

**DISCRETE SOBOLEV INEQUALITIES AND  $L^p$  ERROR ESTIMATES FOR FINITE VOLUME SOLUTIONS OF CONVECTION DIFFUSION EQUATIONS**YVES COUDIERE<sup>1</sup>, THIERRY GALLOUËT<sup>2</sup> AND RAPHAËLE HERBIN<sup>3</sup>

**Abstract.** The topic of this work is to obtain discrete Sobolev inequalities for piecewise constant functions, and to deduce  $L^p$  error estimates on the approximate solutions of convection diffusion equations by finite volume schemes.

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## 1. INTRODUCTION

The aim of this work is to study the discretization by the finite volume method of convection diffusion problems on general structured or non structured grids; these grids may consist of polygonal control volumes satisfying adequate geometrical conditions (which are stated in the sequel) and not necessarily ordered in a Cartesian grid. We shall be concerned here with the so-called “cell-centered” finite volume method. We refer to [1, 18, 25] and references therein for studies on the “vertex-centered” finite volume method, and to [3, 4, 15] and [12] for the related finite volume element and control volume finite element methods.

The analysis of cell centered finite volume schemes has only recently been undertaken. Error estimates were first obtained in the rectangular case [16, 23, 24]. Triangular meshes and Voronoi meshes, which we shall also refer to as “admissible” meshes, were also investigated [13, 19, 22, 27]; convergence results were obtained for Dirichlet boundary conditions and constant diffusion coefficients and were generalized to Neumann and Fourier boundary conditions [17] and to non-homogeneous diffusion matrices [20]. The scheme was also extended to more general “non-admissible” meshes [8, 9, 14], and an error estimate was proven in the case of a quadrangular mesh [9], and in the case of some refined meshes of rectangles [2, 7]. The estimates are obtained in these papers under  $C^2$  or  $H^2$  regularity assumptions on the exact solution.  $L^2$  error estimates between the exact and the approximated solutions are proved to be of order one with respect to the size of the mesh, (and of order two in the case of rectangular meshes).

We shall prove here a  $L^p$  error estimate of order one, with  $p \in [1, +\infty)$  in the two dimensional case and  $p \in [1, 6]$  in the three dimensional case, and derive some lower order  $L^\infty$  estimates as a consequence.

In Section 2 below we present the continuous problem. In Section 3, we describe a finite volume scheme for solving this problem, which was proved to be convergent on families of “admissible meshes” ([13], see Def. 1).

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We also define the discrete spaces and norms which are used to prove the estimates on the schemes. For more general meshes (see Def. 1), we give an extension of the previous scheme which is successfully used in practice [5, 21] and has been proved to converge on quadrangular meshes [9], on rectangular meshes with local refinement [10], and on admissible meshes (in which case it is identical to the previous scheme). Section 4 is concerned with the proof of some discrete inequalities of Sobolev for functions defined on general meshes which yield the final  $L^p$  error estimates for the schemes.

## 2. THE CONTINUOUS PROBLEM

Let us consider the following elliptic equation:

$$-\Delta u(x) + \operatorname{div}(\mathbf{v}u)(x) + bu(x) = f(x), \quad x \in \Omega, \tag{1}$$

with Dirichlet boundary condition:

$$u(x) = g(x), \quad x \in \partial\Omega, \tag{2}$$

where

**Assumption 1** ( $d = 2$  or  $3$ ).

- (i)  $\Omega$  is an open bounded polygonal subset of  $\mathbb{R}^d$ ,
- (ii)  $b \in \mathbb{R}_+^*$ ,
- (iii)  $f : \Omega \rightarrow \mathbb{R}$  is such that  $f \in L^2(\Omega)$ ,
- (iv)  $\mathbf{v} \in C^1(\overline{\Omega}, \mathbb{R}^d)$ ,  $\operatorname{div} \mathbf{v} = 0$ , and  $\exists V \in \mathbb{R}$ ,  $|\mathbf{v}(x)| \leq V$  for all  $x \in \mathbb{R}^d$ , where  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{R}^N$ ,
- (v)  $g \in H^{1/2}(\partial\Omega, \mathbb{R})$ ; let  $\tilde{g} \in H^1(\Omega)$  verifying  $\overline{\gamma}(\tilde{g}) = g$  on  $\partial\Omega$ .

Here, and in the sequel,  $\overline{\gamma}$  denotes the trace operator from  $H^1(\Omega)$  into  $L^2(\partial\Omega)$ .

**Remark 1.** The Laplace operator is considered here for the sake of simplicity, but more general elliptic operators are possible to handle, for instance operators of the form  $-\operatorname{div}(a(u)\nabla u)$  with adequate assumptions on  $a$ .

Let us introduce the weak formulation of problem (1)-(2). A weak solution of (1)-(2) under Assumptions 1 is a function  $u = \tilde{u} + \tilde{g} \in H^1(\Omega)$  satisfying

$$\begin{cases} u = \tilde{u} + \tilde{g} \text{ where } \tilde{u} \in H_0^1(\Omega) \text{ and} \\ \int_{\Omega} (\nabla u(x)\nabla\varphi(x) - \mathbf{v}(x)u(x)\nabla\varphi(x) + u(x)\varphi(x))dx = \int_{\Omega} f(x)\varphi(x)dx, \forall \varphi \in H_0^1(\Omega). \end{cases} \tag{3}$$

By Lax-Milgram's lemma there exists a unique function  $u \in H^1(\Omega)$  which satisfies (3). Furthermore, it is known that if  $\Omega$  and  $\mathbf{v}$  are regular enough (for instance if  $\Omega$  is polygonal and convex), the solution is in  $W^{2,p}(\Omega)$ , for  $f \in L^p(\Omega)$  and  $\tilde{g} \in W^{1,p}(\Omega)$  (for some  $p$ , see [6] for some precisions).

