S. WARDI

A convergence result for an iterative method for the equations of a stationary quasi-newtonian flow with temperature dependent viscosity


<http://www.numdam.org/item?id=M2AN_1998__32_4_391_0>
A CONVERGENCE RESULT FOR AN ITERATIVE METHOD FOR THE EQUATIONS OF A STATIONARY QUASI-NEWTONIAN FLOW WITH TEMPERATURE DEPENDENT VISCOSITY (*)

S. WARDI (')

Abstract — We study a system of equations describing the stationary and incompressible flow of a quasi-Newtonian fluid with temperature dependent viscosity and with a viscous heating. An algorithm which decouples the calculation of the temperature $T$ and the velocity and the pressure $(v, p)$ is presented. It consists in solving iteratively a problem with a non-linear Stokes's operator for $v$ and $p$ and the Poisson's equation with right-hand side in $L^1$ for $T$. We prove, using the method of pseudomonotonicity and under a regularity assumption of Meyers type that the mapping defined by this scheme is a contraction for sufficiently small data.

Résumé — On étudie un système modélisant l'écoulement d'un fluide quasi-Newtonien stationnaire incompressible avec une viscosité dépendant de la température et en tenant compte des effets d'échauffement visqueux. On présente un algorithme découplant le calcul du couple vitesse-pression et de la température $T$. Il s'agit de résoudre itérativement un problème concernant un opérateur de Stokes non linéaire en vitesse et pression, à température donnée, puis une équation de Poisson à second membre $L^1$ en température, à vitesse donnée. On montre à l'aide de la méthode de pseudo-monotonie et sous une hypothèse de régularité de type Meyers que l'application définie par ce schéma est contractante pour des données suffisamment petites.

1. INTRODUCTION

We consider equations describing the incompressible quasi-Newtonian fluid flow with temperature dependent viscosity. Existence for such problem of a weak solution has been recently proved by Baranger and Mikelić, (see [3]), using Schauder fixed point theorem; uniqueness of this solution was left as an open problem.

In numerical simulations one usually uses an iterative decoupled algorithm: here, it will consist in solving iteratively a problem with a non-linear Stokes problem for $v$ and $p$ and the Poisson's equation with right-hand side in $L^1$ for $T$.

We prove in this paper, for small data and under a Meyers's type regularity property of the $r$-Stokesian operator, that this simple algorithm is convergent to the unique weak solution of the problem. In fact, we prove that the operator defined from the iterative method is a contraction and use Banach fixed-point theorem.

Some similar problems, but in the simpler case of two scalar elliptic equations coupling the Laplacian and the heat equation, have been studied by Howinson et al. (see [7]) with uniqueness result for sufficiently small data and sufficiently regular solution, (see also [4]). We will adapt the functional framework and some ideas from [3] in proving existence.

Let us consider a bounded domain $\Omega$ in $\mathbb{R}^n$, $N = 2$ or $3$, with a regular boundary $\Gamma$, and an incompressible quasi-Newtonian fluid flowing in $\Omega$, with temperature dependent viscosity and with a viscous heating. We consider
the steady case and neglect inertia effects. $T$ being the temperature, $v$ the velocity and $p$ the pressure of the fluid, we consider the following problem $(\mathcal{P})$, (see [3] for a derivation of the model from the basic principles of continuum mechanics):

\[
\begin{aligned}
- \text{div} [\mu(T, |D(v)|) D(v)] + \nabla p &= f \text{ in } \Omega, \\
\text{div} v &= 0 \text{ in } \Omega, \\
v &= 0 \text{ on } \Gamma, \\
-k \Delta T + \rho c_p(T) v \nabla T &= \mu(T, |D(v)|) |D(v)|^2 \text{ in } \Omega, \\
T &= T_0 \text{ on } \Gamma.
\end{aligned}
\]

where $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, $c_p(\cdot)$ is a bounded continuous function on $\mathbb{R}$, $k$ is a positive constant, $\rho$ is the constant density of the fluid,

\[
\tau_0 \in L^\infty(\Gamma) \cap \bigcap_{q < \frac{N}{N-1}} W^{1, \frac{q}{q}}(\Gamma); \quad \tau_0 > C_0 > 0 \text{ (a.e.) on } \Gamma (1.1)
\]

This is more realistic than the assumption: $\tau_0 \in H^{3/2}(\Gamma)$, (see [3]).

Furthermore, this assumption on the boundary data ensures the existence of an extension of $\tau_0$, which we will denote by $\tau_0^*$, such that: $\tau_0^* \in W^{1,q}(\Omega)$, $\forall q < N'$, owing to the isomorphism between $W^{1, \frac{q}{q}}(\Gamma)$ and $W^{1,q}(\Omega)/\ker \gamma$, $\gamma$ being the trace operator on $\Gamma$ (see [1], Theorem 7.53).

$\mu$ is supposed continuous on $\mathbb{R}^2$ and satisfies the following properties: $\forall s_1, s_2 \in \mathbb{R}, \forall \xi, \eta \in \mathbb{R}^{N^2}$,

\[
\begin{aligned}
|\mu(s_1, |\xi|) - \mu(s_2, |\xi|)| &\leq K_1 \beta(|s_1 - s_2|)|\xi|^{r-2}, \quad 1 < r \leq 2, \quad (1.2) \\
\text{where } \beta \in C_q(\mathbb{R}), \quad \beta \geq 0 \text{ and } \beta(0) = 0, \quad (1.3) \\
[\mu(s, |\xi|) \xi - \mu(s, |\eta|) \eta] \cdot (\xi - \eta) &\geq K_2 |\xi - \eta|^2 (|\xi| + |\eta|)^{r-2}, \quad (1.4) \\
&\forall s \in \mathbb{R}, \forall \xi, \eta \in \mathbb{R}^{N^2}, \forall \xi_1, \xi_2, \eta \in \mathbb{R}^{N^2}, \quad (1.5)
\end{aligned}
\]

We remark that a classical example of viscosity is the product of an Arrhenius law: $\lambda(T) = C \exp \frac{K}{T}$ and a power law $v(|D(v)|) = v_0 |D(v)|^{r-2}$ (see [2]), the above conditions being satisfied in that case.

