MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

D. CHOI
F. J. PALMA
E. SANCHEZ-PALENCIA
M. A. VILARIÑO

Membrane locking in the finite element computation of very thin elastic shells

Modélisation mathématique et analyse numérique, tome 32, nº 2 (1998), p. 131-152

<http://www.numdam.org/item?id=M2AN_1998__32_2_131_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 32, n° 2, 1998, p. 131 à 152)

MEMBRANE LOCKING IN THE FINITE ELEMENT COMPUTATION OF VERY THIN ELASTIC SHELLS (*)

by D. CHOI (¹), F. J. PALMA (²), E. SANCHEZ-PALENCIA (¹), M. A. VILARIÑO (²)

Abstract. — *The membrane locking phenomenon arises in cases when the middle surface of the shell admits "pure bendings" satisfying the kinematic boundary conditions. It then appears that the discrete approximations by finite elements is unsuited to describe such pure bendings, which are the limit configuration of solutions as the thickness tends to zero. This phenomenon is described in terms of lack of robustness (i.e. lack of uniformity of the finite element convergence $h \searrow 0$ with respect to the thickness of the shell 2ε). We prove that any finite element scheme consisting of in piecewise polynomial functions necessarily exhibits locking for certain shells (and probably for almost any shell admiting pure bendings). Numerical experiments are done for a hyperbolic paraboloid. The superiority of schemes involving high order polynomials (Ganev-Argyris in particular) is shown. It is also seen that reduced integration have very little influence on membrane locking.*

Résumé. — *Le phénomène de verrouillage membranaire intervient lorsque la surface moyenne de la coque admet des « flexions pures » satisfaisant aux conditions aux limites cinématiques. Dans ce cas les approximations discrètes par éléments finis sont inappropriées pour décrire ces flexions pures, qui sont les configurations limites des solutions lorsque l'épaisseur tend vers zéro. Le phénomène est décrit en terme de manque de robustesse (ou manque d'uniformité de la convergence des éléments finis $h \searrow 0$ par rapport à l'épaisseur de la coque 2ε). Nous démontrons que tout schéma éléments finis qui consiste dans des fonctions polynomiales par morceaux, présente nécessairement le phénomène de verrouillage pour certaines surfaces (et probablement pour presque toutes les surfaces admettant des flexions pures). On a fait des exemples numériques pour le paraboloïde hyperbolique, montrant la supériorité des schémas qui font intervenir des polynômes de degré élevé (Ganev-Argyris en particulier). On voit aussi que l'intégration réduite a une très petite influence sur le verrouillage membranaire.*

1. INTRODUCTION

It is known that the phenomenon of membrane locking in numerical computation of thin shells consists in an inadequacy of the finite elements to describe the very peculiar deformations of a shell [15], [10], [6], [9]. Actually, the natural trend of a thin shell is to perform pure bendings, i.e. inextensional displacements of the middle surface (which do not modify the intrinsic metrics of the middle surface), but the explicit description of the phenomenon and the above mentioned inadequacy of the finite element approximation usually lack in the litterature.

The asymptotic study of the behavior of elastic shells as the thickness 2ε tends to zero [15], [16], [17] shows two very different asymptotic behaviors in the cases when the medium surface with the kinematic boundary conditions admits or no inextensional displacements. In the former case, the shell is called to be "non-inhibited" (for "with non-inhibited pure bendings") and in the later it is said to be "inhibited" (for "with inhibited pure bendings"). It should be noticed that the concept of "inhibited" coincides with that of "geometrically rigid", but we adopt the term "inhibited", as "rigidity" is a different concept in mechanics of continua.

It then appears that the membrane locking only occurs in non-inhibited shells. This elementary assertion is nevertheless useful to introduce a little order in the comments on locking which are usually encountered in the literature. For instance in [3], p. 238-239 are considered three examples of shells which are taken as benchmarks

---

(*) Manuscript received September 16, 1996.

(¹) Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05 (France).

(²) Departamento de Analisis Matematico, Facultad de Ciencias, Campus de Teatinos, 29071 Malaga (Spain).

This work is part of the "Action intégrée franco-espagnole PICASSO 95-96". This work is also part of the of the Human Capital and Mobility Program "Shells: Mathematical Modelling and Analysis, Scientific Computing" of the Commission of the European communities (contract ERBCHRXCT 940536).

for testing the adequacy of finite element methods but only the third one (semi-sphere with free boundary) is non-inhibited; the others are not concerned with membrane locking. Moreover, no tests are given for very thin shells, when membrane locking is an asymptotic phenomenon for small thickness. It should also be mentioned that arches, which are the two-dimensional counterparts of shells are always non-inhibited, and the corresponding locking is simpler and better known [6].