## 3. THE FINITE VOLUME SCHEMES

The finite volume scheme is found by integrating equation (1) on a given control volume of a discretization mesh and finding an approximation of the fluxes on the control volume boundary in terms of the discrete unknowns.

Let us first give the assumptions which are needed on the mesh.

### 3.1. Finite volume meshes

We first give the assumptions and notations on the meshes which are used for the discretization of convection diffusion equations by the finite volume scheme.

**Definition 1** (General and admissible meshes). Let  $\Omega$  be an open bounded polygonal subset of  $\mathbb{R}^d$  ( $d = 2$ , or 3). A general finite volume mesh of  $\Omega$  is denoted by  $\mathcal{T}$  and is given by a family of “control volumes” which are open polygonal convex subsets of  $\Omega$  (with positive measure); a family of subsets of  $\overline{\Omega}$  contained in hyperplanes of  $\mathbb{R}^d$ , denoted by  $\mathcal{E}$  (these are the edges (in the two-dimensional case) or sides (in the three-dimensional case) of the control volumes), with strictly positive  $(d - 1)$ -dimensional measure and a family of points of  $\Omega$  denoted by  $\mathcal{P}$  satisfying the following properties (in fact, we shall denote, somewhat incorrectly, by  $\mathcal{T}$  the family of control volumes).

- (i) The closure of the union of all the control volumes is  $\overline{\Omega}$ .
- (ii) For any  $K \in \mathcal{T}$ , there exists a subset  $\mathcal{E}_K$  of  $\mathcal{E}$  such that  $\partial K = \overline{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$ , and we suppose that  $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$ .
- (iii) For any  $(K, L) \in \mathcal{T}^2$  with  $K \neq L$ , either the  $(d - 1)$ -dimensional Lebesgue measure of  $\overline{K} \cap \overline{L}$  is 0, or  $\overline{K} \cap \overline{L} = \overline{\sigma}$  for some  $\sigma \in \mathcal{E}$ , which will then be denoted by  $K|L$ .

An admissible finite volume mesh of  $\Omega$  is a general finite volume mesh of  $\Omega$  which satisfies the following additional condition:

- (iv) The family  $\mathcal{P} = (x_K)_{K \in \mathcal{T}}$  is such that  $x_K \in K$  (for all  $K \in \mathcal{T}$ ) and, if  $\sigma = K|L \in \mathcal{E}_K$ , it is assumed that  $x_K \neq x_L$ , and that the straight line  $\mathcal{D}_{K,L}$  going through  $x_K$  and  $x_L$  is orthogonal to  $K|L$ .

In the sequel, the following notations are used:

- The mesh size is defined by:  $\text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\}$ .
- For any  $K \in \mathcal{T}$  and  $\sigma \in \mathcal{E}$ ,  $m(K)$  is the  $d$ -dimensional Lebesgue measure of  $K$  (*i.e.* area if  $d = 2$ , volume if  $d = 3$ ), and  $m(\sigma)$  the  $(d - 1)$ -dimensional measure of  $\sigma$ .
- The set of interior (resp. boundary) edges is denoted by  $\mathcal{E}_{\text{int}}$  (resp.  $\mathcal{E}_{\text{ext}}$ ), that is  $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E}; \sigma \not\subset \partial\Omega\}$  (resp.  $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}; \sigma \subset \partial\Omega\}$ ).
- The set of the neighbours of  $K$  is denoted by  $\mathcal{N}(K)$ , that is  $\mathcal{N}(K) = \{L \in \mathcal{T}; \exists \sigma \in \mathcal{E}_K, \overline{\sigma} = \overline{K} \cap \overline{L}\}$ .
- For any  $K \in \mathcal{T}$ , and for  $\sigma \in \mathcal{E}_K$ ,  $d_{K,\sigma}$  is the Euclidean distance from  $x_K$  to  $\sigma$ .
- If  $\sigma = K|L \in \mathcal{E}_{\text{int}}$ , we note  $d_\sigma = d_{K|L} = d_{K,\sigma} + d_{L,\sigma}$ . On admissible meshes, it is the Euclidean distance between  $x_K$  and  $x_L$  (which is positive).
- If  $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$ , we note  $d_\sigma = d_{K,\sigma}$ .
- For any  $\sigma \in \mathcal{E}$ ; the “transmissibility” through  $\sigma$  is defined by  $\tau_\sigma = m(\sigma)/d_\sigma$ .
- $\mathcal{S}$  (resp.  $\mathcal{S}_{\text{ext}}$ ) denotes the family of the vertices of the control volumes (resp. the vertices which are on the boundary).
- For any  $\sigma \in \mathcal{E}$ ,  $\mathcal{S}_\sigma$  denotes the set of the vertices of the interface  $\sigma$ .
- For  $K \in \mathcal{T}$  and  $\sigma \in \mathcal{E}_K$ ,  $\mathbf{n}_{K,\sigma}$  denotes the unit normal to  $\sigma$ , outward to  $K$ . Then,  $\mathcal{B}_{K,\sigma} = (\mathbf{t}_{K,\sigma}^i)_{i=1..d-1}$  is a basis of the hyperplane  $\sigma$ , such that  $(\mathbf{n}_{K,\sigma}, \mathbf{t}_{K,\sigma}^i)$  is a direct basis in  $\mathbb{R}^d$ .

**Remark 2.** On admissible meshes, the condition  $x_K \neq x_L$  if  $\sigma = K|L$ , is in fact quite easy to satisfy: two neighbouring control volumes  $K, L$  which do not satisfy it just have to be collapsed into a new control volume  $M$  with  $x_M = x_K = x_L$ , and the edge  $K|L$  removed from the set of edges. The new mesh thus obtained is admissible.

**Remark 3.** The difference between general meshes and admissible meshes is that it is not necessary to be able to construct the family of cell centers ( $\mathcal{P}$ ) such that the edges (or sides)  $K|L$  are perpendicular to the directions  $d_{K|L}$ .