Now, for studying problem $(\mathcal{P})$, we define the following functional spaces: For the velocity $v$, since we have to solve a $r$-Stokes monotone problem:

\[
V_r = \{v \in [W^{1,r}_0(\Omega)]^N / \text{div } v = 0 \text{ in } \Omega\} \quad (1.6)
\]

and for the temperature $T$, since we have a Poisson equation with a right-hand side in $L^1(\Omega)$:

\[
W_N = \bigcap_{1 < q < \frac{N}{N-1}} W^{1,q}_0(\Omega) \quad (1.7)
\]
We say that \((v, T)\), with \(u \in Vr, T \in WN\), \(T > C0\) (a.e.) in \(\Omega\), \(f \in L^{r'}(\Omega)\), is a weak solution of problem \((\mathcal{P})\) if:

\[
\int_\Omega \mu(T, |D(v)|) D(v) : D(\varphi) = \int_\Omega f \varphi, \quad \forall \varphi \in Vr ;
\]

\[
k \int_\Omega \nabla T \nabla \xi - \rho \int_\Omega vC_p(T) \nabla \xi = \int_\Omega \mu(T, |D(v)|) |D(v)|^2 \xi, \quad \forall \xi \in W^{1,\infty}_0(\Omega), \text{ where } C_p(T) = \int_0^T c_p(s) \, ds.
\]

2. THE FIXED POINT ALGORITHM

We introduce the following decoupled algorithm:

We start by \(T^0 = \bar{\tau}_0\), and \((v^0, p^0)\) = the solution in \(Vr \times L^r(\Omega)\) of the Stokes problem, (see [12]):

\[
\begin{align*}
- \text{div} \left[ \mu(\bar{\tau}_0, |D(v^0)|) D(v^0) \right] + \nabla p^0 &= f \quad \text{in } \Omega \\
\text{div} v^0 &= 0 \quad \text{in } \Omega \\
v^0 &= 0 \quad \text{on } \Gamma.
\end{align*}
\]

For \(T^n, v^n, p^n\) given, we search for \(T^{n+1}, v^{n+1}, p^{n+1}\) weak solutions in \(WN \times Vr \times L^{r'}(\Omega)\) of the following homogeneous problem:

\[
(\mathcal{P}_{n+1}) \begin{cases}
- \text{div} \left[ \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) \right] + \nabla p^{n+1} = f & \text{in } \Omega \\
-kA(T^n + \bar{\tau}_0) + \rho c_p(T^n + \bar{\tau}_0) v^{n+1} \nabla (T^n + \bar{\tau}_0) &= \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) |D(v^{n+1})|^2 & \text{in } \Omega
\end{cases}
\]

We define, from this algorithm, the following fixed point operator:

\[
\Phi : Vr \times WN \to Vr \times WN
\]

\[
(u, T_u) \mapsto (v, T_v) = \Phi(u, T_u) \text{ solution of :}
\]

\[
\begin{cases}
- \text{div} \left[ \mu(T_v + \bar{\tau}_0, |D(v)|) D(v) \right] + \nabla p_v = f & \text{in } \Omega, \quad \text{and :}
-kA(T_v + \bar{\tau}_0) + \rho c_p(T_v + \bar{\tau}_0) v \nabla (T_v + \bar{\tau}_0) = \mu(T_v + \bar{\tau}_0, |D(v)|) |D(v)|^2 & \text{in } \Omega.
\end{cases}
\]

where \(p_v \in L^{r'}(\Omega)\) is the pressure associated to \(v\) and is unique up to a constant.

In order to prove that \(\Phi\) is a contracting mapping and hence, to state a convergence theorem for the algorithm \((\mathcal{P}_{n+1})\), we describe a Meyers’s type regularity property of the \(r\)-Stokesian operator used in the first step of \((\mathcal{P}_{n+1})\), i.e. solution of the \(r\)-Stokes problem:

\[
(\mathcal{P}_r) \begin{cases}
- \text{div} \left[ \mu_r(T, |D(v)|) D(v) \right] + \nabla p = f & \text{in } \Omega \\
\text{div} v &= 0 \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \Gamma,
\end{cases}
\]

where \(\mu_r(\ldots) := \mu(\ldots)\) satisfies assumptions (1.2)-(1.5). We can formulate this property as follows:
There exists \( \gamma^* > r \) such that: for \( f \in L^y(\Omega) \) with \( \frac{1}{y} = \frac{1}{r} + \frac{1}{N} \) we have, for each \( v \) solution of the \( r \)-Stokes problem \( (\mathcal{P}_r) \):

\[
D(v) \in L^p(\Omega), \quad \forall r < p \leq \gamma^*, \quad \text{and} \quad \|D(v)\|_{L^p(\Omega)} \leq C\|f\|_{L^y(\Omega)} \tag{2.2}
\]

the constant \( C \) depending only on the data.

Such a regularity result has been proved in [11] for second order equation. See [13] for the case of the \( r \)-Stokesian operator.

