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THE PERIODIC UNFOLDING METHOD FOR A CLASS OF PARABOLIC PROBLEMS WITH IMPERFECT INTERFACES *

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Abstract. In this paper, we use the adapted periodic unfolding method to study the homogenization and corrector problems for the parabolic problem in a two-component composite with ε-periodic connected inclusions. The condition imposed on the interface is that the jump of the solution is proportional to the conormal derivative *via* a function of order ε^γ with $\gamma \leq -1$. We give the homogenization results which include those obtained by Jose in [*Rev. Roum. Math. Pures Appl.* **54** (2009) 189–222]. We also get the corrector results.

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1. Introduction

In this paper, we use the adapted periodic unfolding method to study the homogenization and corrector problem for a linear parabolic problem in a domain $\Omega \subset \mathbb{R}^n$ consisting of two components, a connected component $\Omega_{1\varepsilon}$ and a disconnected component $\Omega_{2\varepsilon}$. The latter is the union of ε-periodic connected inclusions of size ε. The conditions prescribed on the interface $\Gamma^\varepsilon = \partial\Omega_{2\varepsilon}$, separating $\Omega_{1\varepsilon}$ from $\Omega_{2\varepsilon}$, are the continuity of the conormal derivatives and a jump of the solution proportional to the conormal derivatives *via* a function of order ε^γ.

This problem models the heat diffusion in a two-component composite conductor with an ε-periodic interface, where the flux of temperature is proportional to the jump of the temperature field (see Carslaw and Jaeger [7]).

Keywords and phrases. Periodic unfolding method, heat equation, interface problems, homogenization, correctors.

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More precisely, we consider, for the different values of the parameter $\gamma \leq 1$, the homogenization and corrector results for the following problem:

$$\begin{cases} u'_{1\varepsilon} - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = f_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times (0,T), \\ u'_{2\varepsilon} - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = f_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times (0,T), \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times (0,T), \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times (0,T), \\ u_{1\varepsilon} = 0 & \text{on } \partial\Omega \times (0,T), \\ u_{1\varepsilon}(x,0) = U^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ u_{2\varepsilon}(x,0) = U^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}, \end{cases} \tag{1.1}$$

where $A^\varepsilon(x) := A(x/\varepsilon)$, A being a periodic, bounded and positive definite matrix field, $h^\varepsilon(x) := h(x/\varepsilon)$, with h positive, bounded and periodic, $f_\varepsilon = (f_{1\varepsilon}, f_{2\varepsilon})$, $U^0_\varepsilon = (U^0_{1\varepsilon}, U^0_{2\varepsilon})$ and $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$. Here and in what follows, any component indexed by $i(= 1$ *or* $2)$ is defined in $\Omega_{i\varepsilon}$. Denote by $n_{i\varepsilon}$ the unitary outward normal vector of $\partial\Omega_{i\varepsilon}$.

This paper focuses on the study when $\gamma \leq -1$. Indeed, for $\gamma \leq 1$, the homogenization was studied by the oscillating test functions method (see Tartar [21]) in Jose [18]. The corrector results were given in Donato and Jose [11] for $-1 < \gamma \leq 1$. But for the case of $\gamma \leq -1$, to the best knowledge of the author, it was open. The present paper is devoted to solving this problem.

More precisely, we first study the homogenization results for $\gamma \leq -1$, which recover those in [18]. In particular, we give the precise convergences of flux. For the exact statements, see Theorems 3.1–3.2. To obtain the corrector results, it is necessary to impose some stronger assumptions than those of the homogenization results. More precisely, we introduce the assumption on the data f_ε which is slightly weaker than that in [11], and the assumption for the initial condition U^0_ε which is equivalent to that in [11]. Then, we obtain the corrector results (for $\gamma \leq -1$) which are completely new. In particular, for the technical reason, we present them for $\gamma < -1$ and $\gamma = -1$, respectively. For the exact statements, see Theorems 5.3 and 5.5.

The proofs of our results depend mainly on the periodic unfolding method, which was first introduced by Cioranescu *et al.* in [4] for the case of fixed domains (see [5] for more details) and then extended to perforated ones in Cioranescu *et al.* [6]. Later, Cioranescu *et al.* [3] gave a comprehensive presentation of the unfolding method for perforated domains. Subsequently, this method was adapted to two-component domains which are separated by a periodic interface in Donato *et al.* [13], where two unfolding operators over two-component domains were introduced and their properties were discussed.

Concerning the time-dependent periodic unfolding method for fixed domains, we refer to Gaveau [17], where some elementary results were listed without proofs. Recently, Donato and the author adapted some results related to the unfolding method for perforated domains to time-dependent functions in [15], where detailed proofs were given. There, in order to study problem (1.1), we adapt the unfolding method in two-component domains in [13] to time-dependent problems (see Sect. 2). We introduce two unfolding operators: $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$. The operator $\mathcal{T}_1^\varepsilon$, originally denoted by $\mathcal{T}_\varepsilon^*$ in [15], acts on functions defined on $\Omega_{1\varepsilon} \times (0,T)$. The operator $\mathcal{T}_2^\varepsilon$ acts on functions defined on $\Omega_{2\varepsilon} \times (0,T)$. The most important feature of these operators is that they map functions defined on the oscillating domain into functions defined on the fixed domain. Hence, in some sense, they play the role of the extension operators. Also, we list some results related to $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$. In particular, we study the properties of their trace on the common boundary, which will be crucial to the treatment of the interface terms.

For the elliptic problem corresponding to (1.1), Monsurrò [19, 20] gave the homogenization for $\gamma \leq -1$. For $\gamma > -1$, the homogenization was obtained by Donato and Monsurrò [12]. These results are based on the oscillating test functions method. Corresponding corrector results for $-1 < \gamma \leq 1$ were proved by Donato [8]. Recently, Donato *et al.* gave the new proofs of these results by the unfolding method in [13]. For the hyperbolic problem corresponding to (1.1), Donato *et al.* proved the homogenization results for $\gamma \leq 1$ in [9] and the corrector results for $-1 < \gamma \le 1$ in [10]. Our results are also related to those of parabolic problems in perforated domains which were studied in Donato and Nabil [14].

This paper is organized as follows. In Section 2, we adapt the unfolding method for a two-component domain in [13]. In particular, we present some important convergence results. Section 3 is devoted to the homogenization of problem (1.1) according to the different values of γ. In Section 4, we introduce some assumptions on the initial data and give the convergence of the energy. Section 5 focuses on the corrector results.

2. Periodic unfolding method in two-component domains

2.1. Some notations

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary, and let ε be the general term of a sequence of positive real numbers which converges to zero.