It is a remarkable fact that a shell is, by definition, thin, but nevertheless, the equations of classical theory of shells (see, for instance [13], [4]) contain two terms corresponding to the deformation energies of membrane strain and flexion. The corresponding coefficients are in a ratio ε^{-2} as the membrane rigidity is higher than the flexion one. Consequently we may study the asymptotic behavior of a thin shell as $\varepsilon \searrow 0$ ([15], [16], [17] already mentioned). The classical convergence of the finite element approximation as the mesh diameter h tends to zero [4] holds true for fixed ε, but the constants appearing in the convergence estimates depend on ε. Babuska and Suri define the locking [1] as the non-uniformity of the convergence with respect to ε. They also call "robust" an approximation method when the convergence $h \searrow 0$ is uniform with respect to a parameter ε. Locking is in fact lack of robustness. This definition, which we adopt in this paper, seems to be quite fitted as an approximation method in theory of shells, which are naturally thin, should convergence quite well for any value of the small thickness.

Otherwise one must change the mesh diameter when changing the thickness. We shall see that this is the general situation, and the values of h which are necessary to get a good approximation with small ε are impracticable. The paper [10] by Kamoulakos is quite interesting in this context. After giving a review of the work accomplised mainly by L.S.D. Morley, he considers a specific type of quadrangle finite elements and shows that the appropriate h should be of order ε or even ε^2 in certain directions! He also exhibits finite elements computations of shell stresses with errors going from 2 % to 46 000 %!

We also notice that the locking phenomenon seems also be present in a variety of approximation methods. See for instance [9], [6], [11] for the approximation of arches by straight beam elements and [1] for approximation of shells by plane facets.

In this paper we adopt a view point very close to that of Chenais and Paumier [6] but we consider the case of shells instead of that of arches. Our main result (section 4) is that any finite element internal approximation made of functions which are piecewise polynomials leads to locking for certain surfaces such as the hyperbolic paraboloid and the straight helicoid. The proof relies on incompatibilities involving explicit expressions of the equations for these surfaces. We also show in a less rigorous way (begining of section 4) that generally speaking, thereis incompatibility between the pure bendings and the finite dimensional space V_h of the finite element approximation from which the general character of the locking phenomenon follows.

The previous considerations are then extended (section 5) to a widely used kind of non-conforming finite elements, the so-called DKT (= Discrete Kirchhoff Triangle) approximation.

Section 6 is devoted to numerical experiments. We chose a non-inhibited shell (a hyperbolic paraboloid with appropriate boundary conditions) which is not the case in a part of the usual benchmarks. Computations were done for two finite element schemes: Ganev-Argyris (conformal, using high order polynomials) and Sander (non-conformal, with lower oder polynomials). For each one, several different integration schemes were considered. Numerical results show the superiority of higher order polynomials. Indeed, fairly good computations were obtained with an Apollo station for $\varepsilon > 10^{-3}$ (Ganev-Argyris) and $\varepsilon > 10^{-2}$ (Sander), where $2\varepsilon =$ thickness. Otherwise, very little influence of the integration schemes was observed, contrary to the wide-spread opinion that reduced integration diminishes locking effects.

2. SETTING OF THE PROBLEM. ROBUSTNESS AND LOCKING

Let us first state the shell problem in the framework of Koiter theory (see for instance [4]). Let E^3 be the Euclidean space referred to an orthonormed frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and let Ω be a bounded open set of $\mathbb{R}^2$ with boundary Γ. The middle surface of the shell (denoted by S) is the image in E^3 of $\overline{\Omega}$ by the map

$$\varphi : (y^1, y^2) \in \overline{\Omega} \to \varphi(\mathbf{y}) \in E^3 \tag{2.1}$$

At each point of S we consider the two tangent vectors

$$\mathbf{a}_\alpha = \boldsymbol{\varphi}_{,\alpha} \equiv \partial\boldsymbol{\varphi}/\partial y^\alpha \equiv \partial_\alpha \boldsymbol{\varphi} \quad (\alpha = 1, 2) \tag{2.2}$$