Whenever possible, the cell centers should be chosen such that the mesh is admissible; the schemes described below are then identical. This is the case in particular for triangular meshes or Voronoi meshes. Otherwise, they are usually chosen to be the centers of gravity of the control volumes.

### 3.2. Discrete spaces and norms

Let us now introduce the space of piecewise constant functions which are associated to a finite volume mesh and some “discrete  $H_0^1$ ” norm for this space. This discrete norm will be used to obtain some estimates on the approximate solution given by a finite volume scheme.

**Definition 2.** Let  $\Omega$  be an open bounded polygonal subset of  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ , and  $\mathcal{T}$  a general mesh. Define  $X(\mathcal{T})$  as the set of functions from  $\Omega$  to  $\mathbb{R}$  which are constant over each control volume of the mesh.

**Definition 3** (Discrete norms). Let  $\Omega$  be an open bounded polygonal subset of  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ , and  $\mathcal{T}$  a general finite volume mesh in the sense of Definition 1.

For  $u \in X(\mathcal{T})$  such that  $u(x) = u_K$  for a.e.  $x \in K$ , define the discrete  $H_0^1$  norm by

$$\|u\|_{1,\mathcal{T}} = \left( \sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma u)^2 \right)^{\frac{1}{2}} \tag{4}$$

where, for any  $\sigma \in \mathcal{E}$ ,  $\tau_\sigma = m(\sigma)/d_\sigma$  and  $\begin{cases} D_\sigma u = |u_K - u_L|, & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, \\ D_\sigma u = |u_K|, & \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K \end{cases}$  and the sets  $\mathcal{E}$ ,  $\mathcal{E}_{\text{int}}$ ,  $\mathcal{E}_{\text{ext}}$  and  $\mathcal{E}_K$  are given in Definition 1.

### 3.3. A finite volume scheme on admissible meshes

Let  $\mathcal{T}$  be an admissible mesh. Let us now define a finite volume scheme to discretize (1)-(2). In order to describe the scheme in the most general way, one introduces some auxiliary unknowns, namely the fluxes  $F_{K,\sigma}$ , for all  $K \in \mathcal{T}$  and  $\sigma \in \mathcal{E}_K$ , and some (expected) approximations of  $u$  on edge  $\sigma$ , denoted by  $u_\sigma$ , for all  $\sigma \in \mathcal{E}$ .

For  $K \in \mathcal{T}$  and  $\sigma \in \mathcal{E}_K$ ,  $\mathbf{n}_{K,\sigma}$  denote the unit vector normal to  $\sigma$ , outward to  $K$ ,  $f_K$  denote the mean value of  $f$  on  $K$ , and  $v_{K,\sigma}$  denote the integral of  $\mathbf{v} \cdot \mathbf{n}_{K,\sigma}$  on an edge  $\sigma$  of  $K$ :

$$f_K = \frac{1}{m(K)} \int_K f(x) \, dx \quad \text{and} \quad v_{K,\sigma} = \int_\sigma \mathbf{v}(x) \cdot \mathbf{n}_{K,\sigma} \, d\gamma(x) \tag{5}$$

(note that  $d\gamma$  is the integration symbol for the  $(d - 1)$ -dimensional Lebesgue measure on the considered hyper-plane).

We may now write the finite volume scheme for the discretization of problem (1)-(2) under Assumptions 1 as the following set of equations:

$$\sum_{\sigma \in \mathcal{E}_K} \left( F_{K,\sigma} + v_{K,\sigma} u_{\sigma,+} \right) + bm(K) u_K = m(K) f_K, \tag{6}$$

where for the convection term, we use an upstream scheme, *i.e.*

$$\begin{aligned} \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K, \quad u_{\sigma,+} &= \begin{cases} u_K & \text{if } v_{K,\sigma} \geq 0, \\ u_L & \text{otherwise,} \end{cases} \\ \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K, \quad u_{\sigma,+} &= \begin{cases} u_K & \text{if } v_{K,\sigma} \geq 0, \\ u_\sigma & \text{otherwise.} \end{cases} \end{aligned} \tag{7}$$

The flux  $F_{K,\sigma}$  is defined by:

$$\begin{aligned} \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K, \quad F_{K,\sigma} &= -m(K|L) \frac{u_L - u_K}{d_{K|L}}, \\ \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K, \quad F_{K,\sigma} &= -m(\sigma) \frac{u_\sigma - u_K}{d_{K,\sigma}}, \end{aligned} \tag{8}$$

where for any  $\sigma \in \mathcal{E}_{\text{ext}}$ ,

$$u_\sigma = \frac{1}{m(\sigma)} \int_\sigma g(y) d\gamma(y). \tag{9}$$

**Remark 4.** Note that (6)-(9) leads, after an easy elimination of the auxiliary unknowns, to a linear system of  $N$  equations with  $N$  unknowns, namely the  $(u_K)_{K \in \mathcal{T}}$ , with  $N = \text{card}(\mathcal{T})$ . This linear system can be written, using some ordering of the unknowns and equations, as

$$AU = F + D(g), \tag{10}$$

where:

- $U \in \mathbb{R}^N$  is the vector of discrete unknowns (that is the  $(u_K)_{K \in \mathcal{T}}$ ),  $N$  being the number of cells of the mesh  $\mathcal{T}$ ;
- $A$  is a linear application from  $\mathbb{R}^N$  to  $\mathbb{R}^N$ , and  $AU$  corresponds to the discretization of  $-\Delta u(x) + \text{div}vu + bu$ ;
- $F \in \mathbb{R}^N$  corresponds to the discretization of  $f$ ;
- $D(g)$  is a vector of  $\mathbb{R}^N$  which contains all the terms depending on  $g$  (note that  $D$  is an application from  $L^1(\partial\Omega)$  into  $\mathbb{R}^N$ ).