Denoted by $Y = [0, l_1) \times \cdots \times [0, l_n)$ the reference cell with $l_i > 0$, $i = 1, \cdots, n$. We suppose that Y_1 and Y_2 are two nonempty open disjoint subsets of Y such that

$$Y = Y_1 \cup \overline{Y_2},$$

where Y_1 is connected and $\Gamma = \partial Y_2$ is Lipschitz continuous. Let n_i be the unit outward normal to Y_i, $i = 1, 2$.

For any $k \in \mathbb{Z}^n$, we denote

$$Y^k = k_l + Y,\ \Gamma_k = k_l + \Gamma,\ Y_i^k = k_l + Y_i,$$

where $k_l = (k_1 l_1, \cdots, k_n l_n)$ and $i = 1, 2$.

For any fixed ε, let $K_\varepsilon = \{k \in \mathbb{Z}^n \mid \varepsilon Y_i^k \cap \Omega \neq \emptyset, i = 1, 2\}$. We suppose that

$$\partial\Omega \cap \left(\bigcup_{k \in \mathbb{Z}^n} (\varepsilon \Gamma_k) \right) = \emptyset.$$

Write the two components of Ω and the interface, respectively, by:

$$\Omega_{2\varepsilon} = \bigcup_{k \in K_\varepsilon} \varepsilon Y_2^k, \qquad \Omega_{1\varepsilon} = \Omega \backslash \Omega_{2\varepsilon}, \qquad \Gamma^\varepsilon = \partial \Omega_{2\varepsilon}.$$

Notice that $\partial\Omega$ and Γ^ε are disjoint, the component $\Omega_{1\varepsilon}$ is connected and the component $\Omega_{2\varepsilon}$ is the union of ε^{-n} disjoint translated sets of εY_2.

Now we introduce two spaces V^ε and H_γ^ε. Define V^ε by

$$V^\varepsilon := \{v \in H^1(\Omega_{1\varepsilon}) \mid v = 0 \text{ on } \partial\Omega\},$$

endowed with the norm

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_{1\varepsilon})}.$$

For any $\gamma \in \mathbb{R}$, the product space

$$H_\gamma^\varepsilon := \{u = (u_1, u_2) \mid u_1 \in V^\varepsilon,\ u_2 \in H^1(\Omega_{2\varepsilon})\}$$

is equipped with the norm:

$$\|u\|_{H_\gamma^\varepsilon}^2 = \|\nabla u_1\|_{L^2(\Omega_{1\varepsilon})}^2 + \|\nabla u_2\|_{L^2(\Omega_{2\varepsilon})}^2 + \varepsilon^\gamma \|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)}^2.$$

Next we recall the following notations related to the unfolding method in [3,5,13]:

$$\widehat{K}_\varepsilon = \{k \in \mathbb{Z}^n \mid \varepsilon Y^k \subset \Omega\},\quad \widehat{\Omega}_\varepsilon = \operatorname{int} \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon(k_l + \overline{Y}),\quad \Lambda_\varepsilon = \Omega \backslash \widehat{\Omega}_\varepsilon,$$

$$\widehat{\Omega}_{i\varepsilon} = \bigcup_{k\in\widehat{K}_\varepsilon} \varepsilon Y_i^k, \quad \Lambda_{i\varepsilon} = \Omega_{i\varepsilon}\backslash\widehat{\Omega}_{i\varepsilon}, \quad i=1,2, \quad \widehat{\Gamma}^\varepsilon = \partial\widehat{\Omega}_{2\varepsilon}.$$

In what follows, we will use the following notations:

- $\theta_i = |Y_i|/|Y|$, $i=1,2$;
- $\mathcal{M}_{\mathcal{O}}(v) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} v\mathrm{d}x$;
- $\widetilde{g}$ is the zero extension to Ω (resp., $\Omega\times(0,T)$) of any function g defined on $\Omega_{i\varepsilon}$ (resp., $\Omega_{i\varepsilon}\times(0,T)$) for $i=1,2$;
- The letter T is a fixed positive constant in $\mathbb{R}$.

Throughout this paper, we will also use the following general notations:

- c and C denote generic constants which do not depend upon ε.
- δ_{ij} denotes the usual Kronecker symbol.
- The notation $L^p(\mathcal{O})$ will be used both for scalar and vector-valued functions defined on the set $\mathcal{O}$, when no ambiguity arises.

2.2. Time-dependent unfolding operators in two-component domains

In this subsection, we adapt the unfolding method in two-component domains in [13] to time-dependent problems. We introduce two unfolding operators: $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$, which map functions defined on the oscillating domains $\Omega_{1\varepsilon}\times(0,T)$ and $\Omega_{2\varepsilon}\times(0,T)$ into functions defined on the fixed domains $\Omega\times Y_1\times(0,T)$ and $\Omega\times Y_2\times(0,T)$, respectively. As stated in [3,13], this avoids the use of any extension operator. Next, we will recall some properties of $\mathcal{T}_1^\varepsilon$, which is exactly the unfolding operator $\mathcal{T}_\varepsilon^*$ in perforated domains in [15]. We also list some properties of $\mathcal{T}_2^\varepsilon$. Moreover, we study some properties of the traces of $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$ on the common boundary, which will be used to treat the interface term.

For any $z\in\mathbb{R}^n$, we use $[z]_Y$ to denote its integer part $(k_1l_1,\cdots,k_nl_n)$ such that $z-[z]_Y\in Y$, and set

$$\{z\}_Y = z-[z]_Y \quad \text{for} \quad z\in\mathbb{R}^n.$$

Then for each $x\in\mathbb{R}^n$, one has

$$x=\varepsilon\Big(\Big[\frac{x}{\varepsilon}\Big]_Y+\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) \quad \text{for} \quad x\in\mathbb{R}^n.$$

Let us first recall the unfolding operator $\mathcal{T}_\varepsilon$ for the fixed domain $\Omega\times(0,T)$ introduced in [17], where the properties of $\mathcal{T}_\varepsilon$ are shown without proofs.

Definition 2.1. For $p\in[1,+\infty)$ and $q\in[1,\infty]$, let ϕ be in $L^q(0,T;L^p(\Omega))$. The unfolding operator $\mathcal{T}_\varepsilon: L^q(0,T;L^p(\Omega))\mapsto L^q(0,T;L^p(\Omega\times Y))$ is defined as follows:

$$\mathcal{T}^\varepsilon(\phi)(x,y,t)=\begin{cases}\phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y+\varepsilon y,t\Big) & \text{a.e. for } (x,y,t)\in\widehat{\Omega}_\varepsilon\times Y\times(0,T),\\ 0 & \text{a.e. for } (x,y,t)\in\Lambda_\varepsilon\times Y\times(0,T).\end{cases}$$

In a similar way, we extend the unfolding operators in two-component domains in [13] to the following time-dependent unfolding operators in two-component domains.