(greek indices run from 1 to 2, and latin ones from 1 to 3. Note also the different notations used for partial differentiation). The unit normal vector to S is

$$\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2 / |\mathbf{a}_1 \times \mathbf{a}_2| \tag{2.3}$$

The map $\boldsymbol{\varphi}$, and then S are supposed to be smooth; we then consider, in a neighbourhood of S the "normal curvilinear coordinates" y^1, y^2, y^3 where y^3 is the distance to S along the normal. For the sake of simplicity we only consider the case of constant thickness $2\,\varepsilon$. The shell is then the set

$$C = \{M \in E^3, \quad M = \boldsymbol{\varphi}(y^1, y^2) + y^3\,\mathbf{a}_3, \quad (y^1, y^2) \in \overline{\Omega}, \quad |y^3| < \varepsilon\} \tag{2.4}$$

Let $\mathbf{u}(y^1, y^2)$ be the displacement vector of S when the shell is submited to forces $\varepsilon^3\,\mathbf{f}$ by unit surface. We only consider linearized theory for small $\mathbf{u}$. The Koiter theory is then described in terms of the linear term of the *membrane strain tensor*

$$\gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) \tag{2.5}$$

and of the linear terms of the *change of curvature tensor*

$$\rho_{\alpha\beta}(\mathbf{u}) = \bar{b}_{\alpha\beta} - b_{\alpha\beta}\,. \tag{2.6}$$

In the previous expressions, $a_{\alpha\beta}$(resp. $\bar{a}_{\alpha\beta}$) and $b_{\alpha\beta}$(resp. $\bar{b}_{\alpha\beta}$) denote the coefficients of the first and second fundamental forms of S before (resp. after) deformation, i.e.

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = \boldsymbol{\varphi}_{,\alpha} \cdot \boldsymbol{\varphi}_{,\beta} \tag{2.7}$$

$$b_{\alpha\beta} = -\,\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta} = \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta} = \mathbf{a}_3 \cdot \mathbf{a}_{\beta,\alpha} \tag{2.8}$$

where, α denotes *differentiation* with respect to y^α. Correspondingly, the symbol $|\alpha$ will be used for *covariant differentiation.*

The contravariant basis $\mathbf{a}^\alpha$ is defined by

$$\mathbf{a}_\alpha \cdot \mathbf{a}^\beta = \delta_\alpha^\beta \tag{2.9}$$

where δ denotes the Kronecker symbol. The contravariant components of the metric tensor are

$$a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta\,. \tag{2.10}$$

They are used, as well as $a_{\alpha\beta}$ to pass from covariant to contravariant components of vectors and tensors in the usual way. The tensors γ and ρ are then:

$$\gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}\,u_3 \tag{2.11}$$

$$\rho_{\alpha\beta}(\mathbf{u}) = u_{3|\alpha\beta} + b_{\beta|\alpha}^\lambda\,u_\lambda + b_\beta^\lambda\,u_{\lambda|\alpha} + b_\alpha^\lambda\,u_{\lambda|\beta} - b_\alpha^\lambda\,b_{\lambda\beta}\,u_3 \tag{2.12}$$

where

$$\left\{\begin{array}{l} u_{\alpha|\beta} = u_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}\, u_{\lambda} \\ b^{\lambda}_{\beta|\alpha} = b^{\lambda}_{\beta,\alpha} + \Gamma^{\lambda}_{\alpha\nu}\, b^{\nu}_{\beta} - \Gamma^{\nu}_{\beta\alpha}\, b^{\lambda}_{\nu} \\ u_{3|\alpha\beta} = u_{3,\alpha\beta} - \Gamma^{\lambda}_{\alpha\beta}\, u_{3,\lambda} \end{array}\right. \tag{2.13}$$

and Γ are the Christoffel symbols of S:

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} = \mathbf{a}^{\alpha} \,.\, \mathbf{a}_{\gamma,\beta} = \mathbf{a}^{\alpha} \,.\, \mathbf{a}_{\beta,\gamma}\,. \tag{2.14}$$

