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A UNIFIED CONVERGENCE ANALYSIS FOR LOCAL PROJECTION STABILISATIONS APPLIED TO THE OSEEN PROBLEM*

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Abstract. The discretisation of the Oseen problem by finite element methods may suffer in general from two shortcomings. First, the discrete inf-sup (Babuška-Brezzi) condition can be violated. Second, spurious oscillations occur due to the dominating convection. One way to overcome both difficulties is the use of local projection techniques. Studying the local projection method in an abstract setting, we show that the fulfilment of a local inf-sup condition between approximation and projection spaces allows to construct an interpolation with additional orthogonality properties. Based on this special interpolation, optimal *a-priori* error estimates are shown with error constants independent of the Reynolds number. Applying the general theory, we extend the results of Braack and Burman for the standard two-level version of the local projection stabilisation to discretisations of arbitrary order on simplices, quadrilaterals, and hexahedra. Moreover, our general theory allows to derive a novel class of local projection stabilisation by enrichment of the approximation spaces. This class of stabilised schemes uses approximation and projection spaces defined on the same mesh and leads to much more compact stencils than in the two-level approach. Finally, on simplices, the spectral equivalence of the stabilising terms of the local projection method and the subgrid modelling introduced by Guermond is shown. This clarifies the relation of the local projection stabilisation to the variational multiscale approach.

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Introduction

The discretisation of the Oseen problem by finite element methods may suffer in general from two shortcomings. First, the discrete inf-sup (Babuška-Brezzi) condition can be violated. Second, spurious oscillations occur in case of higher Reynolds numbers due to the dominating convection. The idea of streamline upwind Petrov-Galerkin (SUPG) stabilisation has been proposed for the advective term in [12] and extended to the Stokes equations in [27] where a pressure stabilisation Petrov-Galerkin (PSPG) method is considered accommodating low equal-order interpolation to be stable and convergent. This formulation circumvented the need to satisfy the discrete inf-sup condition for many interpolations. In an attempt to get the stability features of these works, a method is proposed in [16] that is at the same time advective stable and overcomes the inf-sup

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restrictions of the standard Galerkin method. A detailed error analysis of these SUPG/PSPG-type stabilisations applied to the incompressible Navier-Stokes equations, including both the case of inf-sup stable and equal-order interpolations, can be found in [41]. Despite the progress of the SUPG/PSPG method in theory and application, an essential drawback of this method is – in particular for higher order interpolations – that various terms need to be added to the weak formulation to guarantee the consistency of the method in a strong way (Galerkin orthogonality holds for smooth solutions). Residual-based stabilisation methods which use inf-sup stable pairs of elements reduce the number of terms which have to be added to the Galerkin formulation [17,33]. However, an additional coupling term between velocity and pressure makes their analysis difficult. Over the last years, several approaches have been developed to relax the strong coupling of velocity and pressure in SUPG/PSPG-type stabilisations and to introduce symmetric versions of the stabilising terms, for an overview see [11, 32].

The local projection method is designed for equal-order interpolation and allows a separation of velocity and pressure in the stabilisation terms. It has been introduced for the Stokes problem in [4], extended to the transport equation in [5], and analysed for low order discretisations of the Oseen equations in [7]. Some variants and applications are discussed in [8, 9]. In the local projection method, the stabilisation term is based on a projection $\pi_h : Y_h \to D_h$ of the finite element space Y_h approximating velocity and pressure into a discontinuous space D_h. Stabilisation of the standard Galerkin method is achieved by adding terms which give a weighted L^2-control over the fluctuations $(id - \pi_h)$ of the gradients of the quantity of interest. The key idea in the error analysis of the local projection scheme is the construction of an interpolant into Y_h which exhibits an additional orthogonality property with respect to the discontinuous space D_h. In [7], the case of low order Q_r-elements $(r = 1, 2)$ on quadrilaterals $(d = 2)$ and hexahedra $(d = 3)$ has been considered. There, π_h has been chosen to be the L^2-projection onto the space of discontinuous Q_{r-1}-elements on a coarser mesh. Unfortunately, this two-level approach leads to a stencil being less compact than for the SUPG/PSPG-type stabilisation.

The main objective of this paper is to give a general convergence theory of local projection schemes leading to *a-priori* error estimates which show the same optimal order of convergence as known for the SUPG/PSPG method. To this end, we study under which conditions an interpolant into Y_h with additional orthogonality properties with respect to D_h can be constructed. We show that an inf-sup condition for the spaces Y_h and D_h is sufficient for the existence of such an interpolant. Our general theory allows us to consider large classes of spaces Y_h and D_h, including the two-level approach on simplices, quadrilaterals, and hexahedra for arbitrary but fixed polynomial degree $r \geq 1$. Moreover, we can derive local projection schemes not only as a two-level approach but also for pairs of spaces Y_h/D_h which are defined on the same mesh family $\mathcal{T}_h$. This opens the way to circumvent the disadvantage of the classical two-level form of the local projection scheme which produces a larger stencil. As we will show, this new approach of enriched approximation spaces works also on simplices, quadrilaterals, and hexahedra for arbitrary polynomial degree $r \geq 1$.

It is well known that stabilised methods can also be derived from a variational multiscale formulation [24–26, 40]. Based on a scale separation of the underlying finite element spaces, it has been shown that it is sufficient to stabilise only the fine scale fluctuations. This results into a stabilising term which gives a weighted L^2-control over the gradient of fluctuations instead of the fluctuations of gradients [15, 19]. We will discuss the relation between this subgrid modelling approach and the local projection scheme in detail. In particular, we show that for linear elements on simplices both approaches lead to the same discrete problem.

The reader will notice that the idea of local projection results in a large number of concrete schemes. They differ in the specification of the solution and projection space Y_h and D_h, respectively, in the way of satisfying the local inf-sup condition (by a two-level approach or by taking enrichments), and in considering fluctuations of the gradients or gradients of fluctuations. Thus, the evaluation of the full potential of each of these methods needs a thorough numerical study which is beyond the scope of this paper. In the following we refer to numerical results obtained by schemes belonging to the considered class of stabilisation methods. The application of subgrid modelling (gradients of fluctuations) to scalar transport equations of convection-diffusion type has been numerically studied in [15, 19–23] in the two-level context for continous piecewise linear and quadratic elements.

The extension to nonconfoming piecewise linear elements enriched by continuous cubic or nonconforming quadratic bubble functions has been considered in [1]. It is remarkable that the local projection stabilisation is also useful for control problems of convection-diffusion type [6]. In [30], the projection of the gradients onto piecewise constants on a coarser mesh for linear and bilinear elements have been studied. Numerical tests showing the evidence of the pressure stabilisation of equal order interpolation for the Stokes problem are presented in [4]. Continuous, piecewise bilinears and biquadratics have been studied there in a two-level context. Stabilisation of both phenomena, dominating convection and violation of the discrete Babuška-Brezzi condition, are investigated in [7–9, 23] for the incompressible Navier-Stokes equation. For the flow around a NACA 0012 airfoil, numerical results are given for Reynolds numbers up to 10^6 in [23]. Convincing are also the result of a benchmark problem of the flow around a cylinder in 2D [9] and 3D [8]. In these papers, a two-level method based on quadratic quadrilateral and hexahedral elements with hanging nodes has been used. In [10], the method is extended to compressible Navier-Stokes equations coupled with further transport equations to describe chemically reacting flows. Finally, turbulent flows including the 3D mixing layer problem are numerically computed by the local projection method in [29, 31].

The plan of the paper is as follows. In Section 1, a weak formulation of the Oseen equations and its standard Galerkin discretisation is given. We formulate the local projection method in an abstract setting. Section 2 is devoted to the convergence analysis of the local projection method in this abstract setting. The basic tool is the construction of a special interpolant based on the fulfilment of a local inf-sup condition. Proving the stability independent of the Reynolds number and the approximated Galerkin orthogonality, we conclude optimal *a-priori* estimates. The application of the theory in the framework of two-level methods is studied in Section 3 where the focus is on defining pairs of finite element spaces satisfying the local inf-sup condition given in Section 2. We extend in Section 4 the analysis to spaces which are defined on the same mesh. Starting from the space D_h, the space Y_h is obtained by enriching standard finite element spaces. Additionally, we study the relation of local projection schemes to the subgrid modelling approach in Section 5. We summarise our results in Section 6.

Notation. Throughout the paper C will denote a generic positive constant which is independent of the Reynolds number and the mesh. Subscripted constants such as C_1 are also independent of the Reynolds number and the mesh, but have a fixed value. We will write shortly $\alpha \sim \beta$, if there are positive constants $\underline{C}$ and $\overline{C}$ such that

$$\underline{C}\beta \leq \alpha \leq \overline{C}\beta$$

holds.

Our Oseen problem will be considered in the domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which is assumed to be a polygonal or polyhedral domain with boundary $\partial\Omega$. For a measurable subset G of Ω, the usual Sobolev spaces $W^{m,p}(G)$ with norm $\|\cdot\|_{m,p,G}$ and semi-norm $|\cdot|_{m,p,G}$ are used. In the case $p = 2$, we have $H^m(G) = W^{m,2}(G)$ and the index p will be omitted. The L^2 inner product on G is denoted by $(\cdot,\cdot)_G$. Note that the index G will be omitted for $G = \Omega$. This notation of norms, semi-norms, and inner products is also used for the vector-valued and tensor-valued case.

1. Local projection stabilisation in an abstract setting

1.1. Weak formulation of the Oseen problem

Let us consider the Oseen problem

$$-\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

as a linearisation of the steady ($\sigma = 0$) and the non-steady ($\sigma > 0$) time-discretised Navier-Stokes equations. Here, we assume $\mathbf{b} \in \mathbf{W}^{1,\infty}(\Omega)$ with $\nabla\cdot\mathbf{b} = 0$. Let $\mathbf{V} = \mathbf{H}_0^1(\Omega)$, $Q = L_0^2(\Omega) = \{q \in L^2(\Omega) \ : \ (q, 1) = 0\}$.

We introduce on the product space $\mathbf{V} \times Q$ the bilinear form A given by

$$A\big((\mathbf{u},p);(\mathbf{v},q)\big) := \nu(\nabla\mathbf{u},\nabla\mathbf{v}) + \big((\mathbf{b}\cdot\nabla)\mathbf{u},\mathbf{v}\big) + \sigma(\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) + (q,\nabla\cdot\mathbf{u}). \tag{1.2}$$

A weak formulation of the Oseen problem (1.1) reads:

$$\text{Find } (\mathbf{u},p)\in\mathbf{V}\times Q \text{ such that for all } (\mathbf{v},q)\in\mathbf{V}\times Q: \qquad A\big((\mathbf{u},p);(\mathbf{v},q)\big) = (\mathbf{f},\mathbf{v}). \tag{1.3}$$

The property

$$\big((\mathbf{b}\cdot\nabla)\mathbf{v},\mathbf{v}\big) = \frac{1}{2}\big(\mathbf{b}\cdot\nabla(\mathbf{v}\cdot\mathbf{v}),1\big) = -\frac{1}{2}(\nabla\cdot\mathbf{b},\mathbf{v}\cdot\mathbf{v}) = 0 \qquad \forall\mathbf{v}\in\mathbf{V} \tag{1.4}$$

allows to apply the Lax-Milgram lemma in the subspace of divergence-free functions and to establish a unique velocity field $\mathbf{u}$. A unique pressure $p\in Q$ such that $(\mathbf{u},p)$ solves (1.3) follows from the Babuška-Brezzi condition for the pair $(\mathbf{V},Q)$ [18].

1.2. Galerkin discretisation

For the finite element discretisation of the Oseen problem (1.3), we introduce a shape regular decomposition $\mathcal{T}_h$ of Ω into d-dimensional simplices, quadrilaterals or hexahedra. The diameter of a cell K will be denoted by h_K and the mesh parameter h represents the maximum diameter of the cells $K\in\mathcal{T}_h$. Let $Y_h\subset H^1(\Omega)$ be a finite element space of continuous, piecewise polynomial functions defined over $\mathcal{T}_h$.

Assumption A1: There is an interpolation operator $i_h : H^1(\Omega)\to Y_h$ such that $i_h : H^1_0(\Omega)\to Y_h\cap H^1_0(\Omega)$ and

$$\|w-i_hw\|_{0,K} + h_K|w-i_hw|_{1,K} \le C\,h_K^l\|w\|_{l,\omega(K)} \qquad \forall w\in H^l(\omega(K)),\ \forall K\in\mathcal{T}_h,\ 1\le l\le r+1, \tag{1.5}$$

where $\omega(K)$ denotes a certain local neighbourhood of K which appears in the definition of these interpolation operators for non-smooth functions, see [14, 38] for more details.

We will also apply this type of interpolation operator to vector-valued functions in a component-wise manner. We indicate this by using boldface notations, for example $\mathbf{i}_h : \mathbf{V}\to Y_h^d\cap\mathbf{V}$.

Remark 1.1. The existence theory of interpolation operators for non-smooth functions satisfying Assumption A1 is well established in the literature, see [2, 14, 38].

For simplicity of presentation, we consider in this paper the case of equal-order interpolation, thus assuming $\mathbf{V}_h := Y_h^d\cap\mathbf{V}$ and $Q_h := Y_h\cap Q$. Now, the standard Galerkin discretisation of (1.3) reads:

$$\text{Find } (\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h \text{ such that for all } (\mathbf{v}_h,q_h)\in\mathbf{V}_h\times Q_h:\ A\big((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h)\big) = (\mathbf{f},\mathbf{v}_h). \tag{1.6}$$

In general, this formulation suffers from two reasons: the violation of the discrete Babuška-Brezzi condition

$$\exists\beta_0>0: \inf_{q_h\in Q_h}\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(q_h,\nabla\cdot\mathbf{v}_h)}{\|q_h\|_0\,|\mathbf{v}_h|_1} \ge \beta_0, \tag{1.7}$$

and the dominating advection in case of $\nu\ll 1$. Both instability phenomena can be handled by the local projection technique which will be the topic of the next subsection.

1.3. Stabilisation by local projection

In order to explain the stabilisation method, we start with some additional notations. By a macro element M we denote the union of one or more neighbouring cells $K \in \mathcal{T}_h$. The local neighbourhood $\omega(K)$ introduced in the definition of the interpolation i_h (*cf.* Assumption A1) leads to a local neighbourhood $\Lambda(M) := \cup_{K\in M}\omega(K)$ of a macro element M. The diameter of a macro element M is denoted by h_M. We assume that the decomposition of Ω into macro elements $M \in \mathcal{M}_h$ is non-overlapping and also shape regular, moreover

$$h_K \sim h_M, \qquad \forall K \subset M, \quad \forall M \in \mathcal{M}_h.$$

One can think of having first the decomposition $\mathcal{M}_h$ into macro elements from which the decomposition $\mathcal{T}_h$ is generated by certain refinement rules. Note that we also allow the case $\mathcal{M}_h = \mathcal{T}_h$.

Let D_h denote a discontinuous finite element space defined on the macro decomposition $\mathcal{M}_h$ and $D_h(M) := \{q_h\big|_M \;:\; q_h \in D_h\}$. Further, let $\pi_M : L^2(M) \to D_h(M)$ be a local projection which defines the projection $\pi_h : L^2(\Omega) \to D_h$ by $(\pi_h w)\big|_M := \pi_M(w\big|_M)$. Associated with the projection π_h is the fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ defined by $\kappa_h := id - \pi_h$, where $id : L^2(\Omega) \to L^2(\Omega)$ is the identity. As in the previous subsection, we apply these operators to vector-valued functions in a component-wise manner and indicate this by using boldface notations, *e.g.* $\boldsymbol{\pi}_h : \mathbf{L}^2(\Omega) \to \mathbf{D}_h$ and $\boldsymbol{\kappa}_h : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$.