Definition 2.2. Let $i=1,2$. For $p\in[1,+\infty)$ and $q\in[1,\infty]$, let ϕ be in $L^q(0,T;L^p(\Omega_{i\varepsilon}))$. The unfolding operator $\mathcal{T}_i^\varepsilon: L^q(0,T;L^p(\Omega_{i\varepsilon}))\to L^q(0,T;L^p(\Omega\times Y_i))$ is defined as follows:

$$\mathcal{T}_i^\varepsilon(\phi)(x,y,t)=\begin{cases}\phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y+\varepsilon y,t\Big) & \text{a.e. for } (x,y,t)\in\widehat{\Omega}_\varepsilon\times Y_i\times(0,T),\\ 0 & \text{a.e. for } (x,y,t)\in\Lambda_\varepsilon\times Y_i\times(0,T).\end{cases}$$

From this definition, the following properties are immediate:

$$\begin{aligned}
&\text{(i)}\quad \mathcal{T}_i^\varepsilon(vw) = \mathcal{T}_i^\varepsilon(v)\mathcal{T}_i^\varepsilon(w), \quad \forall w, v \in L^q(0,T; L^p(\Omega_{i\varepsilon})),\\
&\text{(ii)}\quad \mathcal{T}_i^\varepsilon(\psi\varphi) = \varphi\mathcal{T}_i^\varepsilon(\psi), \quad \forall \psi \in L^p(\Omega_{i\varepsilon}) \text{ and } \varphi \in L^q(0,T),\\
&\text{(iii)}\ \nabla_y(\mathcal{T}_i^\varepsilon(\phi)) = \varepsilon\mathcal{T}_i^\varepsilon(\nabla\phi), \quad \forall \phi \in L^q(0,T; W^{1,p}(\Omega_{i\varepsilon})).
\end{aligned}$$

Lemma 2.3. *Concerning $\mathcal{T}^\varepsilon$ and $\mathcal{T}_i^\varepsilon$, we have the following:*

$$\begin{aligned}
&\mathcal{T}_i^\varepsilon(\omega|_{\Omega_{i\varepsilon}\times(0,T)}) = \mathcal{T}^\varepsilon(\omega)|_{\Omega\times Y_i\times(0,T)},\\
&\mathcal{T}_i^\varepsilon(\psi) = \mathcal{T}^\varepsilon(\widetilde{\psi})|_{\Omega\times Y_i\times(0,T)},
\end{aligned}$$

where ω and ψ are defined on $\Omega\times(0,T)$ and $\Omega_{i\varepsilon}\times(0,T)$, respectively.

In Definitions 2.1 and 2.2, if ϕ is independent of t, then $\mathcal{T}_\varepsilon$ and $\mathcal{T}_i^\varepsilon (i = 1,2)$ are the classical unfolding operators defined in [4] and [13], respectively.

For simplicity, we always write $\mathcal{T}_i^\varepsilon(\phi)$ instead of $\mathcal{T}_i^\varepsilon(\phi|_{\Omega_{i\varepsilon}\times(0,T)})$ for any function ϕ defined in $\Omega\times(0,T)$.

Next we list some properties of $\mathcal{T}_i^\varepsilon$ which are important to the study of the homogenization in Section 3. For $i = 1$, the following results were proved in [15]. For $i = 2$, the proofs are essentially the same. For other properties and related comments, we refer the reader to [3, 13, 15].

Proposition 2.4. *Let $i = 1,2$. For $p \in [1,+\infty)$ and $q \in [1,\infty]$, the operator $\mathcal{T}_i^\varepsilon$ is linear and continuous from $L^q(0,T;L^p(\Omega_{i\varepsilon}))$ to $L^q(0,T;L^p(\Omega\times Y_i))$. Let $\phi \in L^q(0,T;L^1(\Omega_{i\varepsilon}))$ and $w \in L^q(0,T;L^p(\Omega_{i\varepsilon}))$. For a.e. $t \in (0,T)$, we have*

$$\begin{aligned}
&\text{(i)}\quad \frac{1}{|Y|}\int_{\Omega\times Y_i} \mathcal{T}_i^\varepsilon(\phi)(x,y,t)\,\mathrm{d}x\,\mathrm{d}y = \int_{\widehat{\Omega}_{i\varepsilon}} \phi(x,t)\mathrm{d}x = \int_{\Omega_{i\varepsilon}} \phi(x,t)\mathrm{d}x - \int_{\Lambda_{i\varepsilon}} \phi(x,t)\mathrm{d}x,\\
&\text{(ii)}\quad \|\mathcal{T}_i^\varepsilon(w)\|_{L^p(\Omega\times Y_i)} = |Y|^{1/p}\|w\|_{L^p(\widehat{\Omega}_{i\varepsilon})} \le |Y|^{1/p}\|w\|_{L^p(\Omega_{i\varepsilon})}.
\end{aligned}$$

Proposition 2.5. *Let $i = 1,2$. For $q \in [1,+\infty]$, let $\{\phi_\varepsilon\}$ be a sequence in $L^q(0,T;L^1(\Omega_{i\varepsilon}))$ such that*

$$\int_0^T\int_{\Lambda_{i\varepsilon}} |\phi_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{2.1}$$

Then

$$\int_0^T\int_{\Omega_{i\varepsilon}} \phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y_i} \mathcal{T}_i^\varepsilon(\phi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \to 0.$$

As usual, this is denoted by

$$\int_0^T\int_{\Omega_{i\varepsilon}} \phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \simeq \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y_i} \mathcal{T}_i^\varepsilon(\phi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{2.2}$$

In particular, for $p, q \in (1,+\infty)$, let $\{\varphi_\varepsilon\}$ and $\{\psi_\varepsilon\}$ be two sequences in $L^q(0,T;L^p(\Omega_{i\varepsilon}))$ and $L^{q'}(0,T;L^{p'}(\Omega_{i\varepsilon}))$ $(1/p + 1/p' = 1,\ 1/q + 1/q' = 1)$, respectively. Suppose that

$$\begin{aligned}
&\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \to \varphi_i \text{ strongly in } L^q(0,T;L^p(\Omega\times Y_i)),\\
&\mathcal{T}_i^\varepsilon(\psi_\varepsilon) \rightharpoonup \psi_i \text{ weakly in } L^{q'}(0,T;L^{p'}(\Omega\times Y_i)).
\end{aligned} \tag{2.3}$$