We then define the bilinear forms of membrane and flexion energy:

$$a_0(\mathbf{u}, \mathbf{v}) = \int_S a_0^{\alpha\beta\eta\sigma}\, \gamma_{\eta\sigma}(\mathbf{u})\, \gamma_{\alpha\beta}(\mathbf{v})\, dS \tag{2.15}$$

$$a_1(\mathbf{u}, \mathbf{v}) = \int_S a_1^{\alpha\beta\eta\sigma}\, \rho_{\eta\sigma}(\mathbf{u})\, \rho_{\alpha\beta}(\mathbf{v})\, dS \tag{2.16}$$

where the coefficients of a_0 and a_1 are elasticity coefficients satisfying the classical properties of symmetry and positivity

$$a^{\alpha\beta\eta\sigma} = a^{\eta\sigma\alpha\beta} = a^{\alpha\beta\sigma\eta} \tag{2.17}$$

$$a^{\alpha\beta\eta\sigma}\, \xi_{\alpha\beta}\, \xi_{\eta\sigma} \geqslant c \sum_{\alpha\beta} |\xi_{\alpha\beta}|^2 \quad \forall \xi_{\alpha\beta} \text{ symmetric}\,. \tag{2.18}$$

Moreover, the shell is supposed to be clamped by a part Γ_0 of the boundary, simply supported by another one Γ_1 and free by the remainder part. The kinematic boundary conditions to be prescribed are

$$\left\{\begin{array}{ll} \mathbf{u} = 0, \;\; \partial u_3/\partial \nu & \text{on } \Gamma_0 \\ \mathbf{u} = 0 & \text{on } \Gamma_1 \end{array}\right. \tag{2.19}$$

where ν denotes the normal to the boundary. Let V be the space of the kinematically admissible displacements, defined by

$$V = \{ (v_1, v_2, v_3) \in H^1 \times H^1 \times H^2 \,;\, \mathbf{v} \text{ satisfy } (2.19) \} \tag{2.20}$$

The boundary conditions are supposed such that

$$[a_0(\mathbf{v}, \mathbf{v}) + a_1(\mathbf{v}, \mathbf{v})]^{1/2} \tag{2.21}$$

is a norm on V equivalent to the classical one. This is the case if Ω is connected and Γ_0 is not empty [4], [5]. The shell problem is then:

$$\left\{\begin{array}{l} \text{Find } \mathbf{u}^{\varepsilon} \in V \text{ such that} \\ \dfrac{1}{\varepsilon^2} a_0(\mathbf{u}^{\varepsilon}, \mathbf{v}) + a_1(\mathbf{u}^{\varepsilon}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V'V} \quad \forall \mathbf{v} \in V \end{array}\right. \tag{2.22}$$

where $\mathbf{f} \in V'$, $\varepsilon \in (0, \varepsilon_0)$ and the duality product is given by

(2.23) $$\langle \mathbf{f}, \mathbf{v} \rangle_{V'V} = \int_S f^i v_i \, dS .$$

Clearly, by virtue of the hypothesis that (2.21) is a norm equivalent to that of V, the shell problem (2.22) has a unique solution for each fixed value of $\varepsilon \in (0, \varepsilon_0)$, in the framework of the Lax-Milgram theorem.

Our aim is to study the asymptotic behavior of $\mathbf{u}^\varepsilon$ as ε tends to zero and the corresponding Galerkin approximations.

Let V_h (with $h \searrow 0$) be a family of finite-dimensional subspaces of V such that V_h approaches V, i.e. such that

(2.24) $$\forall \mathbf{v} \in V, \quad \lim_{h \searrow 0} \inf_{\mathbf{v}_h \in V_h} \| \mathbf{v} - \mathbf{v}_h \| = 0$$

or, in other words:

(2.25) $$\begin{cases} \forall \mathbf{v} \in V \quad \text{and} \quad h \searrow 0 \\ \text{there exists } \mathbf{v}_h \in V_h \text{ such that} \\ \mathbf{v}_h \in \mathbf{v} \text{ strongly in } V(h \searrow 0) . \end{cases}$$

We then consider the approximate problem associated with (2.22):

(2.26) $$\begin{cases} \text{Find } \mathbf{u}_h^\varepsilon \in V_h \text{ such that} \\ \dfrac{1}{\varepsilon^2} a_0(\mathbf{u}_h^\varepsilon, \mathbf{v}) + a_1(\mathbf{u}_h^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V'V} \quad \forall \mathbf{v} \in V_h . \end{cases}$$

Moreover we shall admit that

(2.27) $$\begin{cases} \forall \varepsilon > 0, \\ \mathbf{u}_h^\varepsilon \to \mathbf{u}^\varepsilon \text{ strongly in } V(h \searrow 0, \text{fixed } \varepsilon) . \end{cases}$$

where $\mathbf{u}_h^\varepsilon$ and $\mathbf{u}^\varepsilon$ are respectively the solution of (2.26) and (2.22). We note that (2.27) holds true for a certain number of finite element approximations of the shell problem (see for instance [4], section II.1).