For technical reason, we introduce:

\[
\gamma_0 = \begin{cases} 
    r & \text{if } N = 2 \\
    \frac{3(r-1)}{2r-3}r & \text{if } N = 3.
\end{cases} \tag{2.3}
\]

We can state:

**THEOREM 2.1**: Assume \((1.1)-(1.5), \frac{N}{2} < r \leq 2, \) and that the exponent \( \gamma^* \) in \ref{2.2} satisfies: \( \gamma^* > \gamma_0 \), where \( \gamma_0 \) is given by \ref{2.3}. Then there exists a constant \( C \), depending only on the data, such that: if \( \|f\|_{L^y(\Omega)} \leq C \), with \( \frac{1}{y} = \frac{1}{r} + \frac{1}{N} \), then the fixed point iteration is a contraction.

**COROLLARY 2.1**: Under the previous assumptions, Problem \( (\mathcal{P}) \) has a unique weak solution and the fixed point algorithm \( (\mathcal{P}_n) \) is convergent.

### 3. PROOF OF THEOREM 2.1

The proof is based on four propositions:

**PROPOSITION 3.1**: Under the assumptions of theorem 2.1, the fixed point operator \( \Phi \) is well defined.

**Proof**: Let us prove existence and uniqueness of a weak solution of \( (\mathcal{P}_{n+1}) \):

The solution \( v^{n+1} \) of the \( r \)-Stokes problem in \( (\mathcal{P}_{n+1}) \) exists in \( V_r \), is unique owing to the assumptions \((1.2)-(1.4)\); and there exists a corresponding pressure \( p^{n+1} \) unique up to a constant, in \( L^r(\Omega) \) (see [12]).

Furthermore, we obtain easily, taking \( v^{n+1} \) as a test-function in the first equation of \( (\mathcal{P}_{n+1}) \), using \((1.4)\) and the Poincaré’s inequality:

\[
\|D(v^{n+1})\|_{L^r(\Omega)^2} \leq \left( \frac{C(\Omega)}{K_2} \right)^{r-1} \|f\|_{L^r(\Omega)}^{1/r} \|D(v^{n+1})\|_{L^r(\Omega)^2} = C(\Omega, r, f). \tag{3.1}
\]

In the second equation in \( (\mathcal{P}_{n+1}) \), the right-hand side is in \( L^1(\Omega) \) since \( v \) is in \( V_r \) and since \( \mu \) satisfies \((1.2), (1.3)\). So we do not have a sufficient regularity for using the classical variational formulation for this problem. Adapting an idea of [3], we decompose this equation in two simpler ones:

Firstly:

\[
\begin{cases}
-k \mathcal{A}T^{n+1} = \mu(T^n + \bar{v}_0, |D(v^{n+1})|) \ |D(v^{n+1})|^2 \text{ in } \Omega, \\
T^{n+1} = 0 \text{ on } \Gamma.
\end{cases} \tag{3.2}
\]
Then, we can apply the results on Poisson’s equation with right-hand side in $L^1$, (see for example [5]) and we obtain existence and uniqueness of a solution to (3 2)

$$T_1^{n+1} \in W_0^1 (\Omega), \quad \forall 1 \leq q < \frac{N}{N-1} = N'.$$

and we have the estimate $\|T_1^{n+1}\|_{W_0^1 (\Omega)} \leq C(\Omega, N, r, \tau_0), \quad \forall 1 \leq q < N'$

In fact, for $N = 3$, we can use some results from [10] (see Theorem 12 1) to get that the solution of (3 2) lies in $W_0^1 (\Omega)$

Indeed, using the first equation of ($\mathcal{P}_{n+1}$), we can write formally the right hand side of (3 2) as follows

$$\text{div} \left( [\mu(T^n + \tilde{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) - p^{n+1} I] v^{n+1} \right) + f v^{n+1},$$

where

$$[\mu(T^n + \tilde{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) - p^{n+1} I] v^{n+1} \in L^N \quad \text{and} \quad f v^{n+1} \in W^{-1, N} (\Omega)$$

This can be easily seen using Holder’s inequality with exponents $p = \frac{(N-1) r}{N(r-1)}$, $p' = \frac{(N-1) r}{N-r}$, (Note that $p > 1$ for $r < N$) Indeed, we obtain, with (1 2)-(1 3)

$$\int_\Omega |\mu(T^n + \tilde{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) v^{n+1}|^N \leq C \int_\Omega \{ |D(v^{n+1})|^{r-1} |v^{n+1}| \}^N$$

$$\leq \|D(v^{n+1})\|_{L^{N(r-1)}} \|v^{n+1}\|_{L^N}^N$$

$$\leq \|D(v^{n+1})\|_{L^N}^N,$$ by Poincare’s inequality and Sobolev Imbedding Theorem, $\leq C(\Omega, r, f)$, by (3 1)

For $f v^{n+1}$, it is easy to see that $\forall \varphi \in W_0^1 (\Omega) (\subset L^p (\Omega), \forall p < \infty)$,

$$\int_\Omega f v^{n+1} \varphi \leq C \|f\|_{L^p} \|v^{n+1}\|_{L^p} \|\varphi\|_{L^\infty (\Omega)}$$

Secondly

$$\begin{cases}
- k A(T_1^{n+1} + \tilde{\tau}_0) + p c_p (T_1^{n+1} + T_2^{n+1} + \tilde{\tau}_0) v^{n+1} \nabla (T_1^{n+1} + T_2^{n+1} + \tilde{\tau}_0) = 0 \quad \text{in } \Omega \\
T_2^{n+1} = 0 \quad \text{on } \Gamma
\end{cases}
$$

