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A-POSTERIORI ERROR ESTIMATES FOR LINEAR EXTERIOR PROBLEMS *VIA* MIXED-FEM AND DTN MAPPINGS

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Abstract. In this paper we combine the dual-mixed finite element method with a Dirichlet-to-Neumann mapping (given in terms of a boundary integral operator) to solve linear exterior transmission problems in the plane. As a model we consider a second order elliptic equation in divergence form coupled with the Laplace equation in the exterior unbounded region. We show that the resulting mixed variational formulation and an associated discrete scheme using Raviart-Thomas spaces are well posed, and derive the usual Cea error estimate and the corresponding rate of convergence. In addition, we develop two different *a-posteriori* error analyses yielding explicit residual and implicit Bank-Weiser type reliable estimates, respectively. Several numerical results illustrate the suitability of these estimators for the adaptive computation of the discrete solutions.

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1. Introduction

The coupling of dual-mixed finite element methods (FEM) and boundary integral equation methods (BEM) has been frequently applied during the last few years to numerically solve a wide class of linear and nonlinear boundary value problems appearing in physics and engineering sciences (see, *e.g.* [6, 9, 18, 22, 24, 25, 32], and the references therein). As it is well known in mechanics, the utilization of dual-mixed FEM allows to compute stresses more accurately than displacements, and the use of BEM is more appropriate for linear homogeneous materials in bounded and unbounded regions. Analogously, according to the terminology in heat conduction problems, the above method combines the advantage of BEM for treating homogeneous domains and that of dual-mixed FEM for getting better approximations of the flux variable in heterogeneous media.

An alternative procedure, when dealing with exterior problems, consists of employing Dirichlet-to-Neumann (DtN) mappings. The combination of this approach with the usual FEM has been applied to several elliptic operators, including the Laplacian and the Lamé system in elasticity (see, *e.g.* [20, 26, 28–30]). In these works

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the corresponding DtN mapping either depends on a boundary integral operator or is expressed in terms of a Fourier-type series expansion. Now, in [16] we utilized the DtN mapping from [29] together with our dual-mixed finite element method from [25] to analyze an exterior transmission problem in hyperelasticity. Then, in [22] we combined a modified dual-mixed FEM with the DtN mapping from [20] and [30] to study the solvability of a nonlinear elliptic equation in divergence form coupled with the Laplace equation in an unbounded region of the plane. This modified dual-mixed method, which is based on the Hu-Washizu principle from elasticity, leads to two-fold saddle point operator equations, which are also called dual-dual mixed formulations (see [17,18]).

On the other hand, in order to guarantee a good rate of convergence of the discrete solutions, one usually applies a mesh-refinement algorithm based on a suitable *a-posteriori* error analysis. To this respect, concerning the combination of the usual FEM with BEM, we may refer to [10,13,14], where mainly reliable *a-posteriori* error estimates are provided. More recently, this kind of result has been extended to the coupling of dual-mixed FEM and BEM for linear and nonlinear problems (see [5,6,12,19,21,23]). Here, the estimates for the linear problems are of explicit residual type, and those for the nonlinear ones are based on the classical Bank-Weiser implicit approach. Up to the authors's knowledge, there is no further contributions in this direction for the combination of dual-mixed FEM with either BEM or DtN mappings.

The main purpose of the present work is to derive explicit and implicit reliable *a-posteriori* error estimates for linear exterior problems in the plane, whose variational formulations are obtained by the combination of dual-mixed FEM with DtN mappings. As a model, we consider the exterior transmission problem from potential theory studied in [32] (see also [12,21,24]). In addition, we use the DtN mapping from [20,30], which is given in terms of the hypersingular boundary integral operator for the Laplacian. The rest of the paper is organized as follows. In Section 2 we introduce the model problem, derive the associated mixed variational formulation, and prove the corresponding solvability and continuous dependence results. Actually, this is done through an equivalent formulation arising from a direct sum decomposition of one of the unknowns. In Section 3 we use Raviart-Thomas spaces to define the discrete scheme, show that it is stable and uniquely solvable, obtain the Cea error estimate, and state the associated rate of convergence. Then, a reliable *a-posteriori* error estimate of explicit residual type is derived in Section 4. Our analysis here follows very closely the techniques from [12,21]. In Section 5 we apply a Bank-Weiser type *a-posteriori* error analysis and provide a reliable estimate that depends on the solution of local problems. An explicit estimate, based on bounds of these local solutions and a suitable averaging technique, is also deduced in this section. Finally, several numerical experiments illustrating the efficiency of these estimators for the adaptive computation of the discrete solutions are given in Section 6.

In what follows, the symbols C, $\tilde{C}$, and $\bar{C}$ are used to denote generic positive constants with different values at different places.

2. The model problem

Let Ω_0 be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ_0. Also, let Ω_1 be the annular domain bounded by Γ_0 and another Lipschitz-continuous closed curve Γ_1 whose interior region contains $\overline{\Omega}_0$. Then, given $f_1 \in L^2(\Omega_1)$, $g \in H^{1/2}(\Gamma_0)$ and a matrix valued function $\boldsymbol{\kappa}_1 \in C(\overline{\Omega}_1)$, we consider the exterior transmission problem: *Find* $u_1 \in H^1(\Omega_1)$ *and* $u_2 \in H^1_{loc}(\mathbb{R}^2 - \overline{\Omega}_0 \cup \overline{\Omega}_1)$ *such that*

$$\begin{gathered} u_1 = g \quad \text{on} \quad \Gamma_0, \quad -\operatorname{div}(\boldsymbol{\kappa}_1 \nabla u_1) = f_1 \quad \text{in} \quad \Omega_1, \\ u_1 = u_2 \quad \text{and} \quad (\boldsymbol{\kappa}_1 \nabla u_1) \cdot \mathbf{n} = \frac{\partial u_2}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma_1, \\ -\Delta u_2 = 0 \quad \text{in} \quad \mathbb{R}^2 - \overline{\Omega}_0 \cup \overline{\Omega}_1, \quad u_2(x) = O(1) \quad \text{as} \quad ||x|| \to +\infty, \end{gathered} \tag{1}$$

where $\mathbf{n} := (n_1, n_2)^{\mathbf{T}}$ denotes the unit outward normal to Γ_1.

We assume that $\boldsymbol{\kappa}_1$ induces a strongly elliptic differential operator, that is there exists $\alpha_1 > 0$ such that

$$\alpha_1 \, ||\boldsymbol{\xi}||^2 \leq (\boldsymbol{\kappa}_1 \, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \qquad \forall \, \boldsymbol{\xi} \in \mathbb{R}^2. \tag{2}$$

We now introduce a sufficiently large circle Γ with center at the origin such that its interior region contains $\overline{\Omega}_0 \cup \overline{\Omega}_1$. Then we let Ω_2 be the annular region bounded by Γ_1 and Γ, put $\Omega := \Omega_1 \cup \Gamma_1 \cup \Omega_2$, and define the global unknown $u := \begin{cases} u_1 & \text{in} \quad \Omega_1 \\ u_2 & \text{in} \quad \Omega_2 \end{cases}$, the data $f := \begin{cases} f_1 & \text{in} \quad \Omega_1 \\ 0 & \text{in} \quad \Omega_2 \end{cases}$, and the flux variable $\boldsymbol{\sigma} := \boldsymbol{\kappa}\nabla u$ in Ω, where $\boldsymbol{\kappa} := \begin{cases} \boldsymbol{\kappa}_1 & \text{in} \quad \Omega_1 \\ \mathbf{I} & \text{in} \quad \Omega_2 \end{cases}$, and $\mathbf{I}$ denotes the identity matrix.

Next, we apply the boundary integral equation method in the region exterior to the circle Γ, and obtain the following Dirichlet-to-Neumann mapping (see [20, 30])

$$\boldsymbol{\sigma} \cdot \boldsymbol{\nu} \;=\; -\,2\,\mathbf{W}(\lambda) \quad \text{on} \quad \Gamma, \tag{3}$$

where $\boldsymbol{\nu}$ is the unit outward normal to $\partial\Omega := \Gamma_0 \cup \Gamma$, $\lambda := u|_\Gamma$ is a further unknown, and $\mathbf{W}$ is the hypersingular boundary integral operator.

We remark that if Γ is choosen as a polygonal boundary instead of a circle, then we would need all the boundary integral operators to express $\boldsymbol{\sigma} \cdot \boldsymbol{\nu}$ in terms of λ. The advantage of using a circle in this case lies on the simplicity of the resulting Dirichlet-to-Neumann mapping (3).

We recall here that $\mathbf{W}$ is the linear operator defined by

$$\mathbf{W}\mu(x) := -\frac{\partial}{\partial\boldsymbol{\nu}(x)} \int_\Gamma \frac{\partial}{\partial\boldsymbol{\nu}(y)}\, E(x,y)\,\mu(y)\,\mathrm{d}s_y \quad \forall\, x \in \Gamma, \quad \forall\, \mu \in H^{1/2}(\Gamma),$$

where $\boldsymbol{\nu}(z)$ stands for the unit outward normal at $z \in \Gamma$, and $E(x,y) := -\frac{1}{2\pi}\log||x-y||$ is the fundamental solution of the two-dimensional Laplacian. It is well known that $\mathbf{W}$ maps continuously $H^{1/2+\delta}(\Gamma)$ into $H^{-1/2+\delta}(\Gamma)$ for all $\delta \in [-1/2, 1/2]$, and that there exists $\alpha_2 > 0$ such that

$$\langle \mathbf{W}(\mu), \mu\rangle_\Gamma \;\geq\; \alpha_2\,||\mu||^2_{H^{1/2}(\Gamma)} \quad \forall \mu \in H_0^{1/2}(\Gamma), \tag{4}$$

where

$$H_0^{1/2}(\Gamma) := \{\mu \in H^{1/2}(\Gamma) : \quad \langle 1, \mu\rangle_\Gamma = 0\,\}\,\cdot$$

In addition, $\mathbf{W}(1) = 0$ and $\mathbf{W}$ is symmetric in the sense that $\langle \mathbf{W}(\mu), \rho\rangle_\Gamma = \langle \mathbf{W}(\rho), \mu\rangle_\Gamma$ for all μ, $\rho \in H^{1/2}(\Gamma)$.

Hereafter, $\langle\cdot,\cdot\rangle_\Gamma$ (resp. $\langle\cdot,\cdot\rangle_{\Gamma_0}$) denotes the duality pairing of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ (resp. $H^{-1/2}(\Gamma_0)$ and $H^{1/2}(\Gamma_0)$) with respect to the $L^2(\Gamma)$ (resp. $L^2(\Gamma_0)$) inner product.

In this way, the model problem (1) is reformulated as a boundary value problem in $\overline{\Omega}$ with the nonlocal boundary condition (3). Hence, by performing the usual integration by parts procedure in Ω, we find that the corresponding mixed variational formulation reads: *Find* $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ *such that*

$$\begin{aligned} A((\boldsymbol{\sigma},\lambda),(\boldsymbol{\tau},\mu)) + B((\boldsymbol{\tau},\mu),u) \;&=\; \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}, \\ B((\boldsymbol{\sigma},\lambda),v) \;&=\; -\int_\Omega f v\,\mathrm{d}x, \end{aligned} \tag{5}$$

for all $((\boldsymbol{\tau},\mu),v) \in H \times Q$, where $H := H(\mathrm{div};\Omega) \times H^{1/2}(\Gamma)$, $Q := L^2(\Omega)$, and the bilinear forms $A : H \times H \to \mathbb{R}$ and $B : H \times Q \to \mathbb{R}$ are defined as follows:

$$A((\boldsymbol{\sigma},\lambda),(\boldsymbol{\tau},\mu)) := \int_\Omega (\boldsymbol{\kappa}^{-1}\,\boldsymbol{\sigma})\cdot\boldsymbol{\tau}\,\mathrm{d}x + 2\langle \mathbf{W}\lambda, \mu\rangle_\Gamma \;-\; \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, \lambda\rangle_\Gamma \;+\; \langle \boldsymbol{\sigma}\cdot\boldsymbol{\nu}, \mu\rangle_\Gamma, \tag{6}$$

$$B((\boldsymbol{\tau},\mu),v) := \int_\Omega v\,\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x, \tag{7}$$

for all $(\boldsymbol{\sigma},\lambda)$, $(\boldsymbol{\tau},\mu) \in H$, for all $v \in Q$.

At this point we recall that $H(\mathrm{div};\Omega)$ is the space of functions $\boldsymbol{\tau} \in [L^2(\Omega)]^2$ such that $\mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)$, which, provided with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{H(\mathrm{div};\Omega)} := \int_\Omega \mathrm{div}\,\boldsymbol{\sigma}\,\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x + \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\,\mathrm{d}x,$$

becomes a Hilbert space. In addition, for all $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, $\boldsymbol{\tau}\cdot\boldsymbol{\nu}|_\Gamma \in H^{-1/2}(\Gamma)$, $\boldsymbol{\tau}\cdot\boldsymbol{\nu}|_{\Gamma_0} \in H^{-1/2}(\Gamma_0)$, and both $||\boldsymbol{\tau}\cdot\boldsymbol{\nu}||_{H^{-1/2}(\Gamma)}$ and $||\boldsymbol{\tau}\cdot\boldsymbol{\nu}||_{H^{-1/2}(\Gamma_0)}$ are bounded above by $||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}$.

On the other hand, each $\mu \in H^{1/2}(\Gamma)$ can be uniquely decomposed as $\mu := \tilde{\mu} + q$, with $\tilde{\mu} := \left(\mu - \frac{1}{|\Gamma|}\int_\Gamma \mu\,\mathrm{d}s\right) \in H_0^{1/2}(\Gamma)$ and $q := \frac{1}{|\Gamma|}\int_\Gamma \mu\,\mathrm{d}s \in \mathbb{R}$, which states that $H^{1/2}(\Gamma) = H_0^{1/2}(\Gamma) \oplus \mathbb{R}$. Further, it is easy to see that $||\mu||^2_{H^{1/2}(\Gamma)} = ||\tilde{\mu}||^2_{H^{1/2}(\Gamma)} + |\Gamma|\,|q|^2$, and hence $||\mu||_{H^{1/2}(\Gamma)}$ and $||(\tilde{\mu}, q)||_{H^{1/2}(\Gamma)\times\mathbb{R}} := ||\tilde{\mu}||_{H^{1/2}(\Gamma)} + |q|$ are equivalent.

Then we write $\lambda = \tilde{\lambda} + p$, with $\tilde{\lambda} \in H_0^{1/2}(\Gamma)$, $p \in \mathbb{R}$, and consider the alternative formulation: *Find* $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ *such that*

$$\begin{aligned} A((\boldsymbol{\sigma}, \tilde{\lambda}), (\boldsymbol{\tau}, \tilde{\mu})) + \tilde{B}((\boldsymbol{\tau}, \tilde{\mu}), (u, p)) &= \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}, \\ \tilde{B}((\boldsymbol{\sigma}, \tilde{\lambda}), (v, q)) &= -\int_\Omega f v\,\mathrm{d}x, \end{aligned} \tag{8}$$

for all $((\boldsymbol{\tau}, \tilde{\mu}), (v, q)) \in \tilde{H} \times \tilde{Q}$, where $\tilde{H} := H(\mathrm{div};\Omega) \times H_0^{1/2}(\Gamma)$, $\tilde{Q} := L^2(\Omega) \times \mathbb{R}$, and the bilinear form $\tilde{B} : \tilde{H} \times \tilde{Q} \to \mathbb{R}$ is defined as

$$\tilde{B}((\boldsymbol{\tau}, \tilde{\mu}), (v, q)) := \int_\Omega v\,\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x - q\,\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma. \tag{9}$$

Then we have the following result.

Theorem 2.1. *Problems* (5) *and* (8) *are equivalent. More precisely:*

1. *If* $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ *is a solution of* (5)*, where* $\lambda := \tilde{\lambda} + p$*, with* $\tilde{\lambda} \in H_0^{1/2}(\Gamma)$ *and* $p \in \mathbb{R}$*, then* $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ *is a solution of* (8)*.*
2. *If* $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ *is a solution of* (8)*, then* $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ *is a solution of* (5) *with* $\lambda := \tilde{\lambda} + p$*.*

Proof. Let $((\boldsymbol{\sigma}, \lambda), u) \in H \times Q$ be a solution of (5), where $\lambda := \tilde{\lambda} + p$, with $\tilde{\lambda} \in H_0^{1/2}(\Gamma)$ and $p \in \mathbb{R}$, and consider $((\boldsymbol{\tau}, \tilde{\mu}), (v, q)) \in \tilde{H} \times \tilde{Q}$. Since $\mathbf{W}(p) = 0$, it follows that

$$A((\boldsymbol{\sigma}, \tilde{\lambda}), (\boldsymbol{\tau}, \tilde{\mu})) + \tilde{B}((\boldsymbol{\tau}, \tilde{\mu}), (u, p)) = A((\boldsymbol{\sigma}, \lambda), (\boldsymbol{\tau}, \tilde{\mu})) + B((\boldsymbol{\tau}, \tilde{\mu}), u) = \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}. \tag{10}$$

Now, taking $\mu = 1$ and $\boldsymbol{\tau} = 0$ in the first equation of (5), and using the symmetry of $\mathbf{W}$ and the fact that $\mathbf{W}(1) = 0$, we find that $\langle \boldsymbol{\sigma}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = 0$, and hence

$$\tilde{B}((\boldsymbol{\sigma}, \tilde{\lambda}), (v, q)) = B((\boldsymbol{\sigma}, \tilde{\lambda}), v) = B((\boldsymbol{\sigma}, \lambda), v) = -\int_\Omega f v\,\mathrm{d}x.$$

This equation and (10) prove that $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ is a solution of (8).