Assumption A2: Let the fluctuation operator κ_h satisfy the following approximation property:

$$\|\kappa_h q\|_{0,M} \le C\, h_M^l |q|_{l,M} \qquad \forall q \in H^l(M),\ \forall M \in \mathcal{M}_h,\ 0 \le l \le r. \tag{1.8}$$

Remark 1.2. We shortly discuss the case in which π_h is the L^2-projection in D_h and the space $D_h(M)$ contains the space $P_{r-1}(M)$ of polynomials of degree less than or equal to $r-1$, $r \ge 1$. This means that

$$(\pi_h w - w, w_h) = 0 \qquad \forall w_h \in D_h,\ w \in L^2(\Omega), \tag{1.9}$$

and

$$\bigoplus_{M\in\mathcal{M}_h} P_{r-1}(M) \subset D_h. \tag{1.10}$$

Since D_h is discontinuous over the macro element faces, (1.9) can be localised and $\pi_M : L^2(M) \to D_h(M)$ is locally defined by

$$(\pi_M w - w, w_h)_M = 0 \qquad \forall w_h \in D_h(M),\ w \in L^2(M). \tag{1.11}$$

In this case, the L^2-projection $\pi_M : L^2(M) \to D_h(M)$ becomes the identity on the subspace $P_{r-1}(M) \subset H^l(M)$. Now, the Bramble-Hilbert lemma gives the approximation properties for $\kappa_h = id - \pi_h$ stated in Assumption A2.

We will modify the discrete problem (1.6) by adding the stabilisation term

$$\begin{aligned} S_h\big((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)\big) := \sum_{M\in\mathcal{M}_h} \Big(&\tau_M(\boldsymbol{\kappa}_h(\mathbf{b}\cdot\nabla)\mathbf{u}_h, \boldsymbol{\kappa}_h(\mathbf{b}\cdot\nabla)\mathbf{v}_h)_M \\ &+ \mu_M(\kappa_h\nabla\cdot\mathbf{u}_h, \kappa_h\nabla\cdot\mathbf{v}_h)_M + \alpha_M(\boldsymbol{\kappa}_h\nabla p_h, \boldsymbol{\kappa}_h\nabla q_h)_M\Big), \end{aligned} \tag{1.12}$$

where τ_M, μ_M, and α_M are user-chosen constants. Their optimal mesh-dependent choice will follow from the error analysis of the method. Now, our stabilised scheme reads:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$:

$$A\big((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)\big) + S_h\big((\mathbf{u}_h, p); (\mathbf{v}_h, q_h)\big) = (\mathbf{f}, \mathbf{v}_h). \tag{1.13}$$

Existence, uniqueness, and convergence properties of the solutions $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ will be studied in the next section.

2. Convergence analysis

2.1. **Special interpolant**

The key ingredient of the error analysis of the local projection method is the construction of an interpolant $j_h : H^1(\Omega) \to Y_h$ such that the error $w - j_h w$ is L^2-orthogonal to D_h without loosing the standard approximation properties. Let $Y_h(M) := \{w_h|_M \ : \ w_h \in Y_h, \ w_h = 0 \text{ on } \Omega\backslash M\}$.

Assumption A3: Let the local inf-sup condition

$$\exists \beta_1 > 0, \quad \forall h > 0 \, \forall M \in \mathcal{M}_h : \qquad \inf_{q_h \in D_h(M)} \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M} \, \|q_h\|_{0,M}} \geq \beta_1 > 0 \tag{2.1}$$

be satisfied.

Remark 2.1. It is clear that $Y_h(M)$ – compared to $D_h(M)$ – has to be rich enough for satisfying A3. In particular, a necessary requirement is

$$\dim Y_h(M) \geq \dim D_h(M). \tag{2.2}$$

On the other hand D_h has to be large enough to guarantee A2. In Section 3, we follow this idea for a given space $Y_h(M)$ by choosing D_h as a discontinuous finite element space on a coarser mesh; its dimension small enough to satisfy A3 but big enough to fulfil A2. A different strategy is used in Section 4 where both spaces are defined on the same mesh, $D_h(M)$ such that A2 holds and $Y_h(M)$ is enriched by additional functions to fulfil A3.

Theorem 2.2. *Let Assumptions* A1 *and* A3 *be satisfied. Then, there are interpolation operators* $j_h : H^1(\Omega) \to Y_h$ *and* $\mathbf{j}_h : \mathbf{V} \to \mathbf{V}_h$ *satisfying the following orthogonality and approximation properties:*

$$(w - j_h w, q_h) = 0 \qquad \forall q_h \in D_h, \ \forall w \in H^1(\Omega), \tag{2.3}$$

$$\|w - j_h w\|_{0,M} + h_M |w - j_h w|_{1,M} \leq C\, h_M^l \|w\|_{l,\Lambda(M)} \qquad \forall w \in H^l(\Omega), \ 1 \leq l \leq r+1, \ \forall M \in \mathcal{M}_h \tag{2.4}$$

$$(\mathbf{w} - \mathbf{j}_h \mathbf{w}, \mathbf{q}_h) = 0 \qquad \forall \mathbf{q}_h \in \mathbf{D}_h, \ \forall \mathbf{w} \in \mathbf{V}, \tag{2.5}$$

$$\|\mathbf{w} - \mathbf{j}_h \mathbf{w}\|_{0,M} + h_M |\mathbf{w} - \mathbf{j}_h \mathbf{w}|_{1,M} \leq C\, h_M^l \|\mathbf{w}\|_{l,\Lambda(M)} \qquad \forall \mathbf{w} \in \mathbf{V} \cap \mathbf{H}^l(\Omega), \ 1 \leq l \leq r+1, \ \forall M \in \mathcal{M}_h. \tag{2.6}$$

Proof. We use a result in [18], Lemma I.4.1: the linear continuous operator $B_h : Y_h(M) \to D_h(M)'$ defined by

$$\langle B_h v_h, q_h \rangle_{D_h(M)} := (v_h, q_h)_M \qquad \forall v_h \in Y_h(M), \ q_h \in D_h(M)$$

is an isomorphism from $W_h(M)^\perp$ onto $D_h(M)'$ with

$$\beta_1 \|v_h\|_{0,M} \leq \|B_h v_h\|_{D_h(M)'} \qquad \forall v_h \in W_h(M)^\perp$$

if and only if A3 holds. Here, $D_h(M)'$ denotes the dual space of $D_h(M)$,

$$W_h(M) := \{v_h \in Y_h(M) \ : \ (v_h, q_h) = 0 \ \ \forall q_h \in D_h(M)\},$$

and $W_h(M)^\perp$ the L^2-orthogonal complement of $W_h(M)$ in $Y_h(M)$. Consequently, for each $w \in H^1(\Omega)$ we have a unique $z_h(w) \in W_h(M)^\perp$ such that

$$\langle B_h z_h(w), q_h \rangle_{D_h(M)} = (z_h(w), q_h)_M = (w - i_h w, q_h)_M \qquad \forall q_h \in D_h(M), \tag{2.7}$$

$$\|z_h(w)\|_{0,M} \le \frac{1}{\beta_1} \|w - i_h w\|_{0,M}. \tag{2.8}$$

We take $j_h w\big|_M := i_h w\big|_M + z_h(w)$ for all $M \in \mathcal{M}_h$. According to $\bigoplus_{M \in \mathcal{M}_h} Y_h(M) \subset Y_h$, a global interpolation $j_h : H^1(\Omega) \to Y_h$ is defined which satisfies for all $M \in \mathcal{M}_h$

$$\|w - j_h w\|_{0,M} \le \left(1 + \frac{1}{\beta_1}\right) \|w - i_h w\|_{0,M} \le C\, h_M^l \|w\|_{l,\Lambda(M)} \qquad \forall w \in H^l(\Omega),\ 1 \le l \le r+1. \tag{2.9}$$

The orthogonality property (2.3) follows from (2.7) and the definition of j_h. It remains to show the approximation property for the H^1-seminorm. To this end, we apply an inverse inequality and (2.8) to get

$$|z_h(w)|_{1,M} \le C h_M^{-1} \|z_h(w)\|_{0,M} \le C h_M^{-1} \|w - i_h w\|_{0,M}. \tag{2.10}$$

Using the triangle inequality, (2.10) and the approximation property (1.5), we conclude

$$|w - j_h w|_{1,M} \le |w - i_h w|_{1,M} + |z_h(w)|_{1,M} \le C\, h_M^{l-1} \|w\|_{l,\Lambda(M)}.$$

Now, (2.5) and (2.6) are extensions to the vector-valued case. Taking into consideration the definition $\mathbf{j}_h \mathbf{w} = \mathbf{i}_h \mathbf{w} + \mathbf{z}_h(w)$, the mapping property $\mathbf{i}_h : \mathbf{V} \to \mathbf{V}_h$, and that $\mathbf{z}_h(\mathbf{w})$ vanishes on the boundary $\partial\Omega$, we get $\mathbf{j}_h : \mathbf{V} \to \mathbf{V}_h$, (2.5), and (2.6). □

Remark 2.3. We see from (2.9) that the constant β_1 should be independent of the mesh size h to guarantee the approximation properties (2.4) and (2.6) of the interpolant j_h. Note that (2.2) does not guarantee that the constant β_1 is independent of the mesh size h.

Remark 2.4. Note that by setting $q_h = 1$ in (2.3) we get $(j_h w, 1) = (w, 1)$ for all $w \in H^1(\Omega)$. This implies in particular $j_h : H^1(\Omega) \cap Q \to Q_h$.

Remark 2.5. Following the ideas in [39] and assuming a family of macro elements which is equivalent to a reference macro element, one can show that (2.1) reduces to show that

$$N_M := \{q_h \in D_h(M) \;:\; (q_h, v_h)_M = 0 \quad \forall v_h \in V_h(M)\} = \{0\}$$

holds true.

2.2. **Stability**

Let us introduce the mesh-dependent norm on the product space $\mathbf{V} \times Q$ by

$$|||(\mathbf{v}, q)||| := \Big(\nu |\mathbf{v}|_1^2 + \sigma \|\mathbf{v}\|_0^2 + (\nu + \sigma)\|q\|_0^2 + S\big((\mathbf{v}, q); (\mathbf{v}, q)\big)\Big)^{1/2}. \tag{2.11}$$

We show that the bilinear form $(A + S_h)$ satisfies an inf-sup condition on $\mathbf{V}_h \times Q_h$.

Lemma 2.6. *Assume* A1, A3, *and* $\max(\nu,\sigma,\tau_M,\mu_M,h_M^2/\alpha_M)\le C$ *for all* $M\in\mathcal{M}_h$. *Then, there is a positive constant* β_2 *independent of* ν *and* h *such that*

$$\inf_{(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times Q_h}\ \sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h}\frac{(A+S_h)\big((\mathbf{v}_h,q_h);(\mathbf{w}_h,r_h)\big)}{|||(\mathbf{v}_h,q_h)|||\ |||(\mathbf{w}_h,r_h)|||}\ge\beta_2>0 \tag{2.12}$$

holds true.

Proof. Let us consider an arbitrary $(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times Q_h$. Choosing $(\mathbf{w}_h,r_h)=(\mathbf{v}_h,q_h)$, we have

$$(A+S_h)\big((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h)\big)=\nu|\mathbf{v}_h|_1^2+\sigma\|\mathbf{v}_h\|_0^2+S_h\big((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h)\big) \tag{2.13}$$

due to property (1.4).

Now we consider another choice to generate an L^2-norm control over the pressure. For any $q_h\in Q_h$, the continuous Babuška-Brezzi condition guarantees the existence of a function $\mathbf{v}_{q_h}\in\mathbf{V}$ such that

$$(\nabla\cdot\mathbf{v}_{q_h},q_h)=-(q_h,q_h),\qquad \|\mathbf{v}_{q_h}\|_1\le C\|q_h\|_0. \tag{2.14}$$

We choose $(\mathbf{w}_h,r_h)=(\mathbf{j}_h\mathbf{v}_{q_h},0)$ where $\mathbf{j}_h$ is the interpolant of Theorem 2.2 satisfying (2.5) and (2.6). Thus, we obtain

$$\begin{aligned}A\big((\mathbf{v}_h,q_h);(\mathbf{j}_h\mathbf{v}_{q_h},0)\big)=\|q_h\|_0^2&-(q_h,\nabla\cdot(\mathbf{j}_h\mathbf{v}_{q_h}-\mathbf{v}_{q_h}))+\big((\mathbf{b}\cdot\nabla)\mathbf{v}_h,\mathbf{j}_h\mathbf{v}_{q_h}\big)\\&+\nu(\nabla\mathbf{v}_h,\nabla\mathbf{j}_h\mathbf{v}_{q_h})+\sigma(\mathbf{v}_h,\mathbf{j}_h\mathbf{v}_{q_h}).\end{aligned} \tag{2.15}$$

We estimate the last four terms on the right hand side. Starting with an integration by parts of the first of them, we get

$$\begin{aligned}-\big(q_h,\nabla\cdot(\mathbf{j}_h\mathbf{v}_{q_h}-\mathbf{v}_{q_h})\big)&=\big(\nabla q_h,(\mathbf{j}_h\mathbf{v}_{q_h}-\mathbf{v}_{q_h})\big)=\big(\boldsymbol{\kappa}_h\nabla q_h,(\mathbf{j}_h\mathbf{v}_{q_h}-\mathbf{v}_{q_h})\big),\\ \big|\big(q_h,\nabla\cdot(\mathbf{j}_h\mathbf{v}_{q_h}-\mathbf{v}_{q_h})\big)\big|&\le\left(\sum_{M\in\mathcal{M}_h}\alpha_M\|\boldsymbol{\kappa}_h\nabla q_h\|_{0,M}^2\right)^{1/2}\left(\sum_{M\in\mathcal{M}_h}\frac{1}{\alpha_M}\|\mathbf{j}_h\mathbf{v}_{q_h}-\mathbf{v}_{q_h}\|_{0,M}^2\right)^{1/2}\\&\le C\ \big(S_h\big((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h)\big)\big)^{1/2}\|\mathbf{v}_{q_h}\|_1\le C\ \big(S_h\big((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h)\big)\big)^{1/2}\|q_h\|_0\\&\le\frac{\|q_h\|_0^2}{8}+C\,S_h\big((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h)\big).\end{aligned} \tag{2.16}$$

Integrating by parts the third term in (2.15), using the H^1 stability of j_h which follows from Theorem 2.2, and (2.14), we obtain

$$|((\mathbf{b}\cdot\nabla)\mathbf{v}_h,\mathbf{j}_h\mathbf{v}_{q_h})|=|(\mathbf{v}_h,(\mathbf{b}\cdot\nabla)\mathbf{j}_h\mathbf{v}_{q_h})|\le C\,\|\mathbf{v}_h\|_0\,|\mathbf{j}_h\mathbf{v}_{q_h}|_1\le\frac{\|q_h\|_0^2}{8}+C\,\|\mathbf{v}_h\|_0^2. \tag{2.17}$$

