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POSTPROCESSING OF A FINITE VOLUME ELEMENT METHOD FOR SEMILINEAR PARABOLIC PROBLEMS *

Min Yang[1], Chunjia Bi[1] and Jiangguo Liu[2]

Abstract. In this paper, we study a postprocessing procedure for improving accuracy of the finite volume element approximations of semilinear parabolic problems. The procedure amounts to solve a source problem on a coarser grid and then solve a linear elliptic problem on a finer grid after the time evolution is finished. We derive error estimates in the L^2 and H^1 norms for the standard finite volume element scheme and an improved error estimate in the H^1 norm. Numerical results demonstrate the accuracy and efficiency of the procedure.

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1. Introduction

We consider the following semilinear parabolic problem:

$$\begin{aligned} u_t - \nabla\cdot(A\nabla u) &= f(\mathbf{x},t,u), \quad \text{in } \Omega\times(0,T],\\ u &= 0, \quad \text{on } \partial\Omega\times(0,T],\\ u &= u_0, \quad \text{in } \Omega\times\{0\}, \end{aligned} \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$, $u_0(\mathbf{x})$ a given smooth function and $A(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{i,j=1}^2$, $(a_{ij}(\mathbf{x}) \in W^{1,\infty}(\Omega))$ a symmetric and positive definite matrix in Ω, *i.e.*, there exists a positive constant a_* such that

$$0 < a_*|\zeta|^2 \le \zeta^T A(\mathbf{x})\zeta, \quad \forall \zeta\in\mathbb{R}^2, \ \mathbf{x}\in\Omega.$$

We assume that $f(\mathbf{x},t,u)$ is a real-valued function defined on $\Omega\times(0,T]\times\mathbb{R}$ satisfying the following condition:

$$|f(\mathbf{x},t,w) - f(\mathbf{x},t,v)| \le C_f|w-v|(1+|w|+|v|)^\gamma, \quad \forall w,v\in\mathbb{R}, \ \text{a.e. } (\mathbf{x},t)\in\Omega\times(0,T]. \tag{1.2}$$

Here C_f is a positive constant and $0 \le \gamma < \infty$. For example, $f(\mathbf{x},t,u)$ could be an arbitrary polynomial of u. The condition (1.2) implies that f is locally Lipschitz continuous [21,30]. The problem (1.1) arises

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[1] Department of Mathematics, Yantai University, Yantai, Shandong 264005, P. R. China. yang@ytu.edu.cn; bicj@ytu.edu.cn
[2] Department of Mathematics, Colorado State University, Fort Collins, CO 80523-1874, USA. liu@math.colostate.edu

in many applications, *e.g.*, combustion modeling, epidemic phenomena, and stochastic controls. As in [19,21,31], we suppose that the initial data u_0 is sufficiently smooth and compatible and the problem (1.1) admits a unique solution satisfying

$$\max_{0\le t\le T} \|u(\cdot,t)\|_{3,q} + \|u_t(\cdot,t)\|_{3,r} \le M, \tag{1.3}$$

where M is a positive constant and $q, r > 1$ are constants to be specified in Section 3. A detailed discussion on the regularity of solutions of nonlinear evolution problems can be found in [20,28,29].

Finite volume element (FVE) methods are discretization tools widely used in engineering applications. The methods possess the advantages of local modeling and simple structures and offer the flexibility to handle complicated geometries. More importantly, the methods ensure local mass conservation, a highly desirable property in many applications. We refer to the monographs [16,22] for general presentations of these methods, and to the papers [2,4,5,11,17,27,32,33] (also the references therein) for more details.

To the best of our knowledge, little progress has been made on the FVE solution of problems of the form (1.1). A reason for this might be that the analysis for the nonlinear term is often very involved. For the linear case, a unified approach is presented in [10] to derive error estimates in the L^2, H^1, and L^∞ norms by connecting FVE methods with finite element (FE) methods. Error estimates and superconvergence results in the L^p norm ($2 \le p < \infty$) are obtained in [11]. In [8], FVE methods for two-dimensional linear parabolic problems in convex polygonal domains are studied and error estimates in the H^1, L^2, and L^∞ norms under limited regularities of exact solutions are established. In order to solve the discrete equations more efficiently, several symmetric FVE schemes are developed in [23,25].

On the other hand, developing efficient algorithms for finite volume element methods is an interesting problem and has been attracting many researchers' attention. The convergence of a V-cycle multigrid algorithm for a FVE method for variable coefficient elliptic problems is considered in [9]. Two-grid FVE methods are presented in [2] for linear and nonlinear elliptic problems and error estimates are derived to justify efficiency of the algorithms. Residual type *a posteriori* error estimates and an adaptive strategy for the finite volume approximation are developed in [6] to treat two- and three- dimensional steady-state convection diffusion reaction problems. In [34], a two-level additive Schwarz domain decomposition FVE method is studied and its convergence rates are shown to be optimal and independent of the number of subregions.

The purpose of this paper is to formulate and analyze a postprocessing FVE procedure for the semilinear parabolic initial boundary value problem (1.1). We first prove the optimal order error estimates in the H^1 and L^2 norm for the standard FVE scheme under certain regularity assumptions on the solution. The main difficulty for this part is to treat the locally Lipschitz continuous nonlinearity and prove the existence of the numerical solution. Furthermore, we develop a postprocessing algorithm to improve efficiency of the methods. The post-processing technique can be seen as a novel two-level or two-grid method, which involves an additional solution on a finer grid after the time evolution is finished. Unlike the traditional two-grid or two-level approaches, there is no communication from fine to coarse meshes until the end of time-marching [13,18,24]. This means that the extra cost of the postprocessing is relatively negligible when compared with the cost of computations from $t = 0$ to $t = T$ on the coarser mesh. In [19], the postprocessed FE methods are proved to have a higher rate of convergence in H^1 and L^2 norms than the standard ones when other than piecewise linear elements are used. A postprocessing linear FE scheme is studied in [14] and the improved H^1 convergence rate is observed. The above analysis is extended to fully discrete case and both temporal and spatial estimates are obtained in [31]. We want to point out that although postprocessing techniques have been studied extensively in the FE framework, how to apply them to FVE methods is still not very well known. There are certain difficulties in handling piecewise constant test functions and nonsymmetric bilinear forms.