Then for any $\eta \in \mathcal{D}(\Omega)$, we have

$$\int_0^T\int_{\Omega_{i\varepsilon}} \varphi_\varepsilon\psi_\varepsilon\eta\,\mathrm{d}x\,\mathrm{d}t \to \frac{1}{|Y_i|}\int_0^T\int_{\Omega\times Y_i} \varphi_i\psi_i\eta\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Proposition 2.6. *Let $i = 1, 2$.*

(i) *For $p, q \in [1, \infty)$, let $w \in L^q(0,T;L^p(\Omega))$. Then*

$$\mathcal{T}_i^\varepsilon(w) \to w \text{ strongly in } L^q(0,T;L^p(\Omega \times Y_i)).$$

(ii) *For $p, q \in [1, \infty)$, let $\{\omega_\varepsilon\}$ be a sequence in $L^q(0,T;L^p(\Omega))$ such that*

$$\omega_\varepsilon \to \omega \text{ strongly in } L^q(0,T;L^p(\Omega)),$$

then we have

$$\mathcal{T}_i^\varepsilon(\omega_\varepsilon) \to \omega \quad \text{strongly in } L^q(0,T;L^p(\Omega \times Y_i)).$$

(iii) *For $p \in (1, \infty)$ and $q \in (1, \infty]$, let $\{\varphi_\varepsilon\}$ be a sequence in $L^q(0,T;L^p(\Omega_{i\varepsilon}))$ such that*

$$\|\varphi_\varepsilon\|_{L^q(0,T;L^p(\Omega_{i\varepsilon}))} \le C.$$

If

$$\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi \text{ weakly in } L^q(0,T;L^p(\Omega \times Y_i)),$$

then we have

$$\widetilde{\varphi}_\varepsilon \rightharpoonup \theta_i \mathcal{M}_{Y_i}(\varphi) \text{ weakly in } L^q(0,T;L^p(\Omega)).$$

For $q = \infty$, the weak convergences above are replaced by the weak convergences, respectively.*

Proposition 2.7. *Let $p, q \in [1, +\infty)$. For $i = 1, 2$, let $\omega_\varepsilon \in L^q(0,T;L^p(\Omega_{i\varepsilon}))$ and $\omega \in L^q(0,T;L^p(\Omega))$, then the following two assertions are equivalent:*

(a) *$\mathcal{T}_i^\varepsilon(\omega_\varepsilon) \to \omega$ strongly in $L^q(0,T;L^p(\Omega \times Y_i))$;*
(b) *$\|\omega_\varepsilon - \omega\|_{L^q(0,T;L^p(\widehat{\Omega}_{i\varepsilon}))} \to 0$.*

Furthermore, (a) *together with $\|\omega_\varepsilon\|_{L^q(0,T;L^p(\Lambda_{i\varepsilon}))} \to 0$ is equivalent to*

$$\|\omega_\varepsilon - \omega\|_{L^q(0,T;L^p(\Omega_{i\varepsilon}))} \to 0.$$

In the following, we are concerned with the action of the unfolding operators on the sequences in $L^2(0,T;H_\gamma^\varepsilon)$. To do that, we first recall the following results related to V^ε and H_γ^ε.

Proposition 2.8 ([11], Rem. 2.3)**.** *There exists a positive constants C (independent of ε) such that*

$$\|u\|_{H^1(\Omega_{1\varepsilon})} \le C\|u\|_{V^\varepsilon}, \quad \forall u \in V^\varepsilon.$$

Proposition 2.9 ([11], Prop. 4.1)**.** *For $\gamma \le 1$, there exist two positive constants c_1, c_2 (independent of ε) such that*

$$c_1\|u\|^2_{V^\varepsilon \times H^1(\Omega_{2\varepsilon})} \le \|u\|^2_{H_\gamma^\varepsilon} \le c_2(1 + \varepsilon^{\gamma-1})\|u\|^2_{V^\varepsilon \times H^1(\Omega_{2\varepsilon})}.$$

Now we show some results related to the jump on the interface. For convenience, we set

$$u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}).$$

By the definition of $\mathcal{T}_i^\varepsilon (i = 1, 2)$, we have the following result (see also the proof of [13], Lem. 2.14).

Proposition 2.10. *For $\gamma \le 1$, suppose that $\{u_\varepsilon\}$ is a sequence in $L^2(0,T;H_\gamma^\varepsilon)$. Then for a.e. $t \in [0,T]$, we have*

$$\frac{1}{\varepsilon|Y|}\int_{\Omega \times \Gamma} |\mathcal{T}_1^\varepsilon(u_{1\varepsilon}) - \mathcal{T}_2^\varepsilon(u_{2\varepsilon})|^2 \, \mathrm{d}x \, \mathrm{d}\sigma_y = \int_{\widehat{\Gamma}^\varepsilon} |u_{1\varepsilon} - u_{2\varepsilon}|^2 \mathrm{d}\sigma_x. \tag{2.4}$$

Remark 2.11. For $\gamma \le 1$, let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(0,T;H^\varepsilon_\gamma)$. By (2.4), Proposition 2.4(ii), Propositions 2.8 and 2.9, we easily get the following uniform estimates:

$$\begin{aligned}
&\|\mathcal{T}^\varepsilon_1(u_{1\varepsilon})\|_{L^2(0,T;L^2(\Omega\times Y_1))} + \|\mathcal{T}^\varepsilon_1(\nabla u_{1\varepsilon})\|_{L^2(0,T;L^2(\Omega\times Y_1))} \le c,\\
&\|\mathcal{T}^\varepsilon_2(u_{2\varepsilon})\|_{L^2(0,T;L^2(\Omega\times Y_2))} + \|\mathcal{T}^\varepsilon_2(\nabla u_{2\varepsilon})\|_{L^2(0,T;L^2(\Omega\times Y_2))} \le c,\\
&\|\mathcal{T}^\varepsilon_1(u_{1\varepsilon}) - \mathcal{T}^\varepsilon_2(u_{2\varepsilon})\|_{L^2(0,T;L^2(\Omega\times\Gamma))} \le c\varepsilon^{\frac{1-\gamma}{2}}.
\end{aligned}$$

If we suppose further that u_ε satisfies

$$\|u_{1\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_{1\varepsilon}))} + \|u_{2\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_{2\varepsilon}))} \le c,$$

then it follows that

$$\|\mathcal{T}^\varepsilon_1(u_{1\varepsilon})\|_{L^\infty(0,T;L^2(\Omega\times Y_1))} + \|\mathcal{T}^\varepsilon_2(u_{2\varepsilon})\|_{L^\infty(0,T;L^2(\Omega\times Y_2))} \le c.$$

The following proposition is a straightforward consequence of Proposition 2.10.