For estimating the remaining terms in (2.15), we use $\max(\nu,\sigma)\le C$ to get

$$\begin{aligned}\big|\nu(\nabla\mathbf{v}_h,\nabla\mathbf{j}_h\mathbf{v}_{q_h})+\sigma(\mathbf{v}_h,\mathbf{j}_h\mathbf{v}_{q_h})\big|&\le\big(\nu|\mathbf{v}_h|_1+\sigma\|\mathbf{v}_h\|_0\big)\|\mathbf{j}_h\mathbf{v}_{q_h}\|_1\le C\big(\nu^{1/2}|\mathbf{v}_h|_1+\sigma^{1/2}\|\mathbf{v}_h\|_0\big)\|q_h\|_0\\&\le\frac{\|q_h\|_0^2}{8}+C\big(\nu|\mathbf{v}_h|_1^2+\sigma\|\mathbf{v}_h\|_0^2\big).\end{aligned} \tag{2.18}$$

The Cauchy-Schwarz inequality and the L^2-stability of $\boldsymbol{\kappa}_h$ give

$$\begin{aligned}\left|S_h\big((\mathbf{v}_h, q_h); (\mathbf{j}_h \mathbf{v}_{q_h}, 0)\big)\right| &\leq C \left(S_h\big((\mathbf{v}_h, 0); (\mathbf{v}_h, 0)\big)\right)^{1/2} |\mathbf{j}_h \mathbf{v}_{q_h}|_1 \leq C \left(S_h\big((\mathbf{v}_h, q_h); (\mathbf{v}_h, q_h)\big)\right)^{1/2} \|q_h\|_0 \\ &\leq \frac{\|q_h\|_0^2}{8} + C\, S_h\big((\mathbf{v}_h, q_h); (\mathbf{v}_h, q_h)\big). \end{aligned} \tag{2.19}$$

Let

$$X := \left(\nu|\mathbf{v}_h|_1^2 + \sigma\|\mathbf{v}_h\|_0^2 + S_h\big((\mathbf{v}_h, q_h); (\mathbf{v}_h, q_h)\big)\right)^{1/2}$$

denote the part of the triple norm without L^2-control over the pressure. Using (2.16)–(2.19), we get from (2.15)

$$(A + S_h)\big((\mathbf{v}_h, q_h); (\mathbf{j}_h \mathbf{v}_{q_h}, 0)\big) \geq \frac{\|q_h\|_0^2}{2} - C\, X^2 - C\, \|\mathbf{v}_h\|_0^2. \tag{2.20}$$

Now, we multiply (2.20) by $2(\nu + \sigma)$ and use the Poincaré inequality to estimate

$$2(\nu + \sigma)\|\mathbf{v}_h\|_0^2 \leq C\big(\nu|\mathbf{v}_h|_1^2 + \sigma\|\mathbf{v}_h\|_0^2\big).$$

Hence, we obtain

$$(A + S_h)\big((\mathbf{v}_h, q_h); 2(\nu + \sigma)(\mathbf{j}_h \mathbf{v}_{q_h}, 0)\big) \geq (\nu + \sigma)\|q_h\|_0^2 - C_1\, X^2 \tag{2.21}$$

with a suitable constant C_1. We define for an arbitrary $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$

$$(\mathbf{w}_h, r_h) := (\mathbf{v}_h, q_h) + \frac{2(\nu + \sigma)}{1 + C_1}(\mathbf{j}_h \mathbf{v}_{q_h}, 0) \in \mathbf{V}_h \times Q_h.$$

Then, we have

$$(A + S_h)\big((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h)\big) \geq \frac{(\nu + \sigma)}{1 + C_1}\|q_h\|_0^2 + \left(1 - \frac{C_1}{1 + C_1}\right) X^2 = \frac{1}{1 + C_1}|||(\mathbf{v}_h, q_h)|||^2 \tag{2.22}$$

and

$$\begin{aligned}|||(\mathbf{w}_h, r_h)||| &\leq |||(\mathbf{v}_h, q_h)||| + \frac{2(\nu + \sigma)}{1 + C_1}|||(\mathbf{j}_h \mathbf{v}_{q_h}, 0)||| \leq |||(\mathbf{v}_h, q_h)||| + C(\nu + \sigma)\|\mathbf{j}_h \mathbf{v}_{q_h}\|_1 \\ &\leq |||(\mathbf{v}_h, q_h)||| + C(\nu + \sigma)\|q_h\|_0 \leq C_2|||(\mathbf{v}_h, q_h)|||. \end{aligned} \tag{2.23}$$

From (2.22) and (2.23) we conclude (2.12) with $\beta_2 = 1/(C_2(1 + C_1))$. □

Remark 2.7. Note that for $\sigma > 0$ we have control over the L^2-norm of pressure and velocity *uniformly with respect to* $\nu > 0$. However, in the case $\sigma = 0$ we lose this control for $\nu \to 0$ due to the presence of the convection term (*cf.* (2.17)). If we consider the Stokes problem (*i.e.* $b = 0$ and $\sigma = 0$), then a careful investigation shows that we still have control over the L^2-norm of the pressure with a constant independent of ν and h.

Remark 2.8. The unique solvability of the stabilised discrete problem (1.13) follows directly from Lemma 2.6.

2.3. Approximated Galerkin orthogonality

In contrast to residual-based stabilisation schemes [11], we do not have the Galerkin orthogonality. Therefore, we investigate in this subsection the consistency error.

Lemma 2.9. *Let* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the solution of* (1.3) *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the solution of* (1.13), *respectively. Then,*

$$A((\mathbf{u} - \mathbf{u}_h, p - p_h); (\mathbf{v}_h, q_h)) = S_h((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) \qquad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h. \tag{2.24}$$

Proof. We get (2.24) simply by subtracting (1.13) from (1.3). □

For estimating the consistency error, we suppose that $\mathbf{b}$ is sufficiently smooth in the sense

$$\mathbf{b}\big|_M \in \mathbf{W}^{r,\infty}(M) \quad \forall M \in \mathcal{M}_h, \qquad \max_{M \in \mathcal{M}_h} \|\mathbf{b}\|_{r,\infty,M} \le C. \tag{2.25}$$

Lemma 2.10. *Let the fluctuation operator* κ_h *satisfy* A2 *and* $\mathbf{b}$ *fulfils* (2.25). *Then, for* $(\mathbf{u}, p) \in \mathbf{H}^{r+1}(\Omega) \times H^{r+1}(\Omega)$ *we have*

$$\big|S_h\big((\mathbf{u}, p); (\mathbf{v}_h, q_h)\big)\big| \le C \Bigg(\sum_{M \in \mathcal{M}_h} h_M^{2r} \Big[\big(\tau_M \|\mathbf{b}\|^2_{r,\infty,M} + \mu_M\big) \|\mathbf{u}\|^2_{r+1,M} + \alpha_M \|p\|^2_{r+1,M} \Big] \Bigg)^{1/2} |||(\mathbf{v}_h, q_h)||| \tag{2.26}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$.

Proof. From the definition of the stabilising term we get

$$|S_h((\mathbf{u}, p); (\mathbf{v}_h, q_h))| \le \Big(S_h((\mathbf{u}, p); (\mathbf{u}, p))\Big)^{1/2} \Big(S_h((\mathbf{v}_h, q_h); (\mathbf{v}_h, q_h))\Big)^{1/2} \le \Big(S_h((\mathbf{u}, p); (\mathbf{u}, p))\Big)^{1/2} |||(\mathbf{v}_h, q_h)|||.$$

Using the approximation properties of $\boldsymbol{\kappa}_h$, we see that

$$S_h\big((\mathbf{u}, p); (\mathbf{u}, p)\big) \le C \sum_{M \in \mathcal{M}_h} h_M^{2r} \big(\tau_M |(\mathbf{b} \cdot \nabla)\mathbf{u}|^2_{r,M} + \mu_M |\nabla \cdot \mathbf{u}|^2_{r,M} + \alpha_M |\nabla p|^2_{r,M} \big)$$

and (2.26) follows. □

Remark 2.11. The assumption $\mathbf{b}\big|_M \in \mathbf{W}^{r,\infty}(M)$ is rather restrictive in the framework of the Navier-Stokes model since $\mathbf{b}$ corresponds to a finite element function which is in general non-smooth across element borders. However, in the case $\mathcal{M}_h = \mathcal{T}_h$ the macro cells are element cells and this assumption should not be a problem. Another way to relax the smoothness assumption on $\mathbf{b}$ is the use of a modified stabilisation term, see Corollary 2.14.

2.4. *A-priori* error estimate

We get from stability and consistency an *a-priori* error estimate in the usual way. The important aspect is that the constant in the error bound will be independent of the viscosity ν and h.

Theorem 2.12. *Assume* A1–A3. *Let* $(\mathbf{u}, p) \in \big(\mathbf{H}^1_0(\Omega) \cap \mathbf{H}^{r+1}(\Omega)\big) \times \big(L^2_0(\Omega) \cap H^{r+1}(\Omega)\big)$ *be the solution of* (1.3) *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the solution of the local projection method* (1.13). *Then, there is a positive constant* C *independent of* ν *and* h *such that*

$$\begin{aligned} |||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \le C \Bigg(\sum_{M \in \mathcal{M}_h} h_M^{2r} \Big[&\nu + h_M^2\, \sigma + h_M^2\, \tau_M^{-1} + \tau_M \|\mathbf{b}\|^2_{r,\infty,M} + h_M^2\, \mu_M^{-1} + \mu_M \\ &+ h_M^2\, \alpha_M^{-1} + \alpha_M \Big] \Big(\|\mathbf{u}\|^2_{r+1,\Lambda(M)} + \|p\|^2_{r+1,\Lambda(M)} \Big) \Bigg)^{1/2} \end{aligned} \tag{2.27}$$

holds true. The choice $\tau_M \sim h_M/\|\mathbf{b}\|_{r,\infty,M}$, $\mu_M \sim h_M$, and $\alpha_M \sim h_M$ is asymptotically optimal and leads to

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \le C\Bigg(\sum_{M\in\mathcal{M}_h}(\nu+h_M)\,h_M^{2r}\Big(\|\mathbf{u}\|^2_{r+1,\Lambda(M)}+\|p\|^2_{r+1,\Lambda(M)}\Big)\Bigg)^{1/2}. \tag{2.28}$$

Proof. Starting with Lemma 2.6, we get an estimate for the error to the interpolants:

$$\begin{aligned}|||(\mathbf{j}_h\mathbf{u}-\mathbf{u}_h, j_hp-p_h)||| &\le \frac{1}{\beta_2}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h}\frac{(A+S_h)\big((\mathbf{j}_h\mathbf{u}-\mathbf{u}_h, j_hp-p_h);(\mathbf{w}_h,r_h)\big)}{|||(\mathbf{w}_h,r_h)|||}\\ &\le \frac{1}{\beta_2}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h}\frac{(A+S_h)\big((\mathbf{u}-\mathbf{u}_h, p-p_h);(\mathbf{w}_h,r_h)\big)}{|||(\mathbf{w}_h,r_h)|||}\\ &\quad+\frac{1}{\beta_2}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h}\frac{(A+S_h)\big((\mathbf{j}_h\mathbf{u}-\mathbf{u}, j_hp-p);(\mathbf{w}_h,r_h)\big)}{|||(\mathbf{w}_h,r_h)|||}.\end{aligned}$$

Using Lemmata 2.9 and 2.10, we estimate the first term by

$$\begin{aligned}&(A+S_h)\big((\mathbf{u}-\mathbf{u}_h,p-p_h);(\mathbf{w}_h,r_h)\big) = S_h\big((\mathbf{u},p);(\mathbf{w}_h,r_h)\big)\\ &\qquad\le C\Bigg(\sum_{M\in\mathcal{M}_h}h_M^{2r}\Big[\big(\tau_M\|\mathbf{b}\|^2_{r,\infty,M}+\mu_M\big)\|\mathbf{u}\|^2_{r+1,M}+\alpha_M\|p\|^2_{r+1,M}\Big]\Bigg)^{1/2}|||(\mathbf{w}_h,r_h)|||.\end{aligned}$$

For the estimation of the second term, we consider each individual term in $(A+S_h)\big((\mathbf{j}_h\mathbf{u}-\mathbf{u}, j_hp-p);(\mathbf{w}_h,r_h)\big)$ separately. The estimation of

$$\nu(\nabla(\mathbf{j}_h\mathbf{u}-\mathbf{u}),\nabla\mathbf{w}_h)+\sigma(\mathbf{j}_h\mathbf{u}-\mathbf{u},\mathbf{w}_h) \le C\Bigg(\sum_{M\in\mathcal{M}_h}h_M^{2r}\big(\nu+\sigma\,h_M^2\big)\|\mathbf{u}\|^2_{r+1,\Lambda(M)}\Bigg)^{1/2}|||(\mathbf{w}_h,r_h)|||$$

is standard. When estimating the next three terms, we use the interpolant constructed in Theorem 2.2. Integrating by parts, we get

$$\begin{aligned}\Big|\Big((\mathbf{b}\cdot\nabla)(\mathbf{j}_h\mathbf{u}-\mathbf{u}),\mathbf{w}_h\Big)\Big| &= \big|(\mathbf{j}_h\mathbf{u}-\mathbf{u},(\mathbf{b}\cdot\nabla)\mathbf{w}_h)\big| = \big|(\mathbf{j}_h\mathbf{u}-\mathbf{u},\boldsymbol{\kappa}_h(\mathbf{b}\cdot\nabla)\mathbf{w}_h)\big|\\ &\le C\Bigg(\sum_{M\in\mathcal{M}_h}h_M^{2r+2}\,\tau_M^{-1}\,\|\mathbf{u}\|^2_{r+1,\Lambda(M)}\Bigg)^{1/2}\Big(S_h\big((\mathbf{w}_h,0);(\mathbf{w}_h,0)\big)\Big)^{1/2}, \qquad (2.29)\\ \big|(p-j_hp,\nabla\cdot\mathbf{w}_h)\big| &= \big|(p-j_hp,\kappa_h\nabla\cdot\mathbf{w}_h)\big|\\ &\le C\Bigg(\sum_{M\in\mathcal{M}_h}h_M^{2r+2}\,\mu_M^{-1}\,\|p\|^2_{r+1,\Lambda(M)}\Bigg)^{1/2}\Big(S_h\big((\mathbf{w}_h,0);(\mathbf{w}_h,0)\big)\Big)^{1/2}, \qquad (2.30)\\ \big|(r_h,\nabla\cdot(\mathbf{j}_h\mathbf{u}-\mathbf{u}))\big| &= \big|(\nabla r_h,\mathbf{j}_h\mathbf{u}-\mathbf{u})\big| = \big|(\boldsymbol{\kappa}_h\nabla r_h,\mathbf{j}_h\mathbf{u}-\mathbf{u})\big|\\ &\le C\Bigg(\sum_{M\in\mathcal{M}_h}h_M^{2r+2}\,\alpha_M^{-1}\,\|\mathbf{u}\|^2_{r+1,\Lambda(M)}\Bigg)^{1/2}\Big(S_h\big((0,r_h);(0,r_h)\big)\Big)^{1/2}. \qquad (2.31)\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\big|S_h\big((\mathbf{j}_h\mathbf{u}-\mathbf{u}, j_hp-p);(\mathbf{w}_h,r_h)\big)\big| \\
&\qquad\leq \Big(S_h\big((\mathbf{j}_h\mathbf{u}-\mathbf{u}, j_hp-p);(\mathbf{j}_h\mathbf{u}-\mathbf{u}, j_hp-p)\big)\Big)^{1/2}\Big(S_h\big((\mathbf{w}_h,r_h);(\mathbf{w}_h,r_h)\big)\Big)^{1/2} \\
&\qquad\leq C\Bigg(\sum_{M\in\mathcal{M}_h} h_M^{2r}\Big[\big(\tau_M\,\|\mathbf{b}\|_{0,\infty,M}^2+\mu_M\big)\,\|\mathbf{u}\|_{r+1,\Lambda(M)}^2+\alpha_M\,\|p\|_{r+1,\Lambda(M)}^2\Big]\Bigg)^{1/2}\,|||(\mathbf{w}_h,r_h)|||.
\end{aligned}$$

