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FORMAL PASSAGE FROM KINETIC THEORY TO INCOMPRESSIBLE NAVIER–STOKES EQUATIONS FOR A MIXTURE OF GASES

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Abstract. We present in this paper the formal passage from a kinetic model to the incompressible Navier−Stokes equations for a mixture of monoatomic gases with different masses. The starting point of this derivation is the collection of coupled Boltzmann equations for the mixture of gases. The diffusion coefficients for the concentrations of the species, as well as the ones appearing in the equations for velocity and temperature, are explicitly computed under the Maxwell molecule assumption in terms of the cross sections appearing at the kinetic level.

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1. Introduction

The formal derivation of the incompressible Navier−Stokes system for a single gas starting from the Boltzmann equation was first described in details in [3]. It was later made rigorous under quite general assumptions on the cross section appearing in the Boltzmann equation (for monoatomic gases) [2,17,18,24,26–28].

Our goal here is to extend the formal derivation of the incompressible Navier−Stokes equations (still starting from equations of Boltzmann type) to the case of a mixture of gases. More precisely, we consider the evolution of a mixture of N elastically scattering monoatomic rarefied gases A^s, $s = 1, \ldots, N$ with particle mass of the sth species denoted by m^s. Let $f^s := f^s(t, \mathbf{x}, \mathbf{v})$ $(s = 1, \ldots, N)$ be the phase space density of each gas. The Boltzmann equation in this setting writes (for $s = 1, \ldots, N$)

$$\partial_t f^s + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^s = \sum_{r=1}^{N} Q^{sr}(f^s, f^r)\,, \tag{1.1}$$

where

$$Q^{sr}(f^s, f^r)(\mathbf{v}) = \iint q\,\sigma^{sr}(q, \chi) \left(f^s(\mathbf{v}')\, f^r(\mathbf{w}') - f^s(\mathbf{v})\, f^r(\mathbf{w}) \right) \mathrm{d}\mathbf{w}\, \mathrm{d}\hat{\boldsymbol{\Omega}}', \tag{1.2}$$

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$$\mathbf{v}' = \frac{m^s}{m^s + m^r}\,\mathbf{v} + \frac{m^r}{m^s + m^r}\,\mathbf{w} + \frac{m^r}{m^s + m^r}\,|\mathbf{v} - \mathbf{w}|\,\hat{\boldsymbol{\Omega}}', \tag{1.3}$$

$$\mathbf{w}' = \frac{m^s}{m^s + m^r}\,\mathbf{v} + \frac{m^r}{m^s + m^r}\,\mathbf{w} - \frac{m^s}{m^s + m^r}\,|\mathbf{v} - \mathbf{w}|\,\hat{\boldsymbol{\Omega}}', \tag{1.4}$$

the quantity $\mathbf{q} = \mathbf{v} - \mathbf{w} = q\,\hat{\boldsymbol{\Omega}}$ is the pre-collision relative velocity (q and $\hat{\boldsymbol{\Omega}}$ are its modulus and direction), $\mathbf{q}' = \mathbf{v}' - \mathbf{w}' = q\,\hat{\boldsymbol{\Omega}}'$ is the post-collision one ($q' = q$ because of momentum and energy conservations), σ^{sr} is the differential cross section (note that $\sigma^{sr} = \sigma^{rs}$) and χ is the angle formed by pre- and post-interaction relative velocity: $\cos\chi = \hat{\boldsymbol{\Omega}} \cdot \hat{\boldsymbol{\Omega}}'$.

It has been shown in the case of a single gas [3] that the scaling of the Boltzmann equations for the distributions $f^s(t, \mathbf{x}, \mathbf{v})$, $s = 1, \ldots, N$ that turns out to be compatible with the incompressible fluid-dynamic limit is

$$\varepsilon\,\partial_t f^s_\varepsilon + \mathbf{v} \cdot \nabla_{\mathbf{x}} f^s_\varepsilon = \frac{1}{\varepsilon} \sum_{r=1}^{N} Q^{sr}(f^s_\varepsilon, f^r_\varepsilon), \tag{1.5}$$

where the small parameter ε stands for the Knudsen number. The dominant process in the evolution is thus the elastic scattering, while the time scale is taken of order ε^{-1}. Analogously to [3], we look for solutions to (1.5) in the form

$$f^s_\varepsilon = \rho^s\; M^s_{(1,\mathbf{0},1)}(1 + \varepsilon\, g^s_\varepsilon), \tag{1.6}$$

where $\rho^s > 0$ are constants and $M^s_{(1,\mathbf{0},1)}$ are absolute normalized Maxwellians with number density equal to 1, mass velocity equal to $\mathbf{0}$, temperature equal to 1, *i.e.*

$$M^s(\mathbf{v}) = \left(\frac{m^s}{2\pi}\right)^{3/2} \mathrm{e}^{-\frac{m^s}{2}\,v^2}. \tag{1.7}$$

Without loss of generality, we may assume

$$\rho = \sum_{s=1}^{N} \rho^s = 1. \tag{1.8}$$

A crucial role in the study of the re-scaled equations (1.5) will be played by the linearized bi-species elastic operator

$$\sum_{r=1}^{N} \rho^s \rho^r \Big[Q^{sr}(g^s_\varepsilon M^s, M^r) + Q^{sr}(M^s, g^r_\varepsilon M^r)\Big]. \tag{1.9}$$

In the case of gases with different particle masses and different cross sections, it is well known that classical Grad's methodology [19, 20] cannot be easily applied to study the formal mean free path limit. However, for the most typical cross sections, that is, hard potentials with cutoff, it has been recently proved [12] that the operator (1.9) has the same good properties as the linearized operator for gases with the same mass (studied for instance in [5]). In particular the non multiplicative part of the operator (1.9) is compact in a suitable L^2-type space, so that linearized systems of Boltzmann equations may be solved owing to the Fredholm alternative. For this reason we expect that the general form of evolution equations that we shall derive in the sequel still holds for a large class of intermolecular potentials. However, since our aim is to build up consistent and completely explicit macroscopic equations, that can be compared with analogous hydrodynamic systems (with coefficients found by means of thermodynamical considerations) used in physical applications, we compute all diffusion coefficients appearing in the macroscopic equations in the case of cross sections of Maxwell molecules type. In this collision frame, our main result is the following:

Proposition 1.1. *Consider a family f_ε^s of solutions of (1.5), with Q^{sr} given by (1.2). Assume also that the intermolecular potential is chosen in such a way that the collision kernels (differential cross section times the relative speed) depend only on the deflection angle χ [15] (that is, the interaction is of Maxwell molecules type):*

$$q\,\sigma^{sr}(q,\chi) = \vartheta^{sr}(\chi), \tag{1.10}$$

and define

$$\begin{aligned} \kappa^{sr} &= 2\pi \int_0^\pi \vartheta^{sr}(\chi)(1-\cos\chi)\sin\chi\,\mathrm{d}\chi, \\ \nu^{sr} &= 2\pi \int_0^\pi \vartheta^{sr}(\chi)(1-\cos^2\chi)\sin\chi\,\mathrm{d}\chi. \end{aligned} \tag{1.11}$$

Then formally, the scaling (1.6) holds, with

$$g_\varepsilon^s(\mathbf{v}) = \alpha^s + m^s\,\mathbf{v}\cdot\mathbf{u} + \left(\frac{1}{2}m^s\,v^2 - \frac{3}{2}\right)T + O(\varepsilon), \tag{1.12}$$

where the parameters α^s, $\mathbf{u}$, T depend on t and $\mathbf{x}$ and satisfy the following Navier–Stokes system for mixtures:

– *Incompressibility condition:*

$$\nabla_{\mathbf{x}}\cdot\mathbf{u} = 0. \tag{1.13}$$

– *Boussinesq identity:*

$$\nabla_{\mathbf{x}}\left(\sum_{s=1}^N \left(\rho^s\,\alpha^s\right) + T\right) = \mathbf{0}. \tag{1.14}$$

– *Convection-diffusion equations for the densities of the species:*

$$\begin{aligned} &\partial_t\left[\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}(\alpha^s-\alpha^r)\right] + \mathbf{u}\cdot\nabla_{\mathbf{x}}\left[\sum_{r\neq s}\rho^r\mu^{sr}\kappa^{sr}(\alpha^s-\alpha^r)\right] \\ &= \Delta_{\mathbf{x}}\left[\sum_{r\neq s}\rho^r(\alpha^s-\alpha^r)\right], \qquad\qquad s=1,\dots,N-1, \end{aligned} \tag{1.15}$$

where $\mu^{sr} = \frac{m^s\,m^r}{m^s+m^r}$ is the reduced mass.

– *Convection-diffusion equation for the momentum:*

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}p = d_1\ \Delta_{\mathbf{x}}\mathbf{u}. \tag{1.16}$$

– *Convection-diffusion equation for the temperature:*

$$\partial_t T + \mathbf{u}\cdot\nabla_{\mathbf{x}}T = d_2\,\Delta_{\mathbf{x}}T. \tag{1.17}$$

In the above equations, $d_1 > 0$, $d_2 > 0$ are diffusion coefficients given by the following formulas:

$$d_1 = \sum_{s=1}^N \rho^s\,\theta^s, \tag{1.18}$$

where the parameters θ^s are the unique solution of the linear system

$$\begin{aligned}&\left[\frac{3}{4}\,\rho^s\nu^{ss}+\sum_{r\neq s}\rho^r\frac{\mu^{sr}}{m^s+m^r}\left(2\kappa^{sr}+\frac{3}{2}\,\frac{m^r}{m^s}\nu^{sr}\right)\right]\theta^s\\&+\sum_{r\neq s}\rho^r\frac{\mu^{sr}}{m^s+m^r}\left(-\,2\kappa^{sr}+\frac{3}{2}\,\nu^{sr}\right)\theta^r=\left(\sum_{s=1}^{N}\rho^s\,m^s\right)^{-1},\qquad s=1,\ldots,N\,;\end{aligned}\tag{1.19}$$

and

$$d_2=\sum_{s=1}^{N}\frac{\rho^s}{\sqrt{m^s}}\,\eta^s,\tag{1.20}$$

where the parameters η^s are the unique solution of the linear system

$$\begin{aligned}&\left\{\frac{1}{2}\,\rho^s(m^s)^{1/2}\nu^{ss}+\sum_{r\neq s}\rho^r\frac{\mu^{sr}}{(m^s+m^r)^2}\Big[(m^s)^{-1/2}\big(3(m^s)^2+(m^r)^2\big)\kappa^{sr}+2(m^s)^{1/2}m^r\nu^{sr}\Big]\right\}\eta^s\\&+\sum_{r\neq s}\rho^r\frac{\mu^{sr}}{(m^s+m^r)^2}\,m^s(m^r)^{1/2}\big(-\,4\kappa^{sr}+2\,\nu^{sr}\big)\eta^r=1,\qquad s=1,\ldots,N.\end{aligned}\tag{1.21}$$

We will see that the expression of the perturbation g^s_ε given in (1.12), and the incompressibility and Boussinesq constraints (1.13), (1.14) hold for any intermolecular potentials.

Note that the system (1.13)–(1.17) is not strongly coupled, in the sense that evolution equation (1.16) could be solved separately (it does not depend on the other unknown fields α^s, T) providing global velocity $\mathbf{u}$ as function of time t and space $\mathbf{x}$. Then, there remains a system of $N+1$ equations for concentrations α^s and temperature T: the Boussinesq condition (1.14) and the N evolution equations (1.15) and (1.17).

Note that the Boussinesq relation becomes $\sum_{s=1}^N\big(\rho^s\,\alpha^s\big)+T=0$ if suitable boundary conditions are imposed, and this yields immediately one of the number densities (for instance α^N) as function of the other ones and of the temperature. Moreover, note that the parameters α^s, T are not necessarily nonnegative, since they are only perturbations at the first order of the coefficients appearing in a Maxwellian function of $\mathbf{v}$.

A self-consistent system coupling number densities and temperature like (1.14), (1.15), (1.17) provides a mathematical justification of the fact, known in physical applications and in extended thermodynamics frame, that in several problems regarding gas mixtures the evolution of concentrations is strongly affected by diffusion of the global temperature, while it depends upon the velocity only through the advection term.

Note that in the present scaling, not all macroscopic degrees of freedom of the fluid appear in the macroscopic equations. Velocities or temperatures specific to each species would appear only if we considered higher orders in expansion (1.12), obtaining Burnett-type equations, or if we took as dominant operator (of order $1/\varepsilon$) in the sth Boltzmann equation only $Q^{ss}(f^s_\varepsilon,f^s_\varepsilon)$, describing elastic collisions between particles of the same species. This scaling, leading of course to a completely different class of macroscopic models (multi-temperatures and multi-velocities), has been studied for instance in [11]. Other formal hydrodynamic limits from kinetic models for (inert or reactive) mixtures have been already performed in compressible asymptotic regimes [8,10]. For binary mixtures, there are also results concerning a Cahn–Hilliard diffusion model coupled with a fluid motion [7,25,30].

The main differences between this work and the incompressible limit performed in [3] in the one-species case, are the following:

- In the one-species case, the Boussinesq condition in strong form is simply $\alpha+T=0$, hence no equation is needed for concentration α since it is completely known from the equation for T. For a mixture the Boussinesq constraint is a link between T and the sum of number densities, so that $N-1$ additional independent evolution equations have to be consistently derived, and these are the ones given in (1.15).

Note that in the case of a mixture of two species, one can directly consider the difference of the two kinetic equations satisfied by the two species in order to get the needed equation. When more than two species are concerned, one has to find the "right" linear combinations between the kinetic equations. These combinations depend on the masses and cross sections, as can be seen in the limiting equation (1.15).

- Also, the assumption of different particle masses complicates the formal derivation of the equations for $\mathbf{u}$ and T, even if at first glance they are exactly the ones expected by physical considerations. Even for Maxwell molecules, the computation of d_1 and d_2, and the proof that these diffusion coefficients are actually strictly positive for any values of masses and collision frequencies require several algebraic manipulations that are not a direct extension of the case of a mixture of two gases.