DEFINITION 2.1: *We say that V_h implies a robust approximation (2.26) of the problem (2.22) if the convergence (2.27) is uniform with respect to ε, i.e. if for given δ there exists γ such that*

(2.28) $$\| \mathbf{u}_h^\varepsilon - \mathbf{u}^\varepsilon \|_V < \delta \text{ for } h \in (0, \gamma) \quad \text{and} \quad \varepsilon \in (0, \varepsilon_0)$$

We also say that the approximation V_h locks when it is not robust.

Remark 2.2: Property (2.28) ensures that an accurate approximation of the solution may be obtained with a sufficiently small mesh diameter h, independently of the thickness. Another possible definition of robutness involves uniformity of the convergence with respect to ε and to $\mathbf{f}$ (for $\mathbf{f}$ belonging to the unit ball of V', for instance), but we shall adopt in this paper the definition 2.1. ■

In practice if we have a locking approximation, we must fix γ in a suited form for each value of ε. In particular, changing the values of ε with a fixed mesh may spoil the quality of the approximation.

The following theorem will be the main tool for the study of locking in the next sections.

THEOREM 2.3: *Under the previous general hypotheses, let us consider the reiterated limits:*

$$\lim_{\varepsilon \searrow 0} \lim_{h \searrow 0} \mathbf{u}_h^\varepsilon \quad \text{strongly in } V \tag{2.29}$$

$$\lim_{h \searrow 0} \lim_{\varepsilon \searrow 0} \mathbf{u}_h^\varepsilon \quad \text{strongly in } V \tag{2.30}$$

and let us suppose that one the two hypotheses a) and b) hereafter is satisfied
a) the limits (2.29), (2.30) exist and are different,
b) the limit (2.29) exists but (2.30) does not.
Then the approximation V_h is not robust.

Remark 2.4: Let us point out that, denoting the solutions of (2.22) and (2.26) by $\mathbf{u}^\varepsilon$ and $\mathbf{u}_h^\varepsilon$ respectively, the limits

i) $\lim_{\varepsilon \to 0} \mathbf{u}^\varepsilon$ strongly in V
ii) $\lim_{\varepsilon \to 0} \mathbf{u}_h^\varepsilon$ strongly in V, for fixed h

always exist, as we shall prove later on in Theorem 3.2. In particular, from (2.27) and i) it follows that the reiterated limit (2.29) always exist, whereas (2.30) may exist or not.■

Proof of Theorem 2.3: It is a classical corollary of the theorem on uniform convergence of continuous functions. For the sake of completeness, let us recall theorem 66 of [20], vol. 1, p. 150: "Let E and F be two metric spaces, A a part of E and $f_0, f_1, \ldots, f_n \ldots$ a sequence of mappings of A into F converging uniformly to f. Let a be an accumulation point of A in E. If for each n, $f_n(x)$ has a limit when for x tending to a by points in A and if F is complete, then $f(x)$ has a limit for x tending to a by points in A and moreover"

$$\lim_{\substack{x \to a \\ x \in A}} f(x) = \lim_{n \to \infty} \left[\lim_{\substack{x \to a \\ x \in A}} f_h(x) \right]. \tag{2.31}$$

Let us apply this theorem taking $E = \mathbb{R}$, $F = V$ (which is a complete space), $x = \varepsilon$, $A = (0, \varepsilon_0)$ $f(x) = \mathbf{u}^\varepsilon$, $f_h(x) = \mathbf{u}_h^\varepsilon$ (with $h = h_n$, a sequence tending to 0 as n tends to infinity, $a = 0$. Let us suppose that the convergence $\mathbf{u}_h^\varepsilon \to \mathbf{u}^\varepsilon$ is uniform with respect to $\varepsilon \in (0, \varepsilon_0)$; then (2.31) holds true, taking the form

$$\lim_{\varepsilon \searrow 0} \mathbf{u}^\varepsilon = \lim_{h \searrow 0} \left[\lim_{\varepsilon \searrow 0} \mathbf{u}_h^\varepsilon \right] \tag{2.32}$$

which is in contraction with a) and b). ■

3. LIMIT PROCESSES AND CONSEQUENCES ON LOCKING

Let us define the closed subspace G of V formed by the pure bendings by:

$$\begin{aligned} G &= \{ \mathbf{v} \in V ; a_0(\mathbf{v}, \mathbf{v}) = 0 \} \\ &= \{ \mathbf{v} \in V ; a_0(\mathbf{v}, \mathbf{w}) = 0 \quad \forall \mathbf{w} \in V \} \\ &= \{ \mathbf{v} \in V ; \gamma_{\alpha\beta}(\mathbf{v}) = 0, \alpha, \beta = 1, 2 \} \end{aligned} \tag{3.1}$$

