)) - (\bar{u} - u(y))'' + (\bar{u} - u(y))^{(4)} \tag{2.15}$$

from which we deduce:

$$u^*(y) \in V \tag{2.16}$$

and we establish the inequalities:

$$\| u^*(y) \|_2 \leqslant \| \bar{u} - u(y) \|_2 \tag{2.17}$$

$$| u^*(y) |_4 \leqslant c\Big(\frac{1}{\varepsilon} \| \bar{u} - u(y) \|_2 + | \bar{u} - u(y) |_4 \Big) \tag{2.18}$$

2.4. Smoothness of the control y

Writing the third equation of (S) in (2.14), for any $z = (0, \dots, z_i, \dots, 0)$ where $z_i \in D(\Omega_i)$ as defined in section 2.1, we get that for $1 \leqslant i \leqslant p$, y_i satisfies on Ω_i:

$$y_i''' = \frac{1}{\tau} ((\alpha_i(u(y) - u^*(y)))'' + (\tau + 1) y_i' - f_i') \tag{2.19}$$

hence:

$$y \in Z \text{ where } Z = \prod_{i=1}^{p} H^3(\Omega_i) \tag{2.20}$$

and

$$\| y \|_3 \leqslant \frac{c}{\tau} (\|f\|_1 + \|y\|_1 + \| u(y) - u^*(y) \|_2)$$

By combining (2.5), (2.9) and (2.17) with this inequality, we deduce:

$$\| y \|_3 \leqslant \frac{c}{\tau\varepsilon} (\|f\|_1 + \|\bar{u}\|_2) \tag{2.21}$$

From (2.5), (2.6), (2.17), (2.18), (2.9) and (2.21), we derive the estimates:

$$\begin{aligned} &\| y \|_1 \leqslant c(\|f\|_1 + \|\bar{u}\|_2) && \text{and} \quad \| y \|_3 \leqslant \frac{c}{\varepsilon\tau} (\|f\|_1 + \|\bar{u}\|_2) \\ &\| u(y) \|_2 \leqslant \frac{c}{\varepsilon} (\|f\|_1 + \|\bar{u}\|_2) && \text{and} \quad | u(y) |_{4,V} \leqslant \frac{c}{\varepsilon^2} (\|f\|_1 + \|\bar{u}\|_2 + |\bar{u}|_4) \\ &\| u^*(y) \|_2 \leqslant \frac{c}{\varepsilon} (\|f\|_1 + \|\bar{u}\|_2) && \text{and} \quad | u^*(y) |_{4,V} \leqslant \frac{c}{\varepsilon^2} (\|f\|_1 + \|\bar{u}\|_2) \end{aligned} \tag{2.22}$$

3. APPROXIMATE MODEL

3.1. Discrete spaces and numerical integration

We built a mesh $\{\theta_j ; 0 \leq j \leq n\}$ on $\Omega =]A_0, B_0[$ such that for any i, $1 \leq i \leq p$, the set $\{\theta_j ; \theta_j \in \Omega_i\}$ is a mesh of $\Omega_i =]A_i, B_i[$ such that each interval $[\theta_j, \theta_{j+1}]$ contains at least three measurement points, in order to deal with the numerical integration. We put:

$$J_j = [\theta_{j-1}, \theta_j] \quad \text{and} \quad h = \sup_{1 \leq j \leq n} (\theta_j - \theta_{j-1}) \tag{3.1}$$

Then, we consider the discrete subspace U_h of U, made up with the cubic splines associated with the mesh of Ω and the discrete space Y_h of Y, made up with the quadratic splines associated with the meshes of the Ω_i, $i \leq 1 \leq p$:

$$\begin{cases} U_h = \{u_h \in C^2(\Omega) \text{ such that } u_h|_{J_j} \in P_3(J_j), 1 \leq j \leq n\} \\ Y_h = \prod_{i=1}^{p} \{y_{h,i} \in C^1(\Omega_i) \text{ such that } y_{h,i}|_{J_j} \in P_2(J_j) \text{ for each } J_j \subset \Omega_i\} \end{cases} \tag{3.2}$$

where $P_k(J_j)$ denotes the space of the polynomials of degree less than or equal to k on J_j. Let $\pi_h v$ denote the interpolated function in U_h of v from V and $\sigma_h z$ the interpolated function in Y_h of z from Z respectively defined by:

$$\begin{cases} \pi_h v(\theta_j) = v(\theta_j), 0 \leq j \leq n \\ (\pi_h v)'(A_0) = v'(A_0) \\ (\pi_h v)'(B_0) = v'(B_0) \end{cases}$$

$$\text{and} \quad \begin{cases} \int_{J_j} (\sigma_h z)_i \, d\theta = \int_{J_j} z_i \, d\theta, \text{ for each } J_j \subset \Omega_i \\ (\sigma_h z)_i (A_i) = z_i(A_i) \\ (\sigma_h z)_i (B_i) = z_i(B_i) \end{cases} , 1 \leq i \leq p$$