Before starting the (formal) proof of Proposition 1.1, we compare the set of equations obtained in Proposition 1.1 to the equations which can be obtained by performing the limit of low Mach number regime in the systems of compressible Navier−Stokes equations for mixtures. We indeed know that incompressible Navier–Stokes equations may be derived also as low Mach number limit of the compressible ones in the case of a single rarefied gas (*cf.* [1, 6, 23]).

We briefly indicate here how the same strategy can be applied for mixtures, starting from the compressible Navier−Stokes equations for mixtures described for example in [13, 14]. Denoting ρ^s, m^s the number density and the particle mass of each species, and $\mathbf{u}$ the global mass velocity, the system writes in the case of a mixture of monoatomic perfect gases:

$$\begin{cases} \partial_t \rho^s + \nabla_{\mathbf{x}} \cdot (\rho^s \mathbf{u}) = \nabla_{\mathbf{x}} \cdot \mathbf{F}^s \qquad s = 1, \dots, N, \\ \partial_t \left(\sum_{s=1}^N \rho^s m^s \mathbf{u} \right) + \nabla_{\mathbf{x}} \cdot \left(\sum_{s=1}^N \rho^s m^s \mathbf{u} \otimes \mathbf{u} \right) + \nabla_{\mathbf{x}} p = \nabla_{\mathbf{x}} \cdot \boldsymbol{\Pi}, \\ \partial_t \left[\sum_{s=1}^N \rho^s \left(m^s \frac{u^2}{2} + \frac{3}{2} T \right) \right] + \nabla_{\mathbf{x}} \cdot \left[\sum_{s=1}^N \rho^s \left(m^s \frac{u^2}{2} + \frac{5}{2} T \right) \mathbf{u} \right] = \nabla_{\mathbf{x}} \cdot (\boldsymbol{\Pi} \cdot \mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}, \end{cases} \tag{1.22}$$

where $p = \sum_{s=1}^N \rho^s T$ is the state law, and $\mathbf{F}^s$, $\boldsymbol{\Pi}$, $\mathbf{q}$ are the diffusion terms given by

$$\begin{aligned} \mathbf{F}^s &= \sum_{j=1}^N L_{sj} \nabla_{\mathbf{x}} \left(\frac{\rho^j}{(2\pi T/m^j)^{3/2}} \right) + L_{sq} \nabla_{\mathbf{x}} (1/T), \\ \boldsymbol{\Pi} &= \left(-\frac{2}{3} \eta \right) (\nabla_{\mathbf{x}} \cdot \mathbf{u}) \mathbf{I} + \eta \left(\nabla_{\mathbf{x}} \mathbf{u} + (\nabla_{\mathbf{x}} \mathbf{u})^T \right), \\ \mathbf{q} &= \sum_{j=1}^N L_{qj} \nabla_{\mathbf{x}} \left(\frac{\rho^j}{(2\pi T/m^j)^{3/2}} \right) - L_{qq} \nabla_{\mathbf{x}} (1/T). \end{aligned} \tag{1.23}$$

Here the diffusion coefficients L_{sj}, L_{sq}, L_{qj}, L_{qq}, η may depend on the temperature T of the mixture. Moreover, they satisfy the constraints of conservation of total mass $\sum_{s=1}^N \rho^s m^s$.

Note that the equation of conservation of energy can be rewritten as an equation for the temperature in the following way:

$$\partial_t T + \mathbf{u} \cdot \nabla_{\mathbf{x}} T + \frac{2}{3} \frac{p}{\sum_{s=1}^N \rho^s} \nabla_{\mathbf{x}} \cdot \mathbf{u} = \frac{2}{3} \frac{1}{\sum_{s=1}^N \rho^s} \nabla_{\mathbf{x}} \mathbf{u} : \boldsymbol{\Pi} + \frac{2}{3} \frac{1}{\sum_{s=1}^N \rho^s} \nabla_{\mathbf{x}} \cdot \mathbf{q}.$$

The scaling of low Mach number corresponds [16] to keeping the first line of (1.22), and rescaling the velocity and temperature equations as

$$\varepsilon^2 \left[\partial_t \left(\sum_{s=1}^N \rho^s m^s \mathbf{u} \right) + \nabla_{\mathbf{x}} \cdot \left(\sum_{s=1}^N \rho^s m^s \mathbf{u} \otimes \mathbf{u} \right) - \nabla_{\mathbf{x}} \cdot \boldsymbol{\Pi} \right] = - \nabla_{\mathbf{x}} p, \tag{1.24}$$

and

$$\partial_t T + \mathbf{u} \cdot \nabla_{\mathbf{x}} T + \frac{2}{3} \frac{p}{\sum_{s=1}^N \rho^s} \nabla_{\mathbf{x}} \cdot \mathbf{u} = \frac{2}{3} \frac{\varepsilon^2}{\sum_{s=1}^N \rho^s} \nabla_{\mathbf{x}} \mathbf{u} : \boldsymbol{\Pi} + \frac{2}{3} \frac{1}{\sum_{s=1}^N \rho^s} \nabla_{\mathbf{x}} \cdot \mathbf{q}. \tag{1.25}$$

We focus on the simple case when all masses are equal (we denote $m = m^s$), and when the Soret and Dufour coefficients L_{qj} and L_{sq} are zero, as is expected for Navier−Stokes equations coming out (by the Chapman–Enskog procedure) from the Boltzmann equation in the case of Maxwell molecules (*cf.* [13]).

Expanding the densities, velocity, pressure and temperature around constant states ρ_0^s, $\mathbf{u}_0$, T_0, p_0, in powers of ε, we end up with

$$\rho^s(t,\mathbf{x}) = \rho_0^s \, (1 + \varepsilon\, \alpha^s(t,\mathbf{x})) + O(\varepsilon^2), \qquad \mathbf{u}(t,\mathbf{x}) = \mathbf{u}_0(t,\mathbf{x}) + \varepsilon\, \mathbf{u}_1(t,\mathbf{x}) + O(\varepsilon^2),$$

$$T(t,\mathbf{x}) = T_0 + \varepsilon\, T_1(t,\mathbf{x}) + O(\varepsilon^2)\,, \qquad p(t,\mathbf{x}) = p_0 + \varepsilon^2 p_2(t,\mathbf{x}) + O(\varepsilon^2).$$

Writing

$$\nabla_{\mathbf{x}} \left(\sum_{s=1}^N \rho^s \, T \right) = \varepsilon \, \nabla_{\mathbf{x}} \left(T_0 \sum_{s=1}^N \rho_0^s \alpha^s + \sum_{s=1}^N \rho_0^s \, T_1 \right) + O(\varepsilon^2), \tag{1.26}$$

and observing that the terms in $O(\varepsilon)$ have to vanish, we get the Boussinesq relation (1.14), where ρ^s is replaced by $\rho_0^s \, \frac{T_0}{\rho_0}$ (with $\rho_0 = \sum_{s=1}^N \rho_0^s$), and T is replaced by T_1, that is

$$\nabla_{\mathbf{x}} \left(\frac{T_0}{\rho_0} \sum_{s=1}^N \rho_0^s \, \alpha^s + T_1 \right) = 0.$$

Considering now the evolution equation for total mass density

$$m \, \partial_t \left(\sum_{s=1}^N \rho^s \right) + m \, \mathbf{u} \cdot \nabla_{\mathbf{x}} \left(\sum_{s=1}^N \rho^s \right) + m \sum_{s=1}^N \rho^s \, \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \tag{1.27}$$

and observing that the first and the second terms in the equation are of order $O(\varepsilon)$, we get in the limit the incompressibility condition (1.13), with $\mathbf{u}$ replaced by $\mathbf{u}_0$, that is,

$$\nabla_{\mathbf{x}} \cdot \mathbf{u}_0 = 0.$$

Using equation (1.27) at next order, we get

$$\partial_t \left(\sum_{s=1}^N \rho_0^s \alpha^s \right) + \mathbf{u}_0 \cdot \nabla_{\mathbf{x}} \left(\sum_{s=1}^N \rho_0^s \alpha^s \right) + \rho_0 \, \nabla_{\mathbf{x}} \cdot \mathbf{u}_1 = 0. \tag{1.28}$$

By using this into the expression of $\mathbf{\Pi}$ given in (1.23), we get that $\nabla_{\mathbf{x}} \cdot \boldsymbol{\Pi}$ tends to $\eta \, \Delta_{\mathbf{x}} \mathbf{u}_0$, so that the momentum equation (1.24) becomes

$$\partial_t \mathbf{u}_0 + \mathbf{u}_0 \cdot \nabla_{\mathbf{x}} \mathbf{u}_0 + \nabla_{\mathbf{x}} \tilde{p} = D_1 \Delta_{\mathbf{x}} \mathbf{u}_0, \tag{1.29}$$

where the diffusion coefficient $D_1 = \eta/(m\, \rho_0)$, and $\tilde{p}$ is a Lagrange multiplier. This corresponds to (1.16), with $\mathbf{u}$ replaced by $\mathbf{u}_0$, p replaced by $\tilde{p}$, and d_1 replaced by D_1.

We now use the expansions in equation (1.25), and get

$$\varepsilon (\partial_t T_1 + \mathbf{u}_0 \cdot \nabla_{\mathbf{x}} T_1) + \frac{2}{3} \varepsilon \, \frac{p_0}{\rho_0} \, \nabla_{\mathbf{x}} \cdot \mathbf{u}_1 = \frac{2}{3} \frac{1}{\rho_0} \, \varepsilon^2 \, \nabla_{\mathbf{x}} \mathbf{u}_0 : \mathbf{\Pi} + \frac{2}{3} \frac{1}{\rho_0} \, \nabla_{\mathbf{x}} \cdot \mathbf{q} + O(\varepsilon^2). \tag{1.30}$$

Using equation (1.28) in order to compute $\nabla_{\mathbf{x}} \cdot \mathbf{u}_1$ in the formula above, and dividing by ε, we end up with

$$\frac{5}{3}\,(\partial_t T_1 + \mathbf{u}_0 \cdot \nabla_{\mathbf{x}} T_1) = \frac{2}{3}\,\frac{L_{qq}(T_0)}{\rho_0 T_0^2}\,\Delta_{\mathbf{x}} T_1.$$

We recover in this way equation (1.17), with T replaced by T_1, $\mathbf{u}$ replaced by $\mathbf{u}_0$, and $d_2 = \frac{2}{5}\,\frac{L_{qq}(T_0)}{\rho_0 T_0^2}$.

Finally, let us consider the evolution equation for number densities. We start from the first equation of (1.22), and use the expansion of ρ^s. We first observe that

$$\partial_t \rho^s + \nabla_{\mathbf{x}} \cdot (\rho^s\,\mathbf{u}) = \varepsilon\,\rho_0^s(\partial_t \alpha^s + \mathbf{u}_0 \cdot \nabla_{\mathbf{x}} \alpha^s + \nabla_{\mathbf{x}} \cdot \mathbf{u}_1) + O(\varepsilon^2).$$

Using (1.28), we get

$$\partial_t \rho^s + \nabla_{\mathbf{x}} \cdot (\rho^s\,\mathbf{u}) = \varepsilon\,(\partial_t + \mathbf{u}_0 \cdot \nabla_{\mathbf{x}}) \left(\rho_0^s \alpha^s - \frac{\rho_0^s}{\rho_0} \sum_{r=1}^{N} \rho_0^r \alpha^r \right) + O(\varepsilon^2).$$

We also expand

$$\begin{aligned} &\sum_{j=1}^{N} L_{sj}(T)\,\nabla_{\mathbf{x}} \left(\frac{\rho^j}{(2\pi\,T/m)^{3/2}} \right) \\ &= \varepsilon \left(\frac{m}{2\pi\,T_0} \right)^{3/2} \sum_{j=1}^{N} L_{sj}(T_0)\,\nabla_{\mathbf{x}} \left(\rho_0^j \alpha^j - \frac{3}{2}\,\frac{\rho_0^j}{T_0}\,T_1 \right) + O(\varepsilon^2) \\ &= \varepsilon \left(\frac{m}{2\pi\,T_0} \right)^{3/2} \sum_{j=1}^{N} L_{sj}(T_0)\,\nabla_{\mathbf{x}} \left(\rho_0^j \alpha^j + \frac{3}{2}\,\frac{\rho_0^j}{\rho_0} \sum_{r=1}^{N} \rho_0^r \alpha^r \right) + O(\varepsilon^2), \end{aligned}$$

thanks to Boussinesq's identity.

Finally, letting ε go to 0, we end up with the identity

$$(\partial_t + \mathbf{u}_0 \cdot \nabla_{\mathbf{x}}) \left(\rho_0^s \alpha^s - \frac{\rho_0^s}{\rho_0} \sum_{r=1}^{N} \rho_0^r \alpha^r \right) = \left(\frac{m}{2\pi\,T_0} \right)^{3/2} \sum_{j=1}^{N} L_{sj}(T_0)\,\Delta_{\mathbf{x}} \left(\rho_0^j \alpha^j + \frac{3}{2}\,\frac{\rho_0^j}{\rho_0} \sum_{r=1}^{N} \rho_0^r \alpha^r \right).$$

We obtain therefore an identity which relates linearly the advection terms $(\partial_t + \mathbf{u}_0 \cdot \nabla_{\mathbf{x}})\alpha^s$ with the diffusion terms $\Delta_{\mathbf{x}} \alpha^j$, as in equation (1.15), with $\mathbf{u}$ replaced by $\mathbf{u}_0$.

As can be seen, the passage from Boltzmann equation to the incompressible Navier−Stokes system gives compatible results when compared with the passage from the compressible Navier−Stokes system towards the incompressible one.

The rest of the paper is devoted to the formal proof of Proposition 1.1 and is organized as follows. In Section 2, the expression (1.12) for the perturbation of a solution is derived. Then, Section 3 concerns the incompressibility and Boussinesq relations (1.13), (1.14). Section 4 is devoted to the evolution equations (1.15) for concentrations, (1.16) for momentum, (1.17) for temperature, respectively. Finally, we report in an Appendix technical lemmas and evaluations of suitable collision contributions used to obtain explicit expressions for diffusion coefficients d_1, d_2. Those computations are specific to the Maxwell molecules case.