The rest of this paper is organized as follows. In Section 2, we describe the FVE method for the semilinear parabolic initial boundary value problem (1.1). In Section 3 we derive optimal order semidiscrete error estimates for finite volume approximation in the H^1 and L^2 norms under certain regularity assumptions. The postprocessing FVE procedure and the improved error estimate in H^1 norm are established in Section 4. Finally numerical experiments are presented in Section 5 to illustrate the theoretical analysis.

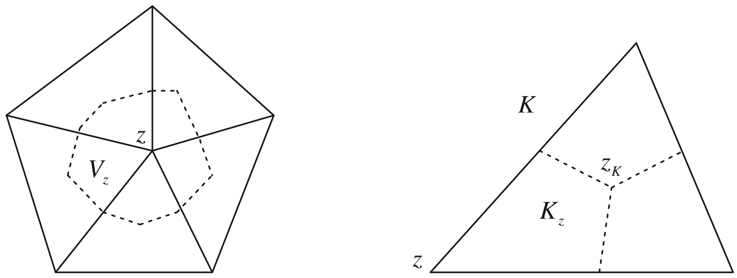

FIGURE 1. *Left*: A sample region with dotted lines indicating the corresponding control volume V_z. *Right*: A triangle K partitioned into three quadrilaterals K_z.

Throughout this paper we use C and ϵ to denote a generic positive constant and a generic small positive constant independent of discretization parameters.

2. Finite volume element scheme

2.1. **Notations**

We shall use the standard notations for the Sobolev space $W_p^m(\Omega)$ with the norm $\|\cdot\|_{m,p,\Omega}$ and the seminorm $|\cdot|_{m,p,\Omega}$, see [1]. To simplify the notations, we denote $W_2^m(\Omega)$ by $H^m(\Omega)$ and skip the index $p = 2$ and Ω, when there is no ambiguity. That is, $\|u\|_{m,p} = \|u\|_{m,p,\Omega}$, $\|u\|_m = \|u\|_{m,2,\Omega}$. The same convention is adopted for the seminorms as well. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with $h = \max h_K$, where h_K is the diameter of any triangle $K \in \mathcal{T}_h$. For this primal triangulation, let S_h be the standard conforming finite element space of piecewise linear functions,

$$S_h = \{v \in C(\Omega) : v|_K \text{ linear}, \ \forall K \in \mathcal{T}_h; \ v|_{\partial\Omega} = 0\}\cdot$$

In order to describe the FVE method, we introduce a dual partition $\mathcal{T}_h^*$ whose elements are called control volumes. We construct the control volumes in the same way as in [7,15]. Let z_K be the barycenter of any $K \in \mathcal{T}_h$. We connect z_K using line segments to the edge midpoints of K, and divide K into three quadrilaterals K_z, $z \in Z_h(K)$, where $Z_h(K)$ is a set of the vertices of K, see Figure 1. For each vertex $z \in Z_h = \cup_{K\in\mathcal{T}_h} Z_h(K)$, we associate a control volume V_z, which consists of the union of the subregions K_z sharing the vertex z. Thus we obtain a group of control volumes covering the domain Ω. This is the dual partition $\mathcal{T}_h^*$. We denote the set of interior vertices of Z_h by Z_h^0.

A dual partition $\mathcal{T}_h^*$ is regular or quasi-uniform, if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \leq \text{meas}(V_z) \leq Ch^2, \ \forall V_z \in \mathcal{T}_h^*.$$

We want to point out that a barycenter-type dual partition can be constructed for any finite element triangulation $\mathcal{T}_h$ and involves relatively simple calculations. In addition, if the primal triangulation $\mathcal{T}_h$ is quasi-uniform, then the dual partition $\mathcal{T}_h^*$ is also quasi-uniform.

2.2. Construction of FVE scheme

We formulate the FVE method for the problem (1.1). Given a vertex $z \in Z_h^0$, we integrate (1.1) over the associated control volume V_z and apply the Green's formula to obtain

$$\int_{V_z} u_t \mathrm{d}x - \int_{\partial V_z} (A\nabla u)\cdot \mathbf{n}\mathrm{d}s = \int_{V_z} f(\mathbf{x}, t, u)\mathrm{d}x, \tag{2.1}$$

where n denotes the unit outer normal vector to ∂V_z. It should be noted that the above formulation is a way of stating that we have an integral conservation form on the control volume.

The integral relation (2.1) can be written in a variational form similar to that of the finite element method with the help of an interpolation operator $I_h^* : S_h \to S_h^*$ defined by

$$I_h^* v = \sum_{z\in Z_h^0} v(z)\Psi_z,$$

where

$$S_h^* = \{v \in L^2(\Omega) : v|_{V_z} \text{ constant},\ \forall z \in Z_h^0;\ v|_{V_z} = 0,\ \forall z \in \partial\Omega\},$$

and Ψ_z is the characteristic function of the control volume V_z. It was shown in [11] that

$$\|v_h - I_h^* v_h\|_{0,p} \le Ch^s \|v_h\|_{s,p}, \quad s = 0, 1, \tag{2.2}$$

and in [7] that

$$\|I_h^* v_h\|_{0,p} \le C\|v_h\|_{0,p} \tag{2.3}$$

for all $v_h \in S_h$ and $p > 1$. Furthermore, $(v_h, I_h^* w_h)$ is symmetric and positive definite for any $v_h, w_h \in S_h$. Therefore, it defines an inner product on S_h, and the corresponding discrete norm is equivalent to the L^2 norm. In other words, there exist two constants $C_* > 0$ and $C^* > 0$ independent of h such that

$$C_*\|v_h\|_0 \le |||v_h|||_0 \le C^*\|v_h\|_0, \quad \forall v_h \in S_h, \tag{2.4}$$

with $|||v_h|||_0 = (v_h, I_h^* v_h)^{1/2}$.