Proposition 2.12. *Let $h^\varepsilon(x) = h(x/\varepsilon)$ with $h \in L^\infty(\Gamma)$ being a Y-periodic function. Suppose that $\phi \in \mathcal{D}(\Omega)$, $\varphi \in \mathcal{D}(0,T)$ and $\{u_\varepsilon\}$ is a sequence in $L^2(0,T;H^\varepsilon_\gamma)$ with $\gamma \le 1$. Then for ε small enough,*

$$\varepsilon\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})\phi\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t = \frac{1}{|Y|}\int_0^T\int_{\Omega\times\Gamma} h(y)\big[\mathcal{T}^\varepsilon_1(u_{1\varepsilon}) - \mathcal{T}^\varepsilon_2(u_{2\varepsilon})\big]\mathcal{T}^\varepsilon_1(\phi)\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t. \tag{2.5}$$

We complete this subsection with some convergence results related to the action of the unfolding operator $\mathcal{T}^\varepsilon_i (i = 1,2)$ on the bounded sequences of $L^2(0,T;H^\varepsilon_\gamma)$, which are crucial to our homogenization results.

Theorem 2.13. *Let $\gamma \le 1$. Suppose that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H^\varepsilon_\gamma)$. Then there exist $u_1 \in L^2(0,T;H^1_0(\Omega))$, $u_2 \in L^2(0,T;L^2(\Omega))$, $\widehat{u}_1 \in L^2(0,T;L^2(\Omega,H^1_{\mathrm{per}}(Y_1)))$ and $\widehat{u}_2 \in L^2(0,T;L^2(\Omega,H^1(Y_2)))$ such that, up to a subsequence (still denoted by ε),*

$$\begin{aligned}
&\text{(i)} \quad \mathcal{T}^\varepsilon_1(u_{1\varepsilon}) \rightharpoonup u_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega,H^1(Y_1))),\\
&\text{(ii)} \quad \mathcal{T}^\varepsilon_1(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y\widehat{u}_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega\times Y_1)),\\
&\text{(iii)} \quad \mathcal{T}^\varepsilon_2(u_{2\varepsilon}) \rightharpoonup u_2 \quad \text{weakly in } L^2(0,T;L^2(\Omega,H^1(Y_2))),\\
&\text{(iv)} \quad \mathcal{T}^\varepsilon_2(\nabla u_{2\varepsilon}) \rightharpoonup \nabla_y\widehat{u}_2 \quad \text{weakly in } L^2(0,T;L^2(\Omega\times Y_2)),
\end{aligned} \tag{2.6}$$

where $\mathcal{M}_\Gamma(\widehat{u}_i) = 0, i = 1,2$. Moreover, if $\gamma < 1$, then $u_1 = u_2$ and

(i) *if $\gamma < -1$, then*

$$\widehat{u}_1 = \widehat{u}_2 - y_\Gamma\nabla u_1 \quad \text{on } (0,T)\times\Omega\times\Gamma, \tag{2.7}$$

where $y_\Gamma = y - \mathcal{M}_\Gamma(y)$.

(ii) *if $\gamma = -1$, then there exists $\zeta \in L^2(0,T;L^2(\Omega\times\Gamma))$ such that*

$$\varepsilon^{-1}\big(\mathcal{T}^\varepsilon_1(u_{1\varepsilon}) - \mathcal{T}^\varepsilon_2(u_{2\varepsilon})\big) \rightharpoonup \widehat{u}_1 - \widehat{u}_2 + y_\Gamma\nabla u_1 + \zeta \text{ weakly in } L^2(0,T;L^2(\Omega\times\Gamma)). \tag{2.8}$$

Proof. The proof can be obtained by following the lines of the proofs of Theorem 2.12 in [3] (see also [15], Thm. 2.19) and Theorem 2.20 in [13]. For the reader's convenience, we repeat some details as follows.

Following the arguments in the proofs of Theorems 2.17–2.20 in [13] and Theorem 2.12 in [3] (see also [15], Thm. 2.19), we obtain that (2.6) holds at least for a subsequence.

If $\gamma < 1$, Remark 2.11 gives

$$\mathcal{T}^\varepsilon_1(u_{1\varepsilon}) - \mathcal{T}^\varepsilon_2(u_{2\varepsilon}) \to 0 \text{ strongly in } L^2(0,T;L^2(\Omega\times\Gamma)). \tag{2.9}$$

On the other hand, thanks to the properties of trace, (2.6)(i) implies

$$\mathcal{T}_1^\varepsilon(u_{1\varepsilon}) \rightharpoonup u_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega\times\Gamma)).$$

Combining this with (2.6)(iii), (2.9) and noticing that u_1 and u_2 do not depend on y, we get

$$u_1 = u_2 \quad \text{for a.e. } (x,t)\in\Omega\times(0,T).$$

At last, (2.7) and (2.8) can be directly proved by following the arguments in the proof of Theorem 2.20 in [13]. □

3. Homogenization results

In this section, we use the adapted unfolding method presented in Section 2 to study the asymptotic behavior of the parabolic problem in a two-component composite with ε-periodic connected inclusions.

To introduce the coefficient matrix, we define, for $\alpha,\beta\in\mathbb{R}$ with $0<\alpha<\beta$, the set $M(\alpha,\beta,\mathcal{O})$ of the $n\times n$ matrix-valued functions $B(x)\in(L^\infty(\mathcal{O}))^{n\times n}$ such that

$$(B(x)\lambda,\lambda)\geq\alpha|\lambda|^2, \qquad |B(x)\lambda|\leq\beta|\lambda|$$

for any $\lambda\in\mathbb{R}^n$ and a.e. on $\mathcal{O}$.

Assume that $A=(a_{ij}(x))_{1\leq i,j\leq n}$ is a Y-periodic matrix such that

$$A\in M(\alpha,\beta,Y).$$

For any $\varepsilon>0$, we set

$$A^\varepsilon(x)=A(x/\varepsilon). \tag{3.1}$$

Let h be a Y-periodic function such that

$$h\in L^\infty(\Gamma) \text{ and } \exists\ h_0\in\mathbb{R}\ s.t.\ 0<h_0<h(y) \text{ a.e. in } \Gamma.$$

Set

$$h^\varepsilon(x)=h(x/\varepsilon). \tag{3.2}$$

In what follows, we always suppose $\gamma\leq-1$ if not otherwise stated. For $T>0$, we will consider the asymptotic behavior, as $\varepsilon\to0$, of the problem (1.1).