2. The weak form of the collision operators

We recall that, in (1.5), Q^{sr} denotes the bi-species elastic operator, describing elastic scattering between particles of species s and r. The most useful tool in the sequel is its weak form:

$$\sum_{r=1}^{N} \int \varphi^s(\mathbf{v})\,Q^{sr}(f_\varepsilon^s, f_\varepsilon^r)\,\mathrm{d}\mathbf{v} = \sum_{r=1}^{N} \iiint q\,\sigma^{sr}(q,\chi)\,\Big[\varphi^s(\mathbf{v}') - \varphi^s(\mathbf{v})\Big] f_\varepsilon^s(\mathbf{v}) f_\varepsilon^r(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\hat{\boldsymbol{\Omega}}'. \tag{2.1}$$

We observe that

$$\int M^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = 1, \qquad \int \mathbf{v}\,M^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = \mathbf{0}, \qquad m^s \int v^2 M^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = 3. \tag{2.2}$$

By inserting distributions (1.6) into the Boltzmann equations (1.5), leading order terms vanish since Maxwellians M^s do not depend on $\mathbf{x}$ and satisfy $Q^{sr}(M^s, M^r) = 0$. There remain the equations

$$\begin{aligned} \varepsilon\,\rho^s \partial_t(g_\varepsilon^s M^s) + \mathbf{v}\cdot\rho^s\nabla_{\mathbf{x}}(g_\varepsilon^s M^s) &= \frac{1}{\varepsilon}\sum_{r=1}^N \rho^s\rho^r \Big[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r)\Big] \\ &\quad + \sum_{r=1}^N \rho^s\rho^r Q^{sr}(g_\varepsilon^s M^s, g_\varepsilon^r M^r). \end{aligned} \tag{2.3}$$

Leading order terms yield, for $s = 1,\dots,N$,

$$\sum_{r=1}^N \rho^s\rho^r \Big[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r)\Big] = O(\varepsilon). \tag{2.4}$$

Defining the linear operator $\mathcal{L}$ (with components $\mathcal{L}^1,..,\mathcal{L}^N$) as

$$\mathcal{L}^s(h^1,..,h^N) = (M^s)^{-1/2}\sum_{r=1}^N \Big[\rho^r\,Q^{sr}(h^s\,(M^s)^{1/2}, M^r) + \rho^s\,Q^{sr}\Big(M^s, h^r\,(M^r)^{1/2}\Big)\Big], \tag{2.5}$$

we know from [12] that (for cross sections of hard potentials type with angular cutoff, including pseudo-Maxwellian molecules and hard spheres) this operator $\mathcal{L}$ is the sum of, on one hand, a compact operator $\mathcal{K}$ from $(L^2(\mathbb{R}^3))^N$ to $(L^2(\mathbb{R}^3))^N$ and, on the other hand, a (component-wise) multiplication operator $(-\nu^s(\mathbf{v})\,Id)_{s=1,..,N}$ with spectrum included in an interval $]-\infty,-Z]$, with $Z>0$.

We also recall that using test functions $\varphi^s(\mathbf{v}) = g_\varepsilon^s(\mathbf{v})$, we get the linearized entropy inequality

$$\begin{aligned} &\sum_{s=1}^N\sum_{r=1}^N \rho^s\rho^r \int g_\varepsilon^s(\mathbf{v})\Big[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r)\Big]\mathrm{d}\mathbf{v} \\ &= -\frac{1}{4}\sum_{s=1}^N\sum_{r=1}^N \rho^s\rho^r \iiint q\,\sigma^{sr}(q,\chi)\,\Big[g_\varepsilon^s(\mathbf{v}') + g_\varepsilon^r(\mathbf{w}') - g_\varepsilon^s(\mathbf{v}) - g_\varepsilon^r(\mathbf{w})\Big]^2 M^s(\mathbf{v})M^r(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\hat{\boldsymbol{\Omega}}' \le 0 \end{aligned} \tag{2.6}$$

with equal sign if and only if the content of the square brackets vanishes $\forall s, r$.

Consequently, the spectrum of $\mathcal{L}$ is included in $\mathbb{R}^-$, and 0 is an eigenvalue of order $4+N$ of $\mathcal{L}$ whose eigenvectors are $[(M^1)^{1/2},\dots,(M^N)^{1/2}, \sum_{s=1}^N m^s\mathbf{v}(M^s)^{1/2}, \sum_{s=1}^N m^s v^2(M^s)^{1/2}]$, *cf.* [15].

Using Weyl's Theorem on compact perturbations of operators [15, 21], we see that the spectrum of $\mathcal{L}$ has a gap between 0 and a strictly negative number $-C$, so that for any $h = (h^1,..,h^N) \in L^2(\mathbb{R}^N)$,

$$||\mathcal{L}h||_{(L^2(\mathbb{R}^3))^N} \ge C\,||h - Ph||_{(L^2(\mathbb{R}^3))^N}, \tag{2.7}$$

where P is the L^2 projector on the vector space spanned by $[(M^s)^{1/2},\, m^s\mathbf{v}(M^s)^{1/2},\, m^s v^2(M^s)^{1/2}]_{s=1,..,N}$.

Using (2.4), we see that $\mathcal{L}([\rho^s g_\varepsilon^s (M^s)^{1/2}]_{s=1,\dots,N}) = O(\varepsilon)$, so that thanks to (2.7), g_ε^s is, up to $O(\varepsilon)$, a linear combination (with $t,\mathbf{x}$-dependent coefficients) of 1, $m^s\mathbf{v}$, $m^s v^2$, and (1.12) holds.

Notice that in (1.12) coefficients have been chosen in such a way that leading order species moments are

$$\begin{gathered} \int (g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = \alpha^s + O(\varepsilon), \qquad \int \mathbf{v}\,(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = \mathbf{u} + O(\varepsilon), \\ m^s \int v^2 (g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = 3\,(\alpha^s + T) + O(\varepsilon). \end{gathered} \tag{2.8}$$

Consequently, putting together (2.2) and (2.8), the moments of distributions f_ε^s are given by

$$\begin{gathered}\int f_\varepsilon^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = \rho^s(1+\varepsilon\,\alpha^s) + O(\varepsilon^2), \qquad \int \mathbf{v}\,f_\varepsilon^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = \varepsilon\,\rho^s\,\mathbf{u} + O(\varepsilon^2),\\ m^s\int v^2 f_\varepsilon^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = 3\,\rho^s + \varepsilon\,3\,\rho^s(\alpha^s+T) + O(\varepsilon^2).\end{gathered} \tag{2.9}$$

3. Conservation equations

By integrating the Boltzmann equations (1.5), we get

$$\varepsilon\,\partial_t\int f_\varepsilon^s(\mathbf{v})\,\mathrm{d}\mathbf{v} + \nabla_{\mathbf{x}}\cdot\int \mathbf{v}f_\varepsilon^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = 0, \qquad s=1,\dots,N, \tag{3.1}$$

representing conservation of single number densities, while by multiplying (1.5) by $m^s\mathbf{v}$ and summing up the N equations, we recover the momentum equation

$$\varepsilon\sum_{s=1}^N m^s\,\partial_t\int \mathbf{v}f_\varepsilon^s(\mathbf{v})\,\mathrm{d}\mathbf{v} + \sum_{s=1}^N m^s\,\nabla_{\mathbf{x}}\cdot\int \mathbf{v}\otimes\mathbf{v}f_\varepsilon^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = \mathbf{0}. \tag{3.2}$$

If we insert the ansatz (1.6) into (3.1) and (3.2), we get

$$\begin{gathered}\varepsilon\,\partial_t\int (g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \nabla_{\mathbf{x}}\cdot\int \mathbf{v}\,(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = 0, \qquad s=1,\dots,N,\\ \varepsilon\sum_{s=1}^N \rho^s\,m^s\,\partial_t\int \mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \sum_{s=1}^N \rho^s\,m^s\,\nabla_{\mathbf{x}}\cdot\int \mathbf{v}\otimes\mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = \mathbf{0}.\end{gathered} \tag{3.3}$$

Keeping the leading order term in the first line of (3.3) provides

$$\nabla_{\mathbf{x}}\cdot\int \mathbf{v}\,(g_\varepsilon^s\,M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = O(\varepsilon),$$

that, bearing in mind the second equality of (2.8), is nothing but the divergence-free condition for global velocity,

$$\nabla_{\mathbf{x}}\cdot\mathbf{u} = 0, \tag{3.4}$$

related to the incompressibility of the mixture.

On the other hand, keeping the leading order term in the second line of (3.3) yields

$$\sum_{s=1}^N \rho^s\,m^s\sum_j \frac{\partial}{\partial x_j}\int v_i\,v_j\,(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = O(\varepsilon),$$

i.e., taking into account the third part of (2.8) and the assumption (1.8),

$$\nabla_{\mathbf{x}}\left(\sum_{s=1}^N\left(\rho^s\,\alpha^s\right) + T\right) = \mathbf{0}, \tag{3.5}$$

which is a natural extension to a mixture of the Boussinesq relation of [3]. If we consider for example a bounded (periodic) space domain such as a torus, condition (3.5) implies [4, 22] the stronger relation

$$\sum_{s=1}^N\left(\rho^s\,\alpha^s\right) + T = 0. \tag{3.6}$$

Note that since α^s and T are perturbations, they are not required to be nonnegative; more precisely, constraint (3.6) implies that, for any fixed time t and position $\mathbf{x}$, at least one of these fields is nonpositive.

4. Equations for concentrations

As already pointed out earlier (see the first line of (3.3)), integrating the Boltzmann equations (1.5) yields

$$\varepsilon\,\partial_t \int (g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \nabla_{\mathbf{x}}\cdot \int \mathbf{v}\,(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = 0, \qquad s = 1,\ldots,N. \tag{4.1}$$

Unlike in the previous papers [3,4,22] dealing with a single rarefied gas, it is now necessary to find a suitable strategy that provides a consistent closure of the streaming part. The sought closure will be built up by resorting to the momentum equation of each species. By multiplying the Boltzmann equations (1.5) by $\mathbf{v}$, we get

$$\varepsilon\,\partial_t \int \mathbf{v} f^s_\varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{v} + \nabla_{\mathbf{x}}\cdot \int \mathbf{v}\otimes\mathbf{v} f^s_\varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{v} = \frac{1}{\varepsilon}\sum_{r\neq s}\int \mathbf{v}\;Q^{sr}(f^s_\varepsilon, f^r_\varepsilon)\,\mathrm{d}\mathbf{v}, \qquad s = 1,\ldots,N, \tag{4.2}$$

since the contributions due to $Q^{ss}(f^s_\varepsilon, f^s_\varepsilon)$ vanish (elastic scattering between particles of the same species preserves species momentum). By inserting (1.6) in the sth equation of (4.2), we obtain

$$\begin{aligned}
&\varepsilon^2\,\rho^s\partial_t \int \mathbf{v}(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \varepsilon\,\rho^s\nabla_{\mathbf{x}}\cdot\int \mathbf{v}\otimes\mathbf{v}(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}\\
&= \sum_{r\neq s}\Big\{\rho^s\rho^r \int \mathbf{v}\,\Big[Q^{sr}(g^s_\varepsilon M^s, M^r) + Q^{sr}(M^s, g^r_\varepsilon M^r)\Big]\,\mathrm{d}\mathbf{v}\\
&\qquad + \varepsilon\,\rho^s\rho^r\int \mathbf{v}\;Q^{sr}(g^s_\varepsilon M^s, g^r_\varepsilon M^r)\,\mathrm{d}\mathbf{v}\Big\}.
\end{aligned} \tag{4.3}$$

Let us evaluate the dominant term

$$\sum_{r\neq s}\rho^s\rho^r\int \mathbf{v}\;\Big[Q^{sr}(g^s_\varepsilon M^s, M^r) + Q^{sr}(M^s, g^r_\varepsilon M^r)\Big]\,\mathrm{d}\mathbf{v},$$

with the Maxwell molecule assumption (1.10). If we recall (1.11), all angular integrations needed here and in the rest of the paper will be amenable to the following ones [9]:

$$\int_{S^2}\vartheta^{sr}(\chi)(\mathbf{q}'-\mathbf{q})\,\mathrm{d}\hat{\boldsymbol{\Omega}}' = -\,\kappa^{sr}\,\mathbf{q}, \tag{4.4a}$$

$$\int_{S^2}\vartheta^{sr}(\chi)\big|\mathbf{q}'-\mathbf{q}\big|^2\,\mathrm{d}\hat{\boldsymbol{\Omega}}' = 2\,\kappa^{sr}\,q^2, \tag{4.4b}$$

$$\int_{S^2}\vartheta^{sr}(\chi)(\mathbf{q}'-\mathbf{q})\otimes(\mathbf{q}'-\mathbf{q})\,\mathrm{d}\hat{\boldsymbol{\Omega}}' = 2\,\kappa^{sr}\mathbf{q}\otimes\mathbf{q} + \frac{1}{2}\,\nu^{sr}\Big(q^2\,\mathbb{I} - 3\,\mathbf{q}\otimes\mathbf{q}\Big), \tag{4.4c}$$

$$\int_{S^2}\vartheta^{sr}(\chi)\big|\mathbf{q}'-\mathbf{q}\big|^2\,(\mathbf{q}'-\mathbf{q})\,\mathrm{d}\hat{\boldsymbol{\Omega}}' = -2\,\Big(2\,\kappa^{sr} - \nu^{sr}\Big)\,q^2\,\mathbf{q}. \tag{4.4d}$$