For any $I_h^* v_h$, we multiply (2.1) by $v_h(z)$ and sum over all $z \in Z_h^0$ to obtain

$$(u_t, I_h^* v_h) + a_h(u, I_h^* v_h) = (f(\mathbf{x}, t, u), I_h^* v_h), \quad \forall v_h \in S_h, \tag{2.5}$$

where the bilinear form $a_h(\cdot, I_h^* \cdot)$ is defined as: for any $u \in H_0^1(\Omega), v_h \in S_h$,

$$a_h(u, I_h^* v_h) = -\sum_{z\in Z_h^0} v_h(z) \int_{\partial V_z} (A\nabla u)\cdot \mathbf{n}\mathrm{d}s.$$

Our semidiscrete FVE method for problem (1.1) is to find $u_h(t) \in S_h$ for all $0 \le t \le T$ such that

$$(u_{h,t}, I_h^* v_h) + a_h(u_h, I_h^* v_h) = (f(\mathbf{x}, t, u_h), I_h^* v_h), \quad \forall v_h \in S_h, \tag{2.6}$$

with the initial approximation given by

$$u_h(0) = R_h u_0,$$

where $R_h : H_0^1(\Omega) \to S_h$ denotes the elliptic projector satisfying

$$a_h(R_h u, I_h^* v_h) = a_h(u, I_h^* v_h), \quad \forall v_h \in S_h. \tag{2.7}$$

In [10,11,15], it was proved that

$$\|u - R_h u\|_1 \le Ch\|u\|_2, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega), \tag{2.8}$$

$$\|u - R_h u\|_{0,p} \le Ch^2\|u\|_{3,q}, \quad \forall u \in W_q^3(\Omega) \cap H_0^1(\Omega), \tag{2.9}$$

where $q > 1$ if $p = 2$, and $q = 2p/(p+2)$ if $p > 2$.

Remark 2.1. Let $\{\Phi_z : z \in Z_h^0\}$ be the standard basis functions of S_h, $\{\Psi_z : z \in Z_h^0\}$ be the associated basis of S_h^*, and $u_h(t) = \sum_{z \in Z_h^0} \alpha_z(t)\Phi_z$. Then scheme (2.6) can be written as a system of ordinary differential equations

$$M\alpha'(t) + S\alpha(t) = \tilde{f}(t, \alpha(t)), \ 0 \le t \le T; \ \alpha(0) = \beta,$$

where $M = ((\Phi_z, \Psi_w))_{zw}$ and $S = (a_h(\Phi_z, \Psi_w))_{zw}$ are the mass and stiffness matrices, respectively, and $\alpha(t)$ and β vectors of the nodal values of $u_h(t)$ and $R_h u_0$. Thus scheme (2.6) represents a non-autonomous system of ordinary differential equations with a locally Lipschitz continuous right-hand side. From (2.4) and Lemma 3.1 below, we know that M is symmetric, and both M and S are positive definite. This implies that there exists a unique local solution u_h on a certain maximal subinterval $[0, t_{**})$ of $[0, T]$. We will show in Lemma 3.4 that $t_{**} = T$ for sufficiently small h.

3. Error analysis of the finite volume element scheme

We will frequently use the following Sobolev's inequality [1]: for $p \in [1, \infty)$, there exists a constant $C = C(\Omega, p)$ such that

$$\|v\|_{0,q} \le C\|v\|_{s,p}, \quad \frac{1}{p} \ge \frac{1}{q} \ge \frac{1}{p} - \frac{s}{2}, \ \forall v \in W_p^s(\Omega). \tag{3.1}$$

Since $\mathcal{T}_h$ is quasi-uniform, the following inverse estimate holds for all $v \in S_h$, see [3,12]:

$$\|v\|_{m,p} \le Ch^{l-m-2(\frac{1}{q}-\frac{1}{p})}\|v\|_{l,q}, \quad 0 \le l \le m \le 1, \ 1 \le q \le p \le \infty. \tag{3.2}$$

The following two lemmas have been proved in [10], where Lemma 3.1 indicates that the bilinear form $a_h(\cdot, I_h^*\cdot)$ is continuous and coercive on S_h, while Lemma 3.2 shows that $a_h(\cdot, I_h^*\cdot)$ is generally unsymmetric but not too far away from being symmetric.

Lemma 3.1. *For h sufficiently small, there exist two positive constants C^* and C_* independent of h such that*

$$|a_h(w_h, I_h^* v_h)| \le C^*\|w_h\|_1\|v_h\|_1, \quad \forall w_h, v_h \in S_h, \tag{3.3}$$

$$a_h(v_h, I_h^* v_h) \ge C_*\|v_h\|_1^2, \quad \forall v_h \in S_h. \tag{3.4}$$

Lemma 3.2. *For h sufficiently small, there exists a constant $C > 0$ such that*

$$|a_h(w_h, I_h^* v_h) - a_h(v_h, I_h^* w_h)| \le Ch\|w_h\|_1\|v_h\|_1, \quad \forall w_h, v_h \in S_h. \tag{3.5}$$

In [21], the following result has been established regarding the local Lipschitz continuity of f as a mapping from $L^{2(\gamma+1)}(\Omega)$ to $L^2(\Omega)$.

Lemma 3.3. *Suppose that f satisfies the condition (1.2). Then there exists a positive constant $C = C(\gamma, C_f, \Omega)$ such that*

$$\|f(\mathbf{x}, t, w) - f(\mathbf{x}, t, v)\|_0 \le C\|w - v\|_{0,2(\gamma+1)}(1 + \|w\|_{0,2(\gamma+1)}^\gamma + \|v\|_{0,2(\gamma+1)}^\gamma) \tag{3.6}$$

for all $w, v \in L^{2(\gamma+1)}(\Omega)$ and a.e. $(\mathbf{x}, t) \in \Omega \times (0, T]$.

Now we state the first main result that estimates the H^1 norm error between the elliptic projection of the exact solution and the FVE approximation. It also asserts the existence of an approximation solution $u_h(t)$ in the whole time period $[0,T]$.