We note that the three spaces appearing in the right side of (3.1) are the same. The equality of the first and the third ones follows from the definition of the form a_0 (2.15) and from the positivity of the coefficients (2.18). The equality of the first and the second spaces follows from the Cauchy-Schwarz inequality for the seminorm $a_0(\mathbf{v}, \mathbf{v})^{1/2}$:

$$|a_0(\mathbf{v}, \mathbf{w})| \leq a_0(\mathbf{v}, \mathbf{v})^{1/2} \, a_0(\mathbf{w}, \mathbf{w})^{1/2}. \tag{3.2}$$

The fact that G is closed follows immediately from (3.1) using for instance the second expression in the right side. Consequently *G is itself a Hilbert space* for the topology induced by that of V, and we may state the *limit problem for $\varepsilon \to 0$*:

$$\begin{cases} \text{Find } \mathbf{u}^0 \in G \text{ such that} \\ a_1(\mathbf{u}^0, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V'V} \quad \forall \mathbf{v} \in G \end{cases} \tag{3.3}$$

We note that, as a_0 vanishes on G, the left side of (3.3) is a form continuous and coercive on G, and consequently $\mathbf{u}^0$ exists and is unique in the classical framework of Lax-Milgram.

Remark 3.1: Let us consider the operators

$$A_\varepsilon \in \mathscr{L}(V, V'), \quad \hat{A} \in \mathscr{L}(G, G') \tag{3.4}$$

defined by

$$\langle A_\varepsilon \mathbf{u}, \mathbf{v} \rangle_{V'V} = \frac{1}{\varepsilon^2} a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in V \tag{3.5}$$

$$\langle \hat{A}\mathbf{u}, \mathbf{v} \rangle_{G'G} = a_1(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in G . \tag{3.6}$$

It classically follows from the Lax-Milgram theorem that A_ε and $\hat{A}$ define isomorphisms of V onto V' and of G onto G' respectively. Moreover, V' may be decomposed as a direct sum:

$$V' = G' \oplus G^p \tag{3.7}$$

where

$$G^p = \{ \mathbf{f} \in V', \langle \mathbf{f}, \mathbf{v} \rangle_{V'V} = 0 \quad \forall \mathbf{v} \in G \}$$

is the polar set of G. We also note that the necessary and sufficient condition for $\mathbf{u}^0 \neq 0$ is that the component of $\mathbf{f}$ on G' does not vanish. ∎

In the same way we define for each h:

$$G_h = G \cap V_h = \{ \mathbf{v} \in V_h \, ; a_0(\mathbf{v}, \mathbf{v}) = 0 \} . \tag{3.8}$$

Let $\mathbf{u}_h^0$ be the solution of

$$\begin{cases} \text{Find } \mathbf{u}_h^0 \in G_h \text{ such that} \\ a_1(\mathbf{u}_h^0, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V'V} \quad \forall \mathbf{v} \in G_h . \end{cases} \tag{3.9}$$

We then have:

THEOREM 3.2: *Let $\mathbf{u}^\varepsilon, \mathbf{u}_h^\varepsilon, \mathbf{u}^0, \mathbf{u}_h^0$ be the solutions of (2.22), (2.26), (3.3) and (3.9) respectively. Then,*

$$\mathbf{u}^\varepsilon \to \mathbf{u}^0 \quad \text{strongly in } V (\varepsilon \searrow 0) \tag{3.10}$$

$$\mathbf{u}_h^\varepsilon \to \mathbf{u}_h^0 \text{ strongly in } V_h (\varepsilon \searrow 0, \text{ fixed } h) \tag{3.11}$$