We have, since $c_p$ is bounded $\forall T \in H^1 (\Omega)$,

$$\int_\Omega v^{n+1} c_p (T_1^{n+1} + \tilde{\tau}_0 + T) \nabla (T_1^{n+1} + \tilde{\tau}_0 + T) \| v^{n+1} \|_{L^r} \|T_1^{n+1} + \tilde{\tau}_0 + T\|_{W^{r, \gamma}} \leq C \|v^{n+1}\|_{L^r} \|T_1^{n+1} + \tilde{\tau}_0 + T\|_{W^{r, \gamma}},$$

$$\leq C \|v^{n+1}\|_{L^r} \|T_1^{n+1} + \tilde{\tau}_0 + T\|_{W^r},$$ since $(r^*)' = \frac{N r}{N r - N + r} < \frac{N}{N-1}$, for $r > \frac{N}{2}$.
and:

\[ \forall \varphi \in H^1_0(\Omega), \quad \left| \int_\Omega v^{n+1} T \nabla \varphi \right| \leq C \| \varphi \|_{H^1_0} \| v^{n+1} \|_{L^2} \| T \|_{L^2(\frac{r}{2}, \cdot)}, \]

\[ \leq C \| \varphi \|_{H^1_0} \| v^{n+1} \|_{W^{1,p}} \| T \|_{H^1}, \]

since \( 2 \left( \frac{r}{2} \right)^p = \frac{6 r}{5 r - 6} < 2^* = 6, \) for \( r > \frac{N}{2}, \)

this for \( N = 3; \) obtaining a same estimate for \( N = 2 \) being more easy due to Sobolev Imbedding Theorem.

Then, we can apply results of pseudomonotone operators theory, (see [9]), to get existence and uniqueness of a solution \( T^{n+1}_2 \) in \( H^1_0(\Omega) \) to problem (3.3) and that: \( \| T^{n+1}_2 \|_{H^1(\Omega)} \leq C, \) where \( C \) depends only on the coefficients of the equation and the data. So, by (3.1), \( C \) depends only on the data.

Note that if \( c_p(T^{n+1}_2 + \tilde{\tau}_0) \) is replaced by \( c_p(T^n + \tilde{\tau}_0) \) in the algorithm, then we can deduce existence and uniqueness of a solution of (3.3) in \( H^1_0(\Omega) \) directly from the results of linear elliptic equations with unbounded coefficients (see [8]) since the coefficient \( v^{n+1} \) satisfies: \( \| v^{n+1} \|_{L^p(\Omega)} \leq C < +\infty, \) with \( p = 2 \) if \( r > N. \)

Finally, taking: \( T^{n+1} = T^{n+1}_1 + T^{n+1}_2, \) we obtain a unique weak solution of \( (\mathcal{P}_{n+1}), \) which satisfies:

\[ \| T^{n+1} \|_{W^p} \leq C(\Omega, N, r, \tau_0). \quad (3.4) \]

We conclude that the mapping \( \Phi \) is well defined.

**PROPOSITION 3.2:** If the iterative method converges to \( (v_0, T_0) \), then \( (v_0, T_0 + \tilde{\tau}_0) \) is a weak solution of \( (\mathcal{P}). \)

**Proof:** From the estimates (3.1) and (3.4), we deduce that there exists a subsequence, still denoted by the same symbol, such that:

- firstly: \( v^n \rightharpoonup v_0 \) in \( V_\rho \) weak. So, by Rellich’s Compactness Theorem,

\[ v^n \rightharpoonup v_0 \text{ in } L^p(\Omega) \text{ strong, for } 1 \leq p < r^* = \frac{Nr}{N-r} \text{ if } r < N, \]

for all \( p < \infty, \) if \( r = N \)

— secondly: \( T^n \rightharpoonup T_0 \) in \( W^{1,q}_0(\Omega) \) weak, \( \forall 1 \leq q < N'. \) Then:

\[ T^n \rightharpoonup T_0 \text{ in } L^m(\Omega) \text{ strong for } 1 \leq m < (N')^* = \frac{N}{N-2}, \text{ if } N = 3, \]

for all \( m < \infty \) if \( N = 2, \) and \( T^n \rightharpoonup T_0 \) a.e. in \( \Omega. \)

Let us now show that: (\( v^n \)) \( \rightharpoonup v_0 \) in \( V_\rho \) strong:

We have by (1.8):

\[ \int_\Omega \mu(T^n + \tilde{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) : D(\psi) = \int_\Omega f \psi, \quad \forall \psi \in V_\rho, \quad (3.7) \]

and taking: \( \psi = \varphi - v^{n+1} : \)

\[ \int_\Omega \mu(T^n + \tilde{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) : D(\varphi - v^{n+1}) = \int_\Omega f(\varphi - v^{n+1}). \quad (3.8) \]
But (1.4) gives:
\[
\int_{\Omega} [\mu(T^n + \tau_0, |D(\varphi)|)D(\varphi) - \mu(T^n + \tau_0, |D(v^n + 1)|)D(v^n + 1)] : D(\varphi - v^n + 1) \geq 0.
\]

Then:
\[
\int_{\Omega} \mu(T^n + \tau_0, |D(\varphi)|)D(\varphi) : D(\varphi - v^n + 1) \geq \int_{\Omega} f(\varphi - v^n + 1). \tag{3.9}
\]

Then, passing to the limit in this inequality, using the continuity of \(\mu\), the a.e. convergence of \(T^n\) to \(T_0\) and the weak convergence of \(v^n\) to \(v_0\), we get:
\[
\int_{\Omega} \mu(T_0 + \tau_0, |D(\varphi)|)D(\varphi) : D(\varphi - v_0) \geq \int_{\Omega} f(\varphi - v_0). \tag{3.10}
\]