### 3.4. A finite volume scheme on general meshes

In the case of a general mesh, the line  $x_Kx_L$  is no longer orthogonal to the edge  $\sigma = K|L$ ; the approximation of the flux by the expression given in (8) is therefore no longer consistent. In order to obtain a consistent approximation of the flux, this expression is modified with a term which involves the tangential derivatives.

Of course, the number of points involved in the discretization on a general mesh is greater than on an admissible mesh (9 points instead of 5 in the quadrangular case).

Let us define a discretization of (1)-(2) on a general mesh  $\mathcal{T}$ , which is still of the form:

$$\sum_{\sigma \in \mathcal{E}_K} \left( F_{K,\sigma} + v_{K,\sigma} u_{\sigma,+} \right) + bm(K) u_K = m(K) f_K, \tag{11}$$

where  $u_{\sigma,+}$  is defined by (7).

The flux  $F_{K,\sigma}$  is given by a Green-Gauss type approximation [8,9]. It consists of discretizing the following Green equality, true for smooth functions:

$$\frac{1}{m(\mathcal{V}_\sigma)} \int_{\mathcal{V}_\sigma} \nabla u dx = \frac{1}{m(\mathcal{V}_\sigma)} \int_{\partial\mathcal{V}_\sigma} u \mathbf{n}_{\text{ext}} d\gamma,$$

where  $\mathcal{V}_\sigma$  is the dual cell associated to  $\sigma$ :

- if  $\sigma = K|L \in \mathcal{E}_{\text{int}}$ , then  $\mathcal{V}_\sigma$  is the diamond shaped cell of vertices  $x_K, x_L$ , and the vertices of  $\mathcal{S}_\sigma$ ;
- if  $\sigma \in \mathcal{E}_{\text{int}}$ , then  $\mathcal{V}_\sigma$  is given by  $K, y_\sigma$  and the vertices of  $\mathcal{S}_\sigma$  (it is a pyramid shaped cell).

The right-hand side should provide a good approximation, denoted by  $\mathbf{p}_\sigma$ , of the gradient along  $\sigma$ . It is discretized by a first order Gauss quadrature, where the vertex values (at the vertices  $M$  of  $\mathcal{S}_\sigma$ ) are interpolated

from the center values (the unknowns). This approximation yields, after some calculations, the following expression for  $F_{K,\sigma}$ :

$$\begin{aligned} \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K, \quad F_{K,\sigma} &= -m(K|L) \left( \frac{u_L - u_K}{d_{K|L}} + \sum_{M \in \mathcal{S}_{K|L}} \lambda_{K|L}(M) u_M \right), \\ \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K, \quad F_{K,\sigma} &= -m(\sigma) \left( \frac{u_\sigma - u_K}{d_{K,\sigma}} + \sum_{M \in \mathcal{S}_\sigma} \lambda_\sigma(M) u_M \right), \end{aligned} \tag{12}$$

where for any  $\sigma \in \mathcal{E}_{\text{ext}}$ ,

$$u_\sigma = g(y_\sigma), \tag{13}$$

where  $y_\sigma$  denotes the center-point of *edge* and the values at the vertices are given by

$$\begin{aligned} \text{if } M \in \mathcal{S} \setminus \mathcal{S}_{\text{ext}}, \quad u_M &= \sum_{K \in \mathcal{N}_M} y_M(K) u_K, \\ \text{if } M \in \mathcal{S}_{\text{ext}}, \quad u_M &= g(x_M), \end{aligned} \tag{14}$$

where  $\mathcal{N}_M$  is the set of the control volumes neighbouring  $M$ :  $\mathcal{N}_M = \{K \in \mathcal{T}, \text{ such that } M \in \overline{K}\}$ .

For a node  $M \in \mathcal{S}$ , the weights  $(y_M(K))_{K \in \mathcal{N}_M}$  must be some barycentric coordinates of  $M$  with respect to the centers  $(x_K)_{K \in \mathcal{N}_M}$ , in order for the scheme to be consistent (see [8]). We may for instance calculate them as follows:

$$y_M(K) = \frac{1}{n_M} \left( 1 + \sum_{i=1}^d \frac{x_G^i (x_G^i - x_K^i)}{\sigma_{ii}^2} \right), \tag{15}$$

where  $n_M = \text{card} \mathcal{N}_M$  is the number of control volumes to which  $M$  belongs,  $(x^i)_{i=1 \dots d}$  are the coordinates of a point  $X$ ,  $G$  is the isobarycenter of  $\mathcal{N}_M$  (i.e.  $n_M x_G^i = \sum_{K \in \mathcal{N}_M} x_K^i$ ), and for  $i, j \in \{1 \dots d\}$ ,  $n_M \sigma_{ij}^2 =$

$$\sum_{K \in \mathcal{N}_M} (x_K^i - x_G^i)(x_K^j - x_G^j).$$

Let us now give a more precise expression of the numerical flux in the two- and three-dimensional cases.

### 3.4.1. The two-dimensional case

Let  $N_{K,\sigma}, S_{K,\sigma}$  denote the two endpoints of  $\sigma$ , such that  $(x_{N_{K,\sigma}} - x_{S_{K,\sigma}}) \cdot \mathbf{t}_{K,\sigma} > 0$ . After calculation of the coefficients  $\lambda_\sigma(M)$ , we find

$$\begin{aligned} \text{if } \sigma = K|L, \quad F_{K,\sigma} &= -m(K|L) \left( \frac{u_L - u_K}{d_{K|L}} - \alpha_{K|L} \frac{u_{N_{K,\sigma}} - u_{S_{K,\sigma}}}{m(K|L)} \right) \\ \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K, \quad F_{K,\sigma} &= -m(\sigma) \left( \frac{u_\sigma - u_K}{d_{K,\sigma}} - \alpha_\sigma \frac{u_{N_{K,\sigma}} - u_{S_{K,\sigma}}}{m(\sigma)} \right), \end{aligned} \tag{16}$$

where  $\alpha_{K|L} = \frac{(x_L - x_K) \cdot \mathbf{t}_{K,\sigma}}{(x_L - x_K) \cdot \mathbf{n}_{K,\sigma}}$  (resp.  $\alpha_\sigma = \frac{(y_\sigma - x_K) \cdot \mathbf{t}_{K,\sigma}}{(y_\sigma - x_K) \cdot \mathbf{n}_{K,\sigma}}$ ) is the tangent of the angle from the normal  $\mathbf{n}_{K,\sigma}$  to the direction  $x_K, x_L$  (resp.  $x_K, y_\sigma$ ).