Lemma 3.4. *Let u and u_h be the solutions of (1.1) and (2.6), respectively. Let R_hu be the elliptic projection of u onto S_h defined in (2.7). Suppose that (1.2) and (1.3) hold. Then, there exists $h_0 > 0$ and a positive constant $C = C(\gamma, C_f, \Omega, M)$ independent of the discretization parameter, such that for all $h \in (0, h_0]$ and $t \in [0,T]$,*

$$\|R_hu - u_h\|_1 \le Ch^2. \tag{3.7}$$

Proof. We decompose the error as $u_h - u = \xi - \eta$, where $\xi = u_h - R_hu$ and $\eta = u - R_hu$. According to Remark 2.1, there exists a maximal interval $[0, t_{**}) \subset [0,T]$ such that $u_h(t)$ exists for all $t \in [0, t_{**})$. Here either $t_{**} = T$ or $t_{**} < T$, and $\lim_{t\to t_{**}} |u_h(t)| = +\infty$. We will show below that $t_{**} = T$, if h is small enough.

From (2.5) and (2.6), we have the following error equation

$$(\xi_t, I_h^* v_h) + a_h(\xi, I_h^* v_h) = (\eta_t, I_h^* v_h) + (f(\mathbf{x}, t, u_h) - f(\mathbf{x}, t, u), I_h^* v_h), \quad t \in (0, t_{**}). \tag{3.8}$$

Taking $v_h = \xi_t$ in (3.8) leads to

$$|||\xi_t|||_0^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a_h(\xi, I_h^*\xi) = \frac{1}{2}[a_h(\xi_t, I_h^*\xi) - a_h(\xi, I_h^*\xi_t)] + (\eta_t, I_h^*\xi_t) + (f(\mathbf{x}, t, u_h) - f(\mathbf{x}, t, u), I_h^*\xi_t).$$

It follows from Lemma 3.2 and the inverse estimate (3.2) that

$$|a_h(\xi_t, I_h^*\xi) - a_h(\xi, I_h^*\xi_t)| \le Ch\|\xi\|_1\|\xi_t\|_1 \le C\|\xi\|_1\|\xi_t\|_0 \le C\|\xi\|_1^2 + \epsilon\|\xi_t\|_0^2.$$

By (2.3),

$$\begin{aligned}|(\eta_t, I_h^*\xi_t)| + |(f(\mathbf{x}, t, u_h) - f(\mathbf{x}, t, u), I_h^*\xi_t)| &\le \|\eta_t\|_0\|I_h^*\xi_t\|_0 + \|f(\mathbf{x}, t, u_h) - f(\mathbf{x}, t, u)\|_0\|I_h^*\xi_t\|_0 \\ &\le C(\|\eta_t\|_0^2 + \|f(\mathbf{x}, t, u_h) - f(\mathbf{x}, t, u)\|_0^2) + \epsilon\|\xi_t\|_0^2.\end{aligned}$$

Thus, using (2.4) and choosing ϵ small enough, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} a_h(\xi, I_h^*\xi) &\le C(\|\eta_t\|_0^2 + \|f(\mathbf{x}, t, u_h) - f(\mathbf{x}, t, u)\|_0^2 + \|\xi\|_1^2) \\ &\le C(\|\eta_t\|_0^2 + \|f(\mathbf{x}, t, u_h) - f(\mathbf{x}, t, R_hu)\|_0^2 + \|f(\mathbf{x}, t, R_hu) - f(\mathbf{x}, t, u)\|_0^2 + \|\xi\|_1^2),\end{aligned} \tag{3.9}$$

for all $t \in (0, t_{**})$. By Lemma 3.3, the Sobolev's inequality (3.1) and (2.8), we have

$$\begin{aligned}\|f(\mathbf{x}, t, R_hu) - f(\mathbf{x}, t, u)\|_0 &\le C\|\eta\|_{0,2(\gamma+1)}(1 + \|u\|_{0,2(\gamma+1)}^\gamma + \|R_hu\|_{0,2(\gamma+1)}^\gamma) \\ &\le C\|\eta\|_{0,2(\gamma+1)}(1 + \|u\|_1^\gamma + \|R_hu\|_1^\gamma) \\ &\le C\|\eta\|_{0,2(\gamma+1)}(1 + \|u\|_1^\gamma + (\|u\|_1 + h\|u\|_2)^\gamma) \\ &\le C\|\eta\|_{0,2(\gamma+1)}.\end{aligned} \tag{3.10}$$

On the other hand, by a similar argument as above, we have

$$\|f(\mathbf{x}, t, R_hu) - f(\mathbf{x}, t, u_h)\|_0 \le C\|\xi\|_1(1 + \|\xi\|_1^\gamma).$$

Note that $\xi(0) = 0$. By the continuity of $\xi(t)$, we set $t_* \in (0, t_{**}]$ to be the largest time such that u_h exists and $\|\xi(t)\|_1 \le 1$ for all $t \in [0, t_*]$. Next we shall show that $t_* = T$, if h is small enough. That also means $t_{**} = T$.

Now for all $t \in (0, t_*]$, we have

$$\|f(\mathbf{x}, t, R_h u) - f(\mathbf{x}, t, u_h)\|_0 \le C\|\xi\|_1, \tag{3.11}$$

where $C > 0$ is a constant depending on the norms of u over the time interval $[0, t^*]$.

By (3.9)–(3.11), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} a_h(\xi, I_h^*\xi) \le C(\|\eta_t\|_0^2 + \|\eta\|_{0,2(\gamma+1)}^2 + \|\xi\|_1^2), \quad t \in (0, t_*]. \tag{3.12}$$

Integrating (3.12) from 0 to $t \le t_*$, noting $\xi(0) = 0$, and using Lemma 3.1 (coercivity), we have

$$\begin{aligned} C_*\|\xi(t)\|_1^2 &\le a_h(\xi(t), I_h^*\xi(t)) \\ &\le C\int_0^t (\|\eta_t(s)\|_0^2 + \|\eta(s)\|_{0,2(\gamma+1)}^2 + \|\xi(s)\|_1^2)\mathrm{d}s, \quad t \in [0, t_*]. \end{aligned}$$