Now, by a usual procedure from Minty's lemma (taking first \(\varphi = v_0 + \alpha \psi,\) with \(\alpha > 0\), in (3.10), then letting \(\alpha \to 0\), and taking \(\psi = - \varphi\), we obtain:
\[
\int_{\Omega} \mu(T_0 + \tau_0, |D(v_0)|)D(v_0) : D(\varphi) = \int_{\Omega} f\varphi, \quad \forall \varphi \in V_r. \tag{3.12}
\]

So in particular:
\[
\int_{\Omega} \mu(T_0 + \tau_0, |D(v_0)|)D(v_0) \geq \int_{\Omega} f\varphi; \quad \text{and, with (3.7):}
\]
\[
\int_{\Omega} \mu(T^n + \tau_0, |D(v^n + 1)|) |D(v^n + 1)|^2 - \int_{\Omega} \mu(T_0 + \tau_0, |D(v_0)|) |D(v_0)|^2
\]
\[
= \left| \int_{\Omega} f(v^n + 1 - v_0) \right| \xrightarrow{n \to \infty} 0. \tag{3.13}
\]

Furthermore, we have:
\[
\int_{\Omega} [\mu(T^n + \tau_0, |D(v_0)|)D(v_0) - \mu(T^n + \tau_0, |D(v^n + 1)|)D(v^n + 1)] : D(v_0 - v^n + 1)
\]
\[
= \int_{\Omega} \mu(T^n + \tau_0, |D(v_0)|)D(v_0) : D(v_0 - v^n + 1)
\]
\[
+ \int_{\Omega} \mu(T_0 + \tau_0, |D(v_0)|)D(v_0) : D(v_0 - v^n + 1)
\]
\[
- \int_{\Omega} \mu(T^n + \tau_0, |D(v^n + 1)|)D(v^n + 1) : D(v_0 - v^n + 1)
\]
\[
= \int_{\Omega} [\mu(T^n + \tau_0, |D(v_0)|) - \mu(T_0 + \tau_0, |D(v_0)|)] D(v_0) : D(v_0 - v^n + 1),
\]
by (3.7) and (3.12). This, with condition (1.4), gives:

\[
K_2 \int_\Omega |D(v_0 - v^{n+1})|^2 \{ |D(v_0)| + |D(v^{n+1})| \}^{r-2} 
\]

\[
\lesssim \int_\Omega \left| \mu(T^n + \bar{\tau}_0, |D(v_0)|) - \mu(T_0 + \bar{\tau}_0, |D(v_0)|) \right| D(v_0) : D(v_0 - v^{n+1}) 
\]

\[
\leq K_1 \int_\Omega \beta(|T^n - T_0|) |D(v_0)|^{r-1} |D(v_0 - v^{n+1})|^r, \text{ by (1.2)} 
\]

\[
\leq C \|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{\nu^2}} \left[ \int_\Omega \beta(|T^n - T_0|) \gamma |D(v_0)| \right]^\frac{1}{r}, \quad (3.14) 
\]

by Holder's inequality (with \(\frac{1}{r'} + \frac{1}{r} = 1\)). But, we have (for \(r < 2\)):

\[
\int_\Omega |D(v_0 - v^{n+1})|^{r'} 
\]

\[
= \int_\Omega |D(v_0 - v^{n+1})|^{r'} \{ |D(v_0)| + |D(v^{n+1})| \}^{\frac{r-2}{2} \gamma |D(v_0)| + |D(v^{n+1})|}^{\frac{2-r}{2}}, 
\]

\[
\lesssim \left[ \int_\Omega |D(v_0 - v^{n+1})|^{2 \gamma |D(v_0)| + |D(v^{n+1})|}^{r-2} \right]^\frac{1}{2} \left[ \int_\Omega \{ |D(v_0)| + |D(v^{n+1})| \}^{\frac{2-r}{2}} \right]. 
\]

and since:

\[
\int_\Omega \{ |D(v_0)| + |D(v^{n+1})| \}^{r'} \lesssim 2^{r-1} (\|D(v_0)\|_{L^r} + \|D(v^{n+1})\|_{L^r}) \leq C, 
\]

by (3.1), then we get (for \(r \leq 2\)):

\[
\|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{\nu^2}} \leq C \left[ \int_\Omega |D(v_0 - v^{n+1})|^{2 \gamma |D(v_0)| + |D(v^{n+1})|}^{r-2} \right]^\frac{1}{2}. 
\]

This gives:

\[
\|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{\nu^2}} 
\]

\[
\leq C \int_\Omega |D(v_0 - v^{n+1})|^{2 \gamma |D(v_0)| + |D(v^{n+1})|}^{r-2}, \quad (3.15) 
\]

\[
\leq C \|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{\nu^2}} \left[ \int_\Omega \beta(|T^n - T_0|) \gamma |D(v_0)| \right]^\frac{1}{r}, \text{ by (3.14)}. 
\]
Therefore:

\[ \left\| D(v_n - v^{n+1}) \right\|_{L^2(\Omega)} \leq C \left[ \int_{\Omega} \beta(\left| T^n - T_0 \right| \right]\left| D(v_n) \right|^r \right]^{1/r}. \]  

(3.16)

Since \( \beta \) is bounded, then we have:

\[ \forall n, \quad \beta(\left| T^n - T_0 \right| ) \leq C \left| D(v_n) \right|^r \quad \text{a.e. in} \ \Omega, \quad \text{with} \quad g \in L^1(\Omega). \]