If we take $\varphi^s(\mathbf{v}) = \mathbf{v}$, from (1.3) we have

$$\varphi^s(\mathbf{v}') - \varphi^s(\mathbf{v}) = -\,\frac{m^r}{m^s+m^r}(\mathbf{v}-\mathbf{w}) + \frac{m^r}{m^s+m^r}\,|\mathbf{v}-\mathbf{w}|\,\hat{\boldsymbol{\Omega}}',$$

hence from (1.10) and (4.4) we get

$$\int q\,\sigma^{sr}(q,\chi)\,\Big[\varphi^s(\mathbf{v}') - \varphi^s(\mathbf{v})\Big]\,\mathrm{d}\hat{\boldsymbol{\Omega}}' = -\,\frac{\mu^{sr}}{m^s}\,\kappa^{sr}\,\mathbf{q},$$

where $\mu^{sr} = \frac{m^s m^r}{m^s+m^r}$ stands for the reduced mass. Consequently, bearing in mind the weak form of the elastic operator (2.1),

$$\begin{aligned}
&\int \mathbf{v}\left[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r)\right]\mathrm{d}\mathbf{v}\\
&= -\frac{\mu^{sr}}{m^s}\,\kappa^{sr}\iint(\mathbf{v}-\mathbf{w})\Big[(g_\varepsilon^s M^s)(\mathbf{v})\,M^r(\mathbf{w}) + M^s(\mathbf{v})\,(g_\varepsilon^r M^r)(\mathbf{w})\Big]\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\\
&= -\frac{\mu^{sr}}{m^s}\,\kappa^{sr}\left[\int \mathbf{v}\,(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} - \int \mathbf{v}\,(g_\varepsilon^r M^r)(\mathbf{v})\,\mathrm{d}\mathbf{v}\right].
\end{aligned} \tag{4.5}$$

In conclusion, the dominant term of the sth equation (4.3) may be rewritten as

$$\begin{aligned}
&\sum_{r\neq s}\rho^s\rho^r\int \mathbf{v}\,\left[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r)\right]\mathrm{d}\mathbf{v}\\
&= -\left(\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\right)\frac{\rho^s}{m^s}\int \mathbf{v}\,(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \frac{\rho^s}{m^s}\left(\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\int \mathbf{v}\,(g_\varepsilon^r M^r)(\mathbf{v})\,\mathrm{d}\mathbf{v}\right).
\end{aligned} \tag{4.6}$$

Coming back to evolution equations for number densities, if we consider the following linear combinations of equations (4.1):

$$\begin{aligned}
&\varepsilon\left(\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\right)\partial_t\int(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} - \varepsilon\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\partial_t\int(g_\varepsilon^r M^r)(\mathbf{v})\,\mathrm{d}\mathbf{v}\\
&+\nabla_{\mathbf{x}}\cdot\left\{\left(\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\right)\int \mathbf{v}\,(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} - \sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\int \mathbf{v}\,(g_\varepsilon^r M^r)(\mathbf{v})\,\mathrm{d}\mathbf{v}\right\} = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s = 1,\dots,N-1,
\end{aligned} \tag{4.7}$$

we note that the content in the curly brackets is directly proportional to the right hand side of (4.6), hence we can insert the sth momentum equation (4.3) into (4.7), ending up with

$$\begin{aligned}
&\left(\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\right)\partial_t\int(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} - \sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\partial_t\int(g_\varepsilon^r M^r)(\mathbf{v})\,\mathrm{d}\mathbf{v}\\
&+\nabla_{\mathbf{x}}\cdot\left\{-m^s\nabla_{\mathbf{x}}\cdot\int \mathbf{v}\otimes\mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + m^s\sum_{r\neq s}\rho^r\int \mathbf{v}\;Q^{sr}(g_\varepsilon^s M^s, g_\varepsilon^r M^r)\,\mathrm{d}\mathbf{v}\right\} = O(\varepsilon)
\end{aligned} \tag{4.8}$$

(all terms have been divided by ε).

Let us recall that distributions g_ε^s take the form (1.12), therefore

$$\begin{aligned}
&\int(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = \alpha^s + O(\varepsilon),\\
&m^s\int \mathbf{v}\otimes\mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = (\alpha^s + T)\,\mathbf{I} + O(\varepsilon).
\end{aligned} \tag{4.9}$$

Moreover,

$$\begin{aligned}\int \mathbf{v}\; Q^{sr}(g^s_\varepsilon M^s, g^r_\varepsilon M^r)\,\mathrm{d}\mathbf{v} &= -\,\frac{\mu^{sr}}{m^s}\,\kappa^{sr}\iint(\mathbf{v}-\mathbf{w})(g^s_\varepsilon M^s)(\mathbf{v})\,(g^r_\varepsilon M^r)(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\\ &= -\,\frac{\mu^{sr}}{m^s}\,\kappa^{sr}\left[\alpha^r\int \mathbf{v}\,(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}-\alpha^s\int \mathbf{v}\,(g^r_\varepsilon M^r)(\mathbf{v})\,\mathrm{d}\mathbf{v}\right]+O(\varepsilon)\\ &= -\,\frac{\mu^{sr}}{m^s}\,\kappa^{sr}(\alpha^r-\alpha^s)\,\mathbf{u}+O(\varepsilon)\end{aligned} \tag{4.10}$$

(in the last two equalities, we have used the first and the second parts of (2.8)). Putting results (4.9) and (4.10) into the macroscopic equation (4.8), we obtain

$$\left(\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\right)\partial_t\alpha^s-\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\partial_t\alpha^r-\,\Delta_{\mathbf{x}}(\alpha^s+T)+\mathbf{u}\cdot\nabla_{\mathbf{x}}\left(\sum_{r\neq s}\rho^r\mu^{sr}\kappa^{sr}(\alpha^s-\alpha^r)\right)=0, \tag{4.11}$$

where we have taken into account that $\nabla_{\mathbf{x}}\cdot\mathbf{u}=0$ (see (3.4)). Using the Boussinesq condition (3.5) and bearing in mind that $\sum_{s=1}^N\rho^s=1$, we get

$$\alpha^s+T=\alpha^s-\sum_{r=1}^N\rho^r\alpha^r=(1-\rho^s)\alpha^s-\sum_{r\neq s}\rho^r\alpha^r=\sum_{r\neq s}\rho^r(\alpha^s-\alpha^r). \tag{4.12}$$

In conclusion, equation (4.11) may be recast as

$$\begin{aligned}&\partial_t\left[\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}(\alpha^s-\alpha^r)\right]+\mathbf{u}\cdot\nabla_{\mathbf{x}}\left[\sum_{r\neq s}\rho^r\mu^{sr}\kappa^{sr}(\alpha^s-\alpha^r)\right]\\ &=\Delta_{\mathbf{x}}\left[\sum_{r\neq s}\rho^r(\alpha^s-\alpha^r)\right],\qquad\qquad s=1,\dots,N-1,\end{aligned} \tag{4.13}$$

or in the equivalent form:

$$\begin{aligned}&\partial_t\left[\alpha^s-\frac{\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}\,\alpha^r}{\sum_{r\neq s}\rho^r\mu^{sr}\,\kappa^{sr}}\right]+\mathbf{u}\cdot\nabla_{\mathbf{x}}\left[\alpha^s-\frac{\sum_{r\neq s}\rho^r\mu^{sr}\kappa^{sr}\,\alpha^r}{\sum_{r\neq s}\rho^r\mu^{sr}\kappa^{sr}}\right]\\ &=\Delta_{\mathbf{x}}\left[\frac{\sum_{r\neq s}\rho^r(\alpha^s-\alpha^r)}{\sum_{r\neq s}\rho^r\mu^{sr}\kappa^{sr}}\right],\qquad\qquad s=1,\dots,N-1.\end{aligned} \tag{4.14}$$

These equations are reported in Proposition 1.1.

5. Momentum equation

We write down again the momentum equation given by the second line of (3.3), but separating the drift term into two parts:

$$\begin{aligned}\varepsilon\sum_{s=1}^N\rho^s\,m^s\,\partial_t\int\mathbf{v}(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}+\sum_{s=1}^N\rho^s\,m^s\,\nabla_{\mathbf{x}}\cdot\int\left(\mathbf{v}\otimes\mathbf{v}-\frac{1}{3}\,v^2\mathbf{I}\right)(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}\\ +\sum_{s=1}^N\rho^s\,m^s\,\nabla_{\mathbf{x}}\int\frac{1}{3}\,v^2(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}=\mathbf{0}.\end{aligned}$$

Dividing by ε and setting

$$p = \frac{1}{\varepsilon}\sum_{s=1}^{N} \rho^s\, m^s \int \frac{1}{3}\, v^2 (g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} \qquad \text{and} \qquad \mathbf{B}(\mathbf{v}) = \mathbf{v}\otimes\mathbf{v} - \frac{1}{3}\, v^2 \mathbf{I}, \tag{5.1}$$

the momentum equation reads as

$$\sum_{s=1}^{N} \rho^s\, m^s\, \partial_t \int \mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \frac{1}{\varepsilon}\sum_{s=1}^{N}\rho^s\, m^s\, \nabla_{\mathbf{x}}\cdot \int \mathbf{B}(\mathbf{v})(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \nabla_{\mathbf{x}} p = \mathbf{0}. \tag{5.2}$$

We multiply the sth Boltzmann equation (2.3) by $m^s\theta^s\,\mathbf{B}(\mathbf{v})$ (where θ^s stand for constants to be determined later), we integrate in $\mathrm{d}\mathbf{v}$ and then we sum over s. We get

$$\begin{aligned} \varepsilon^2 \sum_{s=1}^{N} \rho^s\, m^s\theta^s\, \partial_t \int \mathbf{B}(\mathbf{v})(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \varepsilon \sum_{s=1}^{N} \rho^s\, m^s\theta^s\, \nabla_{\mathbf{x}}\cdot \int \mathbf{B}(\mathbf{v})\otimes\mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} \\ = \sum_{s,r=1}^{N} \rho^s\rho^r m^s\theta^s \int \mathbf{B}(\mathbf{v})\Big[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r)\Big]\mathrm{d}\mathbf{v} \\ + \varepsilon \sum_{s,r=1}^{N} \rho^s\rho^r m^s\theta^s \int \mathbf{B}(\mathbf{v})\, Q^{sr}(g_\varepsilon^s M^s, g_\varepsilon^r M^r)\,\mathrm{d}\mathbf{v}. \end{aligned} \tag{5.3}$$

As concerns the dominant (elastic) contribution, we resort to the following lemma:

Lemma 5.1. *For any constant $C \neq 0$, it is possible to determine θ^s in such a way that*

$$\begin{aligned} \sum_{s,r=1}^{N} \rho^s\rho^r m^s\theta^s \int \mathbf{B}(\mathbf{v})\Big[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r)\Big]\mathrm{d}\mathbf{v} \\ = C \sum_{s=1}^{N} \rho^s\, m^s \int \mathbf{B}(\mathbf{v})(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + O(\varepsilon). \end{aligned} \tag{5.4}$$

For Maxwell molecule interactions, constants θ^s, $s = 1,\ldots,N$, are the unique solution of the following linear system:

$$C = \sum_{r=1}^{N} \rho^r \frac{\mu^{sr}}{m^s + m^r}\left[2\big(-\theta^s + \theta^r\big)\kappa^{sr} - \frac{3}{2}\left(\frac{m^r}{m^s}\theta^s + \theta^r\right)\nu^{sr}\right], \qquad s = 1,\ldots,N. \tag{5.5}$$

The proof of this lemma involves a lot of quite technical computations, and for this reason is postponed to the Appendix.

Remark 5.2. We may achieve another (equivalent) explicit expression for the constant C, that will be useful in the sequel. If we multiply the sth equation of (5.5) by $\rho^s m^s$, *i.e.*

$$\rho^s m^s C = \sum_{r=1}^{N} \rho^s\rho^r \frac{\mu^{sr}}{m^s + m^r}\left[2m^s\big(-\theta^s + \theta^r\big)\kappa^{sr} - \frac{3}{2}\left(m^r\theta^s + m^s\theta^r\right)\nu^{sr}\right], \qquad s = 1,\ldots N,$$

and then we sum over s, we get

$$\begin{aligned}\left(\sum_{s=1}^N \rho^s m^s\right) C &= \sum_{s,r=1}^N \rho^s \rho^r \frac{\mu^{sr}}{m^s+m^r}\left[2m^s\big(-\theta^s+\theta^r\big)\kappa^{sr} - \frac{3}{2}\left(m^r\theta^s + m^s\theta^r\right)\nu^{sr}\right] \\ &= \sum_{s=1}^N \rho^s\theta^s\left[\sum_{r=1}^N \rho^r \frac{\mu^{sr}}{m^s+m^r}\left(-2m^s\kappa^{sr} - \frac{3}{2}m^r\nu^{sr}\right)\right] \\ &\quad + \sum_{r=1}^N \rho^r\theta^r\left[\sum_{s=1}^N \rho^s \frac{\mu^{sr}}{m^s+m^r}\left(2m^s\kappa^{sr} - \frac{3}{2}m^s\nu^{sr}\right)\right] \\ &= -\sum_{s,r=1}^N \rho^s\rho^r \frac{\mu^{sr}}{m^s+m^r}\left[2(m^s-m^r)\kappa^{sr} + 3m^r\nu^{sr}\right]\theta^s,\end{aligned} \tag{5.6}$$

hence

$$C = -\sum_{s,r=1}^N \rho^s\rho^r(\mu^{sr})^2\left[2\frac{m^s-m^r}{m^s m^r}\kappa^{sr} + \frac{3}{m^s}\nu^{sr}\right]\theta^s \Big/ \left(\sum_{s=1}^N \rho^s m^s\right). \tag{5.7}$$

Note that in case of equal masses we would have the much simpler result

$$C = -\frac{3}{4}\sum_{s,r=1}^N \rho^s\rho^r\nu^{sr}\theta^s. \tag{5.8}$$

The right-hand side of (5.4) is the same contribution appearing in the momentum equation (5.2), hence we can insert the equation (5.3) into (5.2), ending up with

$$\begin{aligned}&\sum_{s=1}^N \rho^s\, m^s\, \partial_t \int \mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} + \frac{1}{C}\nabla_{\mathbf{x}}\cdot\left\{\sum_{s=1}^N \rho^s\, m^s\theta^s\,\nabla_{\mathbf{x}}\cdot\int \mathbf{B}(\mathbf{v})\otimes\mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}\right. \\ &\left. -\sum_{s,r=1}^N \rho^s\rho^r m^s\theta^s \int \mathbf{B}(\mathbf{v})\, Q^{sr}(g_\varepsilon^s M^s, g_\varepsilon^r M^r)\,\mathrm{d}\mathbf{v}\right\} + \nabla_{\mathbf{x}} p = O(\varepsilon).\end{aligned} \tag{5.9}$$