We point out the classical interpolation errors (see [5] and [2]):

$$\begin{cases} \|\pi_h v - v\|_m \leq c h^{4-m} |v|_{4,V} ; 0 \leq m \leq 2 \\ \|\sigma_h z - z\|_m \leq c h^{3-m} |z|_3 ; 0 \leq m \leq 1 \end{cases} \tag{3.3}$$

On the reference interval $\hat{J} = [0, 1]$, we write the following quadrature formula where the error $\hat{E}(\hat{v})$ vanishes for any quadratic polynomial $\hat{v}$ on $\hat{J}$:

$$\begin{cases} \int_{\hat{J}} \hat{v}(\hat{\theta}) \, d\hat{\theta} = \hat{I}(\hat{v}) + \hat{E}(\hat{v}) \text{ where } \hat{I}(\hat{v}) = \hat{\lambda}_1 \hat{v}(\hat{t}_1) + \hat{\lambda}_2 \hat{v}(\hat{t}_2) + \hat{\lambda}_3 \hat{v}(\hat{t}_3) \\ \text{with } \hat{t}_1, \hat{t}_2, \hat{t}_3 \in \hat{J} \text{ and } \hat{\lambda}_k = \dfrac{2 - 3(\hat{t}_l + \hat{t}_m) + 6 \hat{t}_l \hat{t}_m}{6(\hat{t}_k - \hat{t}_l)(\hat{t}_k - \hat{t}_m)} \text{ for } k, l, m \in \{1, 2, 3\}, l \neq k, m \neq k \end{cases} \tag{3.4}$$

For $\hat{t}_1 = \hat{t}_3 = 0$ and $\hat{t}_2 = 1/2$, (3.4) is the Simpson quadrature formula. Here, each interval J_j is in correspondence with the reference interval $\hat{J}$ through the mapping Φ_j :

$$\begin{cases} \hat{J} \to J_j \\ \hat{\theta} \to \theta = \Phi_j(\hat{\theta}) = \theta_{j-1} + (\theta_j - \theta_{j-1})\,\hat{\theta} \\ \hat{v} \to v = \hat{v} \circ \Phi_j^{-1} \end{cases} \tag{3.5}$$

and in (3.4) $\hat{t}_1, \hat{t}_2, \hat{t}_3$ will be the inverse image by Φ_j of three measurement points in J_j. We deduce quadrature formulas on the intervals J_j :

$$\int_{J_j} v(\theta)\, d\theta = I_j(v) + E_j(v) \text{ where } \begin{cases} I_j(v) = (\theta_j - \theta_{j-1})\,\hat{I}(\hat{v}) \\ E_j(v) = (\theta_j - \theta_{j-1})\,\hat{E}(\hat{v}) \end{cases} \tag{3.6}$$

and we define approximate forms, for u, v in $C^2(\Omega)$ and y, z in $\prod_{i=1}^{p} C^1(\Omega_i)$:

$$\begin{cases} \langle u, v\rangle_{2,h} = \sum_j I_j(uv + u'v' + u''v'') \text{ where } j \text{ describes } \{1, \dots, n\} \\ \left.\begin{array}{l} a_h(u, v) = \sum_i \sum_j I_j((\alpha_i u)'(\alpha_i v)') \\ b_h(y, u) = \sum_i \sum_j I_j(y_i(\alpha_i u)') \\ \langle y, z\rangle_{0,h} = \sum_i \sum_j I_j(y_i z_i) \\ \langle y, z\rangle_{1,h} = \sum_i \sum_j I_j(y_i z_i + y_i' z_i') \end{array}\right\} \quad \begin{array}{c} \text{where } i \text{ describes } \{1, \dots, p\} \\ \text{and for each } i, \\ j \text{ describes } \{j \text{ such that } J_j \subset \Omega_i\} \end{array} \end{cases} \tag{3.7}$$