Collecting all estimates above, we have shown

$$\begin{aligned}
|||(\mathbf{j}_h\mathbf{u}-\mathbf{u}_h, j_hp-p_h)||| \leq C\Bigg(\sum_{M\in\mathcal{M}_h} h_M^{2r}\Big[\nu+h_M^2\,\sigma+h_M^2\,\tau_M^{-1}+\tau_M\|\mathbf{b}\|_{r,\infty,M}^2+h_M^2\,\mu_M^{-1}+\mu_M \\
+h_M^2\,\alpha_M^{-1}+\alpha_M\Big]\Big(\|\mathbf{u}\|_{r+1,\Lambda(M)}^2+\|p\|_{r+1,\Lambda(M)}^2\Big)\Bigg)^{1/2}. \qquad (2.32)
\end{aligned}$$

By using the triangle inequality

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq |||(\mathbf{u}-\mathbf{j}_h\mathbf{u}, p-j_hp)|||+|||(\mathbf{j}_h\mathbf{u}-\mathbf{u}_h, j_hp-p_h)|||$$

and the approximation property

$$\begin{aligned}
&|||(\mathbf{u}-\mathbf{j}_h\mathbf{u}, p-j_hp)||| \\
&\quad\leq C\Bigg(\sum_{M\in\mathcal{M}_h} h_K^{2r}\Big[\nu+h_M^2\,\sigma+(\nu+\sigma)\,h_M^2+\tau_M\,\|\mathbf{b}\|_{0,\infty,M}^2+\mu_M+\alpha_M\Big]\Big(\|\mathbf{u}\|_{r+1,\Lambda(M)}^2+\|p\|_{r+1,\Lambda(M)}^2\Big)\Bigg)^{1/2},
\end{aligned}$$

we get

$$\begin{aligned}
|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq C\Bigg(\sum_{M\in\mathcal{M}_h} h_M^{2r}\Big[\nu+h_M^2\,\sigma+h_M^2\,\tau_M^{-1}+\tau_M\|\mathbf{b}\|_{r,\infty,M}^2 \\
+h_M^2\,\mu_M^{-1}+\mu_M+h_M^2\,\alpha_M^{-1}+\alpha_M\Big]\Big(\|\mathbf{u}\|_{r+1,\Lambda(M)}^2+\|p\|_{r+1,\Lambda(M)}^2\Big)\Bigg)^{1/2}
\end{aligned}$$

which proves (2.27). Minimizing the upper bound results in the choice $\tau_M \sim h_M/\|\mathbf{b}\|_{r,\infty,M}$, $\mu_M \sim h_M$, and $\alpha_M \sim h_M$, which implies (2.28). □

Remark 2.13. In comparison to the SUPG/PSPG method, we obtain with respect to the norm

$$(v,q) \mapsto \Big(\nu|v|_1^2+\sigma^{1/2}\|v\|_0^2+(\nu+\sigma)\|q\|_0^2\Big)^{1/2}$$

the same rate of convergence for equal-order interpolation [37, 41]. Moreover, the SUPG/PSPG method gives additional control over

$$\Bigg(\sum_{K\in\mathcal{T}_h}\big(\tau_K\|(\mathbf{b}\cdot\nabla)\mathbf{v}+\nabla q\|_{0,K}^2+\mu_K\|\nabla\cdot\mathbf{v}\|_{0,K}^2\big)\Bigg)^{1/2}.$$

Recently, it has been shown [35] that the SUPG/PSPG method also allows a separate control over the terms $\|(\mathbf{b}\cdot\nabla)\mathbf{v}\|_{0,K}$ and $\|\nabla q\|_{0,K}$ if $\sigma > 0$ and the parameters τ_K are chosen appropriately. This behaviour of the SUPG/PSPG method is related to the behaviour of the local projection method where an additional control is only guaranteed over fluctuations of these quantities, *i.e.* with respect to a slightly weaker norm. Moreover, it is well-known that too much stabilisation can lead to smearing out boundary layers. Thus, the right amount of stabilisation seems to be important. Later we will see that the amount of stabilisation in the local projection method can be controlled by choosing the spaces Y_h and D_h, respectively. This flexibility makes the local projection method very attractive.

Finally, we discuss two slightly modified approaches resulting in the same error estimates as those given in Theorem 2.12. The first modification consists in replacing the stabilising term S_h from (1.12) by

$$S_h^1((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h)) := \sum_{M\in\mathcal{M}_h}\Big(\tau_M(\boldsymbol{\kappa}_h(\nabla\mathbf{u}_h),\boldsymbol{\kappa}_h(\nabla\mathbf{v}_h))_M + \alpha_M(\boldsymbol{\kappa}_h\nabla p_h,\boldsymbol{\kappa}_h\nabla q_h)_M\Big) \tag{2.33}$$

which gives control over the fluctuations of the gradients of the velocities instead of separate control over the fluctuations of the derivatives in the streamline direction and the divergence, respectively. In the second modification, we replace the stabilising term S_h from (1.12) by a term S_h^2 which is spectral equivalent, *i.e.*, there are positive constants C_3, C_4, independent of ν and h, such that

$$C_3\, S_h\big((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h)\big) \le S_h^2\big((\mathbf{v}_h,q);(\mathbf{v}_h,q_h)\big) \le C_4\, S_h\big((\mathbf{v}_h,q_h);(\mathbf{v}_h,q_h)\big) \quad \forall(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times Q_h. \tag{2.34}$$

Note that the choice of the parameters τ_M, μ_M, and α_M defining S_h influences the selection of possible stabilising terms S_h^2 satisfying (2.34). When replacing S_h by S_h^i in (2.11), $i=1,2$, two new mesh-dependent norms appear which will be denoted by $|||(\cdot,\cdot)|||_i$, $i=1,2$.

Corollary 2.14. *Assume* A1–A3. *Let* $(\mathbf{u},p)\in\big(\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^{r+1}(\Omega)\big)\times\big(L_0^2(\Omega)\cap H^{r+1}(\Omega)\big)$ *be the weak solution of* (1.3) *and* $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ *be the solution of the local projection method* (1.13) *with* S_h *replaced by* S_h^1. *Then, for* $\sigma>0$ *there is a positive constant* C *independent of* ν *such that*

$$|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_1 \le C\Bigg(\sum_{M\in\mathcal{M}_h} h_M^{2r}\Big[\nu + h_M^2\,(\sigma+\sigma^{-1}|b|_{1,\infty,M}^2) + h_M^2\,\alpha_M^{-1} + \alpha_M + h_M^2\,\tau_M^{-1}\,(1+\|\mathbf{b}\|_{0,\infty,M}^2) + \tau_M\Big]\Big(\|\mathbf{u}\|_{r+1,\Lambda(M)}^2 + \|p\|_{r+1,\Lambda(M)}^2\Big)\Bigg)^{1/2}$$

holds true. The choice $\tau_M\sim h_M\sqrt{1+\|\mathbf{b}\|_{0,\infty,M}^2}$ *and* $\alpha_M\sim h_M$ *is asymptotically optimal and leads to*

$$|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_1 \le C_\sigma\Bigg(\sum_{M\in\mathcal{M}_h}(\nu+h_M)\,h_M^{2r}\,\Big(\|\mathbf{u}\|_{r+1,\Lambda(M)}^2 + \|p\|_{r+1,\Lambda(M)}^2\Big)\Bigg)^{1/2} \tag{2.35}$$

with a constant C_σ *independent of* ν *but depending on* σ.

Proof. A careful check shows that Lemma 2.6 with S_h and $|||(\cdot,\cdot)|||$ replaced by S_h^1 and $|||(\cdot,\cdot)|||_1$, respectively, is valid. Further, the additional smoothness assumption concerning $\mathbf{b}$ in Lemma 2.10 can be omitted since now the approximation properties of the fluctuation already give

$$S_h^1\big((\mathbf{u},p);(\mathbf{u},p)\big) \le C\sum_{M\in\mathcal{M}_h} h_M^{2r}\Big(\tau_M\,|\nabla\mathbf{u}|_{r,M}^2 + \alpha_M\,|\nabla p|_{r,M}^2\Big).$$

However, the estimates (2.29) and (2.30) in the proof of Theorem 2.12 have to be modified. Consider first (2.30):

$$\begin{aligned}\big|(p - j_h p, \nabla\cdot\mathbf{w}_h)\big| = \big|(p - j_h p, \kappa_h\nabla\cdot\mathbf{w}_h)\big| &\le C \sum_{M\in\mathcal{M}_h} h_M^{r+1}\,\tau_M^{-1/2}\|p\|_{r+1,\Lambda(M)}\;\tau_M^{1/2}\|\kappa_h\nabla\cdot\mathbf{w}_h\|_{0,M}\\ &\le C\Bigg(\sum_{M\in\mathcal{M}_h} h_M^{2r+2}\,\tau_M^{-1}\,\|p\|^2_{r+1,\Lambda(M)}\Bigg)^{1/2}\Big(S_h^1((\mathbf{w}_h,0);(\mathbf{w}_h,0))\Big)^{1/2}.\end{aligned}$$

The estimation of (2.29) needs more care. We start as in the proof of Theorem 2.12

$$\begin{aligned}\big|\big((\mathbf{b}\cdot\nabla)(\mathbf{j}_h\mathbf{u}-\mathbf{u}),\mathbf{w}_h\big)\big| = \big|\big(\mathbf{j}_h\mathbf{u}-\mathbf{u},(\mathbf{b}\cdot\nabla)\mathbf{w}_h\big)\big| &= \big|\big(\mathbf{j}_h\mathbf{u}-\mathbf{u},\boldsymbol{\kappa}_h(\mathbf{b}\cdot\nabla)\mathbf{w}_h\big)\big|\\ &\le C\sum_{M\in\mathcal{M}_h} h_M^{r+1}\|\mathbf{u}\|_{r+1,\Lambda(M)}\,\big\|\boldsymbol{\kappa}_h(\mathbf{b}\cdot\nabla)\mathbf{w}_h\big\|_{0,M}.\end{aligned}$$

Let $\overline{\mathbf{b}}$ be the L^2-projection of $\mathbf{b}$ in the space of piecewise constant functions with respect to the macro decomposition $\mathcal{M}_h$. Using the L^2-stability of $\boldsymbol{\kappa}_h$, an inverse inequality, and $\boldsymbol{\kappa}_h(\overline{\mathbf{b}}\cdot\nabla)\mathbf{w}_h = \overline{\mathbf{b}}\cdot\boldsymbol{\kappa}_h(\nabla\mathbf{w}_h)$, we get

$$\begin{aligned}\big\|\boldsymbol{\kappa}_h(\mathbf{b}\cdot\nabla)\mathbf{w}_h\big\|_{0,M} &\le \big\|\boldsymbol{\kappa}_h((\mathbf{b}-\overline{\mathbf{b}})\cdot\nabla)\mathbf{w}_h\big\|_{0,M} + \big\|\boldsymbol{\kappa}_h(\overline{\mathbf{b}}\cdot\nabla)\mathbf{w}_h\big\|_{0,M}\\ &\le C\,h_M|\mathbf{b}|_{1,\infty,M}\|\nabla\mathbf{w}_h\|_{0,M} + \|\mathbf{b}\|_{0,\infty,M}\,\big\|\boldsymbol{\kappa}_h(\nabla\mathbf{w}_h)\big\|_{0,M}\\ &\le C|\mathbf{b}|_{1,\infty,M}\|\mathbf{w}_h\|_{0,M} + \|\mathbf{b}\|_{0,\infty,M}\,\big\|\boldsymbol{\kappa}_h(\nabla\mathbf{w}_h)\big\|_{0,M}.\end{aligned}$$

Since $\sigma > 0$, we end up with

$$\begin{aligned}&\big|\big((\mathbf{b}\cdot\nabla)(\mathbf{j}_h\mathbf{u}-\mathbf{u}),\mathbf{w}_h\big)\big| \le C\sum_{M\in\mathcal{M}_h} h_M^{r+1}\|\mathbf{u}\|_{r+1,\Lambda(M)}\,\Big(|\mathbf{b}|_{1,\infty,M}\|\mathbf{w}_h\|_{0,M} + \|\mathbf{b}\|_{0,\infty,M}\|\boldsymbol{\kappa}_h(\nabla\mathbf{w}_h)\|_{0,M}\Big)\\ &\le C\Bigg(\sum_{M\in\mathcal{M}_h} h_M^{2r}\Big[h_M^2\sigma^{-1}\,|\mathbf{b}|^2_{1,\infty,M} + h_M^2\,\tau_M^{-1}\,\|\mathbf{b}\|^2_{0,\infty,M}\Big]\|\mathbf{u}\|^2_{r+1,\Lambda(M)}\Bigg)^{1/2}\Big(\sigma\|\mathbf{w}_h\|_0^2 + S_h^1\big((\mathbf{w}_h,0);(\mathbf{w}_h,0)\big)\Big)^{1/2}.\end{aligned}$$

The remaining terms can be estimated as in the proof of Theorem 2.12. Finally, we obtain

$$\begin{aligned}|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_1 \le C\Bigg(\sum_{M\in\mathcal{M}_h} h_M^{2r}\Big[\nu + h_M^2\,(\sigma+\sigma^{-1}\,|\mathbf{b}|^2_{1,\infty,M}) + h_M^2\,\alpha_M^{-1} + \alpha_M\\ + h_M^2\,\tau_M^{-1}\,(1+\|\mathbf{b}\|^2_{0,\infty,M}) + \tau_M\Big]\Big(\|\mathbf{u}\|^2_{r+1,\Lambda(M)} + \|p\|^2_{r+1,\Lambda(M)}\Big)\Bigg)^{1/2}\end{aligned}$$

which is the first statement of the corollary. Minimizing the upper bound gives $\tau_M \sim h_M\sqrt{1+\|\mathbf{b}\|^2_{0,\infty,M}}$ and $\alpha_M \sim h_M$, which implies (2.35).

□

We come back to the second modification which replaces S_h by a spectrally equivalent stabilisation term S_h^2. We assume (2.34), the consistency estimate

$$\big|S_h^2\big((\mathbf{u},p);(\mathbf{v}_h,q_h)\big)\big| \le C\,h^{r+1/2}\big(\|\mathbf{u}\|_{r+1}+\|p\|_{r+1}\big)\;|||(\mathbf{v}_h,q_h)|||\quad \forall(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times Q_h \tag{2.36}$$

and the approximation property

$$\left|S_h^2\big((\mathbf{j}_h\mathbf{u}-\mathbf{u}, j_hp-p);(\mathbf{v}_h,q_h)\big)\right| \le C\,h^{r+1/2}\big(\|\mathbf{u}\|_{r+1}+\|p\|_{r+1}\big)\,|||(\mathbf{v}_h,q_h)||| \quad \forall(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times Q_h \tag{2.37}$$

to be satisfied.

Corollary 2.15. *Assume* A1, A3, *and* $\tau_M,\mu_M,\alpha_M\sim h_M$. *Let* $(\mathbf{u},p)\in\big(\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^{r+1}(\Omega)\big)\times\big(L_0^2(\Omega)\cap H^{r+1}(\Omega)\big)$ *be the solution of* (1.3) *and* $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ *be the solution of the local projection method* (1.13) *with* S_h *replaced by* S_h^2 *satisfying* (2.34), (2.36), *and* (2.37). *Then, there is a positive constant* C *independent of* ν *and* h *such that*

$$|||(\mathbf{u}-\mathbf{u}_h,p-p_h)|||_2 \le C(\nu^{1/2}+h^{1/2})\,h^r\,\big(\|\mathbf{u}\|_{r+1}+\|p\|_{r+1}\big) \tag{2.38}$$

holds true.