Conversely, let $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ be a solution of (8), and define $\lambda := \tilde{\lambda} + p$. Taking $v = 0$ and $q = 1$ in the second equation of (8), we deduce that $\langle \boldsymbol{\sigma}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = 0$. Then we consider $((\boldsymbol{\tau}, \mu), v) \in H \times Q$, such that $\mu := \tilde{\mu} + q$, with $\tilde{\mu} \in H_0^{1/2}(\Gamma)$ and $q \in \mathbb{R}$, and observe that

$$A((\boldsymbol{\sigma}, \lambda), (\boldsymbol{\tau}, \mu)) + B((\boldsymbol{\tau}, \mu), u) = A((\boldsymbol{\sigma}, \tilde{\lambda}), (\boldsymbol{\tau}, \tilde{\mu})) + \tilde{B}((\boldsymbol{\tau}, \tilde{\mu}), (u, p)) = \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}. \tag{11}$$

Also, according to the second equation in (8), we find that

$$B((\boldsymbol{\sigma},\lambda),v) \;=\; \tilde{B}((\boldsymbol{\sigma},\tilde{\lambda}),(v,0)) \;=\; -\int_\Omega f v \,\mathrm{d}x,$$

which, together with (11), shows that $((\boldsymbol{\sigma},\lambda),u) \in H \times Q$ is a solution of (5). □

In virtue of Theorem 2.1, from now on we concentrate on the equivalent problem (8). The corresponding continuous and discrete analyses are based on the classical Babuška-Brezzi theory.

At this point we remark, which is easy to prove, that the bilinear forms A, B, and $\tilde{B}$ are all bounded.

We end this section with the following theorem providing the unique solvability and the continuous dependence result for the mixed variational formulation (8) (and hence also for (5)).

Theorem 2.2. *There exists a unique* $((\boldsymbol{\sigma},\tilde{\lambda}),(u,p)) \in \tilde{H} \times \tilde{Q}$ *solution of* (8). *Moreover, there exists* $C > 0$, *independent of the solution, such that*

$$\|((\boldsymbol{\sigma},\tilde{\lambda}),(u,p))\|_{\tilde{H}\times\tilde{Q}} \;\le\; C\,\left\{\,\|f\|_{L^2(\Omega)} \;+\; \|g\|_{H^{1/2}(\Gamma_0)}\,\right\}\cdot$$

Proof. We first prove the continuous inf-sup condition for $\tilde{B}$. Thus, given $(v,q) \in \tilde{Q} := L^2(\Omega)\times\mathbb{R}$, we let $z \in H^1(\Omega)$ be the weak solution of the mixed boundary value problem:

$$-\Delta z = v \quad \text{in} \quad \Omega, \quad z = 0 \quad \text{on} \quad \Gamma_0, \quad \frac{\partial z}{\partial \boldsymbol{\nu}} = q \quad \text{on} \quad \Gamma,$$

for which one can easily show that $\|z\|_{H^1(\Omega)} \le C\,\{\|v\|_{L^2(\Omega)} + |q|\}$. Then we set $\boldsymbol{\tau}_0 := -\nabla z$ and observe that $\operatorname{div}\boldsymbol{\tau}_0 = v$ in Ω, $\boldsymbol{\tau}_0\cdot\boldsymbol{\nu} = -q$ on Γ, and $\|\boldsymbol{\tau}_0\|_{H(\operatorname{div};\Omega)} \le \tilde{C}\,\{\|v\|_{L^2(\Omega)} + |q|\}$. It follows that

$$\sup_{\substack{(\boldsymbol{\tau},\tilde{\mu})\in\tilde{H}\\(\boldsymbol{\tau},\tilde{\mu})\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau},\tilde{\mu}),(v,q))}{\|(\boldsymbol{\tau},\tilde{\mu})\|_{\tilde{H}}} \;\ge\; \frac{\tilde{B}((\boldsymbol{\tau}_0,0),(v,q))}{\|\boldsymbol{\tau}_0\|_{H(\operatorname{div};\Omega)}} \;=\; \frac{\|v\|^2_{L^2(\Omega)} + |\Gamma|\,|q|^2}{\|\boldsymbol{\tau}_0\|_{H(\operatorname{div};\Omega)}} \;\ge\; \beta\,\|(v,q)\|_{\tilde{Q}},$$

where β depends on $|\Gamma|$ and $\tilde{C}$.

We now let $\tilde{V}$ be the kernel of the operator induced by the bilinear form $\tilde{B}$, that is

$$\tilde{V} := \{\,(\boldsymbol{\tau},\tilde{\mu}) \in \tilde{H}: \quad B((\boldsymbol{\tau},\tilde{\mu}),(v,q)) = 0 \quad \forall\,(v,q)\in\tilde{H}\,\},$$

which yields

$$\tilde{V} = \{\,(\boldsymbol{\tau},\tilde{\mu}) \in H(\operatorname{div};\Omega)\times H_0^{1/2}(\Gamma): \quad \operatorname{div}\boldsymbol{\tau} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = 0\,\}\cdot$$

It follows, using (6), (2), and (4), that A is strongly coercive on $\tilde{V}$, that is, for all $(\boldsymbol{\tau},\tilde{\mu}) \in \tilde{V}$ it holds

$$A((\boldsymbol{\tau},\tilde{\mu}),(\boldsymbol{\tau},\tilde{\mu})) \;=\; \int_\Omega (\boldsymbol{\kappa}^{-1}\boldsymbol{\tau})\cdot\boldsymbol{\tau}\,\mathrm{d}x \;+\; 2\,\langle \mathbf{W}(\tilde{\mu}),\tilde{\mu}\rangle_\Gamma \;\ge\; \alpha\,\|(\boldsymbol{\tau},\tilde{\mu})\|^2_{H(\operatorname{div};\Omega)\times H^{1/2}(\Gamma)},$$

where α depends on α_1 and α_2.

Finally, a straightforward application of the abstract Theorem 1.1 in Chapter II of [8] completes the proof. □

3. The discrete scheme

Hereafter we assume, for simplicity, that Γ_0 and Γ_1 are polygonal boundaries. In order to discretize the circle Γ, we proceed similarly as in [22]. This means that given $n \in \mathbb{N}$, we let $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ be a uniform partition of $[0, 2\pi]$ with $t_{j+1} - t_j = \tilde{h} = \frac{2\pi}{n}$ for $j \in \{0, 1, ..., n-1\}$. In addition, we let $\mathbf{z} : [0, 2\pi] \to \Gamma$ be the parametrization of the circle Γ given by $\mathbf{z}(t) := r\,(\cos(t), \sin(t))^T$ for all $t \in [0, 2\pi]$. We denote by $\Omega_{\tilde{h}}$ the annular domain bounded by Γ_0 and the polygonal line $\Gamma_{\tilde{h}}$ whose vertices are $\{\mathbf{z}(t_1), \mathbf{z}(t_2), ..., \mathbf{z}(t_n)\}$·

Then we let $\mathcal{T}_{\tilde{h}}$ be a regular triangulation of $\Omega_{\tilde{h}}$ by triangles T of diameter h_T such that $h := \sup_{T \in \mathcal{T}_{\tilde{h}}} h_T$. We assume that for each $T \in \mathcal{T}_{\tilde{h}}$, either $T \subseteq \overline{\Omega}_1$ or $T \subseteq \overline{\Omega}_2$. Then, we replace each triangle $T \in \mathcal{T}_{\tilde{h}}$ with one side along $\Gamma_{\tilde{h}}$, by the corresponding curved triangle with one side along Γ. In this way, we obtain from $\mathcal{T}_{\tilde{h}}$ a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of straight and curved triangles.

Next, we consider the canonical triangle with vertices $\hat{P}_1 = (0,0)^T$, $\hat{P}_2 = (1,0)^T$ and $\hat{P}_3 = (0,1)^T$ as a reference triangle $\hat{T}$, and introduce a family of bijective mappings $\{F_T\}_{T \in \mathcal{T}_h}$, such that $F_T(\hat{T}) = T$. In particular, if T is a straight triangle of $\mathcal{T}_h$, then F_T is the affine mapping defined by $F_T(\hat{x}) = B_T\,\hat{x} + b_T$, where B_T, a square matrix of order 2, and $b_T \in \mathbb{R}^2$ depend on the vertices of T.

On the other hand, if T is a curved triangle with vertices P_1, P_2 and P_3, such that $P_2 = \mathbf{z}(t_{j-1}) \in \Gamma$ and $P_3 = \mathbf{z}(t_j) \in \Gamma$, then $F_T(\hat{x}) = B_T\,\hat{x} + b_T + G_T(\hat{x})$ for all $\hat{x} := (\hat{x}_1, \hat{x}_2) \in \hat{T}$, where

$$G_T(\hat{x}) = \frac{\hat{x}_1}{1 - \hat{x}_2}\,\{\mathbf{z}(t_{j-1} + \hat{x}_2(t_j - t_{j-1})) - [\mathbf{z}(t_{j-1}) + \hat{x}_2\,(\mathbf{z}(t_j) - \mathbf{z}(t_{j-1}))]\}\,\cdot \tag{12}$$

We now let $\mathbf{J}(F_T)$ and $D(F_T)$ denote, respectively, the Jacobian and the Frêchet differential of the mapping F_T. Then we summarize their main properties in the following lemma.

Lemma 3.1. *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$ F_T is a diffeomorphism of class $\mathcal{C}^\infty$ that maps one-to-one $\hat{T}$ onto the curved triangle T in such a way that $F_T(\hat{P}_i) = P_i$ for all $i \in \{1, 2, 3\}$. In addition, $\mathbf{J}(F_T)$ does not vanish in a neighborhood of $\hat{T}$, and there exist positive constants C_i, $i \in \{1, ..., 5\}$, independent of T and h, such that for all $T \in \mathcal{T}_h$ there hold*

$$C_1\,h_T^2 \,\le\, |\mathbf{J}(F_T)| \,\le\, C_2\,h_T^2, \quad |\mathbf{J}(F_T)^k|_{W^{1,\infty}(\hat{T})} \,\le\, C_3\,h_T^{1+2k} \quad \forall\, k \in \{-1, 1\},$$

and

$$|(DF_T)|_{W^{k,\infty}(\hat{T})} \,\le\, C_4\,h_T^{k+1}, \quad |(DF_T)^{-1}|_{W^{k,\infty}(\hat{T})} \,\le\, C_5\,h_T^{k-1} \quad \forall\, k \in \{0, 1\}\,\cdot$$

Proof. See Theorem 22.4 in [36]. ☐

Herafter, given $s \ge 0$, $\|\cdot\|_{W^{s,\infty}(\hat{T})}$ and $|\cdot|_{W^{s,\infty}(\hat{T})}$ (resp. $\|\cdot\|_{[W^{s,\infty}(\hat{T})]^{2\times 2}}$ and $|\cdot|_{[W^{s,\infty}(\hat{T})]^{2\times 2}}$) denote the norm and semi-norm of the usual Sobolev space $W^{s,\infty}(\hat{T})$ (resp. $[W^{s,\infty}(\hat{T})]^{2\times 2}$). In addition, $|\cdot|_{[H^1(\hat{T})]^2}$ is the semi-norm of $[H^1(\hat{T})]^2$, and given a non-negative integer k and a subset S of $\mathbb{R}$ or $\mathbb{R}^2$, $\mathbf{P}_k(S)$ denotes the space of polynomials defined on S of degree $\le k$.

We now introduce the lowest order Raviart-Thomas spaces. For this purpose, we first let

$$\mathcal{R}T_0(\hat{T}) := \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right\}, \tag{13}$$

and for each triangle $T \in \mathcal{T}_h$, we put

$$\mathcal{R}T_0(T) := \{\, \boldsymbol{\tau} : \quad \boldsymbol{\tau} = \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\boldsymbol{\tau}} \circ F_T^{-1},\ \hat{\boldsymbol{\tau}} \in \mathcal{R}T_0(\hat{T})\,\}\,\cdot \tag{14}$$

Then, we define the finite element subspaces for the unknowns $\boldsymbol{\sigma}$, λ, and u, as follows:

$$H_h^{\boldsymbol{\sigma}} := \{\, \boldsymbol{\tau}_h \in H(\text{div}; \Omega) : \quad \boldsymbol{\tau}_h|_T \in \mathcal{R}T_0(T) \quad \forall\, T \in \mathcal{T}_h \,\}, \tag{15}$$

$$H_h^{\lambda} := \{\, \mu_h : \Gamma \to \mathbb{R}, \quad \mu_h = \hat{\mu}_h \circ \mathbf{z}^{-1},\ \hat{\mu}_h \in H_h^{\lambda}(0, 2\pi) \,\}, \tag{16}$$

with

$$H_h^\lambda(0,2\pi) := \Big\{ \hat{\mu}_h : [0,2\pi] \to \mathbb{R}, \quad \hat{\mu}_h \text{ is continuous and periodic of period } 2\pi, \\ \hat{\mu}_h|_{[t_{j-1},t_j]} \in \mathbf{P}_1(t_{j-1},t_j) \quad \forall\, j \in \{1,...,n\} \Big\},$$

and

$$Q_h := \{ v_h \in L^2(\Omega) : \quad v_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \} \cdot \tag{17}$$

Thus, we set $H_h := H_h^{\boldsymbol{\sigma}} \times H_h^\lambda$ and state the Galerkin scheme associated with the continuous problem (5) as: *Find* $((\boldsymbol{\sigma}_h, \lambda_h), u_h) \in H_h \times Q_h$ *such that*

$$\begin{aligned} A((\boldsymbol{\sigma}_h, \lambda_h), (\boldsymbol{\tau}_h, \mu_h)) + B((\boldsymbol{\tau}_h, \mu_h), u_h) &= \langle \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0}, \\ B((\boldsymbol{\sigma}_h, \lambda_h), v_h) &= -\int_\Omega f v_h \,\mathrm{d}x, \end{aligned} \tag{18}$$

for all $((\boldsymbol{\tau}_h, \mu_h), v_h) \in H_h \times Q_h$.

Next, similarly as for the continuous problem, we introduce an alternative formulation, which is the discrete analogue of (8). To this end, we define

$$H_{h,0}^\lambda := H_h^\lambda \cap H_0^{1/2}(\Gamma), \quad \tilde{H}_h := H_h^{\boldsymbol{\sigma}} \times H_{h,0}^\lambda, \quad \tilde{Q}_h := Q_h \times \mathbb{R}, \tag{19}$$

and consider the Galerkin scheme: *Find* $((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ *such that*

$$\begin{aligned} A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\boldsymbol{\tau}_h, \tilde{\mu}_h)) + \tilde{B}((\boldsymbol{\tau}_h, \tilde{\mu}_h), (u_h, p_h)) &= \langle \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_0}, \\ \tilde{B}((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (v_h, q_h)) &= -\int_\Omega f v_h \,\mathrm{d}x, \end{aligned} \tag{20}$$

for all $((\boldsymbol{\tau}_h, \tilde{\mu}_h), (v_h, q_h)) \in \tilde{H}_h \times \tilde{Q}_h$.

Then we have the following result.

Theorem 3.2. *Problems* (18) *and* (20) *are equivalent. More precisely:*

1. *If* $((\boldsymbol{\sigma}_h, \lambda_h), u_h) \in H_h \times Q_h$ *is a solution of* (18), *where* $\lambda_h := \tilde{\lambda}_h + p_h$, *with* $\tilde{\lambda}_h \in H_{h,0}^\lambda$ *and* $p_h \in \mathbb{R}$, *then* $((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ *is a solution of* (20).
2. *If* $((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ *is a solution of* (20), *then* $((\boldsymbol{\sigma}_h, \lambda_h), u_h) \in H_h \times Q_h$ *is a solution of* (18) *with* $\lambda_h := \tilde{\lambda}_h + p_h$.

Proof. It is similar to the proof of Theorem 2.1 since it is based on the decomposition $H_h^\lambda := H_{h,0}^\lambda \oplus \mathbb{R}$. We omit further details. □

Our next goal is to show that the Galerkin scheme (20) is stable and uniquely solvable. To this end, we consider first the equilibrium interpolation operator $\mathcal{E}_h : [H^1(\Omega)]^2 \to H_h^{\boldsymbol{\sigma}}$, which, according to the Piola transformation used in (14), is given by (see, *e.g.* [8,34])

$$\mathcal{E}_h(\boldsymbol{\tau})|_T := \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}}) \circ F_T^{-1} \qquad \forall\, T \in \mathcal{T}_h,$$

where $\hat{\boldsymbol{\tau}} := \mathbf{J}(F_T)\,(DF_T)^{-1}\boldsymbol{\tau} \circ F_T$ and $\hat{\mathcal{E}} : [H^1(\hat{T})]^2 \to \mathcal{RT}_0(\hat{T})$ is the local equilibrium interpolation operator on the reference triangle $\hat{T}$.