Since the error $\hat{E}$ vanishes for the quadratic polynomials on $\hat{J}$, we establish from the lemma of Bramble-Hilbert (see [3] and [5]), the following estimates, for $u_h, v_h \in U_h$ and $y_h, z_h \in Y_h$:

$$\begin{cases} |a_h(u_h, v_h) - a(u_h, v_h)| \leqslant ch \|u_h\|_2 \; \|v_h\|_2 \\ |\langle w, v_h\rangle_{2,h} - \langle w, v_h\rangle_2| \leqslant ch \|w\|_2 \; \|v_h\|_2 \text{ for } w = u_h \text{ or } w = \bar{u} \\ |b_h(x, u_h) - b(x, u_h)| \leqslant ch \|x\|_1 \; \|u_h\|_2 \text{ for } x = y_h \text{ or } x = f \\ |\langle x, z_h\rangle_{m,h} - \langle x, z_h\rangle_m| \leqslant ch \|x\|_1 \; \|z_h\|_1, 0 \leqslant m \leqslant 1, \text{ for } x = y_h \text{ or } x = f \end{cases} \tag{3.8}$$

3.2. Approximate problem

With the discrete spaces and the approximate forms, we derive from (2.14) the problem:

Find $y_h \in Y_h$, $u_h(y_h) \in U_h$, $\overset{*}{u}_h(y_h) \in U_h$ such that :

$$(S_h) \begin{cases} \forall v_h \in U_h, \; a_h(u_h(y_h), v_h) + \varepsilon\langle u_h(y_h), v_h\rangle_{2,h} + b_h(y_h, v_h) = b_h(f, v_h) + \varepsilon\langle \bar{u}, v_h\rangle_{2,h} \\ \forall v_h \in U_h, \; \varepsilon\langle u_h(y_h), v_h\rangle_{2,h} + a_h(\overset{*}{u}_h(y_h), v_h) + \varepsilon\langle \overset{*}{u}_h(y_h), v_h\rangle_{2,h} = \varepsilon\langle \bar{u}, v_h\rangle_{2,h} \\ \forall z_h \in Y_h, \; b_h(z_h, u_h(y) - \overset{*}{u}_h(y_h)) + \tau\langle y_h, z_h\rangle_{1,h} + \langle y_h, z_h\rangle_{0,h} = \langle f, z_h\rangle_{0,h} \end{cases} \tag{3.9}$$

Existence and uniqueness of the solution $(y_h, u_h, \overset{*}{u}_h)$:

Using (3.8), we write for $v_h \in U_h$:

$$a_h(v_h, v_h) + \varepsilon\langle v_h, v_h\rangle_{2,h} \geqslant (\varepsilon - ch)\; \|v_h\|_2^2 \tag{3.10}$$

Thus there exists $\tilde{h}$ such that for $h \leqslant \tilde{h}$ the function $u_h, v_h \in U_h \to a_h(u_h, v_h) + \varepsilon\langle u_h, v_h\rangle_{2,h}$ is uniformly U_h-elliptic. Therefore, for a given $y_h \in Y_h$, there exists a unique approximate state $u_h(y_h)$ and a unique adjoint state $\overset{*}{u}_h(y_h)$.

Then, we write (2.8) for $y_h, z_h \in Y_h$ where H' is written as in (2.13):

$$b(y_h - z_h, (u(y_h) - u(z_h)) - (u^*(y_h) - u^*(z_h))) + \|y_h - z_h\|_0^2 + \tau\|y_h - z_h\|_1^2 \geqslant \tau\|y_h - z_h\|_1^2$$

Further, using the third equations of (S) and (S_h) and (3.8), we derive:

$$b_h(y_h - z_h, w_h - \overset{*}{w}_h) + \langle y_h - z_h, y_h - z_h\rangle_{0,h} + \tau\langle y_h - z_h, y_h - z_h\rangle_{1,h} \geqslant$$

$$\tau\|y_h - z_h\|_1^2 - c\|y_h - z_h\|_1(\|w - w_h\|_1 + \|w^* - \overset{*}{w}_h\|_1 + h\|w_h\|_2 + h\|\overset{*}{w}_h\|_2 + h\|y_h - z_h\|_1) \quad (3.11)$$

where:

$$\begin{cases} w = u(y_h) - u(z_h) \\ w^* = u^*(y_h) - u^*(z_h) \end{cases} \quad \text{and} \quad \begin{cases} w_h = u_h(y_h) - u_h(z_h) \\ \overset{*}{w}_h = \overset{*}{u}_h(y_h) - \overset{*}{u}_h(z_h) \end{cases}$$