At this point, let us recall that the distributions g_ε^s take the form (1.12). We immediately get

$$\sum_{s=1}^N \rho^s\, m^s\, \partial_t \int \mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = \left(\sum_{s=1}^N \rho^s m^s\right)\partial_t\mathbf{u} + O(\varepsilon). \tag{5.10}$$

Moreover,

$$\begin{aligned}&\sum_{s=1}^N \rho^s m^s\theta^s\,\nabla_{\mathbf{x}}\cdot\int \mathbf{B}(\mathbf{v})\otimes\mathbf{v}(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}|_{ij} \\ &= \sum_{s=1}^N \rho^s m^s\theta^s \sum_k \frac{\partial}{\partial x_k}\int v_k\, B_{ij}(\mathbf{v})(g_\varepsilon^s M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} \\ &= \sum_{s=1}^N \rho^s m^s\theta^s \sum_k \frac{\partial}{\partial x_k}\int \left(v_iv_j - \frac{1}{3}v^2\,\delta_{ij}\right) v_k \sum_h m^s\, v_h u_h\, M^s(\mathbf{v})\,\mathrm{d}\mathbf{v} + O(\varepsilon)\end{aligned}$$

(the other terms of g_ε^s give vanishing contributions by parity arguments). Bearing in mind that

$$\int v_i^4\, M^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = \frac{1}{5}\int v^4\, M^s(\mathbf{v})\,\mathrm{d}\mathbf{v}, \qquad \int v_i^2\, v_j^2\, M^s(\mathbf{v})\,\mathrm{d}\mathbf{v} = \frac{1}{15}\int v^4\, M^s(\mathbf{v})\,\mathrm{d}\mathbf{v} \quad (i\neq j),$$

and

$$\int v^4 \, M^s(\mathbf{v}) \, \mathrm{d}\mathbf{v} = \frac{15}{(m^s)^2},$$

a careful algebra yields

$$\begin{aligned} \sum_{s=1}^{N} \rho^s m^s \theta^s \sum_k \frac{\partial}{\partial x_k} \int v_k \, B_{ij}(\mathbf{v})(g_\varepsilon^s M^s)(\mathbf{v}) \, \mathrm{d}\mathbf{v} &= \sum_{s=1}^{N} \rho^s \theta^s \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \nabla_{\mathbf{x}} \cdot \mathbf{u} \, \delta_{ij} \right] + O(\varepsilon) \\ &= \left(\sum_{s=1}^{N} \rho^s \theta^s \right) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + O(\varepsilon) \end{aligned} \tag{5.11}$$

(where we have taken into account that $\nabla_{\mathbf{x}} \cdot \mathbf{u} = 0$).

Finally, as concerns the collision contribution appearing in (5.9), we get

$$\begin{aligned} &\sum_{s,r=1}^{N} \rho^s \rho^r m^s \theta^s \int \mathbf{B}(\mathbf{v}) \, Q^{sr}(g_\varepsilon^s M^s, g_\varepsilon^r M^r) \, \mathrm{d}\mathbf{v} \\ &= \sum_{s,r=1}^{N} \rho^s \rho^r (\mu^{sr})^2 \theta^s \left(2 \, \frac{m^s - m^r}{m^s m^r} \, \kappa^{sr} + \frac{3}{m^s} \, \nu^{sr} \right) \mathbf{B}(\mathbf{u}) + O(\varepsilon) \end{aligned} \tag{5.12}$$

(see details of computation in the Appendix).

By inserting (5.10), (5.11), (5.12) into the momentum equation (5.9) and taking into account again that $\mathbf{u}$ is divergence-free, we get

$$\begin{aligned} &\left(\sum_{s=1}^{N} \rho^s m^s \right) \partial_t \mathbf{u} + \frac{1}{C} \left\{ \left(\sum_{s=1}^{N} \rho^s \theta^s \right) \Delta_{\mathbf{x}} \mathbf{u} \right. \\ &\left. - \left[\sum_{s,r=1}^{N} \rho^s \rho^r (\mu^{sr})^2 \theta^s \left(2 \, \frac{m^s - m^r}{m^s m^r} \, \kappa^{sr} + \frac{3}{m^s} \, \nu^{sr} \right) \right] \left(\mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} - \frac{1}{3} \nabla_{\mathbf{x}} (u^2) \right) \right\} + \nabla_{\mathbf{x}} p = \mathbf{0}. \end{aligned} \tag{5.13}$$

Now, bearing in mind that C takes the form (5.7), the momentum equation may be cast as

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} \tilde{p} = d_1 \; \Delta_{\mathbf{x}} \mathbf{u}, \tag{5.14}$$

where

$$\tilde{p} = \frac{1}{\sum_{s=1}^{N} \rho^s m^s} \, p - \frac{1}{3} \, u^2,$$

and the diffusion coefficient is

$$d_1 = - \, \frac{1}{C} \left(\sum_{s=1}^{N} \rho^s \, \theta^s \right) \Big/ \left(\sum_{s=1}^{N} \rho^s m^s \right).$$

Lemma 5.3. *The diffusion coefficient d_1 is strictly positive.*

Proof. We recall that constants θ^s arise as solution of the linear system $\mathbf{A} \cdot \boldsymbol{\theta} = \mathbf{b}$ given in (5.5), that can be written in the equivalent form

$$\begin{aligned} &\left[\frac{3}{4} \, \rho^s \nu^{ss} + \sum_{r \neq s} \rho^r \frac{\mu^{sr}}{m^s + m^r} \left(2\kappa^{sr} + \frac{3}{2} \, \frac{m^r}{m^s} \nu^{sr} \right) \right] \theta^s \\ &+ \sum_{r \neq s} \rho^r \frac{\mu^{sr}}{m^s + m^r} \left(- \, 2\kappa^{sr} + \frac{3}{2} \, \nu^{sr} \right) \theta^r = - \, C, \qquad s = 1, \ldots, N. \end{aligned} \tag{5.15}$$

It is possible to prove that the coefficient matrix $\mathbf{A}$ is strictly diagonally dominant. Denoting by a_{sr} the entry of the sth row and the rth column, we have

$$|a_{ss}| = \frac{3}{4}\,\rho^s \nu^{ss} + \sum_{r\neq s} \rho^r \frac{\mu^{sr}}{m^s+m^r}\left(2\kappa^{sr} + \frac{3}{2}\,\frac{m^r}{m^s}\nu^{sr}\right),$$

$$|a_{sr}| = \rho^r \frac{\mu^{sr}}{m^s+m^r}\left|-2\kappa^{sr} + \frac{3}{2}\,\nu^{sr}\right|, \qquad r\neq s.$$

Unfortunately $-2\kappa^{sr} + \frac{3}{2}\,\nu^{sr}$ has not a definite sign. If we set

$$\mathcal{D}_+ = \Big\{ r = 1,\dots,N,\ r\neq s : -2\kappa^{sr} + \frac{3}{2}\,\nu^{sr} \geq 0\Big\},$$
$$\mathcal{D}_- = \Big\{ r = 1,\dots,N,\ r\neq s : -2\kappa^{sr} + \frac{3}{2}\,\nu^{sr} < 0\Big\},$$

then

$$\begin{aligned} |a_{ss}| - \sum_{r\neq s}|a_{sr}| &= \frac{3}{4}\rho^s\nu^{ss} + \sum_{r\in\mathcal{D}_+} \rho^r\frac{\mu^{sr}}{m^s+m^r}\left(4\kappa^{sr} + \frac{3}{2}\frac{m^r-m^s}{m^s}\nu^{sr}\right) \\ &\quad + \frac{3}{2}\sum_{r\in\mathcal{D}_-}\rho^r\frac{\mu^{sr}}{m^s+m^r}\left(\frac{m^r}{m^s}+1\right)\nu^{sr}. \end{aligned} \tag{5.16}$$

Recalling now the definitions of κ^{sr} and ν^{sr} given in (1.11), we get

$$\begin{aligned} 4\kappa^{sr} + \frac{3}{2}\frac{m^r-m^s}{m^s}\nu^{sr} &= \frac{2\pi}{m^s}\int_0^\pi \vartheta^{sr}(\chi)(1-\cos\chi)\left[4m^s + \frac{3}{2}(m^r-m^s) + \frac{3}{2}(m^r-m^s)\cos\chi\right]\sin\chi\,\mathrm{d}\chi \\ &= \frac{\pi}{m^s}\int_0^\pi \vartheta^{sr}(\chi)(1-\cos\chi)\,[5m^s+3m^r+3(m^r-m^s)\cos\chi]\sin\chi\,\mathrm{d}\chi \geq 0, \end{aligned}$$

therefore

$$|a_{ss}| > \sum_{r\neq s}|a_{sr}|, \qquad \forall s = 1,\dots,N.$$

Since $\mathbf{A}$ is diagonally dominant, then it is invertible. Thus for any fixed C, there is a unique solution $\theta^1,\dots,\theta^N$ to the linear system (5.15). If we multiply the sth equation (5.15) by ρ^s, we get the following linear system $\bar{\mathbf{A}}\cdot\boldsymbol{\theta} = \bar{\mathbf{b}}$:

$$\begin{aligned} &\left[\frac{3}{4}\,(\rho^s)^2\nu^{ss} + \sum_{r\neq s}\rho^s\rho^r\frac{\mu^{sr}}{m^s+m^r}\left(2\kappa^{sr} + \frac{3}{2}\,\frac{m^r}{m^s}\nu^{sr}\right)\right]\theta^s \\ &\quad + \sum_{r\neq s}\rho^s\rho^r\frac{\mu^{sr}}{m^s+m^r}\left(-2\kappa^{sr} + \frac{3}{2}\,\nu^{sr}\right)\theta^r = -\rho^s C, \qquad s = 1,\dots,N, \end{aligned} \tag{5.17}$$

which is equivalent to (5.15), in the sense that it admits the same unique solution $\theta^1,\theta^2,\dots\theta^N$. The matrix $\bar{\mathbf{A}}$ is again diagonally dominant ($|\bar{a}_{ss}| > \sum_{r\neq s}|\bar{a}_{sr}|$, $\forall s = 1,\dots,N$), has strictly positive diagonal entries, and moreover is symmetric (unlike $\mathbf{A}$). These properties allow to infer that $\bar{\mathbf{A}}$ is positive definite, *i.e.* $\boldsymbol{\theta}^T\cdot\bar{\mathbf{A}}\cdot\boldsymbol{\theta}\geq 0$. This yields

$$\sum_{s,r=1}^N \theta^s\,\bar{a}_{sr}\,\theta^r = \sum_{s=1}^N \theta^s\,\bar{b}_s = -C\sum_{s=1}^N \rho^s\theta^s > 0 \qquad \forall\boldsymbol{\theta}\neq\mathbf{0},$$

and this proves that $d_1 > 0$. □

The formula for the diffusion coefficient d_1 appearing in Proposition 1.1 is obtained by taking $C = -\left(\sum_{s=1}^{N} \rho^s\, m^s\right)^{-1}$.

In the case of a single species, in [4] the authors prove positivity of diffusion coefficients by abstract arguments, using essentially positivity and symmetry properties of the linearized Boltzmann operator. The generalization of this technique to a mixture seems very awkward since such coefficients arise from a combination of N kinetic equations (hence from a linear system), and moreover possible symmetries are not so evident because of different particle masses. Of course a rigorous proof of this kind for a large class of collision kernels would be an interesting future work.

6. Temperature equation

Let us multiply the sth Boltzmann equation (2.3) by $\frac{1}{2}\, m^s v^2 - \frac{5}{2}$ and then sum over s:

$$\varepsilon \sum_{s=1}^{N} \rho^s \partial_t \int \left(\frac{1}{2}\, m^s v^2 - \frac{5}{2}\right) (g_\varepsilon^s M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} + \sum_{s=1}^{N} \rho^s \nabla_{\mathbf{x}} \cdot \int \mathbf{D}^s(\mathbf{v}) (g_\varepsilon^s M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} = 0, \tag{6.1}$$

where

$$\mathbf{D}^s(\mathbf{v}) = \left(\frac{1}{2}\, m^s v^2 - \frac{5}{2}\right) \mathbf{v}. \tag{6.2}$$

We multiply the sth Boltzmann equation (2.3) by $\varepsilon\, (m^s)^p\, \eta^s\, \mathbf{D}^s(\mathbf{v})$ (where constants η^s and power p will be suitably determined later), we integrate in d$\mathbf{v}$ and then we sum over s. We get

$$\begin{aligned}
&\varepsilon^2 \sum_{s=1}^{N} \rho^s\, (m^s)^p\, \eta^s\, \partial_t \int \mathbf{D}^s(\mathbf{v}) (g_\varepsilon^s M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} + \varepsilon \sum_{s=1}^{N} \rho^s\, (m^s)^p \eta^s\, \nabla_{\mathbf{x}} \cdot \int \mathbf{D}^s(\mathbf{v}) \otimes \mathbf{v} (g_\varepsilon^s M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} \\
&\qquad = \sum_{s,r=1}^{N} \rho^s \rho^r (m^s)^p \eta^s \int \mathbf{D}^s(\mathbf{v}) \Big[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r) \Big] \mathrm{d}\mathbf{v} \\
&\qquad\quad + \varepsilon \sum_{s,r=1}^{N} \rho^s \rho^r (m^s)^p\, \eta^s \int \mathbf{D}^s(\mathbf{v})\, Q^{sr}(g_\varepsilon^s M^s, g_\varepsilon^r M^r)\, \mathrm{d}\mathbf{v}.
\end{aligned} \tag{6.3}$$

We can prove the following property (details of the proof are again postponed to the Appendix):

Lemma 6.1. *For any constant $K \neq 0$, it is possible to determine explicitly a family of constants η^s and a power $p > 0$ (namely $p = 1/2$) in such a way that*

$$\begin{aligned}
&\sum_{s,r=1}^{N} \rho^s \rho^r (m^s)^p\, \eta^s \int \mathbf{D}^s(\mathbf{v}) \Big[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r) \Big] \mathrm{d}\mathbf{v} \\
&= K \sum_{s=1}^{N} \rho^s \int \mathbf{D}^s(\mathbf{v}) (g_\varepsilon^s M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} + O(\varepsilon).
\end{aligned} \tag{6.4}$$

For Maxwell molecules, constants η^s, $s = 1, \ldots, N$, are the unique solution of the following linear system:

$$\begin{aligned}
K = \sum_{r=1}^{N} \rho^r \mu^{sr} \Bigg\{ &\frac{1}{(m^s + m^r)^2} \left[-(m^s)^p \left(3m^s + \frac{(m^r)^2}{m^s}\right) \eta^s + 4\, m^s (m^r)^p\, \eta^r \right] \kappa^{sr} \\
&- \frac{2(\mu^{sr})^2}{m^s m^r} \left[(m^s)^{p-1} \eta^s + (m^r)^{p-1} \eta^r \right] \nu^{sr} \Bigg\}, \qquad s = 1, \ldots, N.
\end{aligned} \tag{6.5}$$

The validity of a proportionality property like (6.4), as well as the corresponding (5.4) for momentum equation, is of course well expected, since the weight functions $\mathbf{D}^s(\mathbf{v})$ and $\mathbf{B}(\mathbf{v})$ are orthogonal to the kernel of the linearized Boltzmann operator. The major difficulty in the present mixture frame is the precise computation of proportionality constants, and for this reason we have tried to treat this point in the most general way, leaving unknowns in the proposed linear combinations, in order to obtain the precise one-to-one relations $(\eta^1, \dots, \eta^N) \leftrightarrow K$ and, in the previous section, $(\theta^1, \dots, \theta^N) \leftrightarrow C$. The point here is that the constants are not guessed but obtained after a systematic computation.