Then, using Lebesgue's Dominated Convergence Theorem and the continuity of \( \beta \) (we have: \( T^n \to T_0 \) a.e.), we deduce from (3.16): \( \left\| D(v_n - v^{n+1}) \right\|_{L^2(\Omega)} \to 0 \). Consequently,

\[ (v^n) \to v_0 \quad \text{in} \quad V, \quad \text{strong}. \]

(3.17)

For \( (T^n) \), we have, by (3.13): \( \forall \xi \in W^{1,\infty}_0(\Omega), \)

\[ \int_{\Omega} \mu(T^n + \tilde{v}_0) |D(v^{n+1})| \, |D(v^{n+1})| \, d\xi \to \int_{\Omega} \mu(T_0 + \tilde{v}_0) \, |D(v_0)| \, |D(v_0)| \, d\xi, \]

and by (3.17) and (3.6):

\[ \int_{\Omega} v^{n+1} C_p(T^{n+1} + \tilde{v}_0) \, \nabla \xi \to \int_{\Omega} v_0 \, C_p(T_0 + \tilde{v}_0) \, \nabla \xi. \]

Indeed:

\[ \rho \int_{\Omega} \left\{ v^{n+1} C_p(T^{n+1} + \tilde{v}_0) - v_0 \, C_p(T_0 + \tilde{v}_0) \right\} \, \nabla \xi \]

\[ \leq C \left\{ \int_{\Omega} \left| v^{n+1} - v_0 \right| \left| C_p(T^{n+1} + \tilde{v}_0) \right| + \int_{\Omega} \left| v_0 \right| \left| C_p(T^{n+1} + \tilde{v}_0) - C_p(T_0 + \tilde{v}_0) \right| \right\} \]

\[ \leq C \left\{ \left\| v^{n+1} - v_0 \right\|_{L^r} \left\| T^{n+1} + \tilde{v}_0 \right\|_{L^q} + \left\| v_0 \right\|_{L^q} \left\| T^{n+1} - T_0 \right\|_{L^r} \right\} \to 0, \]

by (3.6) and (3.17), since we have: \( r' = \frac{r}{r-1} < \frac{N}{N-2} \) for \( r > \frac{N}{2} \).

Furthermore, by (3.6) and the Sobolev Imbedding: \( W^{1,\infty}_0(\Omega) \subset W^{1,q}_0(\Omega), \ \forall q < \frac{N}{N-1} \), we have: \( \forall \xi \in W^{1,\infty}_0(\Omega), \)

\[ k \int_{\Omega} \nabla(T^{n+1} + \tilde{v}_0) \, \nabla \xi \to k \int_{\Omega} \nabla(T_0 + \tilde{v}_0) \, \nabla \xi. \]

So, by uniqueness of the limit, we obtain:

\[ k \int_{\Omega} \nabla(T_0 + \tilde{v}_0) \, \nabla \xi - \rho \int_{\Omega} v_0 \, C_p(T_0 + \tilde{v}_0) \, \nabla \xi \]

\[ = \int_{\Omega} \mu(T_0 + \tilde{v}_0) \, |D(v_0)| \, |D(v_0)| \, \xi; \quad \forall \xi \in W^{1,\infty}_0(\Omega). \]  

(3.18)

vol. 32, n° 4, 1998
Furthermore, the assumption on \( \tau_0 \) implies that the limit \( T_0 + \tau_0 \geq C_0 > 0 \) a.e. in \( \Omega \), (see [3], [6]).

This, (3.12) and (3.18) imply that \( (v_0^*, T_0 + \tau_0) \) is a weak solution of \( (\mathcal{P}) \).

There exists a corresponding pressure \( p_0 \) in \( L^r(\Omega) \), convergence of \( (v^n) \) giving that of \( (p^n) \) in \( W^{-1/r, r}(\Omega) \).

In the sequel, for simplicity, we will take \( c_p(T) = 1 \), this function being of secondary importance in the obtaining of the following estimates, since it is bounded.

**Proposition 3.3:** Under the assumptions of theorem 2.1, the velocities satisfy the following estimate:

\[
\|D(v_1 - v_2)\|_{L^r} \leq C \|f\|_{L^r(\Omega)}^{\frac{r}{r-1}} \|T_{u_1} - T_{u_2}\|_{W^*, r},
\]

where: \( (v_1, T_{u_1}) = \phi(u_1, T_{u_1}) \) and \( (v_2, T_{u_2}) = \phi(u_2, T_{u_2}) \), \( C \) depending only on the data: \( \Omega, N, r, \tau_0, f \).

**Proof:** We easily get from the definition of \( \phi \):

\[
\int_{\Omega} \left[ \mu(T_{u_1} + \tau_0, |D(v_1)|) D(v_1) - \mu(T_{u_1} + \tau_0, |D(v_2)|) D(v_2) \right] : D(v_1 - v_2) = -\int_{\Omega} \left[ \mu(T_{u_1} + \tau_0, |D(v_1)|) - \mu(T_{u_2} + \tau_0, |D(v_2)|) \right] D(v_2) : D(v_1 - v_2).
\]

Therefore, by (1.4) and (1.2):

\[
K_1 \left( \int_{\Omega} |D(v_1 - v_2)|^2 \left[ |D(v_1)| + |D(v_2)| \right] \right)^{-2} \leq K_1 \int_{\Omega} \beta(|T_{u_1} - T_{u_2}|) |D(v_2)|^{r-1} |D(v_1 - v_2)|,
\]

\[
\leq K_1 \|D(v_1 - v_2)\|_{L^r} \left( \int_{\Omega} \beta(|T_{u_1} - T_{u_2}|) |D(v_2)|^r \right)^{\frac{1}{r}}.
\]