On one hand, the functions w and w^* are in V and we get:

$$\begin{cases} \|w\|_2 \leqslant \frac{c}{\varepsilon}\|y_h - z_h\|_0 \\ \|w^*\|_2 \leqslant \frac{c}{\varepsilon}\|y_h - z_h\|_0 \end{cases} \quad \text{and} \quad \begin{cases} |w|_{4,V} \leqslant \frac{c}{\varepsilon^2}\|y_h - z_h\|_1 \\ |w^*|_{4,V} \leqslant \frac{c}{\varepsilon^2}\|y_h - z_h\|_1 \end{cases} \quad (3.12)$$

On the other hand, using (3.10) and (3.8), we write:

$$\|w - w_h\|_2 \leqslant \|w - w_h\|_2 + \|w_h - v_h\|_2 \leqslant c\left(\frac{1}{\varepsilon}\|w - v_h\|_1 + \|w - v_h\|_2 + \frac{h}{\varepsilon}\|y_h - z_h\|_1 + \frac{h}{\varepsilon}\|v_h\|_2\right)$$

which gives for $v_h = \pi_h w$, applying (3.3):

$$\|w - w_h\|_2 \leqslant c\left(\left(\frac{h^3}{\varepsilon} + h^2\right)|w|_{4,V} + \frac{h}{\varepsilon}\|w\|_2 + \frac{h}{\varepsilon}\|y_h - z_h\|_1\right)$$

Whence, with (3.12):

$$\|w - w_h\|_2 \leqslant \frac{h}{\varepsilon^3}\|y_h - z_h\|_1 \quad (3.13)$$

An analogous calculation leads to:

$$\|w^* - \overset{*}{w}_h\|_2 \leqslant \frac{h}{\varepsilon^3}\|y_h - z_h\|_1 \quad (3.14)$$

Finally, combining (3.12), (3.13) and (3.14) with (3.11) yields:

$$b_h(y_h - z_h(u_h(y_h) - u_h(z_h)) - (\overset{*}{u}_h(y_h) - \overset{*}{u}_h(z_h)))$$

$$+ \langle y_h - z_h, y_h - z_h\rangle_{0,h} + \tau\langle y_h - z_h, y_h - z_h\rangle_{1,h} \geqslant \left(\tau - c\frac{h}{\varepsilon^3}\right)\|y_h - z_h\|_1^2 \quad (3.15)$$

Hence there exists $\tilde{h}$ such that for $h \leqslant \tilde{h}$ the approximate control problem admits a unique solution y_h.

Estimate of the error $\|y - y_h\|_1$:
With (3.15) and (3.8), we obtain for $v_h, v_h^* \in U_h$:

$$\|y_h - z_h\|_1 \leqslant \frac{c}{\tau}(\|u(y) - v_h\|_1 + \|u^*(y) - v_h^*\|_1 + \|y - z_h\|_0 + \tau\|y - z_h\|_1$$

$$+ h\|f\|_1 + h\|z_h\|_1 + \|u_h(z_h) - v_h\|_2 + \|u_h^*(z_h) - v_h^*\|_2 + h\|v_h\|_2 + h\|v_h^*\|_2) \tag{3.16}$$

Then, from (3.10) and (3.8), we get:

$$\|u_h(z_h) - v_h\|_2 \leqslant \frac{c}{\varepsilon}(\|u(y) - v_h\|_1 + \varepsilon\|u(y) - v_h\|_2$$

$$+ \|y - z_h\|_0 + h(\|f\|_1 + \|\bar{u}\|_2) + h\|z_h\|_1 + h\|v_h\|_2) \tag{3.17}$$

and:

$$\|u_h^*(z_h) - v_h^*\|_2 \leqslant \frac{c}{\varepsilon}(\|u^*(y) - v_h^*\|_1 + \varepsilon\|u^*(y) - v_h^*\|_2 + \|u(y) - v_h\|_1 + \varepsilon\|u(y) - v_h\|_2$$

$$+ \|y - z_h\|_0 + h(\|f\|_1 + \|\bar{u}\|_2) + h\|z_h\|_1 + h\|v_h\|_2 + h\|v_h^*\|_2) \tag{3.18}$$