We suppose that

$$\begin{aligned} &U_\varepsilon^0 := (U_{1\varepsilon}^0, U_{2\varepsilon}^0)\in L^2(\Omega_{1\varepsilon})\times L^2(\Omega_{2\varepsilon}),\\ &f_\varepsilon := (f_{1\varepsilon}, f_{2\varepsilon})\in L^2\big(0,T;L^2(\Omega_{1\varepsilon})\big)\times L^2\big(0,T;L^2(\Omega_{2\varepsilon})\big). \end{aligned}\tag{3.3}$$

Set

$$\begin{aligned} W^\varepsilon = \Big\{ & v=(v_1,v_2)\in L^2(0,T;V^\varepsilon)\times L^2\big(0,T;H^1(\Omega_{2\varepsilon})\big)\\ &\text{such that } v_1'\in L^2\big(0,T;(V^\varepsilon)'\big),\ v_2'\in L^2\big(0,T;(H^1(\Omega_{2\varepsilon}))'\big)\Big\} \end{aligned}$$

with the norm defined by

$$\|v\|_{W^\varepsilon}=\|v_1\|_{L^2(0,T;V^\varepsilon)}+\|v_2\|_{L^2(0,T;H^1(\Omega_{2\varepsilon}))}+\|v_1'\|_{L^2(0,T;(V^\varepsilon)')}+\|v_2'\|_{L^2(0,T;(H^1(\Omega_{2\varepsilon}))')}.$$

The variational formulation of problem (1.1) is to find $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}) \in W^\varepsilon$ such that

$$\begin{cases} \langle u'_{1\varepsilon}, v_1\rangle_{(V^\varepsilon)', V^\varepsilon} + \langle u'_{2\varepsilon}, v_2\rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} \\ + \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla v_1 \,\mathrm{d}x + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla v_2 \,\mathrm{d}x \\ +\varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon})(v_1 - v_2)\,\mathrm{d}\sigma_x = \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} v_1 \,\mathrm{d}x + \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} v_2 \,\mathrm{d}x \\ \quad \text{in } \mathcal{D}'(0,T) \text{ for every } (v_1, v_2) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}), \\ u_{1\varepsilon}(x,0) = U^0_{1\varepsilon} \text{ in } \Omega_{1\varepsilon}, \\ u_{2\varepsilon}(x,0) = U^0_{2\varepsilon} \text{ in } \Omega_{2\varepsilon}. \end{cases} \tag{3.4}$$

For every fixed ε, the abstract Galerkin method provides the existence and uniqueness of the solution of problem (3.4).

In order to study the homogenization of problem (1.1), we need the following assumptions:

$$\begin{aligned} &\widetilde{U^0_\varepsilon} \rightharpoonup (\theta_1 U^0_1, \theta_2 U^0_2) \text{ weakly in } L^2(\Omega) \times L^2(\Omega), \\ &\widetilde{f_\varepsilon} \rightharpoonup (\theta_1 f_1, \theta_2 f_2) \text{ weakly in } L^2\big(0,T; L^2(\Omega)\big) \times L^2\big(0,T; L^2(\Omega)\big). \end{aligned} \tag{3.5}$$

Under these assumptions, problem (3.4) has a unique solution u_ε with the following estimates (see [9], Prop. 3.4):

$$\begin{cases} \|u_{1\varepsilon}\|_{L^2(0,T;V^\varepsilon)} + \|u_{1\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_{1\varepsilon}))} < C, \\ \|u_{2\varepsilon}\|_{L^2(0,T;H^1(\Omega_{2\varepsilon}))} + \|u_{2\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_{2\varepsilon}))} < C, \\ \varepsilon^{\frac{\gamma}{2}} \|u_{1\varepsilon} - u_{2\varepsilon}\|_{L^2(0,T;L^2(\Gamma^\varepsilon))} < C, \end{cases} \tag{3.6}$$

where the constant C is independent of ε.

The homogenization of problem (1.1) has been studied by the oscillating test functions method in Jose [18]. Here we use the unfolding method to study the homogenization, which will be crucial to get the corrector results. Notice that, up to now, the corrector results for $\gamma \le -1$ can not be achieved by the Tartar's oscillating test function method yet. We also derive the precise convergence of flux.

The study of the homogenization results is carried out according to $\gamma < -1$ or $\gamma = -1$.

3.1. The case $\gamma < -1$

Theorem 3.1. *Let A^ε and h^ε be defined by (3.1) and (3.2), respectively. For $\gamma < -1$, suppose that u_ε is the solution of (1.1) with (3.3) and (3.5). Then, there exist $u_1 \in L^2(0,T; H^1_0(\Omega))$, $\widehat{u}_1 \in L^2(0,T; L^2(\Omega, H^1_{\mathrm{per}}(Y_1)))$ and $\widehat{u}_2 \in L^2(0,T; L^2(\Omega, H^1(Y_2)))$ such that*

$$\begin{aligned} &\text{(i)} \quad \mathcal{T}^\varepsilon_1(u_{1\varepsilon}) \rightharpoonup u_1 \quad \text{weakly in } L^2(0,T; L^2(\Omega, H^1(Y_1))); \\ &\text{(ii)} \quad \mathcal{T}^\varepsilon_1(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(0,T; L^2(\Omega \times Y_1)); \\ &\text{(iii)} \quad \mathcal{T}^\varepsilon_2(u_{2\varepsilon}) \rightharpoonup u_1 \quad \text{weakly in } L^2(0,T; L^2(\Omega, H^1(Y_2))); \\ &\text{(iv)} \quad \mathcal{T}^\varepsilon_2(\nabla u_{2\varepsilon}) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(0,T; L^2(\Omega \times Y_2)); \\ &\text{(v)} \quad \widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i u_1 \quad \text{weakly in } L^2(0,T; L^2(\Omega)). \end{aligned} \tag{3.7}$$

where $\mathcal{M}_\Gamma(\widehat{u}_i) = 0$ for $i = 1,2$. The pair $(u_1, \widehat{u})$ is the unique solution in $L^2(0,T; H^1_0(\Omega)) \times L^2(0,T; L^2(\Omega, H^1_{\mathrm{per}}(Y)))$ with $\mathcal{M}_\Gamma(\widehat{u}) = 0$, of the problem