Proof. A careful check shows that Lemma 2.6 with S_h and $|||(\cdot,\cdot)|||$ replaced by S_h^2 and $|||(\cdot,\cdot)|||_2$, respectively, is valid. Lemma 2.10 is replaced by (2.36). Now following the lines of proof of Theorem 2.12 and bounding S_h by $C_3^{-1}S_h^2$, we get

$$|||(\mathbf{j}_h\mathbf{u}-\mathbf{u}_h, j_hp-p_h)|||_2 \le C(\nu^{1/2}+h^{1/2})\,h^r\,\big(\|\mathbf{u}\|_{r+1}+\|p\|_{r+1}\big).$$

The statement follows from the triangle inequality. □

3. Schemes based on local projection onto coarser meshes

The triangulation $\mathcal{T}_h$ consists of generic cells K whereas the macro mesh $\mathcal{M}_h$ consists of macro cells M. The partition $\mathcal{T}_h$ is formed by a suitable refinement of the macro mesh $\mathcal{M}_h$ which will be indicated by the notation $\mathcal{M}_h=\mathcal{T}_{2h}$.

3.1. Simplices

Let $\widehat{M}$ be the unit d-simplex with the vertices $\hat{a}_i$, $i=1,\dots,d+1$, and the barycenter $\hat{a}_0$. The refinement of $\widehat{M}$ is done in the following way. Each child $\widehat{K}_i$ is given by the vertices $\hat{a}_0$ and $\hat{a}_j$, $j\neq i$, see left picture in Figure 1 for the 2d case. Let $F_M:\widehat{M}\to M$ denote the affine mapping from the reference macro $\widehat{M}$ onto the macro cell $M\in\mathcal{M}_h$. This mapping defines cells $K\in\mathcal{T}_h$ by setting $K=F_M(\widehat{K}_i)$, $i=1,\dots,d+1$, $M\in\mathcal{M}_h$, see right picture in Figure 1 for the 2d case. For a function $v:M\to\mathbb{R}$, we define $\hat{v}:=v\circ F_M:\widehat{M}\to\mathbb{R}$. Furthermore, we consider the affine mapping $F_K:\widehat{K}\to K$ from a reference d-simplex $\widehat{K}$ onto an arbitrary cell $K\in\mathcal{T}_h$.

We choose for the approximation of velocity and pressure the finite element space of continuous, piecewise polynomials of degree $r\in\mathbb{N}$. Let the projection space consist of discontinuous, piecewise polynomials of degree $r-1$ on $\mathcal{T}_{2h}$, *i.e.* shortly

$$Y_h/D_h = P_{r,h}/P^{\rm disc}_{r-1,2h}$$

where

$$P_{r,h} := \{v\in H^1(\Omega)\,:\,v|_K\circ F_K\in P_r(\widehat{K})\quad\forall K\in\mathcal{T}_h\},$$
$$P^{\rm disc}_{r-1,2h} := \{v\in L^2(\Omega)\,:\,v|_M\circ F_M\in P_{r-1}(\widehat{M})\quad\forall M\in\mathcal{T}_{2h}\}.$$

We define the auxiliary space $\widehat{Y}(\widehat{M})$ as a counterpart of $Y_h(M)$ by

$$\widehat{Y}(\widehat{M}) := \{w\in H_0^1(\widehat{M}):\ w|_{\widehat{K}_i}\in P_r(\widehat{K}_i),\quad i=1,\dots,d+1\},$$

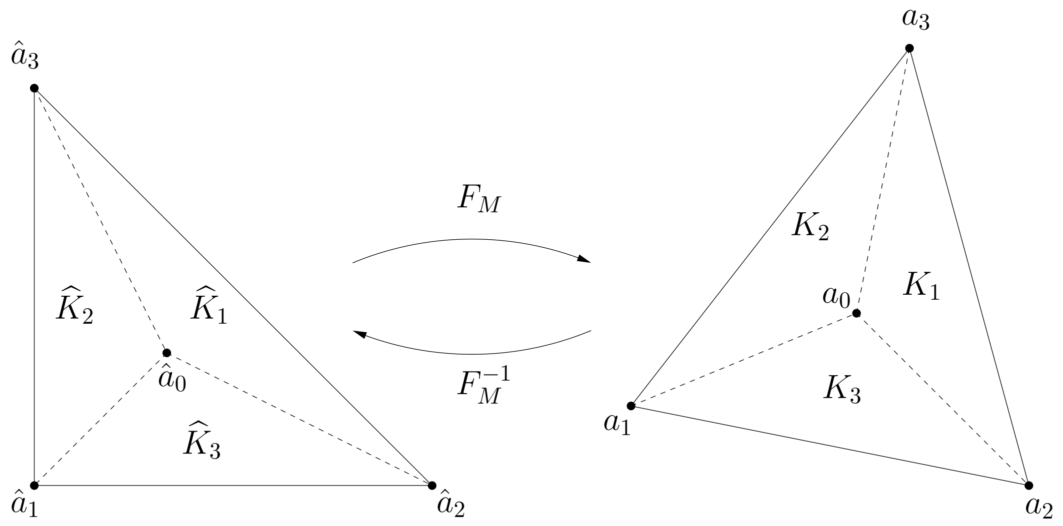

FIGURE 1. Reference macro triangle (left) and macro triangle $M \in \mathcal{M}_h$ (right).

i.e., on the reference d-simplex $\widehat{M}$ we have

$$\widehat{Y}(\widehat{M}) = \{\hat{v} \,:\, \hat{v} \circ F_M^{-1} \in Y_h(M)\}$$

where $\widehat{K}_i$ are children of $\widehat{M}$ resulting from the decomposition of $\widehat{M}$. The existence of the interpolation operators satisfying the Assumption A1 is obvious. Further, we can choose the local projection π_M in such a way that $\pi_M = id$ on the subspace P_{r-1}. This guarantees Assumption A2 too.

Lemma 3.1. *Let the local projection scheme be defined for the pair $Y_h/D_h = P_{r,h}/P^{\text{disc}}_{r-1,2h}$ with an arbitrary but fixed polynomial degree $r \in \mathbb{N}$. Then, on shape regular simplicial meshes the local inf-sup condition* A3 *holds with a constant β_1 independent of h.*

Proof. We make use of the reference transformation $F_M : \widehat{M} \to M$ and observe that the local inf-sup condition A3 can be in virtue of the constant $|\det DF_M|$ investigated merely on the reference macro

$$\inf_{q \in D_h(M)} \sup_{v \in Y_h(M)} \frac{(q,v)_M}{\|q\|_{0,M}\,\|v\|_{0,M}} = \inf_{\hat{q} \in P_{r-1}(\widehat{M})} \sup_{\hat{v} \in \widehat{Y}(\widehat{M})} \frac{(\hat{q},\hat{v})_{\widehat{M}}}{\|\hat{q}\|_{0,\widehat{M}}\,\|\hat{v}\|_{0,\widehat{M}}}. \tag{3.1}$$

Let $\hat{q} \in P_{r-1}(\widehat{M})$ be arbitrarily chosen. By $\hat{b} : \widehat{M} \to \mathbb{R}$ we denote the piecewise linear hat function associated with $\hat{a}_0$, *i.e.*, $\hat{b}|_{\widehat{K}_i}$ is linear, $\hat{b}(\hat{a}_0) = 1$, $\hat{b}(\hat{a}_i) = 0$, $i = 1, \ldots, d+1$, and we set $\hat{v} := \hat{q} \cdot \hat{b}$. We note that $\hat{v} \in \widehat{Y}(\widehat{M})$ since $\hat{v}$ is continuous, $\hat{v}|_{\partial\widehat{M}} = 0$, and $\hat{v}|_{\widehat{K}_i} \in P_r(\widehat{K}_i)$ on each child $\widehat{K}_i$, $i = 1, \ldots, d+1$. Since $\hat{b} > 0$ in $\widehat{M}$, we state that

$$\hat{q} \mapsto (\hat{q}, \hat{q} \cdot \hat{b})^{1/2}_{0,\widehat{M}}$$

is a norm on the space $P_{r-1}(\widehat{M})$. Using the fact that all norms on finite dimensional spaces are equivalent, we get

$$(\hat{q},\hat{v})_{\widehat{M}} = (\hat{q}, \hat{q} \cdot \hat{b})_{\widehat{M}} \ge \beta_1 \|\hat{q}\|^2_{0,\widehat{M}} \tag{3.2}$$

where the constant β_1 is clearly independent of h. Due to $|\hat{b}(\hat{x})| \le 1 \;\forall \hat{x} \in \widehat{M}$, we have on the other hand

$$\|\hat{v}\|_{0,\widehat{M}} \le \|\hat{q}\|_{0,\widehat{M}}\,. \tag{3.3}$$

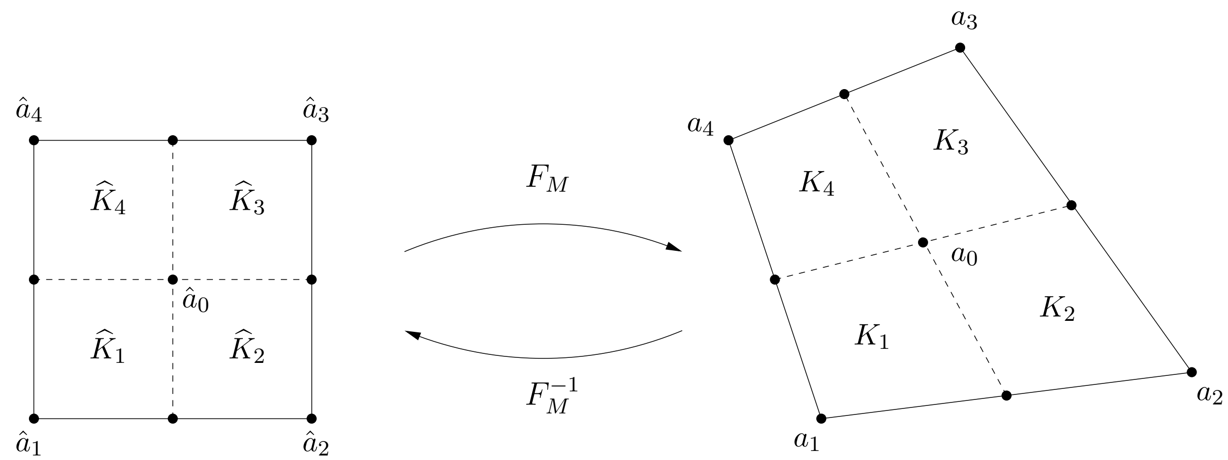

FIGURE 2. Reference macro quadrilateral (left) and macro quadrilateral $M \in \mathcal{M}_h$ (right).

From (3.1)–(3.3) we conclude A3. □

3.2. **Quadrilaterals and hexahedra**

Let $\widehat{M} = (-1,1)^d$ be the reference hyper-cube with the vertices $\hat{a}_i$, $i = 1,\ldots,2^d$, and the barycenter $\hat{a}_0$. $\widehat{M}$ is refined into 2^d congruent cubes $\widehat{K}_i$, $i = 1,\ldots,2^d$. Let $F_M : \widehat{M} \to M$ be the multilinear reference mapping. The refinement of $\widehat{M}$ induces a refinement of M into 2^d cells, see Figure 2 for the 2d case. The union of all these cells forms the principal mesh

$$\mathcal{T}_h = \bigcup_{M\in\mathcal{M}_h} \left\{ F_M(\widehat{K}_i) : \, i = 1,\ldots,2^d \right\} .$$

Furthermore, we consider the multilinear mapping $F_K : \widehat{K} \to K$ from the reference hyper-cube $\widehat{K} = (-1,1)^d$ onto an arbitrary cell $K \in \mathcal{T}_h$.

One can define the projection space D_h in two ways, namely as an image of a space on the reference macro $\widehat{M}$ or directly on the macro M. This leads to different finite element spaces, so we distinguish between both variants of the projection space D_h. The mapped version of D_h takes advantage of fixing the projection space locally on the reference macro. However, the interpolation property A2 turns out to be missing on arbitrary families of meshes [3,34]. To keep the notation clear, we use the extra superscript 'm' for mapped finite element spaces D_h.

3.2.1. *Projection spaces based on mapped finite elements*

First, we consider the following finite element pair

$$Y_h/D_h = Q_{r,h}/Q^{\text{disc,m}}_{r-1,2h}$$

where

$$Q_{r,h} := \{v \in H^1(\Omega) \,:\, v|_K \circ F_K \in Q_r(\widehat{K}) \quad \forall K \in \mathcal{T}_h\},$$
$$Q^{\text{disc,m}}_{r-1,2h} := \{v \in L^2(\Omega) \,:\, v|_M \circ F_M \in Q_{r-1}(\widehat{M}) \quad \forall M \in \mathcal{T}_{2h}\}.$$

The auxiliary space $\widehat{Y}(\widehat{M})$ is defined in analogy to the simplicial macros, *i.e.*, on the reference hyper-cube we have

$$\widehat{Y}(\widehat{M}) = \{w \in H^1_0(\widehat{M}) : \, w|_{\widehat{K}_i} \in Q_r(\widehat{K}_i), \quad i = 1,\ldots,2^d\}$$

where $\widehat{K}_i$ are children of $\widehat{M}$ resulting from the decomposition of $\widehat{M}$. The construction of interpolation operators satisfying the Assumption A1 is standard. Moreover, since $P_{r-1}(M) \subset Q_{r-1}^{\text{disc,m}}(M)$ holds true, the Bramble-Hilbert lemma provides the optimal interpolation properties A2 on families of shape regular meshes.

Following the lines of proof of Lemma 3.1 for establishing Assumption A3, we find that in general $|\det D_F M|$ is no longer constant, and consequently (3.1) holds only in special cases (parallelograms/parallelepipeds). Before we continue our investigation of the Assumption A3, we recall some general properties of the multilinear reference mapping $F_T : \widehat{T} \to T$ from $\widehat{T} = (-1,1)^d$ onto an arbitrary quadrilaterals/hexahedra T. F_T can be expanded as follows

$$F_T(\hat{x}) = m_T + B_T\hat{x} + G_T(\hat{x}) \tag{3.4}$$

where $m_T := F_T(0)$, $B_T := DF_T(0)$, and $G_T(\hat{x}) = F_T(\hat{x}) - F_T(0) - DF_T(0)(\hat{x})$. Furthermore, we define by means of the Euclidian norm the distortion parameter

$$\gamma_T := \sup_{\hat{x}\in\widehat{T}} \|B_T^{-1} DG_T(\hat{x})\|$$

and assume $\gamma_T \le \gamma < 1$. Then, the mapping F_T is one-to-one and from [36], Lemma 2, the estimate

$$C^{-1} d!(1-\gamma_T)^d\, h_T^d \le |\det DF_T(\hat{x})| \le C d!(1+\gamma_T)^d\, h_T^d \tag{3.5}$$

follows. Moreover, on a family of uniformly refined meshes we have

$$\lim_{h\to 0} \gamma_T = 0.$$

For a parallelogram/parallelepiped T, the reference mapping F_T is affine and $\gamma_T = 0$.