Combining these inequalities with (3.16), we derive:

$$\|y - y_h\|_1 \leqslant \|y - z_h\|_1 + \|y_h - z_h\|_1$$

$$\leqslant \frac{c}{\varepsilon\tau}(\|u(y) - v_h\|_1 + \varepsilon\|u(y) - v_h\|_2 + \|u^*(y) - v_h^*\|_1 + \varepsilon\|u^*(y) - v_h^*\|_2$$

$$+ \|y - z_h\|_0 + \varepsilon\tau\|y - z_h\|_1 + h(\|f\|_1 + \|\bar{u}\|_2) + h\|z_h\|_1 + h\|v_h\|_2 + h\|v_h^*\|_2)$$

By choosing $z_h = \sigma_h y$, $v_h = \pi_h u(y)$, $v_h^* = \pi_h u^*(y)$, we get with (3.3):

$$\|y - y_h\| \leqslant \frac{c}{\varepsilon\tau}((h^3 + \varepsilon h^2)(|u(y)|_{4,V} + |u^*(y)|_{4,V}) + (h^3 + \varepsilon\tau h^2)|y|_3$$

$$+ h(\|f\|_1 + \|\bar{u}\|_2) + h\|y\|_1 + h\|u(y)\|_2 + h\|u^*(y)\|_2)$$

Finally with (2.22), we deduce the convergence estimate:

$$\|y - y_h\|_1 \leqslant c\frac{h}{\varepsilon^2\tau}\left(1 + \frac{h^2}{\varepsilon} + \frac{h^2}{\tau}\right)(\|f\|_1 + \|\bar{u}\|_4) \tag{3.19}$$

Estimate of the error $\|u(y) - u_h(y_h)\|_2$:
The inequality (3.17) for $z_h = y_h$ gives:

$$\|u(y) - u_h(y_h)\|_2 \leqslant \|u(y) - v_h\|_2 + \|u_h(y_h) - v_h\|_2$$

$$\leqslant c\Big(\frac{1}{\varepsilon}\|u(y) - v_h\|_1 + \|u(y) - v_h\|_2 + \frac{h}{\varepsilon}\|v_h\|_2 + \frac{1}{\varepsilon}\|y - y_h\|_1 + \frac{h}{\varepsilon}\|y_h\|_1 + \frac{h}{\varepsilon}(\|f\|_1 + \|\bar{u}\|_2)\Big)$$

and for $v_h = \pi_h u(y)$:

$$\|u(y) - u_h(y_h)\|_2 \leq c\left(\left(\frac{h^3}{\varepsilon} + h^2\right) |u(y)|_{4,V} + \frac{h}{\varepsilon}\|u(y)\|_2 + \frac{1}{\varepsilon}\|y - y_h\|_1 + \frac{h}{\varepsilon}\|y\|_1 + \frac{h}{\varepsilon}(\|f\|_1 + \|\bar{u}\|_2)\right)$$

so that with (2.22) and (3.19), we get the estimate:

$$\|u(y) - u_h(y_h)\|_2 \leq c\frac{h}{\varepsilon^3 \tau}\left(1 + \frac{h^2}{\varepsilon} + \frac{h^2}{\tau}\right)(\|f\|_1 + \|\bar{u}\|_4) \tag{3.20}$$

Estimate of the remainder $\sqrt{J(y_h, u_h(y_h))}$ and the norm $\|y_h\|_1$:
Combining (2.10), (2.11), (3.19) and (3.20), we derive:

$$\begin{aligned} \sqrt{J(y_h, u_h(y_h))} &\leq c\left(\sqrt{\tau}\|f\|_1 + \sqrt{\varepsilon}\|\bar{u}\|_2 + \frac{h}{\varepsilon^3 \tau}\left(1 + \frac{h^2}{\varepsilon} + \frac{h^2}{\tau}\right)(\|f\|_1 + \|\bar{u}\|_4)\right) \\ \|y_h\|_1 &\leq c\left(\|f\|_1 + \sqrt{\frac{\varepsilon}{\tau}}\|\bar{u}\|_2 + \frac{h}{\varepsilon^2 \tau}\left(1 + \frac{h^2}{\varepsilon} + \frac{h^2}{\tau}\right)(\|f\|_1 + \|\bar{u}\|_4)\right) \end{aligned} \tag{3.21}$$

4. NUMERICAL RESULTS AND CONCLUSION

Using B-spline bases of U_h and Y_h, we write the problem (3.9) as a linear system. Changing the unknown (u_h, u_h^*, y_h) into $(-u_h^*, u_h - u_h^*, y_h)$, we transform this system into a new one with a symmetric matrix (see [10]) and solve it by a LDL^t factorisation (see [11]). To test our model, many solutions have been computed from ionospheric Doppler shifts measured by the DORIS system.