Lemma 3.3. *There exists $C > 0$, independent of h, such that*

$$||\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})||_{[L^2(\Omega)]^2} \;\leq\; C\, h\, ||\boldsymbol{\tau}||_{[H^1(\Omega)]^2} \tag{21}$$

and

$$\|\mathrm{div}\,(\mathcal{E}_h(\boldsymbol{\tau}))\|_{L^2(\Omega)} \;\leq\; C\, \|\mathrm{div}\,\boldsymbol{\tau}\|_{L^2(\Omega)} \tag{22}$$

for all $\boldsymbol{\tau} \in [H^1(\Omega)]^2$.

Proof. Using the change of variable $x = F_T(\hat{x})$, we find that

$$\begin{aligned}
\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} &= \int_T \|\boldsymbol{\tau}(x) - \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(F_T^{-1}(x))\|_2^2 \,\mathrm{d}x \\
&= \int_{\hat{T}} |\mathbf{J}(F_T)| \, \left\|(\boldsymbol{\tau}\circ F_T)(\hat{x}) - \mathbf{J}(F_T)^{-1}\,(DF_T)\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right\|_2^2 \,\mathrm{d}\hat{x} \\
&= \int_{\hat{T}} |\mathbf{J}(F_T)| \, \left\|\mathbf{J}(F_T)^{-1}\,(DF_T)\,\left[\hat{\boldsymbol{\tau}}(\hat{x}) - \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right]\right\|_2^2 \,\mathrm{d}\hat{x} \\
&\leq \int_{\hat{T}} |\mathbf{J}(F_T)|^{-1}\, \|(DF_T)\|_2^2 \, \left\|\hat{\boldsymbol{\tau}}(\hat{x}) - \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})(\hat{x})\right\|_2^2 \,\mathrm{d}\hat{x},
\end{aligned} \tag{23}$$

where $\|\cdot\|_2$ is the usual euclidean norm for both vectors and matrices in $\mathbb{R}^2$ and $\mathbb{R}^{2\times 2}$, respectively.

Now, since $|\mathbf{J}(F_T)^{-1}| = O(h_T^{-2})$ and $\|(DF_T)\|_2 \leq C_4\, h_T$ (see Lem. 3.1), and because of the approximation property of $\hat{\mathcal{E}}$, we deduce from (23) that

$$\begin{aligned}
\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} &\leq \hat{C}\, \|\hat{\boldsymbol{\tau}} - \hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|^2_{[L^2(\hat{T})]^2} \;\leq\; \hat{C}\, |\,\hat{\boldsymbol{\tau}}\,|^2_{[H^1(\hat{T})]^2} \;=\hat{C}\, |\,\mathbf{J}(F_T)\,(DF_T)^{-1}\,(\boldsymbol{\tau}\circ F_T)\,|^2_{[H^1(\hat{T})]^2} \\
&\leq \hat{C}\, \Big\{\, |\mathbf{J}(F_T)|_{W^{1,\infty}(\hat{T})}\, \|(DF_T)^{-1})\|_{[W^{0,\infty}(\hat{T})]^{2\times 2}}\, \|\boldsymbol{\tau}\circ F_T\|_{[L^2(\hat{T})]^2} \\
&\quad + \|\mathbf{J}(F_T)\|_{W^{0,\infty}(\hat{T})}\, |(DF_T)^{-1}|_{[W^{1,\infty}(\hat{T})]^{2\times 2}}\, \|\boldsymbol{\tau}\circ F_T\|_{[L^2(\hat{T})]^2} \\
&\quad + \|\mathbf{J}(F_T)\|_{W^{0,\infty}(\hat{T})}\, \|(DF_T)^{-1}\|_{[W^{0,\infty}(\hat{T})]^{2\times 2}}\, |\boldsymbol{\tau}\circ F_T|_{[H^1(\hat{T})]^2} \Big\}^2,
\end{aligned} \tag{24}$$

with a constant $\hat{C} > 0$, depending only on $\hat{T}$.

Next, applying the corresponding norm estimates for $\mathbf{J}(F_T)$ and $(DF_T)^{-1}$ (see again Lem. 3.1), changing back the variable $\hat{x}$ by $F_T^{-1}(x)$, and using chain rule in the term $|\boldsymbol{\tau}\circ F_T|_{[H^1(\hat{T})]^2}$, we conclude from (24) that

$$\|\boldsymbol{\tau} - \mathcal{E}_h(\boldsymbol{\tau})\|^2_{[L^2(T)]^2} \;\leq\; \hat{C}\, h^2\, \|\boldsymbol{\tau}\|^2_{[H^1(T)]^2} \qquad \forall\, T \in \mathcal{T}_h. \tag{25}$$

On the other hand, we know from the conmuting diagram property on the reference triangle $\hat{T}$ that

$$\|\mathrm{div}\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|_{L^2(\hat{T})} \leq \|\mathrm{div}\,\hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})}.$$

Then we use the above inequality, identity (1.49) (*cf.* Lem. 1.5) in Chapter III of [8], and Cauchy-Schwarz's inequality, to find that

$$\begin{aligned}
\|\mathrm{div}\,\mathcal{E}_h(\boldsymbol{\tau})\|^2_{L^2(T)} := \int_T \mathrm{div}\,\mathcal{E}_h(\boldsymbol{\tau})\,\mathrm{div}\,\mathcal{E}_h(\boldsymbol{\tau})\,\mathrm{d}x \;&=\; \int_{\hat{T}} \mathrm{div}\,\widehat{\mathcal{E}_h(\boldsymbol{\tau})}\,\mathrm{div}\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\,\mathrm{d}\hat{x} \\
\leq \|\mathrm{div}\,\widehat{\mathcal{E}_h(\boldsymbol{\tau})}\|_{L^2(\hat{T})}\, \|\mathrm{div}\,\hat{\mathcal{E}}(\hat{\boldsymbol{\tau}})\|_{L^2(\hat{T})} \;&\leq\; \|\mathrm{div}\,\widehat{\mathcal{E}_h(\boldsymbol{\tau})}\|_{L^2(\hat{T})}\, \|\mathrm{div}\,\hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})},
\end{aligned} \tag{26}$$

where $\operatorname{div}\widehat{\mathcal{E}_h(\boldsymbol{\tau})}$ stands for $\operatorname{div}\mathcal{E}_h(\boldsymbol{\tau})\circ F_T$.

Then, applying the inequalities (1.40) (*cf.* Lem. 1.4) and (1.54) (*cf.* Lem. 1.6) in Chapter III of [8], and the estimate for $\mathbf{J}(F_T)$ given in Lemma 3.1, we deduce that

$$\|\operatorname{div}\widehat{\mathcal{E}_h(\boldsymbol{\tau})}\|_{L^2(\hat{T})} \leq C\, h_T^{-1}\, \|\operatorname{div}\mathcal{E}_h(\boldsymbol{\tau})\|_{L^2(T)} \quad \text{and} \quad \|\operatorname{div}\hat{\boldsymbol{\tau}}\|_{L^2(\hat{T})} \leq C\, h_T\, \|\operatorname{div}\boldsymbol{\tau}\|_{L^2(T)},$$

which replaced back into (26) yields

$$\|\operatorname{div}(\mathcal{E}_h(\boldsymbol{\tau}))\|_{L^2(T)} \leq C\, \|\operatorname{div}\boldsymbol{\tau}\|_{L^2(T)} \qquad \forall\, T \in \mathcal{T}_h. \tag{27}$$

Hence, summing up over all the triangles $T \in \mathcal{T}_h$ in (25) and (27), we conclude, respectively, (21) and (22). □

We are now in a position to prove the discrete inf-sup condition for the bilinear form $\tilde{B}$.

Lemma 3.4. *There exists $\beta^* > 0$, independent of h, such that for all $(v_h, q_h) \in \tilde{Q}_h$ it holds*

$$\sup_{\substack{(\boldsymbol{\tau}_h,\tilde{\mu}_h)\in\tilde{H}_h \\ (\boldsymbol{\tau}_h,\tilde{\mu}_h)\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}_h,\tilde{\mu}_h),(v_h,q_h))}{||(\boldsymbol{\tau}_h,\tilde{\mu}_h)||_H} \geq \beta^*\, ||(v_h,q_h)||_{\tilde{Q}}.$$

Proof. Given $(v_h, q_h) \in \tilde{Q}_h$, we note that

$$\sup_{\substack{(\boldsymbol{\tau}_h,\tilde{\mu}_h)\in\tilde{H}_h \\ (\boldsymbol{\tau}_h,\tilde{\mu}_h)\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}_h,\tilde{\mu}_h),(v_h,q_h))}{||(\boldsymbol{\tau}_h,\tilde{\mu}_h)||_H} \geq \sup_{\substack{\boldsymbol{\tau}_h\in H_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}_h,0),(v_h,q_h))}{||\boldsymbol{\tau}_h||_{H(\operatorname{div};\Omega)}}.$$

Then, we define $\tilde{v}_h := \begin{cases} v_h & \text{in } \Omega \\ -\dfrac{1}{|\Omega_0|}\left(\displaystyle\int_\Omega v_h\,\mathrm{d}x + q_h\,|\Gamma|\right) & \text{in } \bar{\Omega}_0 \end{cases}$, put $\tilde{\Omega} := \Omega\cup\bar{\Omega}_0$, and let $z \in H^1(\tilde{\Omega})$ be the weak solution of

$$-\Delta z = \tilde{v}_h \quad \text{in} \quad \tilde{\Omega}, \qquad \frac{\partial z}{\partial \boldsymbol{\nu}} = q_h \quad \text{on} \quad \Gamma, \qquad \int_{\tilde{\Omega}} z\,\mathrm{d}x = 0.$$

Since $\tilde{\Omega}$, being the interior region of the circle Γ, is clearly convex, the usual regularity result (see, *e.g.* [27]) implies that $z \in H^2(\tilde{\Omega})$ and

$$||z||_{H^2(\tilde{\Omega})} \leq C\,\{\, ||v_h||_{L^2(\Omega)} + |q_h|\,\}\,.$$

Thus we define $\tilde{\boldsymbol{\tau}} := -\nabla z|_\Omega \in [H^1(\Omega)]^2$, and observe that $\operatorname{div}\tilde{\boldsymbol{\tau}} = v_h$ in Ω, $\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{\nu} = -q_h$ on Γ, and

$$||\tilde{\boldsymbol{\tau}}||_{[H^1(\Omega)]^2} = ||\nabla z||_{[H^1(\Omega)]^2} \leq ||z||_{H^2(\tilde{\Omega})} \leq C\,\{\, ||v_h||_{L^2(\Omega)} + |q_h|\,\}\,. \tag{28}$$

Further, it is easy to see that

$$||\tilde{\boldsymbol{\tau}}||_{H(\operatorname{div};\Omega)} \leq C\,\left\{\, ||v_h||_{L^2(\Omega)} + |q_h|\,\right\}\,. \tag{29}$$

Then, using the approximation property (21) and the estimate (22) (*cf.* Lem. 3.3), we find that

$$\begin{aligned} ||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||^2_{H(\operatorname{div};\Omega)} &= ||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||^2_{[L^2(\Omega)]^2} + ||\operatorname{div}(\mathcal{E}_h(\tilde{\boldsymbol{\tau}}))||^2_{L^2(\Omega)} \\ &\leq C\,\left\{\, \|\tilde{\boldsymbol{\tau}} - \mathcal{E}_h(\tilde{\boldsymbol{\tau}})\|^2_{[L^2(\Omega)]^2} + \|\tilde{\boldsymbol{\tau}}\|^2_{[L^2(\Omega)]^2} + \|\operatorname{div}\tilde{\boldsymbol{\tau}}\|^2_{L^2(\Omega)} \right\} \\ &\leq C\,\left\{\, h^2\,\|\tilde{\boldsymbol{\tau}}\|^2_{[H^1(\Omega)]^2} + \|\tilde{\boldsymbol{\tau}}\|^2_{H(\operatorname{div};\Omega)} \right\}, \end{aligned}$$

which, using (28) and (29), implies

$$||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||_{H(\mathrm{div};\Omega)} \,\leq\, C\,\left\{\, ||v_h||_{L^2(\Omega)} \,+\, |q_h|\,\right\}\cdot \tag{30}$$

We now let $\mathcal{P}_h$ be the orthogonal projection from $L^2(\Omega)$ onto the finite element subspace Q_h. Then, using the identity (1.49) (*cf.* Lem. 1.5) in Chapter III of [8] and the conmuting diagram property on the reference triangle $\hat{T}$, similarly as we did in the proof of Lemma 3.3, we deduce that in this case there also holds $\mathcal{P}_h(\operatorname{div}\mathcal{E}_h(\tilde{\boldsymbol{\tau}})) = \mathcal{P}_h(\operatorname{div}\tilde{\boldsymbol{\tau}})$, which yields

$$\int_\Omega v_h \operatorname{div}\mathcal{E}_h(\tilde{\boldsymbol{\tau}})\,\mathrm{d}x \,=\, \int_\Omega v_h \operatorname{div}\tilde{\boldsymbol{\tau}}\,\mathrm{d}x \,=\, \|v_h\|^2_{L^2(\Omega)}.$$

Next, since $\int_e \mathcal{E}_h(\tilde{\boldsymbol{\tau}})\cdot\boldsymbol{\nu}_e\,\mathrm{d}s = \int_e \tilde{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}_e\,\mathrm{d}s$ for all the edges e of $\mathcal{T}_h$, with $\boldsymbol{\nu}_e$ being the unit outward normal to e, and since $\tilde{\boldsymbol{\tau}}\cdot\boldsymbol{\nu} = -q_h$ on Γ, we deduce that $\langle \mathcal{E}_h(\tilde{\boldsymbol{\tau}})\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = \,-\,q_h\,|\Gamma|$.

According to the above analysis we can write

$$\sup_{\substack{\boldsymbol{\tau}_h\in H_h^{\boldsymbol{\sigma}}\\ \boldsymbol{\tau}_h\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau}_h,0),(v_h,q_h))}{||\boldsymbol{\tau}_h||_{H(\mathrm{div};\Omega)}} \geq \frac{\tilde{B}((\mathcal{E}_h(\tilde{\boldsymbol{\tau}}),0),(v_h,q_h))}{||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||_{H(\mathrm{div};\Omega)}} = \frac{||v_h||^2_{L^2(\Omega)} \,+\, |\Gamma|\,q_h^2}{||\mathcal{E}_h(\tilde{\boldsymbol{\tau}})||_{H(\mathrm{div};\Omega)}}$$

$$\geq \beta^*\,||(v_h,q_h)||_{\tilde{Q}},$$

where the last inequality follows from (30). This finishes the proof. □

We are now in a position to provide the stability and unique solvability of the Galerkin scheme (20), and the corresponding Cea estimate.

Theorem 3.5. *There exists a unique* $((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(u_h,p_h)) \in \tilde{H}_h\times\tilde{Q}_h$ *solution of* (20). *In addition, there exists* $C>0$, *independent of* h, *such that*

$$||((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(u_h,p_h))||_{\tilde{H}\times\tilde{Q}} \,\leq\, C\,\left\{\, ||f||_{L^2(\Omega)} \,+\, ||g||_{H^{1/2}(\Gamma_0)}\,\right\},$$

and

$$||((\boldsymbol{\sigma},\tilde{\lambda}),(u,p)) \,-\, ((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(u_h,p_h))||_{\tilde{H}\times\tilde{Q}} \leq C \min_{((\boldsymbol{\tau}_h,\tilde{\mu}_h),v_h)\in\tilde{H}_h\times Q_h} ||((\boldsymbol{\sigma},\tilde{\lambda}),u) \,-\, ((\boldsymbol{\tau}_h,\tilde{\mu}_h),v_h)||_{\tilde{H}\times Q}.$$

Proof. Let $\tilde{V}_h$ be the discrete kernel of the operator induced by the bilinear form $\tilde{B}$. It is easy to show, according to the definition of $\tilde{B}$ (*cf.* (9)) and Lemma 5.7 in [22], that

$$\tilde{V}_h := \{\,(\boldsymbol{\tau}_h,\tilde{\mu}_h)\in\tilde{H}_h:\quad \langle\boldsymbol{\tau}_h\cdot\boldsymbol{\nu},1\rangle_\Gamma = 0 \quad\text{and}\quad \operatorname{div}\boldsymbol{\tau}_h = 0 \quad\text{in}\quad \Omega\},$$

and hence the bilinear form A is uniformly strongly coercive on $\tilde{V}_h$.

In this way, Lemma 3.4 and direct applications of the abstract Theorems 1.1 and 2.1 in Chapter II of [8] complete the proof. □

We end this section with a result on the rate of convergence of the Galerkin scheme (20). For this purpose, we recall the following approximation properties of the subspaces $H_h^{\boldsymbol{\sigma}}$, $H_{h,0}^{\lambda}$, and Q_h, respectively (see, *e.g.* [2, 8, 31, 34]):

1. $(\mathrm{AP}_h^{\boldsymbol{\sigma}})$: For all $\tau\in[H^1(\Omega)]^2$ with $\operatorname{div}\tau\in H^1(\Omega)$, it holds

$$||\tau - \mathcal{E}_h(\tau)||_{H(\mathrm{div};\Omega)} \,\leq\, C\,h\,\left\{\, ||\tau||_{[H^1(\Omega)]^2} \,+\, ||\operatorname{div}\tau||_{H^1(\Omega)}\,\right\}\cdot$$

2. ($\mathrm{AP}^{\lambda}_{h,0}$): For all $\tilde{\mu} \in H^{3/2}(\Gamma) \cap H^{1/2}_0(\Gamma)$, there exists $\tilde{\mu}_h \in H^{\lambda}_{h,0}$ such that

$$||\tilde{\mu} - \tilde{\mu}_h||_{H^{1/2}(\Gamma)} \leq C\, h\, ||\tilde{\mu}||_{H^{3/2}(\Gamma)}.$$

3. (AP_h): For all $v \in H^1(\Omega)$ it holds

$$||v - \mathcal{P}_h(v)||_{L^2(\Omega)} \leq C\, h\, ||v||_{H^1(\Omega)},$$

where $\mathcal{P}_h$ is the orthogonal projection from $L^2(\Omega)$ onto Q_h.