Lemma 3.2. *Let the local projection scheme be defined for the pair $Y_h/D_h = Q_{r,h}/Q_{r-1,2h}^{\text{disc,m}}$ with an arbitrary but fixed polynomial degree $r \in \mathbb{N}$. Then, the local inf-sup condition* A3 *holds with a constant β_1 independent of h.*

Proof. From (3.5) we get

$$\|q\|_{0,M}^2 \le C d!(1+\gamma_M)^d h_M^d \|\hat{q}\|_{0,\widehat{M}}^2 \qquad \forall q \in D_h(M). \tag{3.6}$$

Let $\hat{b} : \widehat{M} \to \mathbb{R}$ be the piecewise multilinear hat function associated with $\hat{a}_0$, *i.e.*, $\hat{b}|_{\widehat{K}_i} \in Q_1(\widehat{K}_i)$, $i = 1,\ldots,2^d$, $\hat{b}(\hat{a}_0) = 1$, $\hat{b}(\hat{a}_i) = 0$, $i = 1,\ldots,2^d$. For an arbitrary $q \in D_h(M)$ we choose $v(x) := (\hat{q}\cdot\hat{b}) \circ F_M^{-1}(x)$ where $\hat{q} \in Q_{r-1}(\widehat{M})$. Since $\hat{q}\cdot\hat{b}$ is continuous on the closure of $\widehat{M}$, $(\hat{q}\cdot\hat{b})|_{\widehat{K}_i} \in Q_r(\widehat{K}_i)$, $i = 1,\ldots,2^d$, and $\hat{b}|_{\partial\widehat{M}} = 0$, we have $\hat{v}(\hat{x}) := \hat{q}(\hat{x})\hat{b}(\hat{x}) \in \widehat{Y}(\widehat{M})$. Then, it follows from the estimate (3.5)

$$\begin{aligned}(q,v)_M &= \int_M q(x)v(x)\,\mathrm{d}x = \int_{\widehat{M}} \hat{q}(\hat{x})\hat{v}(\hat{x})\,|\det DF_M(\hat{x})|\,\mathrm{d}\hat{x} = \int_{\widehat{M}} \hat{q}(\hat{x})\hat{q}(\hat{x})\hat{b}(\hat{x})\,|\det DF_M(\hat{x})|\,\mathrm{d}\hat{x}\\ &\ge C d!(1-\gamma_M)^d\, h_M^d \int_{\widehat{M}} (\hat{q}(\hat{x}))^2\, \hat{b}(\hat{x})\,\mathrm{d}\hat{x}\,.\end{aligned}$$

The equivalence of norms on the finite dimensional space $Q_{r-1}(\widehat{M})$ implies

$$\|\hat{q}\cdot\sqrt{\hat{b}}\|_{0,\widehat{M}} \ge C\|\hat{q}\|_{0,\widehat{M}} \quad \forall \hat{q} \in Q_{r-1}(\widehat{M})$$

and hence

$$(q,v)_K \ge C d!(1-\gamma_M)^d\, h_M^d\, \|\hat{q}\|_{0,\widehat{M}}^2\,. \tag{3.7}$$

Using $|\hat{b}(\hat{x})| \leq 1 \ \forall \hat{x} \in \widehat{M}$, we get

$$\|v\|_{0,M}^2 \leq \int_{\widehat{M}} (\hat{q}(\hat{x}))^2 \ |\det DF_M(\hat{x})| \, \mathrm{d}\hat{x} \leq C d!(1+\gamma_M)^d \, h_M^d \|\hat{q}\|_{0,\widehat{M}}^2 \,. \tag{3.8}$$

From (3.6)–(3.8) it follows immediately

$$\forall q \in D_h(M) \quad \exists v \in Y_h(M): \qquad \frac{(q,v)_M}{\|q\|_{0,M}\|v\|_{0,M}} \geq C \left(\frac{1-\gamma_M}{1+\gamma_M}\right)^d \geq C \left(\frac{1-\gamma}{1+\gamma}\right)^d =: \beta_1$$

and thus, the local inf-sup condition A3 is proven. □

Alternatively, one can choose a smaller projection spaces D_h as follows

$$P_{r-1,2h}^{\mathrm{disc},m} := \{v \in L^2(\Omega) \, : \, v|_M \circ F_M \in P_{r-1}(\widehat{M}) \quad \forall M \in \mathcal{T}_{2h}\} \,.$$

This results in more stabilisation in the sense that the stabilising term vanishes on the smaller subset $P_{r-1,2h}^{\mathrm{disc},m} \subset Q_{r-1,2h}^{\mathrm{disc},m}$. The Assumption A1 holds without any change. However to guarantee Assumption A2 we have to restrict ourselves to suitably refined families of meshes, see [3] for quadrilaterals and [34] for hexahedra.

Corollary 3.3. *Let the local projection scheme be defined for the pair $Y_h/D_h = Q_{r,h}/P_{r-1,2h}^{\mathrm{disc,m}}$ with an arbitrary but fixed polynomial degree $r \in \mathbb{N}$. Then, the local inf-sup condition* A3 *holds with a constant β_1 independent of h.*

Remark 3.4. To discuss other choices of the projection space D_h, we mention that we have in the case $Y_h/D_h = Q_{r,h}/Q_{r-1,2h}^{\mathrm{disc,m}}$ the relation

$$\dim Y_h(M) = (2r-1)^d \geq r^d = \dim D_h(M),$$

in particular, for $r = 1$ the dimensions of both spaces coincide. In the case $r \geq 2$, one could think of choosing a larger projection space in order to minimise the stabilisation. A possible candidate would be $D_h = Q_{r,2h}^{\mathrm{disc,m}}$ since now

$$\dim Y_h(M) = (2r-1)^d \geq (r+1)^d = \dim D_h(M), \qquad r \geq 2.$$

The Assumption A1 holds without any change, A2 would be satisfied with a higher rate than required. However, whether the inf-sup condition A3 holds, is an open problem.

3.2.2. *Projection spaces based on unmapped finite elements*

We choose for the projection space D_h the space of discontinuous, piecewise polynomials of degree $r-1$ posed directly on the generic cells, *i.e.*

$$Y_h/D_h = Q_{r,h}/P_{r-1,2h}^{\mathrm{disc}}$$

where

$$Q_{r,h} := \{v \in H^1(\Omega) \, : \, v|_K \circ F_K \in Q_r(\widehat{K}) \quad \forall K \in \mathcal{T}_h\} \,,$$
$$P_{r-1,2h}^{\mathrm{disc}} := \{v \in L^2(\Omega) \, : \, v|_M \in P_{r-1}(M) \quad \forall M \in \mathcal{T}_{2h}\} \,.$$

Again, the Assumption A1 holds without any change. In contrast to the mapped version the approximation property A2 is now fulfilled for any shape regular family of meshes. The local inf-sup condition follows from

Lemma 3.5. *Let the local projection scheme be defined for the pair $Y_h/D_h = Q_{r,h}/P_{r-1,2h}^{\mathrm{disc}}$ with an arbitrary but fixed polynomial degree $r \in \mathbb{N}$. Then, on quadrilateral/hexahedral meshes the local inf-sup condition* A3 *holds with a constant β_1 independent of h.*

Proof. Since $P_{r-1,2h}^{\mathrm{disc}}$ is contained in $Q_{r-1,2h}^{\mathrm{disc,m}}$, the proof is a straightforward consequence of Lemma 3.2. □

4. Schemes based on enrichment of approximation spaces

This class of schemes takes advantage of constructing the finite element spaces Y_h and D_h on the same mesh. There are some indications of using only one principal mesh given recently in [11]. However, there the choice of polynomial spaces of order at least $r \geq 2$ and the construction of a nodal fluctuation operator is not optimal with respect to the order of convergence. We propose a novel method based on the enrichment of approximation spaces in order to satisfy the local inf-sup condition A3. The main benefit of this enrichment is the reduced stencil of the stiffness matrix and the avoidance of the difficult handling of data structures in the local projection methods which use coarser meshes. We present the method for two types of mesh geometries.

4.1. Simplices

Let

$$\hat{b}(\hat{x}) := (d+1)^{d+1} \prod_{i=1}^{d+1} \hat{\lambda}_i(\hat{x}) \tag{4.1}$$

be the bubble function which takes the value 1 in the barycentre of the reference simplex $\widehat{K}$. Thereby $\hat{\lambda}_i$, $i = 1, \ldots, d+1$, are barycentric coordinates on $\widehat{K}$. Furthermore, we define the enriched space

$$P_r^{\text{bubble}}(\widehat{K}) := P_r(\widehat{K}) + \hat{b} \cdot P_{r-1}(\widehat{K})\,.$$

Let

$$Y_h/D_h := P_{r,h}^{\text{bubble}}/P_{r-1,h}^{\text{disc}}$$

be the pair of finite element spaces defined *via* the reference mapping

$$\begin{aligned} P_{r,h}^{\text{bubble}} &: = \{v \in H^1(\Omega) \,:\, v|_K \circ F_K \in P_r^{\text{bubble}}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}\,, \\ P_{r-1,h}^{\text{disc}} &: = \{v \in L^2(\Omega) \,:\, v|_K \circ F_K \in P_{r-1}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}\,. \end{aligned}$$

Obviously, the Assumptions A1 and A2 are fulfilled. At a first glance, the enriched space seems to be large, but a more careful look shows

$$P_r(\widehat{K}) + \hat{b} \cdot P_{r-1}(\widehat{K}) = P_r(\widehat{K}) \oplus \left(\hat{b} \cdot \sum_{i=1}^{d} \tilde{P}_{r-i}(\widehat{K}) \right)$$

where $\tilde{P}_r(\widehat{K}) = \operatorname{span}\left(\left\{ \prod_{i=1}^{d} \hat{x}_i^{\alpha_i}\,, \quad \sum_{i=1}^{d} \alpha_i = r\,, \quad (\hat{x}_1, \ldots, \hat{x}_d) \in \widehat{K} \right\} \right)$ is a monomial space. Using the fact that the bubble part of the space $P_r(\widehat{K})$ is $\hat{b} \cdot P_{r-(d+1)}(\widehat{K})$, we have

$$\begin{aligned} \dim \widehat{Y}(\widehat{K}) &= \binom{r-(d+1)+d}{d} + \sum_{i=1}^{d} \left[\binom{r-i+d}{d} - \binom{r-i+d-1}{d} \right] \\ &= \binom{r-1}{d} + \binom{r-1+d}{d} - \binom{r-1}{d} = \dim P_{r-1}(\widehat{K})\,. \end{aligned}$$

The chosen enrichment is minimal with respect to the required inequality (2.2).

Lemma 4.1. *Let the local projection scheme be defined for the pair $Y_h/D_h = P_{r,h}^{\text{bubble}}/P_{r-1,h}^{\text{disc}}$ with an arbitrary but fixed polynomial degree $r \in \mathbb{N}$. Then, the local inf-sup condition* A3 *holds with a constant β_1 independent of h.*

Proof. Since the reference mapping is affine, the proof of the local inf-sup stability can be performed using the identity (3.1) for the reference cell $\widehat{K}$. For an arbitrary $\hat{q} \in P_{r-1}(\widehat{K})$ we choose $\hat{v} := \hat{b} \cdot \hat{q}$ with $\hat{b}$ defined by (4.1). Since $\hat{b} \cdot \hat{q} \in P_r^{\text{bubble}}(\widehat{K})$ and $\big(\hat{b} \cdot \hat{q}\big)|_{\partial \widehat{K}} = 0$, we have $\hat{v} \in \widehat{Y}(\widehat{K})$, and with the analogous argumentation as in the proof of the Lemma 3.1, we get the required positive lower bound β_1 independent of the mesh size h. □

4.2. **Quadrilaterals and hexahedra**

As for the method based on projection onto coarser meshes, we will distinguish between mapped

$$P_{r-1,h}^{\text{disc,m}} := \{v \in L^2(\Omega) \,:\, v|_K \circ F_K \in P_{r-1}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}$$

and unmapped

$$P_{r-1,h}^{\text{disc}} := \{v \in L^2(\Omega) \,:\, v|_K \in P_{r-1}(K) \quad \forall K \in \mathcal{T}_h\}$$

finite element spaces for the projection space D_h. In order to obtain the optimal order of the interpolation error for the mapped projection space, families of uniformly refined quadrilateral/hexahedral meshes are required, see again [3,34], whereas for the unmapped spaces the interpolation property A2 holds on general shape regular meshes. We extend the approximation spaces in order to ensure the local inf-sup condition A3.

4.2.1. *Projection spaces based on mapped finite elements*

Let

$$\hat{b}(\hat{x}) = \prod_{i=1}^{d} (1 - \hat{x}_i^2) \in Q_2(\widehat{K})\,, \quad \hat{x} = (\hat{x}_1, \dots, \hat{x}_d) \in \widehat{K}\,, \quad d = 2, 3, \tag{4.2}$$

be a bubble function associated with the reference cell $\widehat{K} = (-1,1)^d$. The enriched finite element space is set to be

$$Q_r^{\text{bubble},1}(\widehat{K}) := Q_r(\widehat{K}) \oplus \text{span}\left(\hat{b}\,\hat{x}_i^{r-1}\,, i = 1, \dots, d\right).$$

We define a pair of finite element spaces

$$Y_h/D_h := Q_{r,h}^{\text{bubble},1}/P_{r-1,h}^{\text{disc,m}}$$

via the reference mapping

$$Q_{r,h}^{\text{bubble},1} : = \{v \in H^1(\Omega) \,:\, v|_K \circ F_K \in Q_r^{\text{bubble},1}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}\,.$$

We note that in general the functions of spaces $Q_{r,h}^{\text{bubble},1}$, $P_{r-1,h}^{\text{disc,m}}$ are not polynomials. Since $Q_r(\widehat{K}) \subset Q_r^{\text{bubble},1}(\widehat{K})$, the Assumption A1 is satisfied. Assumption A2 holds on uniformly refined meshes, see [3,34].

Lemma 4.2. *Let the local projection scheme be defined for the pair* $Y_h/D_h = Q_{r,h}^{\text{bubble},1}/P_{r-1,h}^{\text{disc,m}}$ *with an arbitrary but fixed polynomial degree* $r \in \mathbb{N}$. *Then, the local inf-sup condition* A3 *holds with a constant* β_1 *independent of* h.

Proof. For an arbitrary $q \in D_h(K)$ we choose $v(x) := (\hat{q} \cdot \hat{b}) \circ F_K^{-1}(x)$ where $\hat{b} \geq 0$ is the bubble function from (4.2), $\hat{q} \in P_{r-1}(\widehat{K})$. Since $\hat{q} = \hat{q}_0 + \hat{q}_1$ with $\hat{q}_0 \in \text{span}\left(x_i^{r-1}, i = 1, \dots, d\right)$ and $\hat{q}_1 \in Q_{r-2}$, we have $\hat{v}(\hat{x}) := \hat{q}(\hat{x})\hat{b}(\hat{x}) \in Q_r^{\text{talbubble},1}(\widehat{K})$. Then, we proceed as in the proof of Lemma 3.2 and get from (3.5)

$$\forall q \in D_h(K) \quad \exists\, v \in Y_h(K): \qquad \frac{(q,v)_K}{\|q\|_{0,K}\|v\|_{0,K}} \geq C\left(\frac{1-\gamma_K}{1+\gamma_K}\right)^d \geq C\left(\frac{1-\gamma}{1+\gamma}\right)^d =: \beta_1.$$

This implies the local inf-sup condition A3. □

TABLE 1. Numbers of non-zero matrix entries in the two-dimensional case for the two-level $(Q_{r,h}/Q^{\text{disc,m}}_{r-1,2h})$ and for the enrichment approach $(Q^{\text{bubble,1}}_{r,h}/P^{\text{disc,m}}_{r-1})$, respectively.