We first evaluate suitable values of the parameters h, ε and τ. On the one hand, we choose h as little as the density of measurement points allows it. On the other hand, the regularization weight ε is assumed to vanish and the weight τ should be fixed so that y_h represents a corrective term of about 10 to 20 % beside the measurement function f. We conclude from numerical experiments that $\tau = 1$ and $\varepsilon = 10^{-3}$ are appropriate weights for the model.

Then, let us comment some graphs (*figs.* 4 to 7). The left-hand graph gives the real ionospheric Doppler shifts f measured with SPOT-2 and this one denoted $\bar{f}$ which would be measured if the electron content was $\bar{u}$. On the right-hand graph, we compare the computed content u_h and $\bar{u}$. We observe that higher or lower Doppler shifts at different latitudes give expected results of u_h (*figs.* 4 and 5) and that similar measurements give close contents

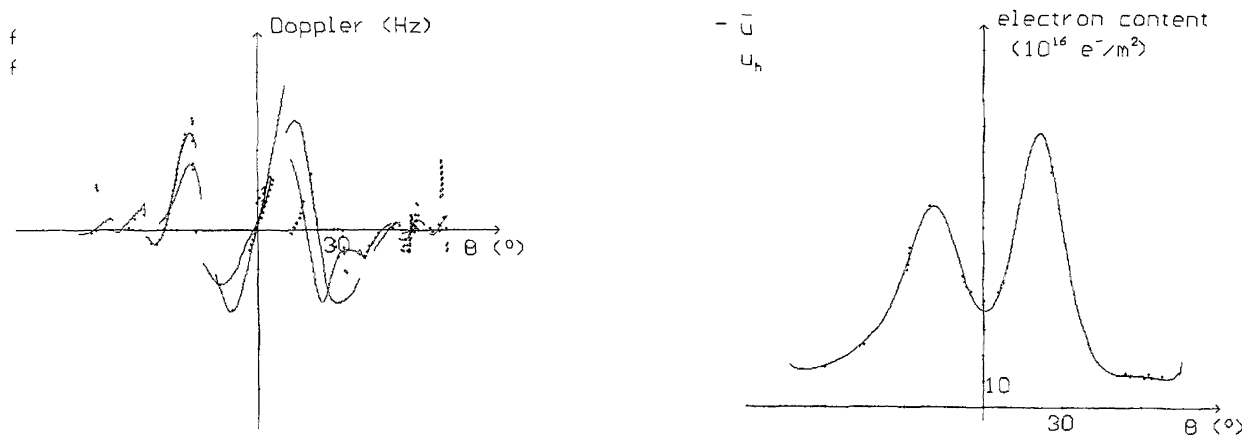

Figure 4. — Example 1.

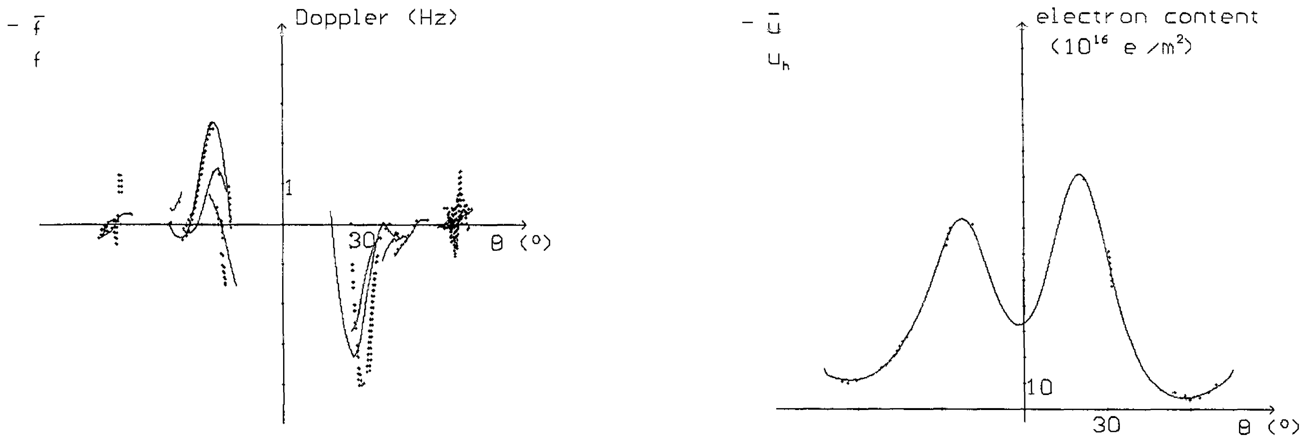

Figure 5. — Example 2.