Then the Gronwall's inequality and (2.9) imply that

$$\begin{aligned} \|\xi(t)\|_1^2 &\le C\int_0^t (\|\eta(s)\|_{0,2(\gamma+1)}^2 + \|\eta_t(s)\|_0^2)\mathrm{d}s \\ &\le Ch^4\int_0^t (\|u(s)\|_{3,q}^2 + \|u_t(s)\|_{3,r}^2)\mathrm{d}s, \quad t \in [0, t_*], \end{aligned}$$

where $q > 1$ if $\gamma = 0$, $q = 2(\gamma+1)/(\gamma+2)$ if $\gamma > 0$, and $r > 1$. Hence,

$$\|\xi(t)\|_1 \le Ch^2, \quad t \in [0, t_*]. \tag{3.13}$$

Therefore, there exists a sufficiently small h_0 such that for $h \in (0, h_0]$, $\|\xi(t)\|_1 \le Ch^2 < 1$, $t \in [0, t_*]$. It follows from the continuity of the function $\|\xi(t)\|_1$ that $t_* = t_{**}$. Otherwise, by continuity there must be a t_*', $t_* < t_*' \le t_{**}$ such that $\|\xi(t)\|_1 \le 1$ for all $t \in [0, t_*']$. But this contradicts the definition of t_*.

Now for $h \in (0, h_0]$, we have $\|\xi(t)\|_1 \le Ch^2 < 1$, $t \in [0, t_{**}]$. We shall show that $t_{**} = T$. Suppose that $t_{**} < T$. Then by the definition of t_{**}, we have $\lim_{t\to t_{**}} |u_h(t)|_\infty = +\infty$. But on the contrary, it follows from a triangle inequality and (3.13) that

$$\begin{aligned} \lim_{t\to t_{**}} |u_h(t)|_\infty &\le \lim_{t\to t_{**}} (|\xi(t)|_\infty + |R_h u(t)|_\infty) \\ &\le \lim_{t\to t_{**}} C|\ln h|^{1/2}\|\xi(t)\|_1 + |R_h u(t_{**})|_\infty \\ &\le C|\ln h|^{1/2}h^2 + |R_h u(t_{**})|_\infty \le \text{const.}, \end{aligned}$$

where we have used the asymptotic Sobolev's inequality $\|v_h\|_{0,\infty} \le C|\ln h|^{1/2}\|v_h\|_1$, $v_h \in S_h$ (see [26]). Thus it must be $t_{**} = T$, *i.e.*, the FVE approximation u_h exists on the whole of $[0, T]$. From the argument above, we have that for $h \in (0, h_0]$,

$$\|R_h u(t) - u_h(t)\|_1 = \|\xi(t)\|_1 \le Ch^2, \quad t \in [0, T],$$

which gives the desired result. □

Remark 3.5. We can see from the above lemma that the presence of a locally Lipschitz nonlinearity, satisfying a certain growth condition, leads to certain difficulties in the error analysis but will not degrade the convergence rate observed in the linear case. Moreover, unlike the linear case, we have to prove the existence of an approximate solution on the entire time interval. Some techniques such as using (asymptotic) Sobolev's inequalities and the bootstrap argument play a crucial role in the proof. A similar proof was presented in [17,21].

By (2.8), (2.9), and Lemma 3.4, we can derive the following H^1 and L^2 error estimates for the finite volume element scheme.

Theorem 3.6. *Let u and u_h be the solutions of (1.1) and (2.6), respectively. Assume the condition in Lemma 3.4 holds. Then for all $h \in (0, h_0]$ and $t \in [0, T]$, we have*

$$\|u - u_h\|_0 + h\|u - u_h\|_1 \le Ch^2, \tag{3.14}$$

where $C = C(\gamma, C_f, \Omega, M)$ is independent of the discretization parameter.

4. Postprocessing and its error analysis

In this section, we present the postprocessing finite volume element algorithm for the semilinear parabolic problem (1.1) based on two finite element spaces. There are two quasi-uniform triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$, with two different mesh sizes H and h ($H > h$). The corresponding finite element spaces S_H and S_h satisfy $S_H \subset S_h$ and are called the coarser and the finer spaces, respectively.

Suppose that we are interested in the solution of (1.1) at time T. Then the idea of our postprocessing technique is to solve the semilinear parabolic problem on a coarser grid T_H from $(0, T]$ and then solve a symmetric linear elliptic problem on a finer grid $\mathcal{T}_h$ only once, at $t = T$.

In order to present the postprocessing FVE scheme, we introduce the following auxiliary bilinear form: for any $u \in H_0^1(\Omega)$, $v_h \in S_h$,

$$\bar{a}_h(u, I_h^* v_h) = -\sum_{z \in Z_h^0} v_h(z) \int_{\partial V_z} (\bar{A}\nabla u) \cdot \mathbf{n} \mathrm{d}s, \tag{4.1}$$

where $\bar{A}|_K = A_K$, and

$$A_K = \frac{1}{\mathrm{meas}(K)} \int_K A(\mathbf{x}) \mathrm{d}x, \quad \forall K \in \mathcal{T}_h.$$

The following lemma has been proved in [10,15].

Lemma 4.1. *For any $w_h, v_h \in S_h$, we have*

$$\bar{a}_h(w_h, I_h^* v_h) = a(w_h, v_h), \tag{4.2}$$

where $a(\cdot, \cdot)$ is the bilinear form related to the finite element method, i.e.*,*

$$a(w_h, v_h) = \int_\Omega A\nabla w_h \cdot \nabla v_h \mathrm{d}x.$$

Our postprocessing FVE procedure reads as:

(1) Find $u_H(t) \in S_H$ such that, for any $v_H \in S_H$,

$$(u_{H,t}, I_H^* v_H) + a_H(u_H, I_H^* v_H) = (f(\mathbf{x}, t, u_H), I_H^* v_H), \quad t \in (0, T]. \tag{4.3}$$

We take $u_H(0) = R_H u_0$ as an initial approximation.

(2) Find $u_h \in S_h$ such that for any $v_h \in S_h$,

$$\bar{a}_h(u_h, I_h^* v_h) = -(u_{H,t}, I_h^* v_h) + (f(\mathbf{x}, t, u_H), I_h^* v_h), \quad t = T. \tag{4.4}$$

We know from Lemma 3.2 that the matrix of $a_h(v_h, I_h^* w_h)$ is generally nonsymmetric. This introduces some difficulties in real implementations and the method suitable for symmetric linear systems cannot be used in this case. From Lemma 4.1, we know that the coefficient matrix of the linear system in the second step of the postprocessing procedure is symmetric and positive definite and hence easier to solve. For example, the conjugate gradient methods can be applied effectively.