### 3.4.2. The three-dimensional case

For  $K \in \mathcal{T}$  and  $\sigma \in \mathcal{E}_K$ , let  $y_{K,\sigma}$  be the perpendicular projection of  $x_K$  on  $\sigma$ . If  $\sigma \in \mathcal{E}_{\text{ext}}$ , we denote by  $y_\sigma$  the centerpoint of  $\sigma$ .

If  $M \in \mathcal{S}_\sigma$ , let  $M^{(i)}$ ,  $i = 1, N$  be the vertices of  $\sigma$ ; then  $m_K(M^{(i)})$  is the algebraic sum of the oriented area of the two triangles  $M^{(i)}y_{K,\sigma}M_\sigma^{(i-1)}$  and  $M^{(i+1)}y_{K,\sigma}M^{(i)}$ , with respect two the normal  $\mathbf{n}_{K,\sigma}$ . Then we have, for any  $\sigma \in \mathcal{E}$ , such that  $M \in \mathcal{S}_\sigma$ ,

$$\begin{aligned} \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K, \quad \lambda_{K|L}(M) &= \frac{m_L(M) - m_K(M)}{d_{K|L}m(K|L)}, \\ \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K, \quad \lambda_\sigma(M) &= \frac{m_{y_\sigma}(M) - m_K(M)}{d_\sigma m(\sigma)}. \end{aligned} \tag{17}$$

**Remark 5.** Again, the scheme may be written in the form (10). For  $\sigma = K|L \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}$ , if  $x_Kx_L$  is perpendicular to  $\sigma$ , then  $y_{K,\sigma} = y_{L,\sigma}$  and the flux along  $\sigma$  is identical to the flux defined in Part 2.

#### 4. DISCRETE SOBOLEV INEQUALITIES AND $L^p$ ERROR ESTIMATES

##### 4.1. Convergence of the finite volume scheme on admissible meshes

The existence and uniqueness of the solution  $(u_K)_{K \in \mathcal{T}}$  to the scheme (6–9) is an easy consequence of the following maximum principle (see [13,19] or [11] for the proof).

**Proposition 1** (Maximum principle). *Under Assumptions 1, let  $\mathcal{T}$  be an admissible mesh in the sense of Definition 1; and  $(f_K)_{K \in \mathcal{T}}$ ,  $(v_{K,\sigma})_{\sigma \in \mathcal{E}_K, K \in \mathcal{T}}$  and  $(u_\sigma)_{\sigma \in \mathcal{E}_{\text{ext}}}$  be defined by (5) and (9). If  $f_K \geq 0$  for all  $K \in \mathcal{T}$ , and  $u_\sigma \geq 0$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}$ , then the solution  $(u_K)_{K \in \mathcal{T}}$  to (6), (7), (8), (9) satisfies  $u_K \geq 0$ , for all  $K \in \mathcal{T}$ .*

Let us define the approximate solution  $u_{\mathcal{T}}: \Omega \in \mathbb{R}^d \rightarrow \mathbb{R}$  by:

$$\forall K \in \mathcal{T}, \quad u_{\mathcal{T}}(x) = u_K, \quad \text{if } x \in K. \tag{18}$$

We now recall the  $L^2$  error estimate which was proven in [13].

**Theorem 1** ( $H^2$  regularity). *Under Assumptions 1, let  $\mathcal{T}$  be an admissible mesh in the sense of Definition 1, and  $\zeta > 0$  be such that,*

$$\forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K, \quad d_{K,\sigma} \geq \zeta d_\sigma, \quad \text{and } d_{K,\sigma} \geq \zeta \text{diam}(K). \tag{19}$$

Let  $u_{\mathcal{T}}$  be the function defined by (18),  $(u_K)_{K \in \mathcal{T}}$  being the solution of (6)-(9), for  $(f_K)_{K \in \mathcal{T}}$  and  $(v_{K,\sigma})_{\sigma \in \mathcal{E}_K, K \in \mathcal{T}}$  defined by (5). Assume, furthermore, that the unique variational solution,  $u$ , to (1)-(2) belongs to  $H^2(\Omega)$ . Finally, let  $e_{\mathcal{T}}$  be defined for all  $K \in \mathcal{T}$  by  $e_{\mathcal{T}}(x) = e_K = u(x_K) - u_K$  if  $x \in K$ . Then, there exists  $C$ , only depending on  $u, g, \mathbf{v}, b, \Omega$  and  $\zeta$ , such that,

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq C \text{size}(\mathcal{T}), \tag{20}$$

where  $\|\cdot\|_{1,\mathcal{T}}$  is the discrete  $H_0^1$  norm given by Definition 3, and,

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq C \text{size}(\mathcal{T}). \tag{21}$$

##### 4.2. Convergence of the scheme on general meshes

In the general case, assuming the scheme to verify a condition of coercitivity, under the regularity Assumptions (19) and a lower bound of  $m(\sigma)$  in each  $K$ , the error between the approximated solution and the mean value of the exact solution on  $K$  verifies estimates (20) and (21) (see [8,9]).

The condition of coercitivity may be interpreted as a local condition on the regularity of the mesh, and on the weights  $(y_M(K))$ .