Then, similarly as in estimate (3.16), we obtain:

\[
\|D(v_1 - v_2)\|_{L^r(\Omega)^{\nu^2}} \leq C \left( \int_{\Omega} |T_{u_1} - T_{u_2}|^{\nu^2} |D(v_2)|^r \right)^{\frac{1}{r}}.
\]

And, by the Meyers’s regularity property of the \( r \)-Stokes problem, using Hölder’s inequality, we obtain:

\[
\|D(v_1 - v_2)\|_{L^r(\Omega)^{\nu^2}} \leq C \|D(v_2)\|_{L^r(\Omega)^{\nu^2}}^{\frac{r}{r-1}} \|T_{u_1} - T_{u_2}\|_{L^{r/(r-1)}(\Omega)}.
\]

Hence, by (2.2):

\[
\|D(v_1 - v_2)\|_{L^r(\Omega)^{\nu^2}} \leq C \|f\|_{L^r(\Omega)}^{\frac{r}{r-1}} \|T_{u_1} - T_{u_2}\|_{L^{r/(r-1)}(\Omega)}^{\frac{r}{r-1}}.
\]
Then, in order to have an estimate of \( \| T_{u_1} - T_{u_2} \|_{L^r(\Omega)} \) with \( r > \frac{N}{2} \), we need to add, for \( N = 3 \), the following regularity assumption: \( \gamma^* > \gamma_0 \), where \( \gamma_0 = \frac{N(r - 1)}{2r - N}r \), which is a necessary and sufficient condition to have: \( \frac{r\gamma^*}{\gamma^* - r} < \frac{N}{N - 2} \). This, with (3.21) gives Proposition 3.3.

**Remark 3.1:** The method used in the previous step does not allow us to prove Proposition 3.3 in the case \( r > 2 \), under a natural assumption on \( \mu \), that is:

\[
[\mu(s, |\xi|) \xi - \mu(s, |\eta|) \eta] : (\xi - \eta) \geq K_4|\xi - \eta|^r.
\]

Indeed, (3.19) and (1.2) would give:

\[
\begin{align*}
K_4 \int_\Omega |D(v_1 - v_2)|^r & \leq K_1 \int_\Omega \beta(|T_{u_1} - T_{u_2}|) |D(v_2)|^{r - 1} |D(v_1 - v_2)|, \\
& \leq C \|D(v_1 - v_2)\|_{L^r(\Omega)} \left( \int_\Omega |T_{u_1} - T_{u_2}|^r |D(v_2)|^r \right)^{\frac{1}{r}}.
\end{align*}
\]

So:

\[
\|D(v_1 - v_2)\|_{L^r(\Omega)}^{r - 1} \leq C \left( \int_\Omega |T_{u_1} - T_{u_2}|^r |D(v_2)|^r \right)^{\frac{1}{r}}.
\]

Finally, we would get, by Hölder’s inequality and for \( \gamma^* > \gamma_0 \),

\[
\|D(v_1 - v_2)\|_{L^r(\Omega)} \leq C \|f\|_{L^r(\Omega)} \|T_{u_1} - T_{u_2}\|_{W^{1,1}}^{\frac{1}{r}}.
\]

Because of the exponent \( \frac{1}{r - 1} < 1 \), for \( r > 2 \), we can not deduce from this estimate that \( \Phi \) is a contracting mapping in that case.

**Proposition 3.4:** Under the assumptions of theorem 2.1, the temperatures satisfy the following estimate:

\[
\|T_{v_1} - T_{v_2}\|_{W^{1,1}} \leq C\{\|f\|_{L^r(\Omega)}^2 + \|f\|_{L^r(\Omega)}^2\} \|T_{u_1} - T_{u_2}\|_{W^{1,1}}
+ C\|f\|_{L^r(\Omega)} \|T_{v_1} - T_{v_2}\|_{W^{1,1}},
\]

where the constant \( C \) depends only of the data: \( \Omega, N, r, \tau_0, f \).

**Proof:** \( (T_{v_1} - T_{v_2}) \) is a solution of the equation:

\[
-k\Delta(T_{v_1} - T_{v_2}) = \left\{ \mu(T_{u_1} - \bar{\tau}_0, |D(v_1)|) |D(v_1)|^2 - \mu(T_{u_1} + \bar{\tau}_0, |D(v_2)|) |D(v_2)|^2 \right\}
- \rho\{v_1 \nabla(T_{v_1} + \bar{\tau}_0) - v_2 \nabla(T_{v_2} + \bar{\tau}_0)\}.
\]

(3.22)
We get, from the definition of $\phi$:

$$
\int_{Q} \left\{ \mu(T_{u_1} + \tau_0, |D(v_1)|) \left| D(v_1) \right|^2 - \mu(T_{u_2} + \tau_0, |D(v_2)|) \left| D(v_2) \right|^2 \right\}
= \int_{Q} \mu(T_{u_2} + \tau_0, |D(v_2)|) D(v_2) : D(v_1 - v_2).
$$

Then:

$$
\left| \int_{Q} \left\{ \mu(T_{u_1} + \tau_0, |D(v_1)|) \left| D(v_1) \right|^2 - \mu(T_{u_2} + \tau_0, |D(v_2)|) \left| D(v_2) \right|^2 \right\} \right|
\leq C \int_{Q} |D(v_2)|^{r-1} |D(v_1 - v_2)|, \text{ by (1.2) - (1.3)},
$$

$$
\leq C \|D(v_2)\|_{L^r(\Omega)} \|D(v_1 - v_2)\|_{L^r(\Omega)},
$$

$$
\leq \|f\|_{L^r(\Omega)} \|T_{u_1} - T_{u_2}\|_{w^r}, \text{ by (2.2) and Proposition 3.3}. \quad (3.23)
$$