Two-level approach			
Object	# object	dofs per object	Matrix entries per dof
Macro vertex	$\mathcal{O}(N^2/4)$	1	$(4r+1)^2$
Macro edge	$\mathcal{O}(N^2/2)$	$2r-1$	$(4r+1)(2r+1)$
Macro cell	$\mathcal{O}(N^2/4)$	$(2r-1)^2$	$(2r+1)^2$
Enrichment approach			
Object	# object	dofs per object	Matrix entries per dof
Vertex	$\mathcal{O}(N^2)$	1	$(2r+1)^2+4\cdot 2$
Edge	$\mathcal{O}(2N^2)$	$r-1$	$(2r+1)(r+1)+2\cdot 2$
Cell	$\mathcal{O}(N^2)$	$(r-1)^2+2$	$(r+1)^2+2$

TABLE 2. Numbers of non-zero matrix entries in the three-dimensional case for the two-level $(Q_{r,h}/Q^{\text{disc,m}}_{r-1,2h})$ and for the enrichment approach $(Q^{\text{bubble,1}}_{r,h}/P^{\text{disc,m}}_{r-1})$, respectively.

Two-level approach			
Object	# object	dofs per object	Matrix entries per dof
Macro vertex	$\mathcal{O}(N^3/8)$	1	$(4r+1)^3$
Macro edge	$\mathcal{O}(3N^3/8)$	$2r-1$	$(4r+1)^2(2r+1)$
Macro face	$\mathcal{O}(3N^3/8)$	$(2r-1)^2$	$(4r+1)(2r+1)^2$
Macro cell	$\mathcal{O}(N^3/8)$	$(2r-1)^3$	$(2r+1)^3$
Enrichment approach			
Object	# object	dofs per object	Matrix entries per dof
Vertex	$\mathcal{O}(N^3)$	1	$(2r+1)^3+8\cdot 3$
Edge	$\mathcal{O}(3N^3)$	$r-1$	$(2r+1)^2(r+1)+4\cdot 3$
Face	$\mathcal{O}(3N^3)$	$(r-1)^2$	$(2r+1)(r+1)^2+2\cdot 3$
Cell	$\mathcal{O}(N^3)$	$(r-1)^3+3$	$(r+1)^3+3$

Remark 4.3. A comparison of the dimensions of the spaces $Y_h(M)$ and $D_h(M)$ shows that

$$\dim \widehat{Y}(\widehat{K}) = (r-1)^d + d \geq \binom{r-1+d}{d} = \dim P_{r-1}(\widehat{K}) \quad \forall r \in \mathbb{N}\ \forall d \in \mathbb{N}.$$

In particular, the enrichment is optimal for biquadratic and bicubic elements on quadrilaterals and for tri-quadratic elements on hexahedra.

Remark 4.4. Note that the space $Q^{\text{bubble,1}}_r(\widehat{K})$ has for $r \geq 2$ exactly d basis functions more than $Q_r(\widehat{K})$, independent of r.

Remark 4.5. To get an impression on the efficiency of the new enrichment approach, we consider the matrix-block corresponding to one scalar component and compare asymptotically the numbers of non-zero entries for a decomposition of $\Omega = (0,1)^d$ into squares/cubes of edge size $1/N$, see Tables 1 and 2. Since the inner degrees of freedom (dofs) dominate for high order elements ($r \gg 1$), we have asymptotically $\mathcal{O}(4N^2r^4)$ and $\mathcal{O}(8N^3r^6)$ non-zero entries for the two-level approach whereas the enrichment technique produces only $\mathcal{O}(N^2r^4)$ and $\mathcal{O}(N^3r^6)$ non-zero entries. This effect is less distinct for moderate r. For example, in the case $r=2$ and $d=2$, we get asymptotically $\mathcal{O}(144N^2)$ compared to $\mathcal{O}(75N^2)$ non-zero entries.

4.2.2. *Projection spaces based on unmapped finite elements*

In order to satisfy A2 on general meshes, we propose an alternative way for setting the finite element pair Y_h/D_h. We choose the space

$$Q_r^{\text{bubble},2}(\widehat{K}) := Q_r(\widehat{K}) + \hat{b} \cdot Q_{r-1}(\widehat{K})$$

with the bubble function $\hat{b}$ from (4.2) and define the enriched space

$$Q_{r,h}^{\text{bubble},2} := \{v \in H^1(\Omega) \,:\, v|_K \circ F_K \in Q_r^{\text{bubble},2}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}\,.$$

Now, our alternative choice is

$$Y_h/D_h = Q_{r,h}^{\text{bubble},2}/P_{r-1,h}^{\text{disc}}.$$

Then, the properties A1 and A2 are naturally fulfilled.

Lemma 4.6. *Let the local projection scheme be defined for the pair* $Y_h/D_h = Q_{r,h}^{\text{bubble},2}/P_{r-1,h}^{\text{disc}}$ *with an arbitrary but fixed polynomial degree* $r \in \mathbb{N}$. *Then, the local inf-sup condition* A3 *holds with a constant* β_1 *independent of* h.

Proof. For an arbitrary $q \in D_h$, we set $v = q \cdot b_K$ where $b_K(x) := (\hat{b} \circ F_K^{-1})(x)$ with the bubble function $\hat{b}$ defined by (4.2) and the reference mapping F_K. Since $q \in P_{r-1}(K)$, we find $\hat{q} \in Q_{r-1}(\widehat{K})$. Consequently, $\hat{b} \cdot \hat{q} \in Q_r^{\text{bubble},2}(\widehat{K})$ and $\big(\hat{b} \cdot \hat{q}\big)|_{\partial\widehat{K}} = 0$. Then, we have $\hat{v} := \hat{b} \cdot \hat{q} \in \widehat{Y}(\widehat{K})$ and $v \in Y_h(K)$. In analogy to the proof of Lemma 4.2, we obtain from (3.5) with $T = K$, the norm equivalence

$$\|\hat{q} \cdot \sqrt{\hat{b}}\|_{0,\widehat{K}} \geq C\|\hat{q}\|_{0,\widehat{K}} \quad \forall \hat{q} \in Q_{r-1}(\widehat{K}),$$

and $\|\hat{v}\|_{0,\widehat{K}} \leq \|\hat{q}\|_{0,\widehat{K}}$, the following

$$\forall q \in D_h(K) \quad \exists\, v \in Y_h(K): \qquad \frac{(q,v)_K}{\|q\|_{0,K}\|v\|_{0,K}} \geq C\left(\frac{1-\gamma_K}{1+\gamma_K}\right)^d \geq C\left(\frac{1-\gamma}{1+\gamma}\right)^d =: \beta_1$$

which is the statement of the lemma. □

Remark 4.7. The space $Q_{r,h}^{\text{bubble},2}$ is more enriched than the space $Q_{r,h}^{\text{bubble},1}$. Comparing the dimensions of spaces $Y_h(K)$ and $D_h(K)$, we can guess that the enriched space could be reduced. However, the question of the validity of the local inf-sup condition is still an open problem.

5. Relation to subgrid modelling

The idea of subgrid modelling goes back to Guermond and has firstly been applied to a scalar transport equation [19]. It is based on a scale separation of the underlying finite element space

$$Y_h = Y_H \oplus Y_h^H$$

where Y_H stands for the space of large scales and Y_h^H for the space of small scales. Associated with the scale separation is a suitable projection operator $P_H : Y_h \to Y_H \subset Y_h$ which is the identity on the subspace Y_H. Let $\overline{\kappa}_h := id - P_H$ denote the fluctuation operator. We assume that the finite element space Y_H is based on a shape regular decomposition of the domain into cells $M \in \mathcal{M}_h$ of diameter h_M. Then, a stabilising term of the form

$$S(u_h, v_h) = \sum_{M \in \mathcal{M}_h} h_M\big(\nabla\overline{\kappa}_h u_h, \nabla\overline{\kappa}_h v_h\big)_M \quad \text{or} \quad S(u_h, v_h) = \sum_{M \in \mathcal{M}_h} h_M\big((b\cdot\nabla)\overline{\kappa}_h u_h, (b\cdot\nabla)\overline{\kappa}_h v_h\big)_M$$

has been proposed to add to the standard Galerkin approach [15,19]. These stabilisation terms can be interpreted as an artificial diffusion term or an artificial diffusion in the streamline direction for the subscales which are represented by Y_h^H. This technique has been developed in different directions, for an extension to time-dependent convection-diffusion problems see *e.g.* [30]. Scale separation plays also an important role in large eddy simulation of turbulent flows, see [28].

The scale separation can be realised in different ways. In the two-level approach, Y_h and Y_H are standard finite element spaces on different refinement levels (we indicate this by writing $Y_H = Y_{2h}$) and Y_h^H consists just of those hierarchical basis functions which are missing in the coarse space Y_{2h} to generate Y_h. Another view is to consider Y_h as a finite element space Y_H enriched by Y_h^H which contains suitable functions, *e.g.* higher order polynomials. However, both variants differ from the local projection approach since the stabilisation term in the subgrid modelling approach is based on gradients of fluctuations $\nabla(id - P_H)u_h$ whereas the local projection method uses fluctuations of the gradients $(id - \pi_h)\nabla u_h$.

In applications, the projection $P_H : Y_h \to Y_{2h}$ in the two-level approach has been often chosen as the global Lagrange interpolant $I_{2h,r}$ into Y_{2h} [4, 7–9, 11, 32]. This leads to a stabilising term of the form

$$\mathcal{S}_h^3\big((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)\big) = \sum_{M\in\mathcal{M}_h} \tau_M(\nabla\overline{\boldsymbol{\kappa}}_h\mathbf{u}_h, \nabla\overline{\boldsymbol{\kappa}}_h\mathbf{v}_h)_M + \alpha_M(\nabla\overline{\kappa}_h p_h, \nabla\overline{\kappa}_h q_h)_M \tag{5.1}$$

instead of S_h given by (1.12). In the following subsections, we study the relations between the stabilising terms S_h^1 given by (2.33) and S_h^3.

5.1. Two-level approach on piecewise linear elements

We consider first the case where $\mathcal{T}_h$ is generated from a refinement of a shape regular triangulation $\mathcal{T}_{2h}$ by connecting the barycentre of each macro cell with its vertices, see Figure 1 for the 2d case. Let Y_h and Y_{2h} denote the spaces of continuous, piecewise linear finite elements associated with the triangulations $\mathcal{T}_h$ and $\mathcal{T}_{2h}$, respectively.

Lemma 5.1. *Let $d \geq 1$, $\pi_{2h,0}$ be the L^2-projection onto the space $P_{0,2h}^{\text{disc}}$ of piecewise constant functions, and $I_{2h,1} : Y_h \to Y_{2h}$ be the Lagrange interpolant into the space $P_{1,2h}$ of continuous, piecewise linear functions. Then, we have*

$$\pi_{2h,0}(\nabla v_h)\Big|_M = \nabla I_{2h,1}(v_h\big|_M) \qquad \forall v_h \in P_{1,h},\ \forall M \in \mathcal{T}_{2h}.$$

Thus, the discrete problems of the local projection method and the subgrid modelling approach coincide.

Proof. We consider the case of an arbitrary dimension $d \geq 1$. We restrict ourselves to the scalar case since the assertion for the vector-valued case follows immediately from the scalar one by a component-wise application.

Let $v_h\big|_M$ be the restriction of a finite element function $v_h \in Y_h$ onto a macro simplex $M \in \mathcal{T}_{2h}$. Furthermore, let λ_i, $i = 1, \ldots, d+1$, denote the barycentric coordinates with respect to the simplex M. Defining the continuous, piecewise linear function

$$\varphi_0(x) = (d+1)\lambda_i(x), \qquad x \in K_i,\ i = 1, \ldots, d+1,$$

we can represent $v_h\big|_M$ by its nodal functionals $N_i(v) = v(a_i)$, $i = 0, \ldots, d+1$, as

$$v_h\big|_M = \sum_{i=1}^{d+1} N_i(v_h)\lambda_i + \widetilde{N}_0(v_h)\varphi_0 \tag{5.2}$$

with

$$\widetilde{N}_0(v) = N_0(v) - \frac{1}{d+1}\sum_{i=1}^{d+1} N_i(v).$$

Since $N_i(\varphi_0) = 0$, $i = 1, \ldots, d+1$, we have

$$I_{2h,1} v_h = \sum_{i=1}^{d+1} N_i(v_h)\lambda_i$$

from which

$$\nabla I_{2h,1} v_h = \sum_{i=1}^{d+1} N_i(v_h)\nabla\lambda_i$$

follows. Let ∇_h denote the piecewise applied gradient operator. Taking into consideration that $\nabla_h v_h$ is constant on each subdomain K_j, $j = 1, \ldots, d+1$, and that $|K_j| = |K|/(d+1)$, we compute the L^2-projection onto $P_0(M)$ to be

$$\pi_{2h,0}(\nabla_h v_h) = \frac{1}{d+1}\sum_{j=1}^{d+1} \nabla_h v_h\big|_{K_j}.$$

Next we get from (5.2) for $j = 1, \ldots, d+1$

$$\nabla v_h\big|_{K_j} = \sum_{i=1}^{d+1} N_i(v_h)\nabla\lambda_i + (d+1)\widetilde{N}_0(v_h)\nabla\lambda_j,$$

$$\frac{1}{d+1}\sum_{j=1}^{d+1} \nabla v_h\big|_{K_j} = \sum_{i=1}^{d+1} N_i(v_h)\nabla\lambda_i + \widetilde{N}_0(v_h)\nabla\sum_{j=1}^{d+1}\lambda_j = \sum_{i=1}^{d+1} N_i(v_h)\nabla\lambda_i.$$

Since $\pi_{2h,0}(\nabla v_h)\Big|_M = \nabla I_{2h,1}(v_h\big|_M)$ holds true, the stabilising terms in both approaches are identical. □

Remark 5.2. In general we have that

$$\pi_{2h,r-1}\nabla_h v_h\Big|_M \neq \nabla I_{2h,r} v_h \qquad \forall v_h \in P_{r,h},\ r \geq 2$$

where $\pi_{2h,r-1}$ is the L^2-projection onto the space $P_{r-1,2h}^{\text{disc}}$ of discontinuous, piecewise polynomials of degree $r-1$ on the coarse mesh $\mathcal{T}_{2h}$ and $I_{2h,r} : Y_h \to P_{r,2h}$ is the Lagrange interpolant into the space of continuous, piecewise polynomials of degree r on the coarse mesh $\mathcal{T}_{2h}$. Similarly, we have that in general on quadrilateral or hexahedral meshes $\mathcal{T}_{2h}$

$$\pi_{2h,r-1}\nabla_h v_h\Big|_M \neq \nabla I_{2h,r} v_h \qquad \forall v_h \in Q_{r,h}^d,\ d \geq 2,\ r \geq 1$$

where $\pi_{2h,r-1}$ is the L^2-projection onto the space $Q_{r-1,2h}^{\text{disc}}$ of discontinuous, piecewise polynomials of degree $r-1$ in each variable and $I_{2h,r} : Y_h \to Q_{r,2h}$ is the Lagrange interpolant into the space of continuous, piecewise polynomials of degree r in each variable. As an example, we consider the case $r = 2$, $d = 1$. We get for the reference macro $\widehat{M} = (-1, +1)$ and the piecewise quadratic function

$$\widehat{v}(\hat{x}) = \begin{cases} 4\hat{x}(1-\hat{x}) & \text{if } 0 \leq \hat{x} \leq 1, \\ 0 & \text{if } -1 \leq \hat{x} < 0, \end{cases}$$

the relation

$$\hat{\pi}_{2h,1}\hat{\nabla}\hat{v} = -\hat{x} \neq 0 = \hat{\nabla}\hat{I}_{2h,2}\hat{v}.$$

Thus, in general the two approaches, subgrid modelling and local projection, do not lead to the same stabilisation term. However, we will see later that this does not automatically exclude the possibility of spectral equivalence of the stabilisation terms.