Then we can establish the following theorem.

Theorem 3.6. *Let $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p))$ and $((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h))$ be the unique solutions of the continuous and discrete mixed formulations* (8) *and* (20)*, respectively. Assume that $\boldsymbol{\sigma} \in [H^s(\Omega)]^2$, $\operatorname{div} \boldsymbol{\sigma} \in H^s(\Omega)$, $\tilde{\lambda} \in H^{s+1/2}(\Gamma)$ and $u \in H^s(\Omega)$, for some $s \in (0, 1]$. Then there exists $C > 0$, independent of h, such that*

$$||((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) - ((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h))||_{\tilde{H} \times \tilde{Q}} \leq C\, h^s \left\{ ||\boldsymbol{\sigma}||_{[H^s(\Omega)]^2} + ||\operatorname{div} \boldsymbol{\sigma}||_{H^s(\Omega)} + ||\tilde{\lambda}||_{H^{s+1/2}(\Gamma)} + ||u||_{H^s(\Omega)} \right\}.$$

Proof. It follows from the Cea estimate in Theorem 3.5, the above approximation properties, and suitable interpolation theorems in the Sobolev spaces. □

4. An explicit residual *a-posteriori* estimate

Let us first introduce some notations. We let $E(T)$ be the set of edges of $T \in \mathcal{T}_h$, and let E_h be the set of all edges of the triangulation $\mathcal{T}_h$. Then we write $E_h = E_h(\Omega) \cup E_h(\Gamma_0) \cup E_h(\Gamma)$, where $E_h(\Omega) := \{e \in E_h : \ e \subseteq \Omega\}$, $E_h(\Gamma) := \{e \in E_h : \ e \subseteq \Gamma\}$, and similarly for $E_h(\Gamma_0)$. In what follows, h_T and h_e stand for the diameters of the triangle $T \in \mathcal{T}_h$ and edge $e \in E_h$, respectively. Also, given a vector-valued function $\boldsymbol{\tau} := (\tau_1, \tau_2)^{\mathbf{T}}$ defined in Ω, an edge $e \in E(T) \cap E_h(\Omega)$, and the unit tangential vector $\mathbf{t}_T$ along e, we let $\boldsymbol{\tau}_T$ be the restriction of $\boldsymbol{\tau}$ to T, and let $J[\boldsymbol{\tau} \cdot \mathbf{t}_T]$ be the corresponding jump across e, that is $J[\boldsymbol{\tau} \cdot \mathbf{t}_T] := (\boldsymbol{\tau}_T - \boldsymbol{\tau}_{T'})|_e \cdot \mathbf{t}_T$, where T' is the other triangle of $\mathcal{T}_h$ having e as edge. Here, the tangential vector $\mathbf{t}_T$ is given by $(-\nu_2, \nu_1)^{\mathbf{T}}$ where $\boldsymbol{\nu}_T := (\nu_1, \nu_2)^{\mathbf{T}}$ is the unit outward normal to ∂T. Finally, we let $\operatorname{curl}(\boldsymbol{\tau})$ be the scalar $\frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}$.

Next, we define the finite element space

$$X_h := \{v_h \in C(\Omega) : \quad v_h|_T = \hat{v} \circ F_T^{-1}, \quad \hat{v} \in \mathbf{P}_1(\hat{T}), \quad \forall\, T \in \mathcal{T}_h\},$$

and let $I_h : H^1(\Omega) \longrightarrow X_h$ be the usual Clément interpolation operator (see [7,15]). The following lemma states the local approximation properties of I_h.

Lemma 4.1. *There exist positive constants C_1 and C_2, independent of h, such that for all $\varphi \in H^1(\Omega)$ there holds*

$$||\varphi - I_h(\varphi)||_{L^2(T)} \leq C_1\, h_T\, ||\varphi||_{H^1(\Delta(T))} \qquad \forall\, T \in \mathcal{T}_h,$$

and

$$||\varphi - I_h(\varphi)||_{L^2(e)} \leq C_2\, h_e^{1/2}\, ||\varphi||_{H^1(\Delta(e))} \qquad \forall\, e \in E_h,$$

where $\Delta(T) := \cup\{T' \in \mathcal{T}_h : \quad T' \cap T \neq \emptyset\}$, and $\Delta(e) := \cup\{T' \in \mathcal{T}_h : \quad T' \cap e \neq \emptyset\}$.

Proof. See Theorem 4.1 in [7]. □

The main goal of the present section is to prove the following theorem providing a reliable *a-posteriori* error estimate.

Theorem 4.2. *Let* $((\boldsymbol{\sigma},\tilde{\lambda}),(u,p)) \in \tilde{H}\times\tilde{Q}$ *and* $((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(u_h,p_h)) \in \tilde{H}_h\times\tilde{Q}_h$ *be the unique solutions of the continuous and discrete formulations* (8) *and* (20)*, respectively. Assume that the Dirichlet data* $g\in H^1(\Gamma_0)$ *and that* $\boldsymbol{\kappa}_1 \in C^1(\Omega_1)$*. Then there exists* $C>0$*, independent of* h*, such that*

$$\left|\left|((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h),(u-u_h,p-p_h))\right|\right|_{\tilde{H}\times\tilde{Q}} \le C\,\left\{\sum_{T\in\mathcal{T}_h}\eta_T^2\right\}^{1/2}, \tag{31}$$

where for any triangle $T\in\mathcal{T}_h$ *we define*

$$\begin{aligned}\eta_T^2 :=& \,||f+\operatorname{div}\boldsymbol{\sigma}_h||^2_{L^2(T)} + h_T^2\,||\operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)||^2_{L^2(T)}\\ &+ h_T^2\,||\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h||^2_{[L^2(T)]^2} + \sum_{e\in E(T)\cap E_h(\Omega)} h_e||\,J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T]\,||^2_{L^2(e)}\\ &+ \sum_{e\in E(T)\cap E_h(\Gamma_0)} h_e\left\{||(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h\cdot\mathbf{t}_T||^2_{L^2(e)} + \left|\left|\frac{\mathrm{d}g}{\mathrm{d}\mathbf{t}_T}\right|\right|^2_{L^2(e)} + ||g-\hat{g}_h||^2_{L^2(e)}\right\}\\ &+ \sum_{e\in E(T)\cap E_h(\Gamma)} h_e\left\{\left|\left|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T-\frac{\mathrm{d}\tilde{\lambda}_h}{\mathrm{d}\mathbf{t}_T}\right|\right|^2_{L^2(e)} + ||\xi_h||^2_{L^2(e)} + \left|\left|\tilde{\lambda}_h-\hat{\lambda}_h\right|\right|^2_{L^2(e)}\right\},\end{aligned} \tag{32}$$

with

$$\xi_h := \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu} + 2\,\mathbf{W}(\tilde{\lambda}_h), \tag{33}$$

$$\hat{g}_h|_e := \frac{1}{h_e}\int_e g\,\mathrm{d}s \qquad \forall\, e\in E_h(\Gamma_0), \tag{34}$$

and

$$\hat{\lambda}_h|_e := \frac{1}{h_e}\int_e \tilde{\lambda}_h\,\mathrm{d}s \qquad \forall\, e\in E_h(\Gamma). \tag{35}$$

In order to prove Theorem 4.2, we need some preliminary results. We begin with the following technical lemma.

Lemma 4.3. *Let* $\hat{\boldsymbol{\sigma}} := \boldsymbol{\sigma}_h + \boldsymbol{\sigma}^* \in H(\operatorname{div};\Omega)$*, where* $\boldsymbol{\sigma}^* := \nabla z$ *and* $z\in H^1(\Omega)$ *is the weak solution of:* $-\Delta z = f + \operatorname{div}\boldsymbol{\sigma}_h$ *in* Ω*,* $z=0$ *on* Γ_0*,* $\frac{\partial z}{\partial\boldsymbol{\nu}}=0$ *on* Γ*. Then there exists* $C>0$*, independent of* h*, such that*

$$\begin{aligned}\left|\left|(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}},\tilde{\lambda}-\tilde{\lambda}_h)\right|\right|^2_{\tilde{H}} \le& \,C\,||f+\operatorname{div}\boldsymbol{\sigma}_h||_{L^2(\Omega)}\,\left|\left|(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}},\tilde{\lambda}-\tilde{\lambda}_h)\right|\right|_{\tilde{H}} + \left|A((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h,\tilde{\lambda}-\tilde{\lambda}_h-\tilde{\mu}_h))\right|\\ &+ |\langle\boldsymbol{\tau}_h\cdot\boldsymbol{\nu},p_h\rangle_\Gamma| + |\langle\,(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}|,\end{aligned}$$

for all $(\boldsymbol{\tau}_h,\tilde{\mu}_h)\in\tilde{H}_h$ *with* $\operatorname{div}\boldsymbol{\tau}_h=0$.

Proof. It follows similarly as the proof of Lemma 4.2 in [21]. We refer to [4] for details. □

Now, we can give an *a-posteriori* error estimate for $(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)$ and $(\tilde{\lambda}-\tilde{\lambda}_h)$ through the following theorem.

Theorem 4.4. *There exists $C > 0$, independent of h, such that*

$$\left|\left|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h)\right|\right|_{\tilde{H}} \leq C \left\{ \sum_{T \in \mathcal{T}_h} \eta_{1,T}^2 \right\}^{1/2}, \tag{36}$$

where for any triangle $T \in \mathcal{T}_h$ we define

$$\begin{aligned}
\eta_{1,T}^2 := & \, ||f + \operatorname{div} \boldsymbol{\sigma}_h||_{L^2(T)}^2 + h_T^2 \, \left|\left|\operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\right|\right|_{L^2(T)}^2 \\
& + \sum_{e \in E(T) \cap E_h(\Omega)} h_e \, || \, J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T] \, ||_{L^2(e)}^2 \\
& + \sum_{e \in E(T) \cap E_h(\Gamma_0)} h_e \left\{ \left|\left|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T\right|\right|_{L^2(e)}^2 + \left|\left| \frac{\mathrm{d}g}{\mathrm{d}\mathbf{t}_T} \right|\right|_{L^2(e)}^2 \right\} \\
& + \sum_{e \in E(T) \cap E_h(\Gamma)} h_e \left\{ \left|\left| (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T - \frac{\mathrm{d}\tilde{\lambda}_h}{\mathrm{d}\mathbf{t}_T} \right|\right|_{L^2(e)}^2 + ||\xi_h||_{L^2(e)}^2 \right\}.
\end{aligned}$$

Proof. From Lemma 4.3 we know that $\boldsymbol{\sigma}^* \cdot \boldsymbol{\nu} = \frac{\partial z}{\partial \boldsymbol{\nu}} = 0$ on Γ and $\operatorname{div}(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) = 0$. In addition, the formulations (8) and (20) yield $\langle \boldsymbol{\sigma} \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = \langle \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = 0$. Then, using Gauss's Theorem we deduce that $\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) \cdot \boldsymbol{\nu}, 1 \rangle_\Gamma = \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_0} = 0$.

Thus, since Ω is connected, there exists a stream function $\varphi \in H^1(\Omega)$, with $\int_\Omega \varphi \, \mathrm{d}x = 0$, such that $\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} = \mathbf{curl}\, \varphi := \left(-\frac{\partial \varphi}{\partial x_2} \;\; \frac{\partial \varphi}{\partial x_1} \right)^{\mathbf{T}}$.

We now introduce the Clément interpolant $\varphi_h := I_h(\varphi) \in X_h$ and take from now on $\boldsymbol{\tau}_h := \mathbf{curl}\, \varphi_h$ in Lemma 4.3. In this way, $\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} - \boldsymbol{\tau}_h = \mathbf{curl}(\varphi - \varphi_h)$, and for all $\tilde{\mu}_h \in H_{h,0}^\lambda$ it holds

$$\begin{aligned}
A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\mathbf{curl}(\varphi - \varphi_h), \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h)) = & \int_\Omega (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{curl}(\varphi - \varphi_h) \, \mathrm{d}x \\
& + \langle \frac{d}{d\mathbf{t}_T}(\varphi - \varphi_h), \tilde{\lambda}_h \rangle_\Gamma + \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h \rangle_\Gamma,
\end{aligned} \tag{37}$$

where $\xi_h := \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} + 2\, \mathbf{W} \tilde{\lambda}_h$. Since $\boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}|_\Gamma \in L^2(\Gamma)$ and $\tilde{\lambda}_h \in H^1(\Gamma)$, it follows easily, using the mapping properties of $\mathbf{W}$, that $\xi_h \in L^2(\Gamma)$.

Now, applying integration by parts, we obtain

$$\begin{aligned}
\int_\Omega (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{curl}(\varphi - \varphi_h) \, \mathrm{d}x = & \sum_{T \in \mathcal{T}_h} \Bigg\{ - \int_T \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \, (\varphi - \varphi_h) \, \mathrm{d}x \\
& + \frac{1}{2} \sum_{e \in E(T) \cap E_h(\Omega)} \langle J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T], \varphi - \varphi_h \rangle_{L^2(e)} + \sum_{e \in E(T) \cap E_h(\Gamma)} \langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T, \varphi - \varphi_h \rangle_{L^2(e)} \\
& + \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T, \varphi - \varphi_h \rangle_{L^2(e)} \Bigg\},
\end{aligned} \tag{38}$$

which, replaced back into (37), yields

$$A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\mathbf{curl}(\varphi - \varphi_h), \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h)) := \sum_{T \in \mathcal{T}_h} \Bigg\{ - \int_T \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\,(\varphi - \varphi_h)\,\mathrm{d}x + \frac{1}{2} \sum_{e \in E(T) \cap E_h(\Omega)} \langle\, J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T], \varphi - \varphi_h \rangle_{L^2(e)} + \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T, \varphi - \varphi_h \rangle_{L^2(e)} + \sum_{e \in E(T) \cap E_h(\Gamma)} \left(\left\langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T - \frac{\mathrm{d}\tilde{\lambda}_h}{\mathrm{d}\mathbf{t}_T}, \varphi - \varphi_h \right\rangle_{L^2(e)} + \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h \rangle_{L^2(e)} \right) \Bigg\}, \tag{39}$$

for all $\tilde{\mu}_h \in H^{\lambda}_{h,0}$, where $\langle \cdot, \cdot \rangle_{L^2(e)}$ stands for the usual $L^2(e)-$inner product.

Next, we define $\mu_h := I_h(w)|_\Gamma$, where $w \in H^1(\Omega)$ is the solution of the boundary value problem: $\Delta w = 0$ in Ω, $w = \tilde{\lambda} - \tilde{\lambda}_h$ on Γ, and $\frac{\partial w}{\partial \boldsymbol{\nu}} = 0$ on Γ_0, and set $\tilde{\mu}_h := \left(\mu_h - \frac{1}{|\Gamma|} \int_\Gamma \mu_h \,\mathrm{d}s \right) \in H^{\lambda}_{h,0}$. It is easy to see that

$$||w||_{H^1(\Omega)} \leq C \left|\left| \tilde{\lambda} - \tilde{\lambda}_h \right|\right|_{H^{1/2}(\Gamma)}, \tag{40}$$

and from Lemma 4.1 it follows that

$$\left|\left| \tilde{\lambda} - \tilde{\lambda}_h - \mu_h \right|\right|_{L^2(e)} = ||w - I_h(w)||_{L^2(e)} \leq C\, h_e^{1/2}\, ||w||_{H^1(\Delta(e))}.$$

Using the property $\langle \xi_h, 1 \rangle_\Gamma = 0$, the above inequality, and the fact that the number of triangles in $\Delta(e)$ is bounded (independently of h), we show that

$$\left| \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Gamma)} \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h \rangle_{L^2(e)} \right| = \left| \sum_{e \in E_h(\Gamma)} \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \tilde{\mu}_h \rangle_{L^2(e)} \right| = \left| \sum_{e \in E_h(\Gamma)} \langle \xi_h, \tilde{\lambda} - \tilde{\lambda}_h - \mu_h \rangle_{L^2(e)} \right| \leq C \left\{ \sum_{e \in E_h(\Gamma)} h_e\, ||\xi_h||^2_{L^2(e)} \right\}^{1/2} ||w||_{H^1(\Omega)}. \tag{41}$$

In order to bound the remaining terms in (39) we apply Cauchy-Schwarz's inequality, Lemma 3.4, and the fact that the number of triangles in $\Delta(T)$ is also bounded. Thus, we find that

$$\left| \sum_{T \in \mathcal{T}_h} \int_T \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\,(\varphi - \varphi_h)\,\mathrm{d}x \right| \leq C \sum_{T \in \mathcal{T}_h} h_T \left|\left| \operatorname{curl}(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \right|\right|_{L^2(T)} ||\varphi||_{1,\Delta(T)}, \tag{42}$$

$$\left| \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Omega)} \langle J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T], \varphi - \varphi_h \rangle_{L^2(e)} \right| \leq C \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Omega)} h_e^{1/2} \left|\left| \, J[(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T] \, \right|\right|_{L^2(e)} ||\varphi||_{1,\Delta(e)}, \tag{43}$$

$$\left| \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T, \varphi - \varphi_h \rangle_{L^2(e)} \right| \leq C \sum_{T \in \mathcal{T}_h} \sum_{e \in E(T) \cap E_h(\Gamma_0)} h_e^{1/2} \left|\left| (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \mathbf{t}_T \right|\right|_{L^2(e)} ||\varphi||_{H^1(\Delta(e))}, \tag{44}$$

and

$$\left|\sum_{T\in\mathcal{T}_h}\sum_{e\in E(T)\cap E_h(\Gamma)}\langle(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T-\frac{\mathrm{d}\tilde{\lambda}_h}{\mathrm{d}\mathbf{t}_T},\varphi-\varphi_h\rangle_{L^2(e)}\right|\leq$$
$$C\sum_{T\in\mathcal{T}_h}\sum_{e\in E(T)\cap E_h(\Gamma)}h_e^{1/2}\left|\left|(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\mathbf{t}_T-\frac{\mathrm{d}\tilde{\lambda}_h}{\mathrm{d}\mathbf{t}_T}\right|\right|_{L^2(e)}||\varphi||_{H^1(\Delta(e))}. \quad (45)$$

Also, we observe that $\langle\boldsymbol{\tau}_h\cdot\boldsymbol{\nu},1\rangle_\Gamma=\langle\mathbf{curl}(\varphi_h)\cdot\boldsymbol{\nu},1\rangle_\Gamma=\langle\frac{\mathrm{d}}{\mathrm{d}\mathbf{t}_T}\varphi_h,1\rangle_\Gamma=0$, which shows that the third term on the right hand side of the inequality in Lemma 4.3 vanishes.