Remark 6.2. With the choice $p = 1/2$, the constant K given in (6.5) becomes

$$\begin{aligned} K = \sum_{r=1}^{N} \rho^r \mu^{sr} \Bigg\{ \frac{1}{(m^s+m^r)^2} \left[-\sqrt{m^s} \left(3m^s + \frac{(m^r)^2}{m^s} \right) \eta^s + 4\, m^s \sqrt{m^r}\, \eta^r \right] \kappa^{sr} \\ - \frac{2(\mu^{sr})^2}{m^s m^r} \left[\frac{\eta^s}{\sqrt{m^s}} + \frac{\eta^r}{\sqrt{m^r}} \right] \nu^{sr} \Bigg\}, \qquad s = 1, \dots, N. \end{aligned} \tag{6.6}$$

If we multiply the sth equation of (6.6) by ρ^s, and then we sum over s, we get an equivalent expression for the constant K:

$$\begin{aligned} K &= \sum_{s,r=1}^{N} \rho^s \rho^r \mu^{sr} \Bigg\{ \frac{1}{(m^s+m^r)^2} \left[-\sqrt{m^s} \left(3m^s + \frac{(m^r)^2}{m^s} \right) \eta^s + 4\, m^s \sqrt{m^r}\, \eta^r \right] \kappa^{sr} \\ &\quad - \frac{2(\mu^{sr})^2}{m^s m^r} \left[\frac{\eta^s}{\sqrt{m^s}} + \frac{\eta^r}{\sqrt{m^r}} \right] \nu^{sr} \Bigg\} \\ &= - \sum_{s,r=1}^{N} \rho^s \rho^r \frac{\mu^{sr} \sqrt{m^s}}{(m^s+m^r)^2} \left[\left(3m^s + \frac{(m^r)^2}{m^s} - 4m^r \right) \kappa^{sr} + 4\, m^r \nu^{sr} \right] \eta^s. \end{aligned} \tag{6.7}$$

Note that in case of equal masses we would have the much simpler result

$$K = -\frac{1}{2}\sqrt{m} \sum_{s,r=1}^{N} \rho^s \rho^r \nu^{sr} \eta^s. \tag{6.8}$$

The right-hand side of (6.4) is the same contribution appearing in the temperature equation (6.1), hence we can insert the equation (6.3) with $p = 1/2$ into (6.1):

$$\begin{aligned} \sum_{s=1}^{N} \rho^s\, \partial_t \!\int \left(\frac{1}{2}\, m^s v^2 - \frac{5}{2} \right) (g^s_\varepsilon M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} + \frac{1}{K}\, \nabla_{\mathbf{x}} \cdot \Bigg\{ \sum_{s=1}^{N} \rho^s \sqrt{m^s}\, \eta^s\, \nabla_{\mathbf{x}} \cdot \!\int \mathbf{D}^s(\mathbf{v}) \otimes \mathbf{v} (g^s_\varepsilon M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} \\ - \sum_{s,r=1}^{N} \rho^s \rho^r \sqrt{m^s}\, \eta^s \int \mathbf{D}^s(\mathbf{v})\, Q^{sr}(g^s_\varepsilon M^s, g^r_\varepsilon M^r)\, \mathrm{d}\mathbf{v} \Bigg\} = O(\varepsilon). \end{aligned} \tag{6.9}$$

Recalling now that the distributions g^s_ε take the form (1.12), we get

$$\begin{aligned} &\sum_{s=1}^{N} \rho^s\, \partial_t \int \left(\frac{1}{2}\, m^s v^2 - \frac{5}{2} \right) (g^s_\varepsilon M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} = \sum_{s=1}^{N} \rho^s \partial_t \left[\frac{3}{2}(\alpha^s + T) - \frac{5}{2}\, \alpha^s \right] + O(\varepsilon) \\ &= \frac{3}{2}\, \partial_t T - \partial_t \left(\sum_{s=1}^{N} \rho^s \alpha^s \right) + O(\varepsilon) = \frac{5}{2}\, \partial_t T + O(\varepsilon), \end{aligned} \tag{6.10}$$

where the last equality holds because of the constraint (3.6). Moreover, by parity arguments,

$$\sum_{s=1}^{N} \rho^s \sqrt{m^s}\, \eta^s \sum_k \frac{\partial}{\partial x_k} \int D_k^s(\mathbf{v}) v_i (g_\varepsilon^s M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v}$$
$$= \sum_{s=1}^{N} \rho^s \sqrt{m^s}\, \eta^s \sum_k \frac{\partial}{\partial x_k} \int \left(\frac{1}{2}\, m^s v^2 - \frac{5}{2}\right) v_i v_k \left[\alpha^s + \left(\frac{1}{2}\, m^s v^2 - \frac{3}{2}\right) T\right] M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + O(\varepsilon)$$

that, bearing in mind

$$m^s \int v^2 M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} = 3, \qquad (m^s)^2 \int v^4 M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} = 15, \qquad (m^s)^3 \int v^6 M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} = 105, \tag{6.11}$$

results in

$$\sum_{s=1}^{N} \rho^s \sqrt{m^s}\, \eta^s \sum_k \frac{\partial}{\partial x_k} \int D_k^s(\mathbf{v}) v_i (g_\varepsilon^s M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} = \left(\sum_{s=1}^{N} \frac{\rho^s}{\sqrt{m^s}}\, \eta^s\right) \frac{5}{2}\, \frac{\partial T}{\partial x_i}\cdot \tag{6.12}$$

Finally,

$$\begin{aligned}\sum_{s,r=1}^{N} \rho^s \rho^r \sqrt{m^s}\, \eta^s \int \mathbf{D}^s(\mathbf{v})\, Q^{sr}(g_\varepsilon^s M^s, g_\varepsilon^r M^r)\, \mathrm{d}\mathbf{v} = \\ = \frac{5}{2} \sum_{s,r=1}^{N} \rho^s \rho^r \eta^s \frac{\mu^{sr} \sqrt{m^s}}{(m^s+m^r)^2} \left\{ \left(3m^s - 4m^r + \frac{(m^r)^2}{m^s}\right) \kappa^{sr} + 4\, m^r \nu^{sr} \right\} \mathbf{u}\, T\, + O(\varepsilon)\end{aligned} \tag{6.13}$$

(see details in the Appendix). Putting all results (6.10), (6.12), (6.13) into equation (6.9), and dividing all terms by 5/2, the temperature equation reads as

$$\begin{aligned}\partial_t T + \frac{1}{K} \nabla_{\mathbf{x}} \cdot \Bigg\{ \left(\sum_{s=1}^{N} \frac{\rho^s}{\sqrt{m^s}}\, \eta^s\right) \nabla_{\mathbf{x}} T \\ - \sum_{s,r=1}^{N} \rho^s \rho^r \eta^s \frac{\mu^{sr} \sqrt{m^s}}{(m^s+m^r)^2} \left[\left(3m^s - 4m^r + \frac{(m^r)^2}{m^s}\right) \kappa^{sr} + 4\, m^r \nu^{sr} \right] \mathbf{u}\, T \Bigg\} = 0.\end{aligned}$$

Bearing in mind that K takes the form (6.7) and that $\mathbf{u}$ is divergence-free, this equation may be recast as

$$\partial_t T + \mathbf{u} \cdot \nabla_{\mathbf{x}} T = d_2 \Delta_{\mathbf{x}} T, \tag{6.14}$$

where the diffusion coefficient is given by

$$d_2 = -\frac{1}{K} \left(\sum_{s=1}^{N} \frac{\rho^s}{\sqrt{m^s}}\, \eta^s\right).$$

Lemma 6.3. *The diffusion coefficient d_2 is strictly positive.*

Proof. The proof is similar to the one of Lemma 5.3 of previous section. We recall that constants η^s arise as the unique solution of the linear system (6.6). If we multiply the sth equation by $\rho^s/\sqrt{m^s}$, we get an equivalent linear system $\hat{\mathbf{A}} \cdot \boldsymbol{\eta} = \hat{\mathbf{b}}$ (it admits the same unique solution $\eta^1,\, \eta^2,\, \ldots\, \eta^N$). The coefficient matrix $\hat{\mathbf{A}}$ is diagonally dominant, has strictly positive diagonal entries, and moreover is symmetric (see the proof of Lem. A.1 in the Appendix A, we skip details here). Therefore it is positive definite, *i.e.* $\boldsymbol{\eta}^T \cdot \hat{\mathbf{A}} \cdot \boldsymbol{\eta} \geq 0$ and this yields

$$\sum_{s,r=1}^{N} \eta^s\, \hat{a}_{sr}\, \eta^r = \sum_{s=1}^{N} \eta^s\, \hat{b}_s = -\, K \sum_{s=1}^{N} \frac{\rho^s}{\sqrt{m^s}}\, \eta^s > 0 \qquad \forall\, \boldsymbol{\eta} \neq \mathbf{0},$$

and this proves that $d_2 > 0$. □

The formula for d_2 appearing in Proposition 1.1 is obtained by specifying $K=-1$. This concludes the proof of Proposition 1.1. $\square$

Appendix A.

In this appendix we will include the detailed proof of technical Lemmas 5.1 and 6.1 and the explicit evaluation of collision contributions $\int \mathbf{B}(\mathbf{v}) Q^{sr}(g^s_\varepsilon M^s, g^r_\varepsilon M^r)\mathrm{d}\mathbf{v}$ and $\int \mathbf{D}^s(\mathbf{v}) Q^{sr}(g^s_\varepsilon M^s, g^r_\varepsilon M^r)\mathrm{d}\mathbf{v}$, useful to compute explicit diffusion coefficients in the Maxwell molecules frame.

Lemma A.1. *For any constant $C \neq 0$, it is possible to determine θ^s in such a way that*

$$\begin{aligned}
&\sum_{s,r=1}^N \rho^s \rho^r m^s \theta^s \int \mathbf{B}(\mathbf{v}) \Big[Q^{sr}(g^s_\varepsilon M^s, M^r) + Q^{sr}(M^s, g^r_\varepsilon M^r) \Big] \mathrm{d}\mathbf{v} \\
&= C \sum_{s=1}^N \rho^s\, m^s \int \mathbf{B}(\mathbf{v}) (g^s_\varepsilon M^s)(\mathbf{v})\, \mathrm{d}\mathbf{v} + O(\varepsilon).
\end{aligned} \tag{A.1}$$

Proof. To evaluate collision contributions, we resort to formula (2.1) and to the same steps sketched in detail in [9] (see Sect. 4.3.1, some details will be omitted here). If we set $\varphi^s_{ij}(\mathbf{v}) = B_{ij}(\mathbf{v}) = v_i v_j - \frac{1}{3} v^2 \delta_{ij}$ (δ_{ij} denotes the Kronecker delta), it may be checked, bearing in mind (1.3), that

$$\begin{aligned}
\varphi^s_{ij}(\mathbf{v}') - \varphi^s_{ij}(\mathbf{v}) &= \frac{\mu^{sr}}{m^s} \Big[(q'_i - q_i) v_j + v_i (q'_j - q_j) \Big] + \left(\frac{\mu^{sr}}{m^s} \right)^2 (q'_i - q_i)(q'_j - q_j) \\
&\quad - \frac{2}{3} \frac{\mu^{sr}}{m^s} \sum_k v_k (q'_k - q_k) \delta_{ij} - \frac{1}{3} \left(\frac{\mu^{sr}}{m^s} \right)^2 |\mathbf{q}' - \mathbf{q}|^2 \delta_{ij},
\end{aligned} \tag{A.2}$$

hence under the usual Maxwell molecules assumption (1.10), owing to (4.4), we have

$$\Upsilon_{ij}(\mathbf{v},\mathbf{w}) = \int q\, \sigma^{sr}(q,\chi) \Big[\varphi^s_{ij}(\mathbf{v}') - \varphi^s_{ij}(\mathbf{v}) \Big] \mathrm{d}\hat{\boldsymbol{\Omega}}' = \kappa^{sr}\, \Upsilon^\kappa_{ij}(\mathbf{v},\mathbf{w}) + \nu^{sr}\, \Upsilon^\nu_{ij}(\mathbf{v},\mathbf{w}), \tag{A.3}$$

where the averaged collision frequencies κ^{sr} and ν^{sr} are given in (1.11), and

$$\begin{aligned}
\Upsilon^\kappa_{ij}(\mathbf{v},\mathbf{w}) &= \frac{\mu^{sr}}{m^s} \left[-2 v_i v_j + v_i w_j + w_i v_j + \frac{2}{3} v^2 \delta_{ij} - \frac{2}{3} \sum_k v_k w_k \delta_{ij} \right] \\
&+ \left(\frac{\mu^{sr}}{m^s} \right)^2 \left[2 v_i v_j - 2 v_i w_j - 2 w_i v_j + 2 w_i w_j - \frac{2}{3} v^2 \delta_{ij} + \frac{4}{3} \sum_k v_k w_k \delta_{ij} - \frac{2}{3} w^2 \delta_{ij} \right], \\
\Upsilon^\nu_{ij}(\mathbf{v},\mathbf{w}) &= \frac{1}{2} \left(\frac{\mu^{sr}}{m^s} \right)^2 \left[v^2 \delta_{ij} + w^2 \delta_{ij} - 2 \sum_k v_k w_k \delta_{ij} - 3 v_i v_j + 3 v_i w_j + 3 w_i v_j - 3 w_i w_j \right].
\end{aligned} \tag{A.4}$$