Indeed, the scheme has been proved to converge on meshes of quadrangles [8] in the following sense:

**Theorem 2** (Convergence on quadrangular meshes). *Under Assumptions 1, let  $\mathcal{T}$  be a mesh of quadrangles (in  $\mathbb{R}^2$ ). For any  $\sigma \in \mathcal{E}$ , let  $\mathcal{C}_\sigma$  denote the union of the six neighbouring cells to  $\sigma$ , and let  $J_\sigma$  be a  $C^2$  mapping from  $\mathcal{C}_\sigma$  onto  $[0, 3\text{size}(\mathcal{T})] \times [0, 2\text{size}(\mathcal{T})]$ . Let  $\xi > 0$  be such that*

$$\forall \sigma \in \mathcal{E}, \quad \sup_{\mathcal{C}_\sigma} |\nabla J_\sigma| \leq \xi, \text{ and } \sup_{\mathcal{C}_\sigma} |\nabla^2 J_\sigma| \leq \xi. \tag{22}$$

Moreover, suppose that the points  $(x_K)_\mathcal{T}$  are the centers of gravity of the control volumes of the mesh  $\mathcal{T}$ . Assume also that the unique variational solution,  $u$ , to (1)-(2), belongs to  $W^{2,p}(\Omega)$ , for  $p > 2$ .

Then there exist a unique solution  $(u_K)_{K \in \mathcal{T}}$  to (11)-(15), and (16), for  $(f_K)_{K \in \mathcal{T}}$  and  $(v_{K,\sigma})_{\sigma \in \mathcal{E}_K, K \in \mathcal{T}}$  defined by (5).

Moreover, let  $u_\mathcal{T}$  be the function defined by (18), and  $e_\mathcal{T}$  be defined by  $e_\mathcal{T}(x) = e_K = \bar{u}_K - u_K$  (if  $x \in K$ ,  $K \in \mathcal{T}$ ),  $\bar{u}_K$  being the mean value of  $u$  on  $K$  (i.e.  $m(K)\bar{u}_K = \int_K u(x)dx$ ).

Then, there exists  $C$  depending on  $u, g, \mathbf{v}, b, \Omega$  and  $\zeta$ , such that,

$$\|e_\mathcal{T}\|_{1,\mathcal{T}} \leq C \text{size}(\mathcal{T}), \text{ and } \|e_\mathcal{T}\|_{L^2(\Omega)} \leq C \text{size}(\mathcal{T}). \tag{23}$$

**Remark 6.** Assumption (22) is actually not restrictive. It only supposes that the mesh is locally (i.e. around an edge) a small deformation of a reference mesh made of parallelograms [9].

Estimate (23) has been proved more recently [7, 10] on meshes of rectangles with geometrically non conform local refinement, only assuming that  $u \in H^2(\Omega)$ .

### 4.3. $L^p$ error estimates

Let us now show an  $L^p$  estimate of the error, for  $2 \leq p < +\infty$  if  $d = 2$ , and for  $1 \leq p \leq 6$  if  $d = 3$ . The error estimate for the  $L^p$  norm is a consequence of the following lemma:

**Lemma 1** (Discrete Sobolev inequality). *Let  $\Omega$  be an open bounded polygonal subset of  $\mathbb{R}^d$  and  $\mathcal{T}$  be a general finite volume mesh of  $\Omega$  in the sense of Definition 1, and let  $\zeta > 0$  be such that*

$$\forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K, \quad d_{K,\sigma} \geq \zeta d_\sigma, \text{ and } d_{K,\sigma} \geq \zeta \text{diam}(K). \tag{24}$$

Let  $u \in X(\mathcal{T})$  (see Def. 2), then, there exists  $C > 0$  only depending on  $\Omega$  and  $\zeta$ , such that for all  $q \in [1, +\infty)$ , if  $d = 2$ , and  $q \in [1, 6]$ , if  $d = 3$ ,

$$\|u\|_{L^q(\Omega)} \leq Cq \|u\|_{1,\mathcal{T}}, \tag{25}$$

where  $\|\cdot\|_{1,\mathcal{T}}$  is the discrete  $H_0^1$  norm defined in Definition 3.

*Proof of Lemma 1.* Let us first prove the two-dimensional case. Assume  $d = 2$  and let  $q \in [2, +\infty)$ . Let  $\mathbf{d}_1 = (1, 0)^t$  and  $\mathbf{d}_2 = (0, 1)^t$ ; for  $x \in \Omega$ , let  $\mathcal{D}_x^1$  and  $\mathcal{D}_x^2$  be the straight lines going through  $x$  and defined by the vectors  $\mathbf{d}_1$  and  $\mathbf{d}_2$ .

Let  $v \in X(\mathcal{T})$ . For all control volume  $K$ , one denotes by  $v_K$  the value of  $v$  on  $K$ . For all control volume  $K$  and a.e.  $x \in K$ , one has

$$v_K^2 \leq \sum_{\sigma \in \mathcal{E}} D_\sigma v \chi_\sigma^{(1)}(x) \sum_{\sigma \in \mathcal{E}} D_\sigma v \chi_\sigma^{(2)}(x), \tag{26}$$

where  $\chi_\sigma^{(1)}$  and  $\chi_\sigma^{(2)}$  are defined by

$$\chi_\sigma^{(i)}(x) = \begin{cases} 1 & \text{if } \sigma \cap \mathcal{D}_x^i \neq \emptyset \\ 0 & \text{if } \sigma \cap \mathcal{D}_x^i = \emptyset \end{cases} \text{ for } i = 1, 2.$$



Recall that  $D_\sigma v = |v_K - v_L|$ , if  $\sigma \in \mathcal{E}_{\text{int}}$ ,  $\sigma = K|L$  and  $D_\sigma v = |v_K|$ , if  $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$ . Integrating (26) over  $K$  and summing over  $K \in \mathcal{T}$  yields

$$\int_{\Omega} v^2(x) dx \leq \int_{\Omega} \left( \sum_{\sigma \in \mathcal{E}} D_\sigma v \chi_\sigma^{(1)}(x) \sum_{\sigma \in \mathcal{E}} D_\sigma v \chi_\sigma^{(2)}(x) \right) dx.$$