For the remaining term on Γ_0 we note that

$$|\langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}|=|\langle\mathbf{curl}(\varphi-\varphi_h)\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}|=\left|\langle\varphi-\varphi_h,\frac{\mathrm{d}g}{\mathrm{d}\mathbf{t}_T}\rangle_{\Gamma_0}\right|,$$

which, applying Lemma 4.1, leads to

$$|\langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}|\leq C\left\{\sum_{e\subseteq E_h(\Gamma_0)}h_e\left|\left|\frac{\mathrm{d}g}{\mathrm{d}\mathbf{t}_T}\right|\right|^2_{L^2(e)}\right\}^{1/2}||\varphi||_{H^1(\Omega)}. \quad (46)$$

Therefore, using (39), (41), (42), (43), (44), (45) and (46), we deduce that

$$\left|A\big((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h,\tilde{\lambda}-\tilde{\lambda}_h-\tilde{\mu}_h)\big)\right|+|\langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}|\leq C\,\hat{\eta}_1\left\{||\varphi||^2_{H^1(\Omega)}+||w||^2_{H^1(\Omega)}\right\}^{1/2}, \quad (47)$$

where $\hat{\eta}_1:=\left\{\sum_{T\in\mathcal{T}_h}\left(\eta^2_{1,T}-||f+\operatorname{div}\boldsymbol{\sigma}_h||^2_{L^2(T)}\right)\right\}^{1/2}$. Now, since $\int_\Omega\varphi\,\mathrm{d}x=0$, we obtain from (47) and (40),

$$\left|A\big((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h,\tilde{\lambda}-\tilde{\lambda}_h-\tilde{\mu}_h)\big)\right|+|\langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}-\boldsymbol{\tau}_h)\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}|\leq C\,\hat{\eta}_1\left\{|\varphi|^2_{H^1(\Omega)}+\left|\left|\tilde{\lambda}-\tilde{\lambda}_h\right|\right|^2_{H^{1/2}(\Gamma)}\right\}^{1/2}$$
$$=C\,\hat{\eta}_1\left|\left|(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}},\tilde{\lambda}-\tilde{\lambda}_h)\right|\right|^2_{\tilde{H}}.$$

Hence, in virtue of Lemma 4.3 and the continuous dependence result given by the estimate $||\boldsymbol{\sigma}^*||_{H(\mathrm{div};\Omega)}\leq\bar{C}\,||f+\operatorname{div}\boldsymbol{\sigma}_h||_{L^2(\Omega)}$, we conclude that

$$\left|\left|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h)\right|\right|_{\tilde{H}}\leq C\left\{||f+\operatorname{div}\boldsymbol{\sigma}_h||^2_{L^2(\Omega)}+\hat{\eta}_1^2\right\}^{1/2}=C\left\{\sum_{T\in\mathcal{T}_h}\eta^2_{1,T}\right\}^{1/2}, \quad (48)$$

which ends the proof. □

In order to complete our *a-posteriori* error estimate, we need to provide the estimate for $(u-u_h)$ and $(p-p_h)$. For this purpose, the following lemma is necessary.

Lemma 4.5. *For any* $\boldsymbol{\tau}\in H(\mathrm{div},\Omega)$ *there exists* $\mathbf{r}_{\boldsymbol{\tau}}\in[H^1(\Omega)]^2$ *such that* $\operatorname{div}(\mathbf{r}_{\boldsymbol{\tau}})=\operatorname{div}\boldsymbol{\tau}$ *in* Ω, $\langle\mathbf{r}_{\boldsymbol{\tau}}\cdot\boldsymbol{\nu},1\rangle_\Gamma=\langle\boldsymbol{\tau}\cdot\boldsymbol{\nu},1\rangle_\Gamma$, *and*

$$||\mathbf{r}_{\boldsymbol{\tau}}||_{[H^1(\Omega)]^2}\leq\bar{C}\,||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)},$$

with a constant $\bar{C} > 0$, independent of $\boldsymbol{\tau}$.

Proof. We proceed similarly as the proof of Lemma 4.4 in [21]. Let $\mathcal{O}$ be the convex domain whose boundary is the circle Γ, that is $\mathcal{O} := \bar{\Omega}_0 \cup \Omega$. Then, given $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$ we consider the function $\tilde{f} \in L^2(\mathcal{O})$ defined by

$$\tilde{f} := \begin{cases} \mathrm{div}\,\boldsymbol{\tau} \quad \text{in} \quad \Omega \\ -\dfrac{1}{|\Omega_0|}\left\{\displaystyle\int_\Omega \mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x \,-\, \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma\right\} \quad \text{in} \quad \Omega_0. \end{cases}$$

Since $\int_{\mathcal{O}} \tilde{f}\,\mathrm{d}x - \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = 0$, we deduce that the weak solution $z \in H^1(\mathcal{O})$ of: $\Delta z = \tilde{f}$ in $\mathcal{O}$, $\frac{\partial z}{\partial \boldsymbol{\nu}} = \frac{1}{|\Gamma|}\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma$ on Γ, and $\int_{\mathcal{O}} z\,\mathrm{d}x = 0$, is uniquely determined. In addition, a classical regularity result and the trace theorem in $H(\mathrm{div};\Omega)$ imply that $z \in H^2(\mathcal{O})$ and

$$||z||_{H^2(\mathcal{O})} \,\le\, C\left\{\,||\tilde{f}||_{L^2(\mathcal{O})} \,+\, |\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma|\,\right\} \,\le\, \bar{C}\,||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}. \tag{49}$$

Thus, we put $\mathbf{r}_{\boldsymbol{\tau}} := \nabla z|_\Omega$ and observe that $\mathbf{r}_{\boldsymbol{\tau}} \in [H^1(\Omega)]^2$, $\mathrm{div}\,(\mathbf{r}_{\boldsymbol{\tau}}) = \tilde{f} = \mathrm{div}\,\boldsymbol{\tau}$ in Ω, and $\langle \mathbf{r}_{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = \langle \frac{\partial z}{\partial \boldsymbol{\nu}}, 1\rangle_\Gamma = \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma$. Finally, (49) yields

$$||\mathbf{r}_{\boldsymbol{\tau}}||_{[H^1(\Omega)]^2} \,\le\, ||z||_{H^2(\Omega)} \,\le\, ||z||_{H^2(\mathcal{O})} \,\le\, \bar{C}\,||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)},$$

which completes the proof of the lemma. □

The *a-posteriori* error estimate for $(u - u_h, p - p_h) \in \tilde{Q}$ is established now.

Theorem 4.6. *There exists $C > 0$, independent of h, such that*

$$||(u - u_h, p - p_h)||_{\tilde{Q}} \le C\left\{\sum_{T\in\mathcal{T}_h} \eta_T^2\right\}^{1/2}.$$

Proof. The continuous inf-sup condition for $\tilde{B}$ (*cf.* proof of Th. 2.2) yields the inequality

$$||(u - u_h, p - p_h)||_{\tilde{Q}} \,\le\, \tilde{C} \sup_{\substack{\boldsymbol{\tau}\in H(\mathrm{div};\Omega)\\ \boldsymbol{\tau}\neq 0}} \frac{\tilde{B}((\boldsymbol{\tau},0),(u,p)) - \tilde{B}((\boldsymbol{\tau},0),(u_h,p_h))}{||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}}\,. \tag{50}$$

Now, given $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$ we consider the function $\mathbf{r}_{\boldsymbol{\tau}}$ provided by Lemma 4.5 and note that

$$\begin{aligned}\tilde{B}((\boldsymbol{\tau},0),(u,p)) &:= \int_\Omega u\,\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x - p\,\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma \\ &= \int_\Omega u\,\mathrm{div}\,(\mathbf{r}_{\boldsymbol{\tau}})\,\mathrm{d}x - p\,\langle \mathbf{r}_{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = \tilde{B}((\mathbf{r}_{\boldsymbol{\tau}},0),(u,p)),\end{aligned}$$

which, according to the first equation of (8), gives

$$\tilde{B}((\boldsymbol{\tau},0),(u,p)) = -\,A((\boldsymbol{\sigma},\tilde{\lambda}),(\mathbf{r}_{\boldsymbol{\tau}},0)) + \langle \mathbf{r}_{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}. \tag{51}$$

Similarly, using now the properties of the operator $\mathcal{E}_h$, we easily deduce that

$$\begin{aligned}\tilde{B}((\boldsymbol{\tau},0),(u_h,p_h)) &:= \int_\Omega u_h\,\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x - p_h\,\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma \\ &= \int_\Omega u_h\,\mathrm{div}\,(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}))\,\mathrm{d}x - p_h\,\langle \mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = \tilde{B}((\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}),0),(u_h,p_h)),\end{aligned}$$

which, in virtue of the first equation of (20), yields

$$\tilde{B}((\boldsymbol{\tau},0),(u_h,p_h)) = -\,A((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}),0)) + \langle \mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}. \tag{52}$$

Then, by replacing (51) and (52) back into (50), we obtain

$$||(u-u_h,p-p_h)||_{\tilde{Q}} \le C \sup_{\substack{\boldsymbol{\tau}\in H(\mathrm{div};\Omega)\\ \boldsymbol{\tau}\neq 0}} \left\{ \frac{\begin{array}{c} A((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})-\mathbf{r}_{\boldsymbol{\tau}},0)) \;-\; A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h),(\mathbf{r}_{\boldsymbol{\tau}},0)) \\ -\;\langle(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})-\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}\end{array}}{|||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}} \right\}. \tag{53}$$

We now bound the terms on the right hand side of (53). First, the boundedness of A, Theorem 4.4, and Lemma 4.5 imply that

$$|\,A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{\lambda}-\tilde{\lambda}_h),(\mathbf{r}_{\boldsymbol{\tau}},0))\,| \;\le\; C\left\{\sum_{T\in\mathcal{T}_h}\eta_{1,T}^2\right\}^{1/2} ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}. \tag{54}$$

Next, since $\mathcal{E}_h$ satisfies $\int_e \mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu}\,\mathrm{d}s = \int_e \mathbf{r}_{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}\,\mathrm{d}s$ for all $e\in E_h$, we deduce that

$$\langle(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})-\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0} \;=\; \langle(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})-\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu},g-s_h\rangle_{\Gamma_0} \quad \forall\, s_h\in S_h,$$

where S_h is the space of piecewise constant functions on the partition of Γ_0 induced by the triangulation $\mathcal{T}_h$, and hence

$$|\,\langle(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})-\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}\,| \;\le \sum_{e\in E_h(\Gamma_0)} ||g-s_h||_{L^2(e)}\; ||(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})-\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu}||_{L^2(e)} \tag{55}$$

for all $s_h\in S_h$. But, with the same interpolation results used in the proof of Theorem 4.5 in [21], we can prove that

$$||(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})-\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu}||_{L^2(e)} \;\le\; \tilde{C}\,h_e^{1/2}\,||\mathbf{r}_{\boldsymbol{\tau}}||_{[H^1(T_e)]^2}, \tag{56}$$

where T_e is the triangle to which e belongs, and $\tilde{C}$, a constant independent of h, may depend on the minimum angle of $\mathcal{T}_h$.

In this way, (55), (56), Cauchy-Schwarz's inequality, and Lemma 4.5 lead to

$$\begin{aligned} |\,\langle(\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}})-\mathbf{r}_{\boldsymbol{\tau}})\cdot\boldsymbol{\nu},g\rangle_{\Gamma_0}\,| &\le C \inf_{s_h\in S_h}\left\{\sum_{e\in E_h(\Gamma_0)} h_e\,||g-s_h||_{L^2(e)}^2\right\}^{1/2} ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)} \\ &= C\left\{\sum_{e\in E_h(\Gamma_0)} h_e\,||g-\hat{g}_h||_{L^2(e)}^2\right\}^{1/2} ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)}, \end{aligned} \tag{57}$$

where $\hat{g}_h|_e := \frac{1}{h_e}\int_e g\,\mathrm{d}s$ for all $e\in E_h(\Gamma_0)$.

In order to bound the first term on the right hand side of (53), we recall from (25) that

$$||\mathcal{E}_h(\boldsymbol{\zeta})-\boldsymbol{\zeta}||_{[L^2(T)]^2} \;\le\; C\,h_T\,||\boldsymbol{\zeta}||_{[H^1(T)]^2} \quad \forall\,\boldsymbol{\zeta}\in[H^1(\Omega)]^2, \quad \forall\, T\in\mathcal{T}_h. \tag{58}$$

Thus, applying Cauchy-Schwarz's inequality, (58) with $\boldsymbol{\zeta} = \mathbf{r}_{\boldsymbol{\tau}}$, Lemma 4.5, and following a similar analysis to the one yielding (57), we can show that

$$\begin{aligned}\Big| A((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}, 0)) \Big| &\leq \Big| \int_\Omega (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \, \mathrm{d}x \Big| + \Big| \langle (\mathcal{E}_h(\mathbf{r}_{\boldsymbol{\tau}}) - \mathbf{r}_{\boldsymbol{\tau}}) \cdot \boldsymbol{\nu}, \tilde{\lambda}_h \rangle_\Gamma \Big| \\ &\leq C \left\{ \sum_{T \in \mathcal{T}_h} h_T^2 ||\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h||^2_{[L^2(T)]^2} + \sum_{e \in E_h(\Gamma)} h_e ||\tilde{\lambda}_h - \hat{\lambda}_h||^2_{L^2(e)} \right\}^{1/2} ||\boldsymbol{\tau}||_{H(\mathrm{div};\Omega)},\end{aligned} \tag{59}$$

where $\hat{\lambda}_h|_e := \frac{1}{h_e} \int_e \tilde{\lambda}_h \, \mathrm{d}s$ for all $e \in E_h(\Gamma)$.

Therefore, by inserting (54), (57), and (59) back into (53), we conclude the required estimate. ☐

Finally, the proof of Theorem 4.2, which is the main contribution of this section, follows straightforward from Theorems 4.4 and 4.6.

5. An implicit *a-posteriori* estimate

In this section we apply a Bank-Weiser type procedure (similarly as in [19] and [23]) to our model problem. For the classical Bank-Weiser's approach we refer to [3]. As a result of our analysis we obtain a second reliable *a-posteriori* error estimate of implicit type, which depends on the solution of local problems. In addition, we bound these local solutions, introduce a suitable averaging technique, and transform the original estimate into an explicit one.