In [2], the following lemma reveals the difference between the bilinear form of FVE method and that of finite element method.

Lemma 4.2. *For any $w_h, v_h \in S_h$, $w \in H^2(\Omega)$, $p \geq 1$, $1/p + 1/q = 1$, we have*

$$|a(w_h, v_h) - a_h(w_h, I_h^* v_h)| \leq Ch^2(h^{-1}|w - w_h|_{1,p} + \|w\|_{2,p})\|v_h\|_{1,q}. \tag{4.5}$$

Next we next state and prove three more technical lemmas to be used in the error analysis of the postprocessing FVE scheme.

Lemma 4.3. *Let u and u_h be the solutions of (1.1) and (2.6), respectively. Suppose that (1.2) and (1.3) hold. Then there exists a positive constant $C = C(\gamma, C_f, \Omega, M)$ such that*

$$\|u_t - u_{h,t}\|_0 \leq Ch, \quad t \in [0, T]. \tag{4.6}$$

Proof. Taking $v_h = \xi_t$ in (3.8), and using (2.3) and the inverse estimate (3.2), we obtain

$$\begin{aligned} |||\xi_t|||_0^2 &\leq C\|\xi\|_1\|\xi_t\|_1 + \|\eta_t\|_0\|I_h^*\xi_t\|_0 + \|f(\mathbf{x}, t, u) - f(\mathbf{x}, t, u_h)\|_0\|I_h^*\xi_t\|_0 \\ &\leq C\|\xi_t\|_0(\|\xi\|_1 h^{-1} + \|\eta_t\|_0 + \|f(\mathbf{x}, t, u) - f(\mathbf{x}, t, u_h)\|_0). \end{aligned} \tag{4.7}$$

By Theorem 3.6,

$$\|u_h\|_1 \leq \|u\|_1 + \|u - u_h\|_1 \leq \text{const.} \tag{4.8}$$

Taking into account Lemma 3.3 and the Sobolev's inequality, we have

$$\begin{aligned} \|f(\mathbf{x}, t, u) - f(\mathbf{x}, t, u_h)\|_0 &\leq C\|u - u_h\|_{0,2(\gamma+1)}(1 + \|u\|_{0,2(\gamma+1)}^\gamma + \|u_h\|_{0,2(\gamma+1)}^\gamma) \\ &\leq C\|u - u_h\|_1(1 + \|u\|_1^\gamma + \|u_h\|_1^\gamma) \\ &\leq C\|u - u_h\|_1. \end{aligned} \tag{4.9}$$

Combining (4.7) with (4.9), and using (2.4), (2.9), Lemma 3.4, and Theorem 3.6, we obtain

$$\|\xi_t\|_0 \leq C(\|\xi\|_1 h^{-1} + \|\eta_t\|_0 + \|u - u_h\|_1) \leq C(h + h^2).$$

Combined with (2.9), this finishes the proof. □

Lemma 4.4. *Let u and u_h be the solutions of (1.1) and (2.6), respectively. Suppose that (1.2) and (1.3) hold. Then there exists a positive constant $C = C(\gamma, C_f, \Omega, M)$ such that*

$$|(f(\mathbf{x}, t, u) - f(\mathbf{x}, t, u_h), I_h^* v_h)| \leq Ch^2\|v_h\|_1, \tag{4.10}$$

for any $v_h \in S_h$ and $t \in [0, T]$.

Proof. Let $1/p + 1/q = 1/2$ with $p = 2(1+\gamma)$. By (1.2), the Hölder's inequality, and (2.3), we obtain

$$\begin{aligned}|(f(\mathbf{x},t,u) - f(\mathbf{x},t,u_h), I_h^* v_h)| &\le C_f \int_\Omega |u-u_h|(1+|u|+|u_h|)^\gamma |I_h^* v_h| \mathrm{d}x \\ &\le C_f \|u-u_h\|_0 \|(1+|u|+|u_h|)^\gamma\|_{0,q} \|I_h^* v_h\|_{0,p} \\ &\le C \|u-u_h\|_0 \|(1+|u|+|u_h|)^\gamma\|_{0,q} \|v_h\|_{0,p}.\end{aligned}$$

Since $1/q = (p-2)/2p = \gamma/p$, we have

$$\begin{aligned}\|(1+|u|+|u_h|)^\gamma\|_{0,q} &= \|1+|u|+|u_h|\|_{0,p}^\gamma \\ &\le (1+\|u\|_{0,p} + \|u_h\|_{0,p})^\gamma.\end{aligned}$$

Note that $p \ge 2$. Using the above two estimates and the Sobolev's inequality (3.1), we obtain

$$\begin{aligned}|(f(\mathbf{x},t,u) - f(\mathbf{x},t,u_h), I_h^* v_h)| &\le C\|u-u_h\|_0 (1+\|u\|_{0,p} + \|u_h\|_{0,p})^\gamma \|v_h\|_{0,p} \\ &\le C\|u-u_h\|_0 \|(1+\|u\|_1 + \|u_h\|_1)^\gamma \|v_h\|_1.\end{aligned}$$

Combining Theorem 3.6 and (4.8) leads to

$$|(f(\mathbf{x},t,u) - f(\mathbf{x},t,u_h), I_h^* v_h)| \le C\|u-u_h\|_0 \|v_h\|_1 \le Ch^2 \|v_h\|_1,$$

which gives the desired result. □

Lemma 4.5. *Let u and u_h be the solutions of (1.1) and (2.6), respectively. Suppose that (1.2) and (1.3) hold. Then there exists a positive constant $C = C(\gamma, C_f, \Omega, M)$ such that*

$$|(u_t - u_{h,t}, I_h^* v_h)| \le Ch^2 \|v_h\|_1, \tag{4.11}$$

for any $v_h \in S_h$ and $t \in (0,T]$.