Note that  $\chi_\sigma^{(1)}$  (resp.  $\chi_\sigma^{(2)}$ ) only depends on the second component  $x_2$  (resp. the first component  $x_1$ ) of  $x$  and that both functions are non zero on a region the width of which is less than  $m(\sigma)$ ; hence

$$\int_{\Omega} v^2(x) dx \leq \left( \sum_{\sigma \in \mathcal{E}} m(\sigma) D_\sigma v \right)^2. \tag{27}$$

Applying the inequality (27) to  $v = |u|^\alpha \text{sign}(u)$ , where  $u \in X(\mathcal{T})$  and  $\alpha > 1$  yields

$$\int_{\Omega} |u(x)|^{2\alpha} dx \leq \left( \sum_{\sigma \in \mathcal{E}} m(\sigma) D_\sigma v \right)^2.$$

Now, since  $|v_K - v_L| \leq \alpha(|u_K|^{\alpha-1} + |u_L|^{\alpha-1})|u_K - u_L|$ , if  $\sigma \in \mathcal{E}_{\text{int}}$ ,  $\sigma = K|L$  and  $|v_K| \leq \alpha(|u_K|^{\alpha-1})|u_K|$ , if  $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$ ,

$$\left( \int_{\Omega} |u(x)|^{2\alpha} dx \right)^{\frac{1}{2}} \leq \alpha \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) |u_K|^{\alpha-1} D_\sigma u.$$

Using Hölder’s inequality with  $p, p' \in \mathbb{R}_+$  such that  $\frac{1}{p} + \frac{1}{p'} = 1$  yields that

$$\left( \int_{\Omega} |u(x)|^{2\alpha} dx \right)^{\frac{1}{2}} \leq \alpha \left( \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} |u_K|^{p(\alpha-1)} m(\sigma) d_{K,\sigma} \right)^{\frac{1}{p}} \left( \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{|D_\sigma u|^{p'}}{d_{K,\sigma}^{p'}} m(\sigma) d_{K,\sigma} \right)^{\frac{1}{p'}}.$$

Since  $\sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_{K,\sigma} = 2m(K)$ , this gives

$$\left( \int_{\Omega} |u(x)|^{2\alpha} dx \right)^{\frac{1}{2}} \leq \alpha 2^{\frac{1}{p}} \left( \int_{\Omega} |u(x)|^{p(\alpha-1)} dx \right)^{\frac{1}{p}} \left( \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{|D_\sigma u|^{p'}}{d_{K,\sigma}^{p'}} m(\sigma) d_{K,\sigma} \right)^{\frac{1}{p'}},$$

which yields, choosing  $p$  such that  $p(\alpha - 1) = 2\alpha$ , i.e.  $p = \frac{2\alpha}{\alpha-1}$  and  $p' = \frac{2\alpha}{\alpha+1}$ ,

$$\|u\|_{L^q(\Omega)} = \left( \int_{\Omega} |u(x)|^{2\alpha} dx \right)^{\frac{1}{2\alpha}} \leq \alpha 2^{\frac{1}{p}} \left( \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{|D_\sigma u|^{p'}}{d_{K,\sigma}^{p'}} m(\sigma) d_{K,\sigma} \right)^{\frac{1}{p'}}, \tag{28}$$

where  $q = 2\alpha$ . Let  $r = \frac{2}{p'}$  and  $r' = \frac{2}{2-p'}$ , using Hölder’s inequality yields

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{|D_\sigma u|^{p'}}{d_{K,\sigma}^{p'}} m(\sigma) d_{K,\sigma} \leq \left( \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{|D_\sigma u|^2}{d_{K,\sigma}^2} m(\sigma) d_{K,\sigma} \right)^{\frac{p'}{2}} \left( \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_{K,\sigma} \right)^{\frac{1}{r'}},$$

replacing in (28) gives

$$\|u\|_{L^q(\Omega)} \leq \alpha 2^{\frac{1}{p}} \left( \frac{2}{\zeta} \right)^{\frac{1}{2}} (2m(\Omega))^{\frac{1}{p'r'}} \|u\|_{1,\mathcal{T}}$$

and then (25) with, for instance,  $C = (\frac{2}{\xi})^{\frac{1}{2}}((2m(\Omega))^{\frac{1}{2}} + 1)$ .

Let us now prove the three-dimensional case. Let  $d = 3$ . Using the same notations as in the two-dimensional case, let  $\mathbf{d}_1 = (1, 0, 0)^t$ ,  $\mathbf{d}_2 = (0, 1, 0)^t$  and  $\mathbf{d}_3 = (0, 0, 1)^t$ ; for  $x \in \Omega$ , let  $\mathcal{D}_x^1$ ,  $\mathcal{D}_x^2$  and  $\mathcal{D}_x^3$  be the straight lines going through  $x$  and defined by the vectors  $\mathbf{d}_1$ ,  $\mathbf{d}_2$  and  $\mathbf{d}_3$ . Let us again define the functions  $\chi_\sigma^{(1)}$ ,  $\chi_\sigma^{(2)}$  and  $\chi_\sigma^{(3)}$  by

$$\chi_\sigma^{(i)}(x) = \begin{cases} 1 & \text{if } \sigma \cap \mathcal{D}_x^i \neq \emptyset \\ 0 & \text{if } \sigma \cap \mathcal{D}_x^i = \emptyset \end{cases} \text{ for } i = 1, 2, 3.$$

Let  $v \in X(\mathcal{T})$  and let  $A \in \mathbb{R}_+$  such that  $\Omega \subset [-A, A]^3$ ; we also denote by  $v$  the function defined on  $[-A, A]^3$  which equals  $v$  on  $\Omega$  and 0 on  $[-A, A]^3 \setminus \Omega$ . By the Cauchy-Schwarz inequality, one has:

$$\int_{-A}^A \int_{-A}^A |v(x_1, x_2, x_3)|^{\frac{3}{2}} dx_1 dx_2 \leq \left( \int_{-A}^A \int_{-A}^A |v(x_1, x_2, x_3)| dx_1 dx_2 \right)^{\frac{1}{2}} \left( \int_{-A}^A \int_{-A}^A |v(x_1, x_2, x_3)|^2 dx_1 dx_2 \right)^{\frac{1}{2}}. \tag{29}$$