We first need a symmetric, bounded, and strongly coercive bilinear form $\mathbf{A}$ on the space $\tilde{H} := H(\mathrm{div};\Omega) \times H_0^{1/2}(\Gamma)$. In particular, from now on we consider

$$\mathbf{A}((\boldsymbol{\zeta}, \rho), (\boldsymbol{\tau}, \mu)) := \langle \boldsymbol{\zeta}, \boldsymbol{\tau} \rangle_{H(\mathrm{div};\Omega)} + \langle \mathbf{W}(\rho), \mu \rangle_\Gamma \quad \forall\, (\boldsymbol{\zeta}, \rho),\, (\boldsymbol{\tau}, \mu) \in \tilde{H}. \tag{60}$$

Then, given the solutions $((\boldsymbol{\sigma}, \tilde{\lambda}), (u, p)) \in \tilde{H} \times \tilde{Q}$ and $((\boldsymbol{\sigma}_h, \tilde{\lambda}_h), (u_h, p_h)) \in \tilde{H}_h \times \tilde{Q}_h$ of the continuous and Galerkin schemes (8) and (20), respectively, we define the $\tilde{H}$-Ritz projection of the error with respect to $\mathbf{A}$, as the unique $(\bar{\boldsymbol{\sigma}}, \bar{\lambda}) \in \tilde{H}$ such that

$$\mathbf{A}((\bar{\boldsymbol{\sigma}}, \bar{\lambda}), (\boldsymbol{\tau}, \mu)) = A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (\boldsymbol{\tau}, \mu)) + \tilde{B}((\boldsymbol{\tau}, \mu), (u - u_h, p - p_h)) \tag{61}$$

for all $(\boldsymbol{\tau}, \mu) \in \tilde{H}$. The existence of such a $(\bar{\boldsymbol{\sigma}}, \bar{\lambda})$ is guaranteed by the fact that the right hand side of (61) (as a mapping acting on $(\boldsymbol{\tau}, \mu)$) constitutes a linear and bounded functional on $\tilde{H}$.

Now, given $T \in \mathcal{T}_h$ and $e \in E(T)$, we denote by $\langle \cdot, \cdot \rangle_{H(\mathrm{div};T)}$ the inner product of $H(\mathrm{div};T)$ and let $\langle \cdot, \cdot \rangle_{\partial T}$ be the duality pairing between $H^{-1/2}(\partial T)$ and $H^{1/2}(\partial T)$ with respect to the $L^2(\partial T)$-inner product. In addition, we let $H_{00}^{1/2}(e)$ be the space of functions in $H^{1/2}(e)$ whose extensions by zero to the rest of ∂T are in $H^{1/2}(\partial T)$, and denote by $\langle \cdot, \cdot \rangle_e$ the duality pairing between $H_{00}^{-1/2}(e)$ and $H_{00}^{1/2}(e)$ with respect to the $L^2(e)$-inner product. As before, $\boldsymbol{\nu}_T$ stands for the unit outward normal to ∂T.

The following theorem provides an important upper bound for the Ritz projection $(\bar{\boldsymbol{\sigma}}, \bar{\lambda}) \in \tilde{H}$.

Theorem 5.1. *Assume there exists $s > 2$ such that $g \in H^{1/2}(\Gamma_0) \cap W^{1-1/s,s}(\Gamma_0)$ and let $\tilde{\varphi}_h$ be a function in $H^1(\Omega) \cap W^{1,s}(\Omega)$ such that $\tilde{\varphi}_h(\bar{x}) = g(\bar{x})$ for each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ_0, and $\tilde{\varphi}_h(\bar{x}) = \tilde{\lambda}_h(\bar{x}) + p_h$ for each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ. Further, for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div};T)$ be the unique solution of the local problem*

$$\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathrm{div};T)} = G_{h,T}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in H(\mathrm{div};T), \tag{62}$$

where $G_{h,T} \in H(\mathrm{div};T)'$ *is defined by*

$$G_{h,T}(\boldsymbol{\tau}) := -\int_T (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h)\cdot\boldsymbol{\tau}\,\mathrm{d}x - \int_T u_h \,\mathrm{div}\,\boldsymbol{\tau}\,\mathrm{d}x + \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\varphi}_h\rangle_{\partial T} + \sum_{e\in E(T)\cap E_h(\Gamma)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h\rangle_e + \sum_{e\in E(T)\cap E_h(\Gamma_0)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, g - \tilde{\varphi}_h\rangle_e. \tag{63}$$

Then there holds

$$\mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\bar{\boldsymbol{\sigma}},\bar{\lambda})) \le \sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} + ||\mathbf{W}^{-1}||\,||\,2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}\,||^2_{H^{-1/2}(\Gamma)}. \tag{64}$$

Proof. We first observe from (8) that

$$A((\boldsymbol{\sigma},\tilde{\lambda}),(\boldsymbol{\tau},\mu)) + \tilde{B}((\boldsymbol{\tau},\mu),(u,p)) = \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0},$$

and hence

$$\mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\boldsymbol{\tau},\mu)) = \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0} - A((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\boldsymbol{\tau},\mu)) - \tilde{B}((\boldsymbol{\tau},\mu),(u_h,p_h)) \tag{65}$$

for all $(\boldsymbol{\tau},\mu)\in\tilde{H}$. Thus, since $\mathbf{A}$ is symmetric and strongly coercive on $\tilde{H}$, we have that

$$-\frac{1}{2}\,\mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\bar{\boldsymbol{\sigma}},\bar{\lambda})) = \min_{(\boldsymbol{\tau},\mu)\in\tilde{H}} \left\{ \frac{1}{2}\,\mathbf{A}((\boldsymbol{\tau},\mu),(\boldsymbol{\tau},\mu)) - \mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\boldsymbol{\tau},\mu)) \right\}, \tag{66}$$

which, according to (65), becomes

$$-\frac{1}{2}\,\mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\bar{\boldsymbol{\sigma}},\bar{\lambda})) = \min_{(\boldsymbol{\tau},\mu)\in\tilde{H}} \mathcal{J}(\boldsymbol{\tau},\mu), \tag{67}$$

with

$$\mathcal{J}(\boldsymbol{\tau},\mu) := \frac{1}{2}\,\mathbf{A}((\boldsymbol{\tau},\mu),(\boldsymbol{\tau},\mu)) + A((\boldsymbol{\sigma}_h,\tilde{\lambda}_h),(\boldsymbol{\tau},\mu)) + \tilde{B}((\boldsymbol{\tau},\mu),(u_h,p_h)) - \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_{\Gamma_0}. \tag{68}$$

On the other hand, the hypotheses on g and $\tilde{\varphi}_h$ imply, according to the Sobolev imbedding theorems, that $(g-\tilde{\varphi}_h)|_e \in H^{1/2}_{00}(e)$ for each $e\in E_h(\Gamma_0)$ and $(\tilde{\lambda}_h + p_h - \tilde{\varphi}_h)|_e \in H^{1/2}_{00}(e)$ for each $e\in E_h(\Gamma)$, whence

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g-\tilde{\varphi}_h\rangle_{\Gamma_0} = \sum_{e\in E_h(\Gamma_0)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g-\tilde{\varphi}_h\rangle_e \text{ and } \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h\rangle_\Gamma = \sum_{e\in E_h(\Gamma)} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h\rangle_e. \tag{69}$$

Further, we also get $-\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_T, \tilde{\varphi}_h\rangle_{\partial T} + \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, \tilde{\varphi}_h\rangle_\Gamma + \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, \tilde{\varphi}_h\rangle_{\Gamma_0} = 0$, which is then added to the quadratic functional $\mathcal{J}$.

In this way, recalling the definitions of $\mathbf{A}$, A, and $\tilde{B}$, and using (69), we obtain

$$\mathcal{J}(\boldsymbol{\tau},\mu) = \sum_{T\in\mathcal{T}_h} \mathcal{J}_{1,T}(\boldsymbol{\tau}_T) + \mathcal{J}_2(\mu), \tag{70}$$

where $\boldsymbol{\tau}_T$ is the restriction $\boldsymbol{\tau}|_T$,

$$\mathcal{J}_{1,T}(\boldsymbol{\tau}_T) := \frac{1}{2}\,||\boldsymbol{\tau}_T||^2_{H(\mathrm{div};T)} - G_{h,T}(\boldsymbol{\tau}_T), \tag{71}$$

and

$$\mathcal{J}_2(\mu) := \frac{1}{2}\,\langle \mathbf{W}(\mu), \mu\rangle_\Gamma \,+\, \langle 2\,\mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}, \mu\rangle_\Gamma. \tag{72}$$

We observe here that

$$\min_{\boldsymbol{\tau}_T \in H(\mathrm{div};T)} \mathcal{J}_{1,T}(\boldsymbol{\tau}_T) \,=\, -\,\frac{1}{2}\,||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)}, \tag{73}$$

where $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div};T)$ is the unique solution of the local problem (62).

Hence, replacing (70) up to (72) back into (67), noting that $H(\mathrm{div};\Omega)$ is contained in the *broken* space

$$H(\mathrm{div};\Omega)^{br} := \{\boldsymbol{\tau} \in [L^2(\Omega)]^2 : \quad \boldsymbol{\tau}_T \in H(\mathrm{div};T) \quad \forall\, T \in \mathcal{T}_h\},$$

and using (73), we deduce that

$$\begin{aligned}
-\,\frac{1}{2}\,\mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\bar{\boldsymbol{\sigma}},\bar{\lambda})) &= \min_{\boldsymbol{\tau} \in H(\mathrm{div};\Omega)} \left\{ \sum_{T\in\mathcal{T}_h} \mathcal{J}_{1,T}(\boldsymbol{\tau}_T) \right\} + \min_{\mu \in H_0^{1/2}(\Gamma)} \mathcal{J}_2(\mu) \\
&\geq \sum_{T\in\mathcal{T}_h} \min_{\boldsymbol{\tau}_T \in H(\mathrm{div};T)} \mathcal{J}_{1,T}(\boldsymbol{\tau}_T) + \min_{\mu \in H_0^{1/2}(\Gamma)} \mathcal{J}_2(\mu) \\
&= -\frac{1}{2} \sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} - \frac{1}{2}\,\langle \mathbf{W}(\rho), \rho\rangle_\Gamma,
\end{aligned} \tag{74}$$

where $\rho \in H_0^{1/2}(\Gamma)$ is the unique solution to the equation

$$\langle \mathbf{W}(\rho), \mu\rangle_\Gamma \,=\, -\,\langle\, 2\,\mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}, \mu\rangle_\Gamma \qquad \forall\, \mu \in H_0^{1/2}(\Gamma). \tag{75}$$

It follows from (75) that

$$-\,\frac{1}{2}\,\langle \mathbf{W}(\rho), \rho\rangle_\Gamma \,\geq\, -\,\frac{1}{2}\,||\mathbf{W}^{-1}||\;||2\,\mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||^2_{H^{-1/2}(\Gamma)},$$

whence (74) yields

$$-\,\frac{1}{2}\,\mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\bar{\boldsymbol{\sigma}},\bar{\lambda})) \,\geq\, -\,\frac{1}{2} \sum_{T\in\mathcal{T}_h} ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} \,-\, \frac{1}{2}\,||\mathbf{W}^{-1}||\;||2\,\mathbf{W}(\tilde{\lambda}_h) \,+\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||^2_{H^{-1/2}(\Gamma)},$$

which completes the proof. □

It is important to remark that the above theorem does not require any further condition on $\tilde{\varphi}_h$, and hence, in principle, this function can be chosen in many different ways. However, we will prove below that the proposed *a-posteriori* error estimate becomes efficient up to a term depending on $(u - \tilde{\varphi}_h)$. This property is called *quasi-efficiency*. Therefore, one should try to choose $\tilde{\varphi}_h$ as close as possible, at least empirically, to the exact solution u.

We now give the main reliable *a-posteriori* error estimate for the Galerkin scheme (20), which makes use of the $\tilde{H}$-Ritz projection $(\bar{\boldsymbol{\sigma}},\bar{\lambda})$ and the associated upper bound provided by Theorem 5.1.

Theorem 5.2. *Let $\tilde{\varphi}_h$ be as indicated in Theorem 5.1, and for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div};T)$ be the unique solution of the local problem (62). Then there exists $C > 0$, independent of h, such that*

$$\Big|\Big|((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))\Big|\Big|_{\tilde{H}\times\tilde{Q}} \le C \left\{ \sum_{T\in\mathcal{T}_h} \theta_T^2 + R_\Gamma^2 \right\}^{1/2},$$

where

$$\theta_T^2 := ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} + ||\, f + \mathrm{div}\, \boldsymbol{\sigma}_h \,||^2_{L^2(T)}$$

and

$$R_\Gamma^2 := ||\, 2\, \mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu} \,||^2_{H^{-1/2}(\Gamma)}.$$

Proof. The continuous dependence result given by Theorem 2.2 is equivalent to stating that the variational formulation (8) satisfies a global inf-sup condition, which means that there exists $\tilde{C} > 0$ such that

$$||((\boldsymbol{\zeta}, \rho), (w, r))||_{\tilde{H}\times\tilde{Q}} \le \tilde{C} \sup_{\substack{((\boldsymbol{\tau},\mu),(v,q))\in\tilde{H}\times\tilde{Q} \\ ||((\boldsymbol{\tau},\mu),(v,q))||\le 1}} \Big\{ A((\boldsymbol{\zeta},\rho),(\boldsymbol{\tau},\mu)) + \tilde{B}((\boldsymbol{\tau},\mu),(w,r)) + \tilde{B}((\boldsymbol{\zeta},\rho),(v,q)) \Big\}$$

for all $((\boldsymbol{\zeta}, \rho), (w, r)) \in \tilde{H} \times \tilde{Q}$.

In particular, taking $((\boldsymbol{\zeta}, \rho), (w, r)) := ((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))$ in the above inequality, and using the definition of the Ritz projection $(\bar{\boldsymbol{\sigma}}, \bar{\lambda}) \in \tilde{H}$ (*cf.* (61)), and the statements of the continuous and Galerkin schemes (8) and (20), we obtain that

$$||((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))||_{\tilde{H}\times\tilde{Q}} \le \tilde{C} \sup_{\substack{((\boldsymbol{\tau},\mu),(v,q))\in\tilde{H}\times\tilde{Q} \\ ||((\boldsymbol{\tau},\mu),(v,q))||\le 1}} \left\{ \mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\boldsymbol{\tau},\mu)) - \int_\Omega (f + \mathrm{div}\,\boldsymbol{\sigma}_h)\, v \,\mathrm{d}x \right\}.$$

Hence, using the properties of $\mathbf{A}$, and applying Cauchy-Schwarz's inequality, we deduce that there exists $\bar{C} > 0$ such that

$$||((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))||_{\tilde{H}\times\tilde{Q}} \le \bar{C} \left\{ \mathbf{A}((\bar{\boldsymbol{\sigma}},\bar{\lambda}),(\bar{\boldsymbol{\sigma}},\bar{\lambda})) + \sum_{T\in\mathcal{T}_h} ||f + \mathrm{div}\,\boldsymbol{\sigma}_h||^2_{L^2(T)} \right\}^{1/2},$$

which, together with the upper bound (64), finishes the proof. □

The following lemma provides *a-priori* estimates for the solution of the local problem (62). They will be used to show the *quasi-efficiency* of the estimate provided by Theorem 5.2, and also to deduce an explicit reliable *a-posteriori* error estimate based on a suitable averaging technique.

Lemma 5.3. *Let $\tilde{\varphi}_h$ be as indicated in Theorem 5.1, and for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div};T)$ be the unique solution of the local problem (62). Then there exists $C > 0$, independent of h and T, such that*

$$\begin{aligned} ||\hat{\boldsymbol{\sigma}}_T||_{H(\mathrm{div};T)} \le C \Big\{ &||\,(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) - \nabla\tilde{\varphi}_h\,||^2_{[L^2(T)]^2} + ||\,u_h - \tilde{\varphi}_h\,||^2_{L^2(T)} \\ &+ \sum_{e\in E(T)\cap E_h(\Gamma)} ||\tilde{\lambda}_h + p_h - \tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)} + \sum_{e\in E(T)\cap E_h(\Gamma_0)} ||g - \tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)} \Big\}^{1/2}. \end{aligned} \quad (76)$$

In addition, for any $z \in H^1(\Omega) \cap W^{1,s}(\Omega)$*, with* $s > 2$*, such that* $z = g$ *on* Γ_0*, we get*

$$||\hat{\boldsymbol{\sigma}}_T||_{H(\mathrm{div};T)} \leq C \left\{ ||(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) - \nabla z||^2_{[L^2(T)]^2} + ||u_h - z||^2_{L^2(T)} + ||\mathbf{J}_{h,T}(z)||^2_{H^{1/2}(\partial T)} \right\}^{1/2}, \tag{77}$$

where $\mathbf{J}_{h,T}(z) := \begin{cases} 0 & on \quad \partial T \cap \Gamma_0 \\ z - (\tilde{\lambda}_h + p_h) & on \quad \partial T \cap \Gamma. \\ z - \tilde{\varphi}_h & otherwise \end{cases}$

Proof. We first recall from (62) that $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div};T)$ and $\langle \hat{\boldsymbol{\sigma}}_T, \boldsymbol{\tau} \rangle_{H(\mathrm{div};T)} = G_{h,T}(\boldsymbol{\tau})$ for all $\boldsymbol{\tau} \in H(\mathrm{div};T)$, where

$$\begin{aligned} G_{h,T}(\boldsymbol{\tau}) := & - \int_T (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) \cdot \boldsymbol{\tau} \,\mathrm{d}x - \int_T u_h \,\mathrm{div}\, \boldsymbol{\tau} \,\mathrm{d}x + \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\varphi}_h \rangle_{\partial T} \\ & + \sum_{e \in E(T) \cap E_h(\Gamma)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h \rangle_e + \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, g - \tilde{\varphi}_h \rangle_e. \end{aligned} \tag{78}$$

Since $\tilde{\varphi}_h \in H^1(\Omega)$, we apply Gauss's formula to obtain

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\varphi}_h \rangle_{\partial T} = \int_T \nabla \tilde{\varphi}_h \cdot \boldsymbol{\tau} \,\mathrm{d}x + \int_T \tilde{\varphi}_h \,\mathrm{div}\, \boldsymbol{\tau} \,\mathrm{d}x.$$

Then, replacing this expression back into (78), applying Cauchy-Schwarz's inequality, and using the fact that $||\hat{\boldsymbol{\sigma}}_T||_{H(\mathrm{div};T)} = ||G_{h,T}||_{H(\mathrm{div};T)'}$, we arrive to (76).