Proof: It is classical as (3.10) and (3.11) are penalty limit processes. Let us prove for instance (3.11). Taking $\mathbf{v} = \mathbf{u}_h^\varepsilon$ in (2.26) it follows from the coerciveness of $a_0 + a_1$:

$$c \| \mathbf{u}_h^\varepsilon \|_V^2 \leqslant a_0(\mathbf{u}_h^\varepsilon, \mathbf{u}_h^\varepsilon) + a_1(\mathbf{u}_h^\varepsilon, \mathbf{u}_h^\varepsilon) \leqslant \tag{3.12}$$

$$\leqslant \frac{1}{\varepsilon^2} a_0(\mathbf{u}_h^\varepsilon, \mathbf{u}_h^\varepsilon) + a_1(\mathbf{u}_h^\varepsilon, \mathbf{u}_h^\varepsilon) = \langle \mathbf{f}, \mathbf{u}_h^\varepsilon \rangle_{V'V} \leqslant C \| \mathbf{u}_h^\varepsilon \|_V$$

for two non-vanishing constants c and C. We then see that the $\mathbf{u}_h^\varepsilon$ remain in a ball of the finite-dimensional space V_h. After extraction of a subsequence (in fact it is the whole sequence because we shall see later that the limit is unique)

$$\mathbf{u}_h^\varepsilon \to \mathbf{u}^* \text{ weakly and strongly in } V_h(\varepsilon \searrow 0, \text{fixed } h) . \tag{3.13}$$

It follows from (3.12) that

$$a_0(\mathbf{u}_h^\varepsilon, \mathbf{u}_h^\varepsilon)^{1/2} \leqslant K\varepsilon . \tag{3.14}$$

On the other hand, for any fixed $\mathbf{v} \in V_h$, we have from (3.13):

$$a_0(\mathbf{u}_h^\varepsilon, \mathbf{v}) \to a_0(\mathbf{u}^*, \mathbf{v}) \quad (\varepsilon \searrow 0, \text{fixed } h) , \tag{3.15}$$

and, using (3.14) and an estimate analogous to (3.2) we see that the left side of (3.15) converges to zero in modulus, and consequently $a_0(\mathbf{u}^*, \mathbf{v}) = 0$. Taking $\mathbf{v} = \mathbf{u}^*$ we see that $\mathbf{u}^* \in G_h$. On the other hand, taking in (2.26) $\mathbf{v} \in G_h$ we have:

$$a_1(\mathbf{u}_h^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V'V} \quad \forall \mathbf{v} \in G_h . \tag{3.16}$$

and letting $\varepsilon \searrow 0$, we have, according to (3.13):

$$a_1(\mathbf{u}^*, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V'V} \quad \forall \mathbf{v} \in G_h . \tag{3.17}$$

As we saw that $\mathbf{u}^* \in G_h$, $\mathbf{u}^* = \mathbf{u}_h^0$ is the only solution of (3.9) and (3.11) is proven.

The proof of (3.10) follows exactly the same steps, showing the weak convergence in V. The strong convergence in V needs and ulterior reasoning. As we pointed out before, the proof is classical, and may be seen for instance in [6], Theorem 1. ■

The result (3.10) constitutes the fundamental property of thin shells. When the thickness tends to zero, the solutions converge to a solution which is an inextensional displacement, i.e. a pure bending. This property is meaningfull in the case when the shell is "*non-inhibited*", i.e. S admits non zero pure bendings, or

$$G \neq \{0\} . \tag{3.18}$$

In the *inhibited* case, i.e. $G = \{0\}$, (3.10) always hold true, but evidently $\mathbf{u}^0 = 0$. In order to have a better description of the asymptotic behavior of $\mathbf{u}^\varepsilon$ we write $\mathbf{u}^\varepsilon = \varepsilon^2 \mathbf{v}^\varepsilon$ and we study the limit behavior of $\mathbf{v}^\varepsilon$. This asymptotic process, which has nothing to do with the membrane locking may be seen in [15] or [16], [17].

According to (3.11) an analogous behavior holds for the approximate problem in the space V_h. *The membrane locking occurs when G_h does not approach G.* More precisely:

PROPOSITION 3.3: *Let us admit the general hypotheses of this paper, in particular V_h approaches V, (see (2.25)). Moreover, let us assume that the subspaces $G_h = G \cap V_h$ defined in (3.8) do not approach G, defined in (3.1) (in other words, there exists at least a non-zero element $\hat{\mathbf{v}}$ of G which cannot be approached in the form (2.25), with G and G_h instead of V and V_h, respectively).*