Proof. From (2.5), (2.6), and Lemma 3.1 (continuity), we have

$$\begin{aligned}|(u_t - u_{h,t}, I_h^* v_h)| &= |a_h(R_h u - u_h, I_h^* v_h) + (f(\mathbf{x},t,u_h) - f(\mathbf{x},t,u), I_h^* v_h)| \\ &\le |a_h(R_h u - u_h, I_h^* v_h)| + |(f(\mathbf{x},t,u_h) - f(\mathbf{x},t,u), I_h^* v_h)| \\ &\le C^* \|R_h u - u_h\|_1 \|v_h\|_1 + |(f(\mathbf{x},t,u_h) - f(\mathbf{x},t,u), I_h^* v_h)|. \end{aligned} \tag{4.12}$$

Therefore, Lemmas 3.4 and 4.4 together with (4.12) yield the desired estimate. □

Now comes the main result of this section.

Theorem 4.6. *Let $u(T)$ be the solution of (1.1) at time T and u_h be the solution of (4.3) and (4.4). Assume that (1.2) and (1.3) hold. Then, there exists a positive constant $C = C(\gamma, C_f, \Omega, M)$ independent of the discretization parameters such that*

$$\|u(T) - u_h\|_1 \le C(h + H^2). \tag{4.13}$$

Proof. We drop the explicit dependence on $\mathbf{x}$ and T for conciseness. From (2.5), (4.3), and (4.4), we have

$$\begin{aligned}\bar{a}_h(R_h u - u_h, I_h^* v_h) &= (\bar{a}_h(R_h u, I_h^* v_h) - a_h(R_h u, I_h^* v_h)) + a_h(R_h u, I_h^* v_h) - \bar{a}_h(u_h, I_h^* v_h)\\ &= (\bar{a}_h(R_h u, I_h^* v_h) - a_h(R_h u, I_h^* v_h)) + a_h(u, I_h^* v_h) - \bar{a}_h(u_h, I_h^* v_h)\\ &= (\bar{a}_h(R_h u, I_h^* v_h) - a_h(R_h u, I_h^* v_h)) - (u_t - u_{H,t}, I_h^* v_h) + (f(u) - f(u_H), I_h^* v_h)\\ &= S_1 - S_2 + S_3.\end{aligned}$$

By Lemmas 4.1, 4.2, and (2.8), we have

$$\begin{aligned}|S_1| &= |\bar{a}_h(R_h u, I_h^* v_h) - a_h(R_h u, I_h^* v_h)|\\ &= |a(R_h u, v_h) - a_h(R_h u, I_h^* v_h)|\\ &\le Ch^2(h^{-1}\|u - R_h u\|_1 + \|u\|_2)\|v_h\|_1\\ &\le Ch^2\|u\|_2\|v_h\|_1.\end{aligned} \tag{4.14}$$

To estimate S_2, we rewrite S_2 as follows

$$\begin{aligned}S_2 &= (u_t - u_{H,t}, I_h^* v_h)\\ &= (u_t - u_{H,t}, I_h^*(v_h - I_H v_h)) + (u_t - u_{H,t}, I_H^*(I_H v_h)) + (u_t - u_{H,t}, I_h^*(I_H v_h) - I_H^*(I_H v_h))\\ &= S_{21} + S_{22} + S_{23},\end{aligned}$$

where $I_H : C(\Omega) \to S_H$ is the general linear interpolation operator satisfying (see, *e.g.*, [3,12])

$$\|v - I_H v\|_{m,p} \le CH^{s-m}\|v\|_{s,p} \tag{4.15}$$

for $0 \le m \le s \le 2$ and $v \in W_p^s(\Omega), 1 \le p \le \infty$. It is easy to see that $\|I_H v\|_1 \le C\|v\|_1$. From (2.3), Lemma 4.3, and (4.15), we have

$$\begin{aligned}|S_{21}| &= |(u_t - u_{H,t}, I_h^*(v_h - I_H v_h))|\\ &\le \|u_t - u_{H,t}\|_0\|I_h^*(v_h - I_H v_h)\|_0\\ &\le CH\|v_h - I_H v_h\|_0\\ &\le CH^2\|v_h\|_1.\end{aligned}$$

Using Lemma 4.5, we obtain

$$\begin{aligned}|S_{22}| &= |(u_t - u_{H,t}, I_H^*(I_H v_h))|\\ &\le CH^2\|I_H v_h\|_1 \le CH^2\|v_h\|_1.\end{aligned}$$

It follows from Lemma 4.3, a triangle inequality and (2.2) that

$$\begin{aligned}|S_{23}| &= |(u_t - u_{H,t}, I_h^*(I_H v_h) - I_H^*(I_H v_h))|\\ &\le \|u_t - u_{H,t}\|_0\|I_h^*(I_H v_h) - I_H^*(I_H v_h)\|_0\\ &\le CH(\|I_h^*(I_H v_h) - I_H v_h\|_0 + \|I_H^*(I_H v_h) - I_H v_h\|_0)\\ &\le CH(h\|I_H v_h\|_1 + H\|I_H v_h\|_1)\\ &\le CH(h + H)\|v_h\|_1.\end{aligned}$$

From the above inequalities, we obtain an estimate of S_2:

$$|S_2| \le CH(h+H)\|v_h\|_1 \le CH^2\|v_h\|_1. \tag{4.16}$$

Similarly, we have

$$\begin{aligned} |S_3| &= |(f(u)-f(u_H), I_h^* v_h)| \\ &\le |(f(u)-f(u_H), I_h^*(v_h - I_H v_h))| + |(f(u)-f(u_H), I_H^*(I_H v_h))| \\ &\quad + |(f(u)-f(u_H), I_h^*(I_H v_h) - I_H^*(I_H v_h))| \\ &\le CH\|f(u)-f(u_H)\|_0\|v_h\|_1 + CH^2\|I_H v_h\|_1 \\ &\quad + C(h+H)\|f(u)-f(u_H)\|_0\|v_h\|_1 \\ &\le CH\|u-u_H\|_1\|v_h\|_1 + CH^2\|v_h\|_1 + C(h+H)\|u-u_H\|_1\|v_h\|_1 \\ &\le CH^2\|v_h\|_1, \end{aligned} \tag{4.17}$$