Now remark that

$$\int_{-A}^A \int_{-A}^A |v(x_1, x_2, x_3)| dx_1 dx_2 \leq \sum_{\sigma \in \mathcal{E}} D_\sigma v \int_{-A}^A \int_{-A}^A \chi_\sigma^{(3)}(x) dx_1 dx_2 \leq \sum_{\sigma \in \mathcal{E}} m(\sigma) D_\sigma v.$$

Moreover, computations which were already performed in the two-dimensional case give that

$$\int_{-A}^A \int_{-A}^A |v(x_1, x_2, x_3)|^2 dx_1 dx_2 \leq \int_{-A}^A \int_{-A}^A \sum_{\sigma \in \mathcal{E}} D_\sigma v \chi_\sigma^{(1)}(x) \sum_{\sigma \in \mathcal{E}} D_\sigma v \chi_\sigma^{(2)}(x) dx_1 dx_2 \leq \left( \sum_{\sigma \in \mathcal{E}} m(\sigma_{x_3}) D_\sigma v \right)^2,$$

where  $\sigma_{x_3}$  denotes the intersection of  $\sigma$  with the plane which contains the point  $(0, 0, x_3)$  and is orthogonal to  $\mathbf{d}_3$ . Therefore, integrating (29) in the third direction yields:

$$\int_{\Omega} |v(x)|^{\frac{3}{2}} dx \leq \left( \sum_{\sigma \in \mathcal{E}} m(\sigma) D_\sigma v \right)^{\frac{3}{2}} \tag{30}$$

Now let  $v = |u|^4 \text{sign}(u)$ , since  $|v_K - v_L| \leq 4(|u_K|^3 + |u_L|^3)|u_K - u_L|$ , Inequality (30) yields:

$$\int_{\Omega} |u(x)|^6 dx \leq \left[ 4 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} |u_K|^3 D_\sigma u m(\sigma) \right]^{\frac{3}{2}}.$$

By Cauchy-Schwarz' inequality and since  $\sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_{K,\sigma} = 3m(K)$ , this yields

$$\int_{\Omega} |u(x)|^6 dx \leq \left[ 4 \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} |u_K|^3 D_\sigma u m(\sigma) \right]^{\frac{3}{2}}.$$

□

**Remark 7** (Discrete Poincaré inequality). In the above proof, Inequality (27) leads to a proof of some discrete Poincaré inequality. Indeed, let  $\Omega$  be an open bounded polygonal subset of  $\mathbb{R}^2$ . Let  $\mathcal{T}$  be a general finite volume mesh of  $\Omega$  in the sense of Definition 1. Let  $v \in X(\mathcal{T})$ . Then, using (27), the Cauchy-Schwarz inequality and the fact that  $\sum_{\sigma \in \mathcal{E}} m(\sigma) d_\sigma = d m(\Omega)$  yields

$$\|v\|_{L^2(\Omega)}^2 \leq d m(\Omega) \|v\|_{1,\mathcal{T}}^2.$$

**Corollary 1** (Error estimate). *Under the same assumptions and with the same notations as in Theorem 1 or Theorem 2, there exists  $C > 0$  only depending on  $u$ ,  $\zeta$  and  $\Omega$  such that*

$$\|e_{\mathcal{T}}\|_{L^q(\Omega)} \leq Cq\text{size}(\mathcal{T}); \text{ for any } q \in \begin{cases} [1, +\infty) & \text{if } d = 2, \\ [1, 6] & \text{if } d = 3. \end{cases} \quad (31)$$

Furthermore, there exists  $C \in \mathbb{R}_+$  only depending on  $u$ ,  $\zeta$ ,  $\zeta_{\mathcal{T}} = \min\{\frac{m(K)}{\text{size}(\mathcal{T})^d}, K \in \mathcal{T}\}$ , and  $\Omega$ , such that

$$\|e_{\mathcal{T}}\|_{L^\infty(\Omega)} \leq C\text{size}(\mathcal{T})(|\ln(\text{size}(\mathcal{T}))| + 1), \quad \text{if } d = 2. \quad (32)$$

$$\|e_{\mathcal{T}}\|_{L^\infty(\Omega)} \leq C\text{size}(\mathcal{T})^{1/2}, \quad \text{if } d = 3. \quad (33)$$

*Proof.* Estimate (20) of Theorem 1 and (23) of Theorem 2 and Inequality (25) of Lemma 1 immediately yield Estimate (31). Let us now prove (32) and (33).

Remark that

$$\|e_{\mathcal{T}}\|_{L^\infty(\Omega)} = \max\{|e_K|, K \in \mathcal{T}\} \leq \left(\frac{1}{\zeta_{\mathcal{T}}\text{size}(\mathcal{T})^d}\right)^{\frac{1}{q}} \|e_{\mathcal{T}}\|_{L^q}. \quad (34)$$

In the two-dimensional case, a study of the real function defined, for  $q \geq 2$ , by  $q \mapsto \ln q + (1 - \frac{2}{q}) \ln h$  (with  $h = \text{size}(\mathcal{T})$ ) shows that its minimum is attained for  $q = -2 \ln h$ , if  $\ln h \leq -\frac{1}{2}$ . And therefore (31) and (34) yield (32).

The three-dimensional case (33) is an immediate consequence of (34), (31) with  $q = 6$ , and (20) or (23). with  $q = 6$ .  $\square$

**Remark 8.** Similar error estimates hold in the case of locally refined rectangular meshes (see [2] for the scheme (7–8), and [9] for the scheme (12–17)).

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