Let us first consider

$$\Phi_{ij} := \iint \Upsilon^\kappa_{ij}(\mathbf{v},\mathbf{w}) \Big[g^s_\varepsilon(\mathbf{v}) + g^r_\varepsilon(\mathbf{w}) \Big] M^s(\mathbf{v})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w}.$$

Using a parity argument (actually two parity arguments, $\mathbf{v} \mapsto -\mathbf{v}$, and $\mathbf{w} \mapsto -\mathbf{w}$), we see that

$$\begin{aligned}\Phi_{ij} = &\left[\left(\frac{\mu^{sr}}{m^s}\right)^2 - \frac{\mu^{sr}}{m^s}\right] \iint \left[2v_iv_j - \frac{2}{3}\, v^2\delta_{ij}\right] \left[g_\varepsilon^s(\mathbf{v}) + g_\varepsilon^r(\mathbf{w})\right] M^s(\mathbf{v})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w} \\ &+ \left(\frac{\mu^{sr}}{m^s}\right)^2 \iint \left[2w_iw_j - \frac{2}{3}\, w^2\delta_{ij}\right] \left[g_\varepsilon^s(\mathbf{v}) + g_\varepsilon^r(\mathbf{w})\right] M^s(\mathbf{v})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w} + O(\varepsilon) \\ = &- 2\left(\frac{\mu^{sr}}{m^s}\right)^2 \frac{m^s}{m^r} \int \left[v_iv_j - \frac{1}{3}\, v^2\delta_{ij}\right] \left[g_\varepsilon^s(\mathbf{v}) + \alpha^r\right] M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} \\ &+ 2\left(\frac{\mu^{sr}}{m^s}\right)^2 \int \left[w_iw_j - \frac{1}{3}\, w^2\delta_{ij}\right] \left[\alpha^s + g_\varepsilon^r(\mathbf{w})\right] M^r(\mathbf{w})\, \mathrm{d}\mathbf{w} + O(\varepsilon) \\ = &\,2\left(\frac{\mu^{sr}}{m^s}\right)^2 \left\{-\frac{m^s}{m^r} \int B_{ij}(\mathbf{v})\, g_\varepsilon^s(\mathbf{v})\, M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + \int B_{ij}(\mathbf{w})\, g_\varepsilon^r(\mathbf{w})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{w}\right\} + O(\varepsilon).\end{aligned}$$

We now turn to

$$\Psi_{ij} := \iint \Upsilon_{ij}^\nu(\mathbf{v}, \mathbf{w}) \left[g_\varepsilon^s(\mathbf{v}) + g_\varepsilon^r(\mathbf{w})\right] M^s(\mathbf{v})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w}.$$

Owing to the usual parity arguments, it's easy to see that

$$\begin{aligned}\Psi_{ij} := &\,\frac{1}{2}\left(\frac{\mu^{sr}}{m^s}\right)^2 \iint \left[v^2\,\delta_{ij} + w^2\,\delta_{ij} - 3v_iv_j - 3w_iw_j\right] \left[g_\varepsilon^s(\mathbf{v}) + g_\varepsilon^r(\mathbf{w})\right] M^s(\mathbf{v})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w} + O(\varepsilon) \\ = &\,\frac{1}{2}\left(\frac{\mu^{sr}}{m^s}\right)^2 \left\{\int \left[v^2\,\delta_{ij} - 3v_iv_j\right]\left[g_\varepsilon^s(\mathbf{v}) + \alpha^r\right] M^s(\mathbf{v})\, \mathrm{d}\mathbf{v}\right. \\ &\left. + \int \left[w^2\,\delta_{ij} - 3w_iw_j\right]\left[\alpha^s + g_\varepsilon^r(\mathbf{w})\right] M^r(\mathbf{w})\, \mathrm{d}\mathbf{w}\right\} + O(\varepsilon) \\ = &-\frac{3}{2}\left(\frac{\mu^{sr}}{m^s}\right)^2 \left\{\int B_{ij}(\mathbf{v})\, g_\varepsilon^s(\mathbf{v})\, M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + \int B_{ij}(\mathbf{w})\, g_\varepsilon^r(\mathbf{w})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{w}\right\} + O(\varepsilon).\end{aligned}$$

In conclusion,

$$\begin{aligned}&\sum_{s,r=1}^N \rho^s\rho^r m^s\theta^s \int \mathbf{B}(\mathbf{v})\left[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r)\right]\mathrm{d}\mathbf{v} \\ &= 2\sum_{s,r=1}^N \rho^s\rho^r m^s\theta^s \left(\frac{\mu^{sr}}{m^s}\right)^2 \kappa^{sr} \left\{-\frac{m^s}{m^r}\int \mathbf{B}(\mathbf{v})\, g_\varepsilon^s(\mathbf{v})\, M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + \int \mathbf{B}(\mathbf{w})\, g_\varepsilon^r(\mathbf{w})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{w}\right\} \\ &\quad - \frac{3}{2}\sum_{s,r=1}^N \rho^s\rho^r m^s\theta^s \left(\frac{\mu^{sr}}{m^s}\right)^2 \nu^{sr} \left\{\int \mathbf{B}(\mathbf{v})\, g_\varepsilon^s(\mathbf{v})\, M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + \int \mathbf{B}(\mathbf{w})\, g_\varepsilon^r(\mathbf{w})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{w}\right\} + O(\varepsilon) \\ &= \sum_{s,r=1}^N \rho^s\rho^r \frac{\mu^{sr}}{m^s + m^r}\left[2\, m^s\big(-\theta^s + \theta^r\big)\kappa^{sr} - \frac{3}{2}\big(m^r\theta^s + m^s\theta^r\big)\nu^{sr}\right] \int \mathbf{B}(\mathbf{v})\, g_\varepsilon^s(\mathbf{v})\, M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + O(\varepsilon).\end{aligned}$$

The sought relation (A.1) is satisfied if we may choose θ^s, $s = 1, \ldots, N$, such that

$$C = \sum_{r=1}^N \rho^r \frac{\mu^{sr}}{m^s + m^r}\left[2\big(-\theta^s + \theta^r\big)\kappa^{sr} - \frac{3}{2}\left(\frac{m^r}{m^s}\theta^s + \theta^r\right)\nu^{sr}\right], \qquad s = 1, \ldots, N. \tag{A.5}$$

This is a linear system of the kind $\mathbf{A}\cdot\boldsymbol{\theta}=\mathbf{b}$ for the N unknowns θ^s, where coefficients matrix $\mathbf{A}$ has already been proved to be diagonally dominant (see the proof of Lem. 5.3), then it is non-singular (the determinant is different from zero). Thus for any fixed C, there is a unique solution $\theta^1,\ldots,\theta^N$ to the linear system (A.5). □

Lemma A.2. *As concerns the collision contribution appearing in* (5.9)*, we have*

$$\sum_{s,r=1}^{N}\rho^s\rho^r m^s\theta^s\int \mathbf{B}(\mathbf{v})\,Q^{sr}(g^s_\varepsilon M^s,g^r_\varepsilon M^r)\,\mathrm{d}\mathbf{v}=\sum_{s,r=1}^{N}\rho^s\rho^r(\mu^{sr})^2\theta^s\left(2\,\frac{m^s-m^r}{m^sm^r}\,\kappa^{sr}+\frac{3}{m^s}\,\nu^{sr}\right)\mathbf{B}(\mathbf{u})+O(\varepsilon). \tag{A.6}$$

Proof. Notice that

$$\int B_{ij}(\mathbf{v})Q^{sr}(g^s_\varepsilon M^s,g^r_\varepsilon M^r)\mathrm{d}\mathbf{v}=\iint\Big[\kappa^{sr}\,\Upsilon^{\kappa}_{ij}(\mathbf{v},\mathbf{w})+\nu^{sr}\,\Upsilon^{\nu}_{ij}(\mathbf{v},\mathbf{w})\Big](g^s_\varepsilon M^s)(\mathbf{v})\,(g^r_\varepsilon M^r)(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}, \tag{A.7}$$

where Υ^{κ}_{ij} and Υ^{ν}_{ij} are given in (A.4). Bearing in mind the properties of the distributions g^s_ε, and in particular that

$$\int v_i\,(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}=u_i+O(\varepsilon),\qquad\qquad \int\left(v_i\,v_j-\frac{1}{3}\,v^2\,\delta_{ij}\right)(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}=O(\varepsilon),$$

we get

$$\begin{aligned}&\iint\Upsilon^{\kappa}_{ij}(\mathbf{v},\mathbf{w})(g^s_\varepsilon M^s)(\mathbf{v})\,(g^r_\varepsilon M^r)(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\\&=2\,\frac{\mu^{sr}}{m^s}\,B_{ij}(\mathbf{u})-4\left(\frac{\mu^{sr}}{m^s}\right)^2B_{ij}(\mathbf{u})+O(\varepsilon)=2\left(\frac{\mu^{sr}}{m^s}\right)^2\frac{m^s-m^r}{m^r}\,B_{ij}(\mathbf{u})+O(\varepsilon)\end{aligned} \tag{A.8}$$

(of course this term would vanish in case of equal masses), and

$$\iint\Upsilon^{\nu}_{ij}(\mathbf{v},\mathbf{w})(g^s_\varepsilon M^s)(\mathbf{v})\,(g^r_\varepsilon M^r)(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}=3\left(\frac{\mu^{sr}}{m^s}\right)^2B_{ij}(\mathbf{u})+O(\varepsilon). \tag{A.9}$$

□

Lemma A.3. *For any constant $K\neq 0$, it is possible to determine explicitly a family of constants η^s and a power $p>0$ (namely $p=1/2$) in such a way that*

$$\sum_{s,r=1}^{N}\rho^s\rho^r(m^s)^p\,\eta^s\int\mathbf{D}^s(\mathbf{v})\Big[Q^{sr}(g^s_\varepsilon M^s,M^r)+Q^{sr}(M^s,g^r_\varepsilon M^r)\Big]\mathrm{d}\mathbf{v}=K\sum_{s=1}^{N}\rho^s\int\mathbf{D}^s(\mathbf{v})(g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v}+O(\varepsilon). \tag{A.10}$$

Proof. The strategy of proof is similar to the one of Lemma A.1, but for readers' convenience we report the details of computation. If we set $\varphi^s_i(\mathbf{v})=D^s_i(\mathbf{v})=\frac{1}{2}(m^sv^2-5)\,v_i$, from (1.3), we get

$$\begin{aligned}\varphi^s_i(\mathbf{v}')-\varphi^s_i(\mathbf{v})&=\frac{1}{2}\,\mu^{sr}v^2(q'_i-q_i)+\frac{1}{2}\,m^s\left(\frac{\mu^{sr}}{m^s}\right)^2v_i|\mathbf{q}'-\mathbf{q}|^2+\frac{1}{2}\,m^s\left(\frac{\mu^{sr}}{m^s}\right)^3(q'_i-q_i)|\mathbf{q}'-\mathbf{q}|^2\\&\quad+\mu^{sr}v_i\sum_j v_j(q'_j-q_j)+m^s\left(\frac{\mu^{sr}}{m^s}\right)^2\sum_j v_j(q'_i-q_i)(q'_j-q_j)-\frac{5}{2}\frac{\mu^{sr}}{m^s}(q'_i-q_i),\end{aligned} \tag{A.11}$$

hence under the Maxwell molecules assumption (1.10), we get (see (4.4))

$$\Theta_i(\mathbf{v},\mathbf{w})=\int q\,\sigma^{sr}(q,\chi)\Big[\varphi^s_i(\mathbf{v}')-\varphi^s_i(\mathbf{v})\Big]\mathrm{d}\hat{\boldsymbol{\Omega}}'=\kappa^{sr}\,\Theta^{\kappa}_i(\mathbf{v},\mathbf{w})+\nu^{sr}\,\Theta^{\nu}_i(\mathbf{v},\mathbf{w}), \tag{A.12}$$

where averaged collision frequencies κ^{sr} and ν^{sr} are given in (1.11), and

$$\begin{aligned}\Theta_i^\kappa(\mathbf{v},\mathbf{w}) = \mu^{sr}\Big[&-\frac{1}{2}\,v^2 q_i + \frac{\mu^{sr}}{m^s} v_i q^2 - 2\left(\frac{\mu^{sr}}{m^s}\right)^2 q_i q^2 - v_i \sum_j v_j q_j \\ &+ 2\,\frac{\mu^{sr}}{m^s}\sum_j v_j q_i q_j + \frac{5}{2}\,\frac{1}{m^s}\,q_i\Big], \\ \Theta_i^\nu(\mathbf{v},\mathbf{w}) = \frac{(\mu^{sr})^2}{m^s}\Big[&\frac{\mu^{sr}}{m^s} q^2 q_i + \frac{1}{2}\,v_i q^2 - \frac{3}{2}\sum_j v_j q_i q_j\Big].\end{aligned} \tag{A.13}$$