5.2. Enriched piecewise linear elements

In the previous section we have seen that both methods, the local projection scheme and the subgrid modelling, results for the two-level approach with $Y_h = P_{1,h}$ and $D_h = P_{0,2h}^{\rm disc}$ in the same stabilisation term. Now we will show that the same is also true for enriched piecewise linear elements, *i.e.*, $Y_h = P_{1,h}^{\rm bubble}$ and $D_h = P_{0,h}^{\rm disc}$.

Lemma 5.3. *Let $d \geq 1$, $\pi_{h,0}$ be the L^2-projection onto the space $P_{0,h}^{\rm disc}$ of piecewise constant functions, and $I_{h,1} : Y_h \to P_{1,h}$ be the Lagrange interpolant into the space $P_{1,h}$ of continuous, piecewise linear functions. Then, we have*

$$\pi_{h,0}(\nabla v_h)\Big|_K = \nabla I_{h,1}(v_h\big|_K) \qquad \forall v_h \in P_{1,h}^{\rm bubble}, \forall K \in \mathcal{T}_h.$$

Thus, the discrete problems of the local projection method and the subgrid modelling approach coincide.

Proof. For simplicity of notation we present the proof for the scalar case. The extension to the vector-valued case in the space Y_h^d is straightforward. We consider a simplex $K \in \mathcal{T}_h$ with the vertices a_i, $i = 1, \ldots, d+1$, the barycentre a_0, and the barycentric coordinates λ_i, $i = 1, \ldots, d+1$. The restriction $v_h\big|_K$ of a finite element function $v_h \in Y_h$ onto K can be represented by its nodal functionals $N_i(v) = v(a_i)$, $i = 0, \ldots, d+1$, as

$$v_h\big|_K = \sum_{i=1}^{d+1} N_i(v_h)\lambda_i + \widetilde{N}_0(v_h)\, b = I_{h,1}v_h + \widetilde{N}_0(v_h)\, b$$

where

$$\widetilde{N}_0(v) = N_0(v) - \frac{1}{d+1}\sum_{i=1}^{d+1} N_i(v) \quad \text{and} \quad b = (d+1)^{d+1}\prod_{i=1}^{d+1}\lambda_i$$

is the scaled product of all barycentric coordinates λ_i, $i = 1, \ldots, d+1$. Applying the gradient gives

$$\nabla v_h\big|_K = \nabla(I_{h,1}v_h) + N_0(v_h)\,\nabla b.$$

Since $\nabla(I_{h,1}v_h)$ is constant on K, we have $\pi_{h,0}\nabla(I_{h,1}v_h) = \nabla(I_{h,1}v_h)$. Hence, it remains to show

$$\pi_{h,0}(\nabla b) = \frac{1}{|K|}\int_K \nabla b\,\mathrm{dx} = \mathbf{0}.$$

This follows immediately from the Gaussian theorem since b vanishes on ∂K. Consequently, $\pi_{h,0}(\nabla v_h)\Big|_K = \nabla I_{h,1}(v_h\big|_K)$ holds and the stabilising terms in both approaches are identical. □

5.3. Spectral equivalence of the stabilising terms on simplices

Now, we will show on simplices the spectral equivalence between the stabilising terms S_h^3 given by (5.1) and S_h^1 given by (2.33). To this end, it is sufficient to show the existence of positive constants C_3, C_4 such that

$$C_3\|\kappa_h\nabla w_h\|_{0,M} \leq \|\nabla\overline{\kappa}_h w_h\|_{0,M} \leq C_4\|\kappa_h\nabla w_h\|_{0,M} \qquad \forall w_h \in Y_h, \quad \forall M \in \mathcal{M}_h. \tag{5.3}$$

First we consider the two-level approach.

Lemma 5.4. *Let $Y_h = P_{r,h}$, $D_h = P_{r-1,2h}^{\rm disc}$, $\pi_{2h,r-1}$ be the L^2-projection onto D_h, $\kappa_h = id - \pi_{2h,r-1}$, $I_{2h,r}$ be the Lagrange interpolant in $P_{r,2h}$, and $\overline{\kappa}_h = id - I_{2h,r}$. Then, the stabilising terms S_h^3 and S_h^1 are spectrally equivalent.*

Proof. For $M \in \mathcal{T}_{2h}$ let $F_M : \widehat{M} \to M$ be the affine mapping from the reference macro cell $\widehat{M}$ onto the cell M and $B_M = DF_M$. The L^2-projection $\pi_{2h,r-1}$ and the Lagrange interpolant $I_{2h,r}$ are invariant with respect to affine transformations, *i.e.*, denoting the corresponding operators on the reference cell by $\hat{\pi}$ and $\hat{I}$, we have

$$(\widehat{\pi_{2h,r-1}w}) = \hat{\pi}\hat{w}, \qquad \widehat{I_{2h,r}w} = \hat{I}\hat{w}$$

and the corresponding relations for the fluctuation operators

$$\widehat{\kappa_h \nabla w} = \hat{\kappa}\, \widehat{\nabla w}, \qquad \widehat{\overline{\kappa}_h w} = \hat{\overline{\kappa}}\, \widehat{\nabla w}.$$

Now using the transformation formulas $\widehat{\nabla v} = B_M^{-T}\hat{\nabla}\hat{v}$ and $\hat{\nabla}\hat{v} = B_M^T \widehat{\nabla v}$ [13], Chapter 3.1, we obtain

$$\|\kappa_h \nabla w\|_{0,M} = |\det B_M|^{1/2}\, \|\widehat{\kappa_h \nabla w}\|_{0,\widehat{M}} = |\det B_M|^{1/2}\, \|\hat{\kappa} B_M^{-T}\hat{\nabla}\hat{w}\|_{0,\widehat{M}} \le |\det B_M|^{1/2}\, \|B_M^{-1}\|\, \|\hat{\kappa}\hat{\nabla}\hat{w}\|_{0,\widehat{M}}, \quad (5.4)$$

$$\|\hat{\nabla}\hat{\overline{\kappa}}\hat{w}\|_{0,\widehat{M}} = \|\hat{\nabla}\widehat{\overline{\kappa}_h w}\|_{0,\widehat{M}} = |\det B_M|^{-1/2}\|B_M^T \nabla \overline{\kappa}_h w\|_{0,M} \le |\det B_M|^{-1/2}\|B_M\|\, \|\nabla\overline{\kappa}_h w\|_{0,M}, \quad (5.5)$$

where $\|B_M\|$ and $\|B_M^{-1}\|$ are the matrix norms of B_M and B_M^{-1} which are induced by the Euclidean vector norm.

If there is a constant C such that $\|\hat{\kappa}\hat{\nabla}\hat{w}\|_{0,\widehat{M}} \le C\, \|\hat{\nabla}\hat{\overline{\kappa}}\hat{w}\|_{0,\widehat{M}}$, we get from (5.4), (5.5), and $\|B_M^{-1}\|\, \|B_M\| \le C$ which holds for shape regular meshes,

$$\|\kappa_h \nabla w\|_{0,M} \le C_3^{-1}\, \|\nabla \overline{\kappa}_h w\|_{0,M}$$

which is the left hand side of (5.3). The proof of the right hand side follows from $\|\hat{\nabla}\hat{\overline{\kappa}}\hat{w}\|_{0,\widehat{M}} \le C\, \|\hat{\kappa}\hat{\nabla}\hat{w}\|_{0,\widehat{M}}$ by the same arguments.

To show the missing inequalities on the reference element, we consider the mappings

$$\hat{w} \mapsto \|\hat{\kappa}\hat{\nabla}\hat{w}\|_{0,\widehat{M}}, \qquad \hat{w} \mapsto \|\hat{\nabla}\hat{\overline{\kappa}}\hat{w}\|_{0,\widehat{M}}$$

which are norms on the finite dimensional factor spaces

$$P_r(\widehat{M}) \,/\, \{\hat{w} \;:\; \hat{\kappa}\hat{\nabla}\hat{w} = \mathbf{0}\} \qquad \text{and} \qquad P_r(\widehat{M}) \,/\, \{\hat{w} \;:\; \hat{\nabla}\hat{\overline{\kappa}}\hat{w} = \mathbf{0}\},$$

respectively. Let us assume that $\hat{\kappa}\hat{\nabla}\hat{w} = \mathbf{0}$. Then, we conclude

$$\hat{\nabla}\hat{w} = \hat{\pi}\hat{\nabla}\hat{w} \in \left(P_{r-1}(\widehat{M})\right)^d \quad \Longrightarrow \quad \hat{w} \in P_r(\widehat{M}) \quad \Longrightarrow \quad \hat{I}\hat{w} = \hat{w} \quad \Longrightarrow \quad \hat{\nabla}\hat{\overline{\kappa}}\hat{w} = \mathbf{0}.$$

Conversely, assuming $\hat{\nabla}\hat{\overline{\kappa}}\hat{w} = \mathbf{0}$, and taking into consideration that $\hat{w}$ is continuous on $\widehat{M}$, we obtain

$$\hat{w} = \hat{I}\hat{w} + \text{const} \in P_r(\widehat{M}) \quad \Longrightarrow \quad \hat{\nabla}\hat{w} = \hat{\nabla}\hat{I}\hat{w} \in \left(P_{r-1}(\widehat{M})\right)^d \quad \Longrightarrow \quad \hat{\pi}\hat{\nabla}\hat{w} = \hat{\nabla}\hat{w}. \quad \Longrightarrow \quad \hat{\kappa}\hat{\nabla}\hat{w} = \mathbf{0}.$$

Thus, the two factor spaces coincide and the missing inequalities follow from the equivalence of norms on finite dimensional spaces. □

Let us turn to the case of enriched finite element spaces Y_h.

Lemma 5.5. *Let* $Y_h = P_{r,h}^{\text{bubble}}$ *defined in Section* 4.1, $D_h = P_{r-1,h}^{\text{disc}}$, $\pi_{h,r-1}$ *be the* L^2*-projection onto* D_h, $I_{h,r}$ *be the Lagrange interpolant in* $P_{r,h}$, $\kappa_h = id - \pi_{h,r-1}$, *and* $\overline{\kappa}_h = id - I_{h,r}$. *Then, the stabilising terms* S_h^3 *and* S_h^1 *are spectrally equivalent.*

Proof. First, by using the affine transformation $F_K : \widehat{K} \to K$ from the reference cell $\widehat{K}$ onto K, we can show, as in the proof of Lemma 5.4, that it suffices to establish the corresponding estimations on the reference cell. As before, we will do it by showing that the mappings

$$\hat{w} \mapsto \|\hat{\kappa}\hat{\nabla}\hat{w}\|_{0,\widehat{K}}, \qquad \hat{w} \mapsto \|\hat{\nabla}\hat{\overline{\kappa}}\hat{w}\|_{0,\widehat{K}}$$

are norms on the corresponding factor spaces

$$P_r(\widehat{M}) \big/ \{\hat{w} \ : \ \hat{\kappa}\hat{\nabla}\hat{w} = \mathbf{0}\} \qquad \text{and} \qquad P_r(\widehat{M}) \big/ \{\hat{w} \ : \ \hat{\nabla}\hat{\overline{\kappa}}\hat{w} = \mathbf{0}\}.$$

Let us assume that $\hat{\kappa}\hat{\nabla}\hat{w} = \mathbf{0}$. Then,

$$\hat{\nabla}\hat{w} = \hat{\pi}\hat{\nabla}\hat{w} \in \big(P_{r-1}(\widehat{K})\big)^d \quad \Longrightarrow \quad \hat{w} \in P_r(\widehat{K}) \quad \Longrightarrow \quad \hat{I}\hat{w} = \hat{w} \quad \Longrightarrow \quad \hat{\nabla}\hat{\overline{\kappa}}\hat{w} = \mathbf{0}.$$

Conversely, assuming $\hat{\nabla}\hat{\overline{\kappa}}\hat{w} = \mathbf{0}$, we obtain

$$\hat{w} = \hat{I}\hat{w} + \text{const.} \in P_r(\widehat{K}) \quad \Longrightarrow \quad \hat{\nabla}\hat{w} = \hat{\nabla}\hat{I}\hat{w} \in \big(P_{r-1}(\widehat{K})\big)^d \quad \Longrightarrow \quad \hat{\pi}\hat{\nabla}\hat{w} = \hat{\nabla}\hat{w} \quad \Longrightarrow \quad \hat{\kappa}\hat{\nabla}\hat{w} = \mathbf{0}.$$

Thus, we conclude the existence of two constants C_3 and C_4 such that

$$C_3\|\kappa_h \nabla w_h\|_{0,K} \le \|\nabla \overline{\kappa}_h w_h\|_{0,M} \le C_4\|\kappa_h \nabla w_h\|_{0,K} \qquad \forall w_h \in Y_h, \quad \forall K \in \mathcal{T}_h$$

and the stabilising terms S_h^3 and S_h^1 are spectrally equivalent. □

Remark 5.6. For quadrilateral and hexahedral elements we do not have in general the spectral equivalence of the stabilising terms. As an example we consider the case $d = 2$, $r = 1$. In the two-level approach we have on the macro element $\widehat{M} = (-1, +1)^2$ for the function $\hat{w}(\hat{x}) = \hat{x}_1\hat{x}_2$

$$\hat{\nabla}\hat{w} - \hat{\pi}\hat{\nabla}\hat{w} = \hat{\nabla}\hat{w} = (\hat{x}_2, \hat{x}_1)^T,$$

but the Lagrange interpolant $\hat{I}$ in $Q_1(\widehat{M})$ leads to

$$\hat{I}\hat{w} = \hat{w} \quad \Longrightarrow \quad \hat{\nabla}(\hat{w} - \hat{I}\hat{w}) = (0, 0)^T.$$

The same argument holds for enriched approximation spaces Y_h on a reference cell $\widehat{K}$.

6. Conclusions

We have investigated in this paper the local projection stabilisation for the Oseen equations on simplices, quadrilaterals, and hexahedra in a general manner.

Starting with an abstract framework, we have seen that three ingredients are essential for stable discretisations with optimal error estimates. The first two conditions are approximation properties of the approximation space Y_h, *cf.* Assumption A1, and the projection space D_h, *cf.* Assumption A2, respectively. These assumptions are fulfilled for a large class of spaces. The third condition requires the inf-sup stability of the local approximation space $Y_h(M)$ and the local projection space $D_h(M)$, *cf.* Assumption A3.

We have considered two different types of local projection schemes. The projection space in the two-level approach is based on a coarser macro triangulation while the projection space in the enrichment method is defined on the same mesh as the approximation space. The enrichment method is easier to implement and generates a stencil which is much more compact than the stencil for the two-level approach, *cf.* Remark 4.5. Moreover, one can construct an enrichment method on quadrilaterals and hexahedra such that the local approximation space contains compared to the usual Q_r-elements just d functions more where d is the space dimension.

Finally, we have seen that the local projection scheme and the subgrid modelling introduced by Guermond result in the same discrete problems provided that piecewise linears on simplices are considered. Moreover, we have shown that the stabilisation terms of the local projection scheme and the subgrid modelling are spectrally equivalent for higher order elements on simplices. Unfortunately, this equivalence doesn't hold on quadrilaterals and hexahedra.

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