On the other hand, given $z \in H^1(\Omega) \cap W^{1,s}(\Omega)$, with $s > 2$, such that $z = g$ on Γ_0, we obtain

$$\begin{aligned} & \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\varphi}_h \rangle_{\partial T} + \sum_{e \in E(T) \cap E_h(\Gamma)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h \rangle_e + \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, g - \tilde{\varphi}_h \rangle_e = \\ & \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, z \rangle_{\partial T} - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, z - \tilde{\varphi}_h \rangle_{\partial T} + \sum_{e \in E(T) \cap E_h(\Gamma)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\lambda}_h + p_h - \tilde{\varphi}_h \rangle_e - \sum_{e \in E(T) \cap E_h(\Gamma_0)} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \tilde{\varphi}_h - z \rangle_e \\ & = \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, z \rangle_{\partial T} - \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T, \mathbf{J}_{h,T}(z) \rangle_{\partial T}, \end{aligned}$$

which, replaced back into (78), yields (77) and ends the proof. ◻

We show next that the reliable *a-posteriori* error estimate from Theorem 5.2 is *quasi-efficient*, that is, it is efficient up to a term depending on the traces of $(u - \tilde{\varphi}_h)$ on the edges of $\mathcal{T}_h$. Indeed, we have the following lemma.

Lemma 5.4. *Let* $\tilde{\varphi}_h$ *be as indicated in Theorem 5.1, and assume that* $u \in W^{1,s}(\Omega)$*, with* $s > 2$*. Then there exists* $C > 0$*, independent of* h*, such that for all* $T \in \mathcal{T}_h$

$$\theta_T^2 \leq C \left\{ ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{H(\mathrm{div};T)} + ||u - u_h||^2_{L^2(T)} + \|\mathbf{J}_{h,T}(u)\|^2_{H^{1/2}(\partial T)} \right\}, \tag{79}$$

where $\mathbf{J}_{h,T}(u) := \begin{cases} 0 & on \quad \partial T \cap \Gamma_0 \\ \lambda - \lambda_h & on \quad \partial T \cap \Gamma. \\ u - \tilde{\varphi}_h & otherwise \end{cases}$ *Further, there exists* $\tilde{C} > 0$*, independent of* h*, such that*

$$\sum_{T \in \mathcal{T}_h} \theta_T^2 + R_\Gamma^2 \leq \tilde{C} \left\{ ||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))||^2_{\tilde{H} \times \tilde{Q}} + \sum_{T \in \mathcal{T}_h} \|\mathbf{J}_{h,T}(u)\|^2_{H^{1/2}(\partial T)} \right\}. \tag{80}$$

Proof. From the second equation of (8) we get $\operatorname{div}\boldsymbol{\sigma} = -f$ in Ω and $\langle \boldsymbol{\sigma}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = 0$. In addition, from the first equation of (8) we deduce that $\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma} = \nabla u$ in Ω, $u = \tilde{\lambda} + p$ on Γ, $u = g$ on Γ_0, and $2\,\mathbf{W}(\tilde{\lambda}) + \boldsymbol{\sigma}\cdot\boldsymbol{\nu} = 0$ on Γ.

Then, applying Lemma 5.3 (*cf.* (77)) with $z = u$, we obtain that

$$||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} \leq C \left\{ ||(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) - (\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma})||^2_{[L^2(T)]^2} + ||u_h - u||^2_{L^2(T)} + ||\mathbf{J}_{h,T}(u)||^2_{H^{1/2}(\partial T)} \right\}. \tag{81}$$

Hence, (79) follows from (81) and the fact that

$$\theta_T^2 := ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} + ||f + \operatorname{div}\boldsymbol{\sigma}_h||^2_{L^2(T)} = ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} + ||\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||^2_{L^2(T)}.$$

On the other hand, using that $(2\,\mathbf{W}(\tilde{\lambda}) + \boldsymbol{\sigma}\cdot\boldsymbol{\nu}) = 0$ on Γ, and applying the boundedness of $\mathbf{W}$ and the trace theorem in $H(\mathrm{div};\Omega)$, we obtain that

$$R_\Gamma^2 := ||2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}||^2_{H^{-1/2}(\Gamma)} \leq C \left\{ ||\tilde{\lambda}_h - \tilde{\lambda}||^2_{H^{1/2}(\Gamma)} + ||\boldsymbol{\sigma}_h - \boldsymbol{\sigma}||^2_{H(\mathrm{div};\Omega)} \right\}. \tag{82}$$

Finally, summing up in (79) over all $T \in \mathcal{T}_h$, and adding (82), we conclude (80) and finish the proof. □

The *quasi-efficiency* provided by Lemma 5.4 is in agreement with the properties of the classical Bank-Weiser approach. In fact, it is well known that this *a-posteriori* error analysis only yields reliability, and that it is possible to obtain an explicit lower bound of the error through the utilization of a different estimator, usually of residual type.

Our next purpose is to bound the global quantity R_Γ by computable local indicators on the edges $e \in E_h(\Gamma)$. Indeed, we have the following lemma.

Lemma 5.5. *There exists $C > 0$, independent of h, such that*

$$R_\Gamma^2 \leq C \log[1 + C_h(\Gamma)] \sum_{e\in E_h(\Gamma)} h_e\, ||2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}||^2_{L^2(e)}, \tag{83}$$

where

$$C_h(\Gamma) := \max\left\{ \frac{h_e}{h_{e'}} : \quad e \text{ and } e' \text{ are neighbour edges of } \Gamma \right\}.$$

Proof. We first observe from the definitions of the finite element subspaces $H_h^{\boldsymbol{\sigma}}$ and $H_{h,0}^{\lambda}$ (*cf.* (15) and (19)) that $(\boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})|_\Gamma \in L^2(\Gamma)$ and $\tilde{\lambda}_h \in H^1(\Gamma)$, and hence, a mapping property of $\mathbf{W}$ implies that $(2\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}) \in L^2(\Gamma)$.

Now, taking $\boldsymbol{\tau}_h = 0$ in the first equation of (20), and $(v_h, q_h) = (0,1)$ in the second one, we deduce, respectively, that $\langle 2\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}, \tilde{\mu}_h\rangle_\Gamma = 0$ for all $\tilde{\mu}_h \in H_{h,0}^{\lambda}$, and $\langle \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = 0$.

Therefore, using the decomposition $H_h^{\lambda} = H_{h,0}^{\lambda} \oplus \mathbb{R}$, the symmetry of $\mathbf{W}$, and the fact that $\mathbf{W}(1) = 0$, we conclude that $(2\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu})$ is $L^2(\Gamma)$-orthogonal to H_h^{λ}. Thus, a straightforward application of Theorem 2 in [11] yields the estimate (83) and ends the proof. □

As a consequence of Theorem 5.2 and Lemma 5.5, we obtain the following result.

Theorem 5.6. *Let $\tilde{\varphi}_h$ be as indicated in Theorem 5.1, and for each $T \in \mathcal{T}_h$ let $\hat{\boldsymbol{\sigma}}_T \in H(\mathrm{div};T)$ be the unique solution of the local problem* (62). *Then there exists $C > 0$, independent of h, such that*

$$\left|\left|((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))\right|\right|_{\tilde{H}\times\tilde{Q}} \leq C \left\{ \sum_{T\in\mathcal{T}_h} \tilde{\theta}_T^2 \right\}^{1/2},$$

where

$$\tilde{\theta}_T^2 := ||\hat{\boldsymbol{\sigma}}_T||^2_{H(\mathrm{div};T)} + ||\, f + \operatorname{div} \boldsymbol{\sigma}_h \,||^2_{L^2(T)} + \log[1 + C_h(\Gamma)] \sum_{e \in E(T)\cap E_h(\Gamma)} h_e \, ||2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h \cdot \boldsymbol{\nu}||^2_{L^2(e)}.$$

It is important to remark here that the local problem defining $\hat{\boldsymbol{\sigma}}_T$ lives in the infinite dimensional space $H(\mathrm{div};T)$, and therefore, it can only be solved approximately by considering suitable finite dimensional subspaces. To this respect, as indicated in [1], we suggest to apply the p or the $h-p$ version.

Alternatively, we propose to utilize the upper bound (76) from Lemma 5.3 to derive a fully explicit reliable *a-posteriori* error estimate that does not require neither the exact nor any approximate solution of the local problem (62). More precisely, our main explicit reliable *a-posteriori* error estimate for the Galerkin scheme (20) is stated as follows.

Theorem 5.7. *Let $\tilde{\varphi}_h$ be as indicated in Theorem 5.1. Then there exists $C > 0$, independent of h, such that*

$$\Big|\Big|((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \tilde{\lambda} - \tilde{\lambda}_h), (u - u_h, p - p_h))\Big|\Big|_{\tilde{H}\times\tilde{Q}} \le C \left\{ \sum_{T\in\mathcal{T}_h} \hat{\theta}_T^2 \right\}^{1/2}, \tag{84}$$

where

$$\begin{aligned} \hat{\theta}_T^2 := & ||\,(\boldsymbol{\kappa}^{-1}\boldsymbol{\sigma}_h) - \nabla\tilde{\varphi}_h\,||^2_{[L^2(T)]^2} + ||\,u_h - \tilde{\varphi}_h\,||^2_{L^2(T)} + ||\,f + \operatorname{div}\boldsymbol{\sigma}_h\,||^2_{L^2(T)} \\ & + \sum_{e\in E(T)\cap E_h(\Gamma)} ||\tilde{\lambda}_h + p_h - \tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)} + \sum_{e\in E(T)\cap E_h(\Gamma_0)} ||g - \tilde{\varphi}_h||^2_{H^{1/2}_{00}(e)} \\ & + \log[1 + C_h(\Gamma)] \sum_{e\in E(T)\cap E_h(\Gamma)} h_e \, ||2\,\mathbf{W}(\tilde{\lambda}_h) + \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}||^2_{L^2(e)}. \end{aligned} \tag{85}$$

Proof. It follows directly from Theorem 5.6 and Lemma 5.3. □

We end this section by setting an appropriate choice for $\tilde{\varphi}_h$. As suggested by the quasi-efficiency result provided by Lemma 5.4, this function needs to be as close as possible to the exact solution u. Hence, we follow an averaging technique and define $\tilde{\varphi}_h : \bar{\Omega} \to \mathbb{R}$ as the unique continuous function satisfying the following conditions.

1. $(\tilde{\varphi}_h|_T \circ F_T) \in \mathbf{P}_1(\hat{T})$ for all $T \in \mathcal{T}_h$, where F_T is the diffeomorphism mapping the reference triangle $\hat{T}$ onto T (*cf.* Section 3).
2. For each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ_0: $\tilde{\varphi}_h(\bar{x}) = g(\bar{x})$.
3. For each vertex $\bar{x}$ of $\mathcal{T}_h$ lying on Γ: $\tilde{\varphi}_h(\bar{x}) = \tilde{\lambda}_h(\bar{x}) + p_h$.
4. For each vertex $\bar{x}$ of $\mathcal{T}_h$ not lying on $\Gamma_0 \cup \Gamma$: $\tilde{\varphi}_h(\bar{x})$ is the weighted average of the constant values of u_h on all the triangles $T \in \mathcal{T}_h$ to which $\bar{x}$ belongs. Here, the weighting is according to the relative area of each triangle.

Finally, we observe that for implementation purposes, the $H^{1/2}$-norms appearing in the definition of the local indicators $\hat{\theta}_T$ can be bounded using an interpolation theorem. More precisely, given an edge $e \in E_h(\Gamma)\cup E_h(\Gamma_0)$, and a function $\rho \in H^1_0(e)$, we have

$$||\rho||^2_{H^{1/2}_{00}(e)} \le ||\rho||_{L^2(e)} \, ||\rho||_{H^1_0(e)}.$$

6. Numerical results

We now provide several numerical results illustrating the performance of the discrete scheme (18), and supporting the quality and efficiency of the *a-posteriori* error estimates given by (31, 32) and (84, 85). We emphasize, according to Theorem 3.2, that it suffices to solve (18) instead of the equivalent Galerkin scheme (20).

For the geometry of the problem, we let Γ_0 ($\partial\Omega_0$) and Γ_1 be the boundaries of the squares with center at $(0,0)$ and side lengths given by 1 and 4, respectively. In other words, Γ_0 is the polygonal curve determined by the vertices $(1/2,1/2)$, $(-1/2,1/2)$, $(-1/2,-1/2)$, and $(1/2,-1/2)$, and Γ_1 is the one determined by $(2,2)$, $(-2,2)$, $(-2,-2)$, and $(2,-2)$. In all our computations we consider $\boldsymbol{\kappa}_1$ equals the identity matrix $\mathbf{I}$, and choose the data f and g so that the exact solution of (1) is

$$u(x,y) := \frac{x}{(x-0.45)^2+y^2}\,\chi\left(\sqrt{x^2+y^2}\right) \qquad \forall\,(x,y)\in\mathbb{R}^2-\bar{\Omega}_0,$$

where $\chi \in C^2([\frac{1}{2},+\infty))$ is the cut-off function defined by

$$\chi(r) := \begin{cases} r^3-3r^2+3r, & \text{if } \frac{1}{2}\le r\le 1 \\ 1, & \text{if } 1\le r. \end{cases}$$

Hence, we take Γ as the circle with center at $(0,0)$ and radius 4, and recall that the computational domain Ω is the annular region bounded by Γ_0 and Γ.

We observe that u has a singularity at $(0.45,0)$, $u\in C^2(\Omega)$, and $u\notin C^3(\Omega)$. In fact, because of the definition of χ, the third order derivatives of u are not continuous on the unit circle.

We let N be the number of degrees of freedom defining the subspaces H_h and Q_h, that is $N :=$ number of edges of $\mathcal{T}_h$ + number of nodes on Γ + number of triangles of $\mathcal{T}_h$. Also, we use the following notations for the individual and global errors

$$\mathbf{e}(u) := ||u-u_h||_{L^2(\Omega)}, \quad \mathbf{e}(\lambda) := ||\lambda-\lambda_h||_{L^2(\Gamma)}, \quad \mathbf{e}(\boldsymbol{\sigma}) := ||\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||_{H(\mathrm{div};\Omega)},$$

and

$$\mathbf{e} := \left\{ [\mathbf{e}(u)]^2 + [\mathbf{e}(\lambda)]^2 + [\mathbf{e}(\boldsymbol{\sigma})]^2 \right\}^{1/2},$$

where $((\boldsymbol{\sigma},\lambda),u)\in H\times Q$ and $((\boldsymbol{\sigma}_h,\lambda_h),u_h)\in H_h\times Q_h$ are the solutions of (5) and (18), respectively. In addition, we consider the error estimates given by

$$\eta := \left\{ \sum_{T\in\mathcal{T}_h} \eta_T^2 \right\}^{1/2} \quad \text{and} \quad \hat{\theta} := \left\{ \sum_{T\in\mathcal{T}_h} \hat{\theta}_T^2 \right\}^{1/2},$$

where η_T and $\hat{\theta}_T$ are defined by (32) and (85).

The adaptive algorithm used in our computations follows a standard approach from [35] (see also [33]). More precisely, given a parameter $\boldsymbol{\gamma}\in(0,1)$, we apply the following scheme:

1. Start with a coarse mesh $\mathcal{T}_h$.
2. Solve the discrete problem (18) for the actual mesh $\mathcal{T}_h$.
3. Compute η_T ($\hat{\theta}_T$) for each triangle $T\in\mathcal{T}_h$.
4. Evaluate stopping criterion and decide to finish or go to next step.
5. Use *blue-green* procedure to refine each T whose indicator η_T ($\hat{\theta}_T$) is among the $100\boldsymbol{\gamma}\,\%$ of the largest indicators. Define resulting mesh as actual mesh $\mathcal{T}_h$, update h and go to step 2.

In Tables 6.1 throughout 6.5 we display the errors for each unknown, the error estimates η and $\hat{\theta}$, and the effectivity indices $\mathbf{e}/\eta$ and $\mathbf{e}/\hat{\theta}$, for uniform and adaptive refinements. In addition, Figures 6.1 and 6.2 show the global error $\mathbf{e}$ *versus* the degrees of freedom N. We consider here two choices of the refinement parameter $\boldsymbol{\gamma}$, namely 0.1 and 0.25. We remark that the errors on each triangle $T\in\mathcal{T}_h$ are computed using a 16 points Gaussian quadrature rule.