Taking into account that $q = |\mathbf{v}-\mathbf{w}|$, a careful algebra yields

$$\begin{aligned}\Theta_i^\kappa(\mathbf{v},\mathbf{w}) = \frac{\mu^{sr}}{(m^s+m^r)^2}\Big[&-\left(\frac{3}{2}\,(m^s)^2 + \frac{1}{2}\,(m^r)^2\right) v_i v^2 + m^r(m^s-m^r) v_i w^2 \\ &+ 2m^r(m^s-m^r)\sum_j v_j w_i w_j + 2(m^r)^2 w_i w^2 + \frac{1}{2}\,(m^s-m^r)^2 w_i v^2 \\ &+ (m^s-m^r)^2 v_i \sum_j v_j w_j + \frac{5}{2}\frac{(m^s+m^r)^2}{m^s}(v_i - w_i)\Big],\end{aligned} \tag{A.14}$$

$$\begin{aligned}\Theta_i^\nu(\mathbf{v},\mathbf{w}) = \frac{(\mu^{sr})^3}{(m^s)^2 m^r}\Big[&- m^s v_i v^2 + \left(\frac{1}{2}\,m^s + \frac{3}{2}\,m^r\right) v_i w^2 + \left(-\frac{3}{2}\,m^s + \frac{1}{2}\,m^r\right)\sum_j v_j w_i w_j \\ &- m^r w_i w^2 + \left(\frac{3}{2}\,m^s + \frac{1}{2}\,m^r\right) w_i v^2 + \left(\frac{1}{2}\,m^s - \frac{3}{2}\,m^r\right) v_i \sum_j v_j w_j\Big].\end{aligned} \tag{A.15}$$

Let us first consider

$$\tilde{\Phi}_i := \iint \Theta_i^\kappa(\mathbf{v},\mathbf{w})\left[g_\varepsilon^s(\mathbf{v}) + g_\varepsilon^r(\mathbf{w})\right] M^s(\mathbf{v})\, M^r(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}.$$

Using parity arguments ($\mathbf{v}\mapsto -\mathbf{v}$ and $\mathbf{w}\mapsto -\mathbf{w}$), we see that some integrals vanish, and we have

$$\begin{aligned}\tilde{\Phi}_i = \frac{\mu^{sr}}{(m^s+m^r)^2}\Big\{ &\iint\Big[-\left(\frac{3}{2}\,(m^s)^2 + \frac{1}{2}\,(m^r)^2\right) v_i v^2 + m^r(m^s-m^r) v_i w^2 \\ &+ 2m^r(m^s-m^r)\sum_j v_j w_i w_j + \frac{5}{2}\frac{(m^s+m^r)^2}{m^s}\,v_i\Big] g_\varepsilon^s(\mathbf{v}) M^s(\mathbf{v})\, M^r(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w} \\ &+ \iint\Big[2(m^r)^2 w_i w^2 + \frac{1}{2}\,(m^s-m^r)^2 w_i v^2 + (m^s-m^r)^2 v_i \sum_j v_j w_j \\ &- \frac{5}{2}\frac{(m^s+m^r)^2}{m^s}\,w_i\Big] g_\varepsilon^r(\mathbf{w}) M^s(\mathbf{v})\, M^r(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\Big\} + O(\varepsilon).\end{aligned} \tag{A.16}$$

Recalling (2.2), we finally get (skipping all intermediate details)

$$\begin{aligned}\tilde{\Phi}_i = &-\frac{\mu^{sr}}{(m^s+m^r)^2}\left(3m^s + \frac{(m^r)^2}{m^s}\right)\int D_i^s(\mathbf{v}) g_\varepsilon^s(\mathbf{v}) M^s(\mathbf{v})\,\mathrm{d}\mathbf{v} \\ &+ \frac{4\mu^{sr} m^r}{(m^s+m^r)^2}\int D_i^r(\mathbf{w}) g_\varepsilon^r(\mathbf{w}) M^r(\mathbf{w})\,\mathrm{d}\mathbf{w} + O(\varepsilon).\end{aligned} \tag{A.17}$$

We now turn to

$$\tilde{\Psi}_i := \iint \Theta_i^\nu(\mathbf{v},\mathbf{w}) \Big[g_\varepsilon^s(\mathbf{v}) + g_\varepsilon^r(\mathbf{w}) \Big] M^s(\mathbf{v})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w}.$$

By usual parity arguments, integrals providing non-vanishing contributions are

$$\begin{aligned} \tilde{\Psi}_i = \frac{(\mu^{sr})^3}{(m^s)^2 m^r} \Bigg\{ \iint \Bigg[&- m^s v_i v^2 + \left(\frac{1}{2}\, m^s + \frac{3}{2}\, m^r \right) v_i w^2 \\ &+ \left(-\frac{3}{2}\, m^s + \frac{1}{2}\, m^r \right) \sum_j v_j w_i w_j \Bigg] g_\varepsilon^s(\mathbf{v}) M^s(\mathbf{v})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w} \\ &+ \iint \Bigg[-\, m^r w_i w^2 + \left(\frac{3}{2}\, m^s + \frac{1}{2}\, m^r \right) w_i v^2 \\ &+ \left(\frac{1}{2}\, m^s - \frac{3}{2}\, m^r \right) v_i \sum_j v_j w_j \Bigg] g_\varepsilon^r(\mathbf{w}) M^s(\mathbf{v})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{w} \Bigg\} + O(\varepsilon), \end{aligned} \tag{A.18}$$

from which we obtain

$$\tilde{\Psi}_i = -\frac{2(\mu^{sr})^3}{(m^s)^2 m^r} \left\{ \int D_i^s(\mathbf{v}) g_\varepsilon^s(\mathbf{v}) M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + \int D_i^r(\mathbf{w}) g_\varepsilon^r(\mathbf{w}) M^r(\mathbf{w})\, \mathrm{d}\mathbf{w} \right\} + O(\varepsilon). \tag{A.19}$$

In conclusion,

$$\begin{aligned} &\sum_{s,r=1}^N \rho^s \rho^r (m^s)^p\, \eta^s \int \mathbf{D}^s(\mathbf{v}) \Big[Q^{sr}(g_\varepsilon^s M^s, M^r) + Q^{sr}(M^s, g_\varepsilon^r M^r) \Big] \mathrm{d}\mathbf{v} \\ &= \sum_{s,r=1}^N \rho^s \rho^r (m^s)^p\, \eta^s\, \kappa^{sr} \left\{ -\frac{\mu^{sr}}{(m^s+m^r)^2} \left(3m^s + \frac{(m^r)^2}{m^s} \right) \int \mathbf{D}^s(\mathbf{v}) g_\varepsilon^s(\mathbf{v}) M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} \right. \\ &\quad \left. + \frac{4\mu^{sr} m^r}{(m^s+m^r)^2} \int \mathbf{D}^r(\mathbf{w}) g_\varepsilon^r(\mathbf{w}) M^r(\mathbf{w})\, \mathrm{d}\mathbf{w} \right\} \\ &\quad - \sum_{s,r=1}^N \rho^s \rho^r (m^s)^p\, \eta^s\, \nu^{sr} \frac{2(\mu^{sr})^3}{(m^s)^2 m^r} \left\{ \int \mathbf{D}^s(\mathbf{v})\, g_\varepsilon^s(\mathbf{v})\, M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + \int \mathbf{D}^r(\mathbf{w})\, g_\varepsilon^r(\mathbf{w})\, M^r(\mathbf{w})\, \mathrm{d}\mathbf{w} \right\} + O(\varepsilon) \\ &= \sum_{s,r=1}^N \rho^s \rho^r \mu^{sr} \left\{ \frac{1}{(m^s+m^r)^2} \left[-(m^s)^p \left(3m^s + \frac{(m^r)^2}{m^s} \right) \eta^s + 4\, m^s (m^r)^p\, \eta^r \right] \kappa^{sr} \right. \\ &\quad \left. - \frac{2(\mu^{sr})^2}{m^s m^r} \left[(m^s)^{p-1} \eta^s + (m^r)^{p-1} \eta^r \right] \nu^{sr} \right\} \int \mathbf{D}^s(\mathbf{v})\, g_\varepsilon^s(\mathbf{v})\, M^s(\mathbf{v})\, \mathrm{d}\mathbf{v} + O(\varepsilon). \end{aligned}$$

The sought relation (A.10) is satisfied if we can choose $p > 0$ and η^s, $s = 1, \dots, N$, such that

$$\begin{aligned} K = \sum_{r=1}^N \rho^r \mu^{sr} &\left\{ \frac{1}{(m^s+m^r)^2} \left[-(m^s)^p \left(3m^s + \frac{(m^r)^2}{m^s} \right) \eta^s + 4\, m^s (m^r)^p\, \eta^r \right] \kappa^{sr} \right. \\ &\left. - \frac{2(\mu^{sr})^2}{m^s m^r} \left[(m^s)^{p-1} \eta^s + (m^r)^{p-1} \eta^r \right] \nu^{sr} \right\}, \qquad s = 1, \dots, N. \end{aligned} \tag{A.20}$$

This is a linear system $\tilde{\mathbf{A}} \cdot \boldsymbol{\eta} = \tilde{\mathbf{b}}$ for the N unknowns η^s, that may be rewritten as

$$\left\{ \frac{1}{2}\,\rho^s (m^s)^p \nu^{ss} + \sum_{r\neq s} \rho^r \frac{\mu^{sr}}{(m^s+m^r)^2}\Big[(m^s)^{p-1}\big(3(m^s)^2+(m^r)^2\big)\kappa^{sr} + 2(m^s)^p m^r \nu^{sr}\Big] \right\} \eta^s$$
$$+ \sum_{r\neq s} \rho^r \frac{\mu^{sr}}{(m^s+m^r)^2}\, m^s (m^r)^p \big(-\,4\kappa^{sr} + 2\,\nu^{sr}\big)\eta^r = -\,K, \qquad s = 1,\dots,N. \tag{A.21}$$

If we are able to find a value for the power p such that the corresponding coefficient matrix $\tilde{\mathbf{A}}$ is diagonally dominant, then the existence of a solution η^s, $s = 1,\dots,N$, is guaranteed. Recalling now the definitions of κ^{sr} and ν^{sr} given in (1.11), we get

$$-\,4\kappa^{sr} + 2\nu^{sr} = -\,4\pi \int_0^\pi \vartheta^{sr}(\chi)(1-\cos\chi)^2 \sin\chi \,\mathrm{d}\chi \leq 0,$$

therefore

$$|\tilde{a}_{ss}| - \sum_{r\neq s} |\tilde{a}_{sr}| = \frac{1}{2}\,\rho^s (m^s)^p \nu^{ss} + \sum_{r\neq s} \rho^r \frac{\mu^{sr}}{(m^s+m^r)^2}\Big\{\Big[(m^s)^{p-1}\big(3(m^s)^2+(m^r)^2\big)$$
$$-\,4\,m^s (m^r)^p\Big]\kappa^{sr} + 2\Big[(m^s)^p m^r + m^s (m^r)^p\Big]\nu^{sr}\Big\}. \tag{A.22}$$

We can prove that if $p = 1/2$ the coefficient in the square brackets in front of κ^{sr} turns out to be nonnegative for all values of m^s, m^r. In fact, it may be written as

$$(m^s)^{3/2}\left[\left(\frac{m^r}{m^s}\right)^2 - 4\left(\frac{m^r}{m^s}\right)^{1/2} + 3\right],$$

and it's easy to check that the function

$$f(y) = y^2 - 4y^{1/2} + 3$$

takes its minimum for $y = 1$ and $f(1) = 0$, hence $f(y) \geq 0$ $\forall y \geq 0$. In conclusion, for $p = 1/2$

$$|\tilde{a}_{ss}| > \sum_{r\neq s} |\tilde{a}_{sr}|, \qquad \forall s = 1,\dots,N,$$

hence for any fixed K there is a unique solution $\eta^1, \dots, \eta^N$ to the linear system (A.21). ☐

Lemma A.4. *As concerns the collision contribution appearing in* (6.9)*, we have*

$$\sum_{s,r=1}^N \rho^s \rho^r \sqrt{m^s}\,\eta^s \int \mathbf{D}^s(\mathbf{v})\, Q^{sr}(g^s_\varepsilon M^s, g^r_\varepsilon M^r)\,\mathrm{d}\mathbf{v}$$
$$= \frac{5}{2}\sum_{s,r=1}^N \rho^s\rho^r\eta^s \frac{\mu^{sr}\sqrt{m^s}}{(m^s+m^r)^2}\left\{\left(3m^s - 4m^r + \frac{(m^r)^2}{m^s}\right)\kappa^{sr} + 4\,m^r\nu^{sr}\right\}\mathbf{u}\,T + O(\varepsilon). \tag{A.23}$$

Proof. Notice that

$$\int D^s_i(\mathbf{v})\, Q^{sr}(g^s_\varepsilon M^s, g^r_\varepsilon M^r)\,\mathrm{d}\mathbf{v} = \iint \Big[\kappa^{sr}\,\Theta^\kappa_i(\mathbf{v},\mathbf{w}) + \nu^{sr}\,\Theta^\nu_i(\mathbf{v},\mathbf{w})\Big](g^s_\varepsilon M^s)(\mathbf{v})\,(g^r_\varepsilon M^r)(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}, \tag{A.24}$$

where Θ^κ_i and Θ^ν_i are given in (A.14)–(A.15). Taking into account moments of the distributions g^s_ε given in (2.8), and moreover the third order moment

$$\int v_i\, v^2 (g^s_\varepsilon M^s)(\mathbf{v})\,\mathrm{d}\mathbf{v} = \frac{5}{m^s}\,u_i + O(\varepsilon),$$

we get

$$\iint \Theta_i^\kappa(\mathbf{v},\mathbf{w})(g_\varepsilon^s M^s)(\mathbf{v})\,(g_\varepsilon^r M^r)(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w} = \frac{5}{2}\,\frac{\mu^{sr}}{m^s(m^s+m^r)^2}\Big(3(m^s)^2 - 4m^s m^r + (m^r)^2\Big)u_i T + O(\varepsilon), \tag{A.25}$$

and

$$\iint \Theta_i^\nu(\mathbf{v},\mathbf{w})(g_\varepsilon^s M^s)(\mathbf{v})\,(g_\varepsilon^r M^r)(\mathbf{w})\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w} = 10\,\frac{(\mu^{sr})^3}{(m^s)^2 m^r}\,u_i T + O(\varepsilon). \tag{A.26}$$

□

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