Then the approximation of problem (2.22) by V_h is not robust (locking holds at least for certain **f**, *for instance for* $\mathbf{f} = \hat{A}\hat{\mathbf{v}}$ *where* $\hat{A}$ *denotes the operator defined in (3.6)).*

Proof: We consider problems (2.22) and (2.26) with $\mathbf{f} = \hat{A}\hat{\mathbf{v}}$. We are showing that we then are in the hypotheses of theorem 2.3. From (2.27) we see that the limit $h \searrow 0$ of $\mathbf{u}_h^{\varepsilon}$ exists and is $\mathbf{u}^{\varepsilon}$. Next, the limit $\varepsilon \searrow 0$ of $\mathbf{u}^{\varepsilon}$ exists by (3.10) and is equal to $\mathbf{u}^0$, which is in fact $\hat{\mathbf{v}}$ for the considered force **f** (see (3.3) and (3.6) if necessary). It then follows that the reiterated limit (2.29) exists and is equal to $\hat{\mathbf{v}}$. On the other hand, the reiterated limit (2.30) becomes by virtue of (3.11)

$$\lim_{h \searrow 0} \mathbf{u}_h^0 \quad \text{strongly in } V \tag{3.19}$$

but $\mathbf{u}_h^0 \in G_h \subset G$, and the convergence in (3.19) is in fact "strongly in G". This limit may exist or not exist, but in any case by hypothesis, it is not equal to $\hat{\mathbf{v}}$, and theorem 2.3 applies. ■

Remark 3.4: Locking holds in particular when $G \neq \{0\}$ but $G_h = G \cap V_h = \{0\}$ for any h. In that case we have a total locking, for any **f** with non-zero component in G'. ■

Remark 3.5: We shall see in the next sections examples where $\mathbf{v} \in G_h$ implies $v_3 = 0$, but G contain elements with non-zero third component. Clearly in that case the G_h do not approach G (see (2.25) with G and G_h instead of V and V_h, respectively) and locking holds. ■

4. ANY PIECEWISE POLYNOMIAL INTERNAL APPROXIMATION IS NECESSARILY NOT ROBUST (OR LOCKS)

In this section we are proving that any finite element internal approximation of the shell problem such that the elements of V_h are in each triangle of the mesh polynomial functions (i.e. practically any finite element method) locks for certain shapes of the middle surface. Clearly we may wonder if there is no locking for other surfaces. This is the reason why we first give some heuristic reasons showing that locking is probably a generic phenomenon, holding for almost any surface, with a very few exceptions.

As we saw in proposition 3.3, locking appears when the spaces $G_h = G \cap V_h$ do not approach G where

$$G_h = \{\mathbf{v} \in V_h\,;\, \gamma_{\alpha\beta}(\mathbf{v}) = 0, \quad \alpha, \beta = 1, 2\} \tag{4.1}$$

In order to get an intuitive idea of the nature of this problem, let us consider to simplify and fix ideas, the case when the $b_{\alpha\beta}$ are constant and the Christoffel symbols vanish. The constraints appearing in (4.1) are then:

$$\begin{cases} \gamma_{11}(\mathbf{v}) \equiv \partial_1 v_1 - b_{11} v_3 = 0 \\ \gamma_{22}(\mathbf{v}) \equiv \partial_2 v_2 - b_{22} v_3 = 0 \\ \gamma_{12}(\mathbf{v}) \equiv \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) - b_{12} v_3 = 0 \end{cases} \tag{4.2}$$

Let us admit that one of the coefficients b_{11} and b_{22} does not vanish. This condition is generally satisfied, and certainly it does if the surface is either elliptic or parabolic. Let $b_{11} \neq 0$. The first condition (4.2) shows that

$$\mathbf{v} \in G_h \Rightarrow v_3 = \frac{1}{b_{11}} \partial_1 v_1 \tag{4.3}$$

Let us consider the restriction of (4.3) to a triangle of the mesh. Then, as v_1 is a polynomial of some degree m, v_3 is necessarily a polynomial of degree $m - 1$. This is perfectly incoherent with the fact that the components v_α ($\alpha = 1, 2$) belong to H^1 but $v_3 \in H^2$: any finite element scheme classically used in shell theory (cf. [4] for instance) uses polynomials of higher order in the discretization of v_3 than in that of v_1, v_2. In these conditions, the convergence of the approximation is seriously compromised.

It should be noticed that the above reasoning is very close to that of [6], theorem 4, where it is shown that in the case of a circular arch, there is a robust non-classical approximation by finite elements where v_α