$$\begin{cases} -\int_0^T \int_\Omega u_1 \Psi \varphi' \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|Y|} \int_0^T \int_{\Omega\times Y} A(\nabla u_1 + \nabla_y \widehat{u})(\nabla\Psi + \nabla_y \Phi)\varphi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ = \int_0^T \int_\Omega (\theta_1 f_1 + \theta_2 f_2)\Psi\varphi \,\mathrm{d}x\,\mathrm{d}t, \\ \text{for all } \varphi \in \mathcal{D}(0,T), \Psi \in H^1_0(\Omega) \text{ and } \Phi \in L^2(\Omega, H^1_{\mathrm{per}}(Y)), \\ u_1(x,0) = \theta_1 U^0_1 + \theta_2 U^0_2 \qquad \text{in } \Omega, \end{cases} \tag{3.8}$$

where $\widehat{u} \in L^2(0,T;L^2(\Omega, H^1_{\mathrm{per}}(Y)))$ *is the extension by periodicity of the following function (still denoted by* $\widehat{u}$*):*

$$\widehat{u}(\cdot,y,\cdot) = \begin{cases} \widehat{u}_1(\cdot,y,\cdot) & \text{when } \ y \in Y_1, \\ \widehat{u}_2(\cdot,y,\cdot) - y_\Gamma \nabla u_1 & \text{when } \ y \in Y_2, \end{cases} \tag{3.9}$$

with $y_\Gamma = y - \mathcal{M}_\Gamma(y)$*. Also, we have*

$$\widehat{u} = \sum_{j=1}^{n} \frac{\partial u_1}{\partial x_j} \widehat{\chi}_j, \tag{3.10}$$

where $\widehat{\chi}_j \in H^1_{\mathrm{per}}(Y)(j=1,\cdots,n)$ *is the solution of the cell problem*

$$\begin{cases} -\operatorname{div}\big(A(y)\nabla(\widehat{\chi}_j + y_j)\big) = 0 & \text{in } Y, \\ \mathcal{M}_Y(\widehat{\chi}_j) = 0, \quad \widehat{\chi}_j \ \text{is } Y\text{-periodic.} \end{cases} \tag{3.11}$$

And u_1 *is the unique solution of the homogenized problem*

$$\begin{cases} u_1' - \operatorname{div}(A^0_\gamma \nabla u_1) = \theta_1 f_1 + \theta_2 f_2 & \text{in } \Omega \times (0,T), \\ u_1 = 0 & \text{on } \partial\Omega \times (0,T), \\ u_1(x,0) = \theta_1 U_1^0 + \theta_2 U_2^0 & \text{in } \Omega \end{cases} \tag{3.12}$$

with $A^0_\gamma = (a^0_{ij})_{1\le i,j\le n}$ *defined by*

$$a^0_{ij} = \mathcal{M}_Y\left(a_{ij} + \sum_{k=1}^{n} a_{ik} \frac{\partial \widehat{\chi}_j}{\partial y_k}\right). \tag{3.13}$$

Moreover, we have the following convergences:

$$\begin{aligned} A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} &\rightharpoonup A^1_\gamma \nabla u_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \\ A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} &\rightharpoonup A^2_\gamma \nabla u_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \end{aligned} \tag{3.14}$$

where $A^l_\gamma = (a^l_{ij})_{n\times n}$ $(l=1,2)$ *is defined by*

$$a^l_{ij} = \theta_l \mathcal{M}_{Y_l}\left(a_{ij} + \sum_{k=1}^{n} a_{ik} \frac{\partial \widehat{\chi}_j}{\partial y_k}\right). \tag{3.15}$$

Proof. In view of (3.6), Theorem 2.13 implies that convergences (3.7)(i)$-$(iv) hold at least for a subsequence (still denoted by ε). By (3.6) and Proposition 2.6(iii), we further obtain that

$$\begin{cases} \text{(i)} \quad \widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i \mathcal{M}_{Y_i}(u_1) \ \text{ weakly in } L^2(0,T;L^2(\Omega)) \ \text{ for } \ i=1,2, \\ \text{(ii)} \ \ A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \rightharpoonup \theta_1 \mathcal{M}_{Y_1}[A(\nabla u_1 + \nabla_y \widehat{u}_1)] \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \\ \text{(iii)} \ A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup \theta_2 \mathcal{M}_{Y_2}[A(\nabla_y \widehat{u}_2)] \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{3.16}$$

Notice that u_1 is independent of y, we get convergence (3.7)(v) from (3.16)(i).

Let $\Psi \in \mathcal{D}(\Omega)$. For $i=1,2$, let $\phi_i \in \mathcal{D}(\Omega)$ and $\psi_i \in H^1_{\mathrm{per}}(Y_i)$. Define $v_{i\varepsilon}$ by

$$v_{i\varepsilon}(x) = \Psi(x) + \varepsilon\phi_i(x)\psi_i^\varepsilon(x) \text{ and } \psi_i^\varepsilon(x) = \psi_i\left(\frac{x}{\varepsilon}\right). \tag{3.17}$$

Then

$$\nabla v_{i\varepsilon} = \nabla\Psi + \varepsilon\psi_i^\varepsilon \nabla\phi_i + \phi_i \cdot (\nabla_y \psi_i)\Big(\frac{\cdot}{\varepsilon}\Big).$$

By Proposition 2.6(ii),

$$\begin{aligned}&\mathcal{T}_i^\varepsilon(v_{i\varepsilon}) \to \Psi,\ \mathcal{T}_i^\varepsilon(\nabla v_{i\varepsilon}) \to \nabla\Psi + \nabla_y \Phi_i \text{ strongly in } L^2(\Omega\times Y_i),\\ &\mathcal{T}_i^\varepsilon(\phi_i\psi_i^\varepsilon) \to \Phi_i \quad \text{strongly in } L^2(\Omega\times Y_i) \text{ with } \Phi_i(x,y)=\phi_i(x)\psi_i(y).\end{aligned} \tag{3.18}$$