Furthermore,

$$
\rho \left| \int_{Q} v_1 \nabla(T_{v_1} + \tau_0) - v_2 \nabla(T_{v_2} + \tau_0) \right|
\leq \rho \int_{Q} |(v_1 - v_2) \nabla(T_{v_1} + \tau_0)| + \rho \int_{Q} |v_2 (\nabla T_{v_1} - \nabla T_{v_2})|
\leq C \|v_1 - v_2\|_{L^r/N - r} \|\nabla(T_{v_1} + \tau_0)\|_{L^{N_r/N - N + r}}
+ C \|v_2\|_{L^{N_r/N - r}} \|\nabla T_{v_1} - \nabla T_{v_2}\|_{L^{N_r/N - N + r}}, \text{ for } r < N;
$$

$$
\leq C \|D(v_1 - v_2)\|_{L^r} \|T_{v_1} + \tau_0\|_{w^r} + C \|D(v_2)\|_{L^r} \|T_{v_1} - T_{v_2}\|_{w^r},
$$

by Poincaré's inequality and Sobolev imbedding theorem (Recall that: $\frac{N_r}{N_r - N + r} < \frac{N}{N - 1}$, for $r > \frac{N}{2}$). Then, by Proposition 3.3, estimates (2.2) and (3.4), we obtain:

$$
\rho \left| \int_{Q} \left\{ v_1 \nabla(T_{v_1} + \tau_0) - v_2 \nabla(T_{v_2} + \tau_0) \right\} \right|
\leq C \|f\|_{L^r(\Omega)} \|T_{u_1} - T_{u_2}\|_{w^r} + C \|f\|_{L^q(\Omega)} \|T_{v_1} - T_{v_2}\|_{w^r} \quad (3.24)
$$
Then, (3.22)-(3.24) imply that: \( T_{v_1} - T_{v_2} \) is a solution of the equation: 
\[-A(T_{v_1} - T_{v_2}) = F, \]
where \( F \in L^1(\Omega) \) and consequently the following estimate holds (see [5]):
\[
\| T_{v_1} - T_{v_2} \| W^{1,q}(\Omega) \leq C \| F \|_{L^1(\Omega)} \quad \forall q < \frac{N}{N-1}.
\]
This, with estimates (3.23) and (3.24) gives Proposition 3.4.

**End of proof of Theorem 2.1:** We can now deduce that there exists a closed ball \( B_R \) nonempty in \( V_r \times W_N \) such that: \( \phi(B_R) \subset B_R \); and \( \phi \) is a contracting mapping on \( B_R \), for \( r > \frac{N}{2} \) and \( \|f\|_{L^p} \) sufficiently small:

By the definition of \( v^0 \) and \( T^0 \), we can easily choose \( R > 0 \) such that: \( \|D(v^0)\|_{L^r(\Omega)} + \|\tau^0\|_{L^s(\Omega)} \leq R \), and consequently \( (v^0, T^0) \in B_R \).

Our aim is to prove that there exists \( \delta, \ 0 < \delta < 1 \), such that:
\[
\| (v_1, T_{v_1}) - (v_2, T_{v_2}) \|_{V_r \times W_N} \leq \delta \| (u_1, T_{u_1}) - (u_2, T_{u_2}) \|_{V_r \times W_N}.
\]
Using Proposition 3.4, we obtain that if \( \|f\|_{L^r(\Omega)} \) is sufficiently small, that is:
\[
C \max \{ \|f\|_{L^r(\Omega)}^{2/r'}, \|f\|_{L^r(\Omega)} \} < \delta < \frac{1}{2},
\]
then:
\[
(1 - \tilde{\delta}) \| T_{v_1} - T_{v_2} \|_{W_N} \leq \tilde{\delta} \| T_{u_1} - T_{u_2} \|_{W_N}.
\]
Finally, taking: \( \delta = \frac{\tilde{\delta}}{1 - \tilde{\delta}} \), we get:
\[
\| T_{v_1} - T_{v_2} \|_{W_N} \leq \tilde{\delta} \| T_{u_1} - T_{u_2} \|_{W_N} \quad \text{with} \quad 0 < \delta < 1.
\]  
(3.25)

Analogously, in proposition 3.3, if \( f \) is sufficiently small, then:
\[
\| D(v_1 - v_2) \|_{L^r} \leq \tilde{\delta} \| T_{u_1} - T_{u_2} \|_{W_N}.
\]  
(3.26)

Finally, (3.25) and (3.26) imply that \( \phi \) is a contraction mapping, for \( r > \frac{N}{2} \), \( f \) sufficiently small and, for \( N = 3, v \) sufficiently regular: \( D(v) \in L^{r'}; \gamma < \gamma'' \). This gives Theorem 2.1.

Then, under the above assumptions, we can apply the Banach fixed-point theorem to get that \( \phi \) admits a unique fixed point \( (v_0, T_0) \) in \( B_R \). Furthermore, there exists a corresponding pressure \( p \) unique up to a constant. Then, the algorithm \( (\emptyset_\alpha) \) converges to this solution. Since, a solution of \( (\emptyset) \) corresponds to a fixed point of \( \phi \), then, using Proposition 3.2, we obtain that \( (v_0, T_0 + \tau_0) \) is the unique weak solution of problem \( (\emptyset) \). Therefore, Corollary 2.1 is proved.

**ACKNOWLEDGEMENTS**

The author is grateful to Pr. J. Baranger and Pr. A. Mikelic for helpful discussions and suggestions and to the Referee for several constructive remarks.

**REFERENCES**