TABLE 6.1. Individual errors, error estimates η and $\hat{\theta}$, and effectivity indices for the uniform refinement.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	η	$\mathbf{e}/\eta$	$\hat{\theta}$	$\mathbf{e}/\hat{\theta}$
1222	0.9602	0.3206	21.0856	39.2721	0.5375	29.4685	0.7164
4764	0.6674	0.1793	18.0624	36.2539	0.4986	25.1351	0.7191
18808	0.5291	0.1765	12.4059	34.1627	0.3635	17.7766	0.6986
74736	0.4847	0.1796	6.8630	25.9430	0.2653	10.2980	0.6683

TABLE 6.2. Individual errors, error estimate, and effectivity index for the adaptive refinement based on η, with $\boldsymbol{\gamma} = 0.1$.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	η	$\mathbf{e}/\eta$
1222	0.9602	0.3206	21.0856	39.2721	0.5375
1705	0.6835	0.1922	18.0697	36.2829	0.4984
2339	0.5532	0.1837	12.4267	34.2311	0.3634
3226	0.5135	0.1842	6.9190	26.0892	0.2660
4344	0.5046	0.1842	3.7602	17.0105	0.2233
5913	0.5027	0.1843	2.3152	11.3346	0.2097
8340	0.4947	0.1813	1.7358	8.2160	0.2208
12553	0.4907	0.1809	1.5224	6.3317	0.2542
19094	0.4846	0.1798	1.3737	4.9846	0.2945
28893	0.4826	0.1801	1.2419	3.9749	0.3382

TABLE 6.3. Individual errors, error estimate, and effectivity index for the adaptive refinement based on $\hat{\theta}$, with $\boldsymbol{\gamma} = 0.1$.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	$\hat{\theta}$	$\mathbf{e}/\hat{\theta}$
1222	0.9602	0.3206	21.0856	29.4685	0.7164
1802	0.6806	0.1889	18.0673	25.1435	0.7191
2587	0.5460	0.1796	12.4188	17.8093	0.6981
3676	0.5054	0.1811	6.8987	10.3820	0.6665
5215	0.4944	0.1807	3.7113	6.0897	0.6155
11078	0.4885	0.1803	1.5939	3.2497	0.5160
17558	0.4834	0.1795	1.3610	2.9870	0.4873
27947	0.4819	0.1794	1.2214	2.8579	0.4637
45060	0.4798	0.1795	1.1351	2.7639	0.4506
73261	0.4777	0.1796	1.1031	2.6731	0.4547

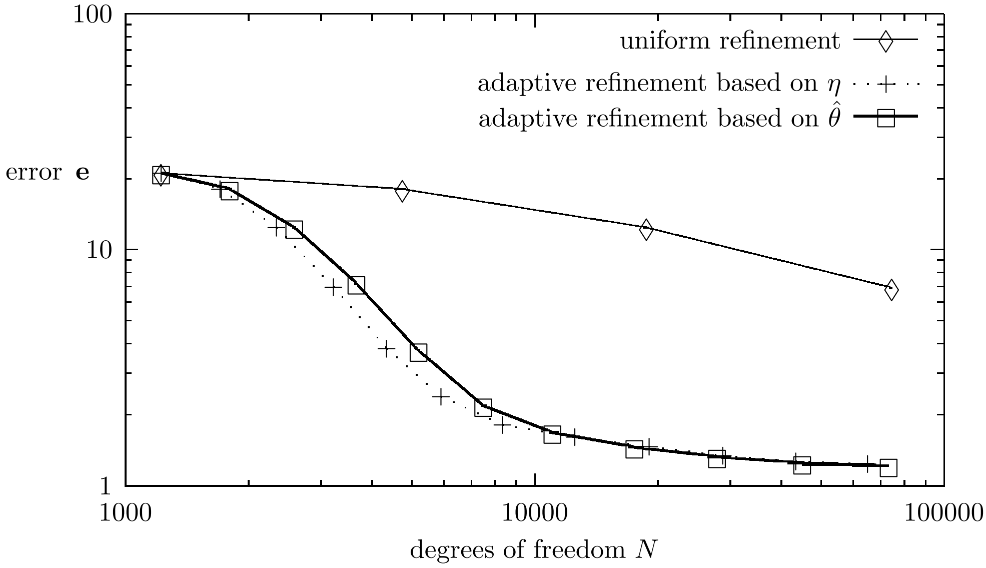

FIGURE 6.1. Error $\mathbf{e}$ for uniform and adaptive refinements (with $\boldsymbol{\gamma} = 0.1$).

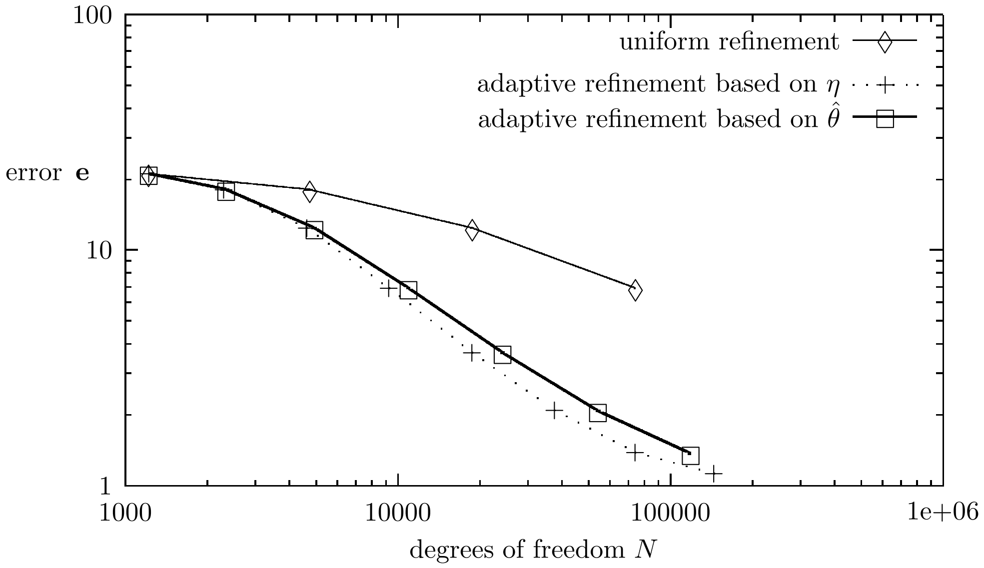

FIGURE 6.2. Error $\mathbf{e}$ for uniform and adaptive refinements (with $\boldsymbol{\gamma} = 0.25$).

TABLE 6.4. Individual errors, error estimate, and effectivity index for the adaptive refinement based on η, with $\boldsymbol{\gamma} = 0.25$.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	η	$\mathbf{e}/\eta$
1222	0.9602	0.3206	21.0856	39.2721	0.5375
2305	0.6725	0.1823	18.0628	36.2611	0.4985
4654	0.5356	0.1772	12.4080	34.1781	0.3634
9264	0.4899	0.1786	6.8699	25.9641	0.2654
18693	0.4769	0.1789	3.6320	16.6864	0.2198
37606	0.4736	0.1795	2.0264	10.5919	0.1972

TABLE 6.5. Individual errors, error estimate, and effectivity index for the adaptive refinement based on $\hat{\theta}$, with $\boldsymbol{\gamma} = 0.25$.

N	$\mathbf{e}(u)$	$\mathbf{e}(\lambda)$	$\mathbf{e}(\boldsymbol{\sigma})$	$\hat{\theta}$	$\mathbf{e}/\hat{\theta}$
1222	0.9602	0.3206	21.0856	29.4685	0.7164
2345	0.6725	0.1825	18.0628	25.1380	0.7191
4941	0.5355	0.1772	12.4075	17.7864	0.6983
10986	0.4887	0.1785	6.8666	10.3175	0.6674
24320	0.4758	0.1790	3.6255	5.9069	0.6198
54177	0.4729	0.1798	2.0168	3.6833	0.5645

As expected, the errors $\mathbf{e}$ for the adaptive refinements decrease considerably faster than for the uniform one. Also, it is observed in all cases that $\mathbf{e}$ is mainly dominated by the individual error $\mathbf{e}(\boldsymbol{\sigma})$. Further, the indices $\mathbf{e}/\eta$ and $\mathbf{e}/\hat{\theta}$ are always bounded above, which provides experimental evidences for the estimates (31) and (84). We note, at least for this example, that the adaptive algorithm based on $\hat{\theta}$ is more efficient than the one based on η. Nevertheless, as shown in Figures 6.1 and 6.2 the adaptive refinement using η converges a bit faster than the one using $\hat{\theta}$. Now, it is also clear from Figures 6.1 and 6.2 that the adaptive meshes generated with $\boldsymbol{\gamma} = 0.1$ yield a much faster decreasing of $\mathbf{e}$ than with $\boldsymbol{\gamma} = 0.25$. However, after about $N = 15000$ degrees of freedom, this process saturates and no further significant improvement is obtained. On the other hand, the decreasing obtained with $\boldsymbol{\gamma} = 0.25$ shows a closer behaviour to the expected quasi-optimal linear rate of convergence. These facts can also be verified from Tables 6.2 up to 6.5 by computing the experimental rates of convergence, that is the quantities $-\frac{2\,\log(\mathbf{e}/\mathbf{e}')}{\log(N/N')}$, where $\mathbf{e}$ and $\mathbf{e}'$ are the global errors associated with two consecutive adaptive meshes with N and N' degrees of freedom, respectively.

Next, in Figures 6.3 and 6.4 we display initial and intermediate meshes obtained with the refinement strategies. We observe that the adaptive algorithms, based on both η and $\hat{\theta}$, are able to recognize a neighborhood of $(0.5, 0)$, which is close to the singular point $(0.45, 0)$. Also, they clearly identify the unit circle, on which, as mentioned before, the exact solution u looses smoothness.

Finally, we emphasize that the numerical results presented in this section provide enough support for the adaptive methods being much more efficient than a uniform discretization when solving linear exterior problems.

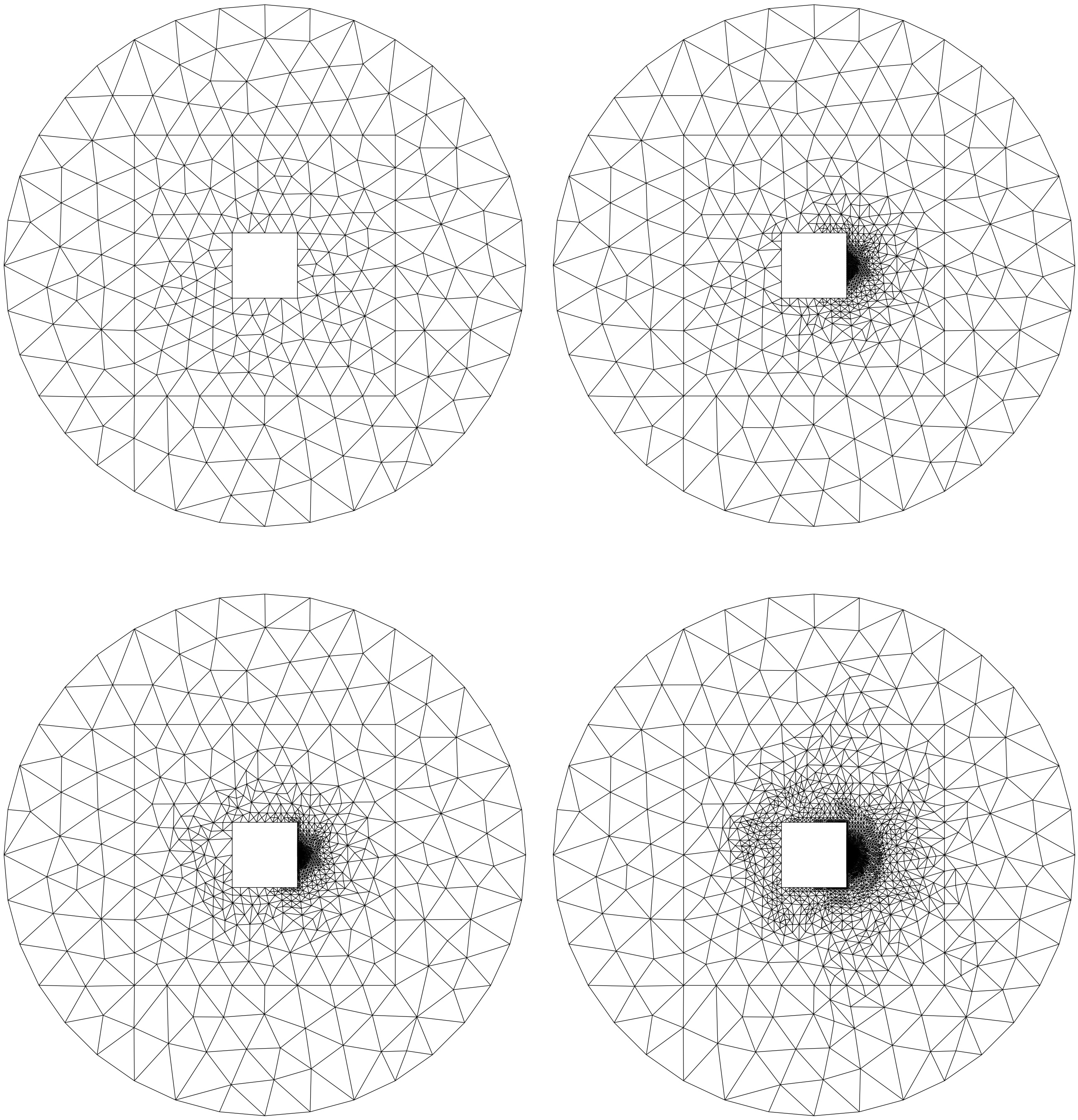

FIGURE 6.3. Initial and intermediate meshes with 1222, 4344, 8340, and 28893 degrees of freedom, respectively, for the adaptive refinement based on η, with $\boldsymbol{\gamma} = 0.1$.

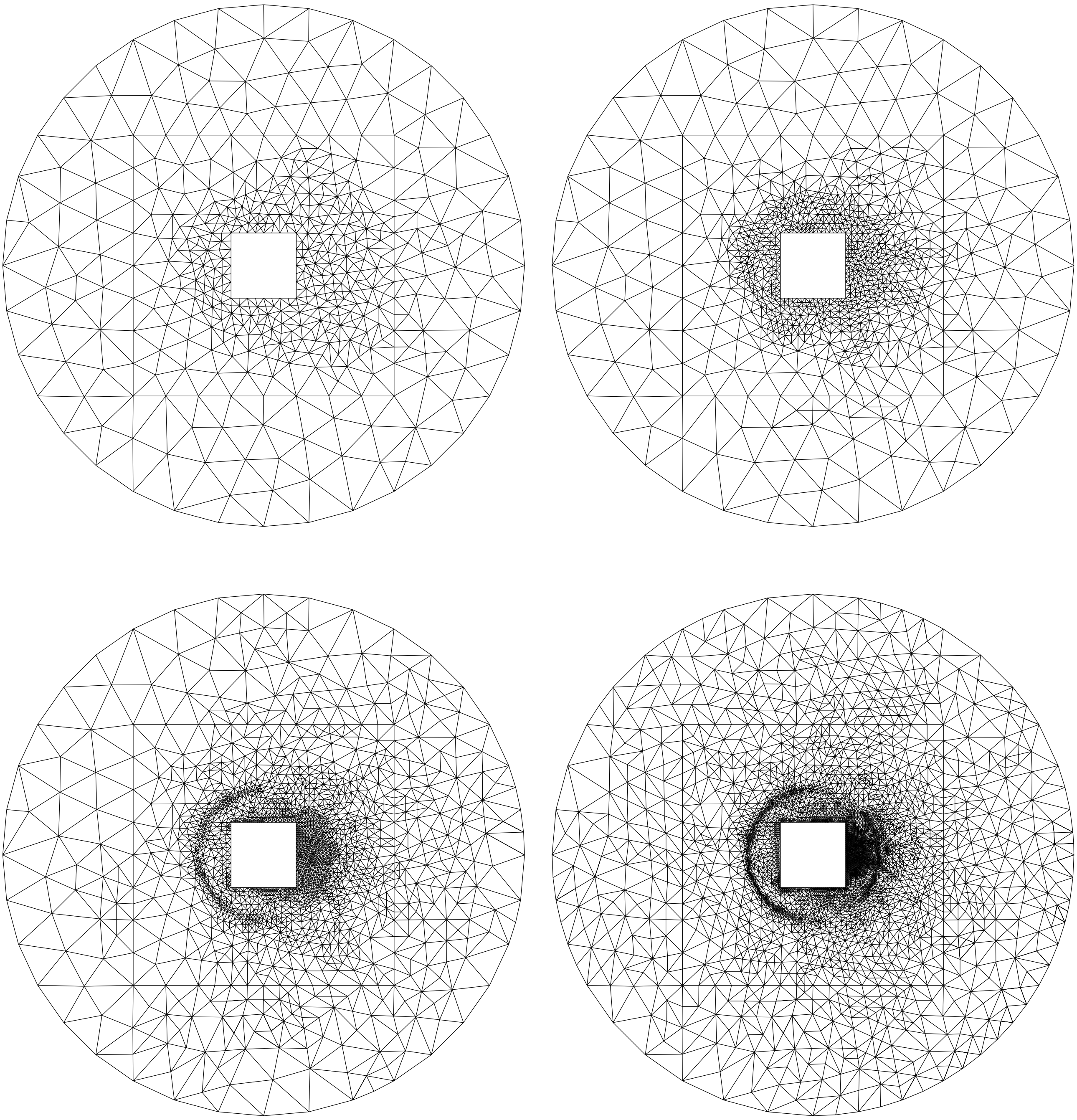

FIGURE 6.4. Intermediate meshes with 2345, 4941, 10986, and 24320 degrees of freedom, respectively, for the adaptive refinement based on $\hat{\theta}$, with $\boldsymbol{\gamma} = 0.25$.

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