where we have used (4.9), Lemma 4.4, and Theorem 3.6. Combining (4.14), (4.16) and (4.17), we have

$$|\bar{a}_h(R_h u - u_h, I_h^* v_h)| \le CH^2\|v_h\|_1.$$

Taking $v_h = R_h u - u_h$ and using Lemma 3.1 (coercivity), we obtain

$$\|R_h u - u_h\|_1 \le CH^2.$$

It follows from (2.8) and a triangle inequality that

$$\|u-u_h\|_1 \le \|R_h u - u_h\|_1 + \|u - R_h u\|_1 \le C(H^2 + h),$$

which yields the desired result. □

Remark 4.7. From Theorem 4.6, we see that if $h = \mathcal{O}(H^2)$, then the highest possible convergence rate in the H^1 norm for the postprocessing FVE method is $\mathcal{O}(H^2)$. Thus the postprocessing procedure improves the convergence rate over the standard FVE method error estimate in the H^1 norm, which is only $\mathcal{O}(H)$ (see Thm. 3.6), by one order. Since the mesh refinement is performed only at the final time T, the method increases the accuracy of the standard FVE approximation at low extra computational costs.

Remark 4.8. We consider the spatial discretization to focus on postprocessing. In practical computations, the method should be combined with a time-stepping algorithm. Let N be a positive integer. Consider a temporal discretization $0 = t_0 < t_1 < \ldots < t_N = T$ and set $u^n = u(\cdot, t_n)$ $(0 \le n \le N)$. Then an implicit backward Euler postprocessing procedure is given by

(1) Find $u_H^n \in S_H$ such that for any $v_H \in S_H$,

$$\left(\frac{u_H^n - u_H^{n-1}}{t_n - t_{n-1}}, I_H^* v_H\right) + a_H(u_H^n, I_H^* v_H) = (f^n(\mathbf{x}, u_H^n), I_H^* v_H), \quad 1 \le n \le N$$

with $u_H^0 = R_H u_0$.

TABLE 1. H^1 and L^2 errors and convergence rates on the coarser grid.

N	H^1-norm error	Rate	L^2-norm error	Rate
2	1.1909×10^{-1}	–	6.7219×10^{-2}	–
4	9.0843×10^{-2}	0.40	1.5746×10^{-2}	2.09
8	5.6665×10^{-2}	0.68	3.8616×10^{-3}	2.03
16	3.1461×10^{-2}	0.85	9.5043×10^{-4}	2.02
32	1.6556×10^{-2}	0.93	2.7318×10^{-4}	1.80
64	8.5047×10^{-3}	0.96	1.0567×10^{-2}	1.37

TABLE 2. Postprocessed H^1 errors and convergence rates on the finer grids.

N	$h = 1/N^2$	Rate	$h = 1/(3N)$	Rate
2	9.0458×10^{-2}	–	7.0077×10^{-2}	–
4	3.1494×10^{-2}	1.52	4.0542×10^{-2}	0.79
8	8.5010×10^{-3}	1.89	2.1721×10^{-2}	0.90
16	2.2445×10^{-3}	1.92	1.1236×10^{-2}	0.95
32	5.7212×10^{-4}	1.97	5.7134×10^{-3}	0.98

(2) Find $u_h \in S_h$ such that for any $v_h \in S_h$,

$$\bar{a}_h(u_h, I_h^* v_h) = -\left(\frac{u_H^N - u_H^{N-1}}{t_N - t_{N-1}}, I_h^* v_h\right) + (f^N(\mathbf{x}, u_H^N), I_h^* v_h).$$

Of course, higher order temporal discretization methods such as the Runge-Kutta methods or multistep methods can also be used. On the other hand, from a practical point of view, we just need to choose $h < H$ to obtain a considerable error reduction in spite of the demanding requirement $h = \mathcal{O}(H^2)$.

5. Numerical experiments

In this section, we present numerical experiments to illustrate the theoretical results presented in the previous sections. In particular, our main interest is to verify Theorems 3.6 and 4.6. We consider the following parabolic equation with a quadratic nonlinearity,

$$u_t - \nabla \cdot \left(\frac{1}{1 + |\mathbf{x}|^2} \nabla u\right) + u^2 = f(\mathbf{x}, t), \quad \text{in } \Omega \times (0, T]$$

with a homogeneous Dirichlet boundary condition. The domain is $\Omega = [0, 1]^2$, the final time is $T = 1$, the exact solution is $u(x, y, t) = \mathrm{e}^{-t/2} \sin(\pi x) \sin(\pi y)$, and the right hand side $f(\mathbf{x}, t)$ is computed accordingly.

The domain Ω is partitioned into N uniform pieces in each direction and then each rectangle is divided into two triangles, resulting in a mesh with size $H = 1/N$. The finite element space S_H is built on the coarser grid $\mathcal{T}_H$ with $N = 2, 4, 8, 16, 32, 64$. We use the backward Euler temporal formula with a relatively small time step $\Delta t = 10^{-3}$, so that the dominant error will be the spatial error. Table 1 lists the errors in the H^1 and L^2 norms and also the convergence rates at $t = T$, respectively. The results are in accordance with the estimates obtained in Theorem 3.6 for $N \leq 32$. It is not a surprise to see that the L^2 error convergence rate for $N = 64$ drops significantly. This is due to the fact that the expected L^2 error for the backward Euler fully discrete scheme is of order $O(\Delta t + H^2)$ and the error is dominated by the temporal approximation as N increases.

To illustrate the theoretical findings in Theorem 4.6, we compute the postprocessing FVE approximation at T on two finer grids with $h = H^2$ and $h = H/3$, respectively. Table 2 shows that, if $h = H^2$, then the H^1 convergence rate is close to second order; but if $h = H/3$, the H^1 convergence rate is nearly unchanged, although the error is smaller when compared with the results in Table 1. We also find that the H^1 errors in Table 1 for $N = 4, 16, 64$ are approximately the same as those in Table 2 for $N = 2, 4, 8$, when $h = H^2$ is used. This means that one step postprocessing on a finer grid can yield the same accuracy as a standard all-time-level FVE computation on the same grid. Therefore, these results confirm Theorem 4.6 and Remark 4.7.

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