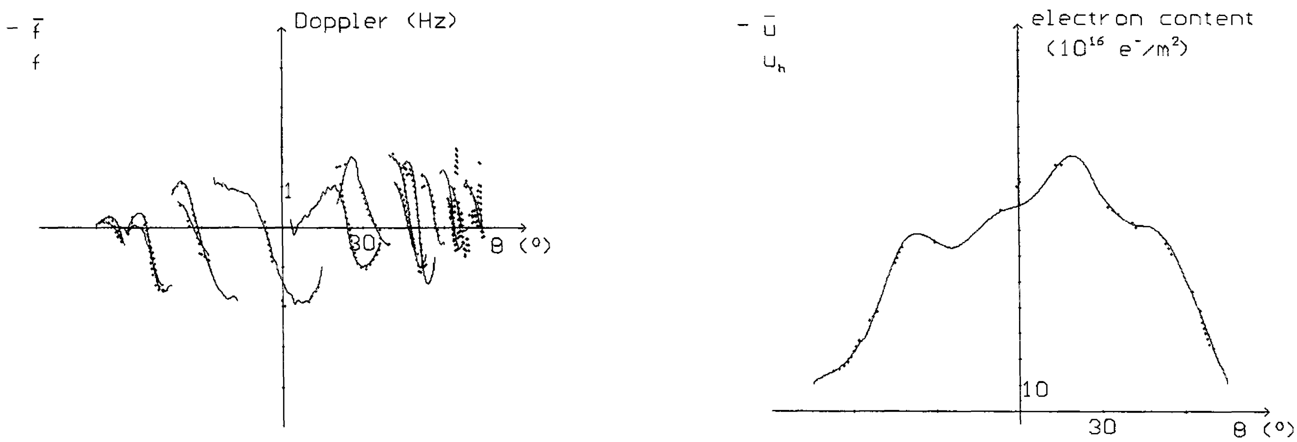

Figure 6. — Example 3.

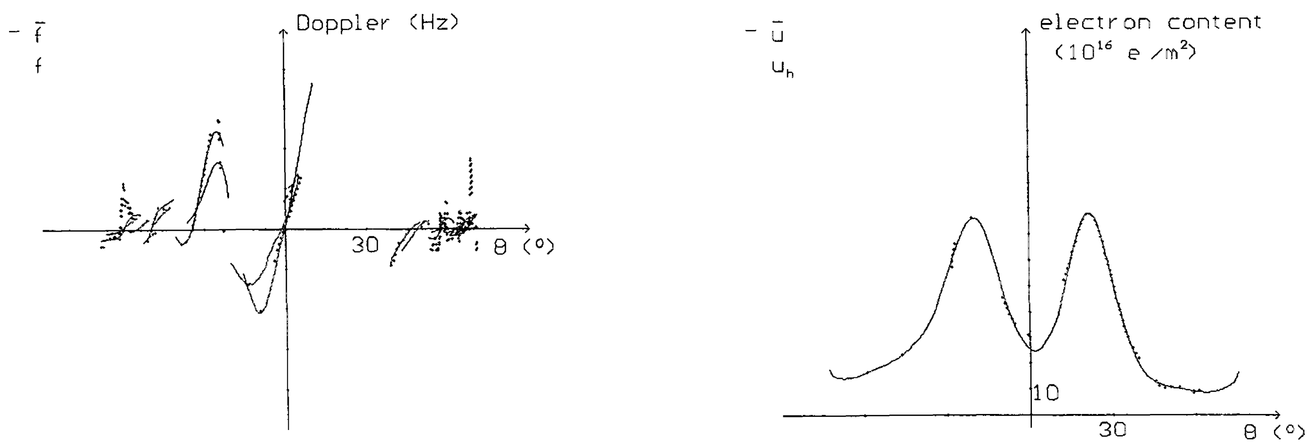

Figure 7. — Example 4.