Let $\phi \in \mathcal{D}(\Omega)$ and $\psi \in H^1_{\mathrm{per}}(Y)$. During the proof of Theorem 3.1, we suppose $\phi_i = \phi,\ \psi_i = \psi|_{Y_i}$ for $i = 1, 2$. Let $\varphi \in \mathcal{D}(0,T)$. From (3.7) and (3.18), we use Proposition 2.5 to obtain that

$$\begin{aligned}&\int_0^T\int_{\Omega_{i\varepsilon}} u_{i\varepsilon}v_{i\varepsilon}\varphi'\,\mathrm{d}x\,\mathrm{d}t \to \theta_i\int_0^T\int_\Omega u_1\Psi\varphi'\,\mathrm{d}x\,\mathrm{d}t,\\ &\int_0^T\int_{\Omega_{1\varepsilon}} A^\varepsilon\nabla u_{1\varepsilon}\nabla v_{1\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t \to \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y_1} A(\nabla u_1+\nabla_y\widehat{u}_1)(\nabla\Psi+\nabla_y\Phi)\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,\\ &\int_0^T\int_{\Omega_{2\varepsilon}} A^\varepsilon\nabla u_{2\varepsilon}\nabla v_{2\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t \to \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y_2} A(\nabla_y\widehat{u}_2)(\nabla\Psi+\nabla_y\Phi)\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,\\ &\int_0^T\int_{\Omega_{i\varepsilon}} f_{i\varepsilon}v_{i\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t \to \theta_i\int_0^T\int_\Omega f_i\Psi\varphi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.19}$$

where $\Phi(x,y) = \phi(x)\psi(y)$. Choosing $(v_{1\varepsilon}\varphi, v_{2\varepsilon}\varphi)$ as test function in the variational formulation (3.4), we get

$$\begin{aligned}&-\int_0^T\int_{\Omega_{1\varepsilon}} u_{1\varepsilon}v_{1\varepsilon}\varphi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_{1\varepsilon}} A^\varepsilon\nabla u_{1\varepsilon}\nabla v_{1\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad-\int_0^T\int_{\Omega_{2\varepsilon}} u_{2\varepsilon}v_{2\varepsilon}\varphi'\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_{2\varepsilon}} A^\varepsilon\nabla u_{2\varepsilon}\nabla v_{2\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_{\Omega_{1\varepsilon}} f_{1\varepsilon}v_{1\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_{2\varepsilon}} f_{2\varepsilon}v_{2\varepsilon}\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.20}$$

Passing to the limit, then making use of (3.9) and (3.19), we obtain the equation in (3.8). Here we also used the density of $\mathcal{D}(\Omega)$ in $H^1_0(\Omega)$ and the density of $\mathcal{D}(\Omega)\otimes H^1_{\mathrm{per}}(Y_1)$ in $L^2(\Omega, H^1_{\mathrm{per}}(Y_1))$.

Setting $\Psi = 0$ in (3.8), we obtain

$$\mathrm{div}_y A(\nabla u_1 + \nabla_y\widehat{u}) = 0.$$

Notice that u_1 is independent of y and $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$. Hence we get (3.10). Then by a standard computation, we get the convergence (3.14) from (3.16) and the following identity:

$$\frac{1}{|Y|}\int_Y A(\nabla u_1 + \nabla_y\widehat{u})\nabla\Psi\mathrm{d}y = A^0_\gamma\nabla u_1\nabla\Psi \tag{3.21}$$

with A^0_γ defined by (3.13).

Moreover, we obtain the equation in (3.12). By a similar argument as that in [15], we know the initial condition is satisfied. Consequently, u_1 solves problem (3.12) with A^0_γ defined by (3.13).

Standard arguments give the ellipticity of A^0_γ and the uniqueness of the solution of the homogenized problem. Hence we get that the pair $(u_1, \widehat{u})$ with $\mathcal{M}_\Gamma(\widehat{u}) = 0$ is the unique solution of problem (3.8) due to (3.10). This implies that all convergences in Theorem 3.1 hold for the whole sequence. □

3.2. The case $\gamma = -1$

Theorem 3.2. *Let A^ε and h^ε be defined by (3.1) and (3.2), respectively. For $\gamma = -1$, suppose that u_ε is the solution of (1.1) with (3.3) and (3.5). Then there exist $u_1 \in L^2(0,T;H^1_0(\Omega))$, $\widehat{u}_1 \in L^2(0,T;L^2(\Omega, H^1_{\mathrm{per}}(Y_1)))$ and $\widehat{u}_2 \in L^2(0,T;L^2(\Omega, H^1(Y_2)))$ such that (3.7), where $\mathcal{M}_\Gamma(\widehat{u}_i) = 0$ $(i = 1, 2)$. The triple $(u_1, \widehat{u}_1, \check{u}_2)$ is the* *unique solution in* $L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;L^2(\Omega,H^1_{\rm per}(Y_1)))\times$ $L^2(0,T;$ $L^2(\Omega,H^1_{\rm per}(Y_2)))$ *with* $\mathcal{M}_\Gamma(\widehat{u}_1)=0$, *of the problem*

$$\begin{cases} -\displaystyle\int_0^T\int_\Omega u_1\Psi\varphi'\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y_1}A(\nabla u_1+\nabla_y\widehat{u}_1)(\nabla\Psi+\nabla_y\Phi_1)\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ +\displaystyle\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y_2}A(\nabla u_1+\nabla_y\check{u}_2)(\nabla\Psi+\nabla_y\Phi_2)\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ +\displaystyle\frac{1}{|Y|}\int_0^T\int_{\Omega\times\Gamma}h(y)(\widehat{u}_1-\check{u}_2)(\Phi_1-\Phi_2)\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t\\ =\displaystyle\int_0^T\int_\Omega(\theta_1f_1+\theta_2f_2)\Psi\varphi\,\mathrm{d}x\,\mathrm{d}t,\\ \textit{for all } \varphi\in\mathcal{D}(0,T),\Psi\in H_0^1(\Omega) \textit{ and } \Phi_i\in L^2(\Omega,H^1_{\rm per}(Y_i)),\ i=1,2,\\ u_1(x,0)=\theta_1U_1^0+\theta_2U_2^0 \qquad \textit{in } \Omega. \end{cases} \tag{3.22}$$

Here $\check{u}_2\in L^2(0,T;L^2(\Omega,H^1_{\rm per}(Y_2)))$ *is the extension by periodicity of the following function (still denoted by* $\check{u}_2$*):*

$$\check{u}_2=\widehat{u}_2-y_\Gamma\nabla u_1-\zeta, \tag{3.23}$$

where $y_\Gamma=y-\mathcal{M}_\Gamma(y)$ *and* ζ *is some function in* $L^2(0,T;L^2(\Omega))$.

Moreover, we have

$$\widehat{u}_1=\sum_{j=1}^n\frac{\partial u_1}{\partial x_j}\chi_1^j \quad \textit{and} \quad \check{u}_2=\sum_{j=1}^n\frac{\partial u_1}{\partial x_j}\chi_2^j, \tag{3.24}$$

where $(\chi_1^j,\chi_2^j)\in H^1_{\rm per}(Y_1)\times H^1(Y_2)(j=1,\dots,n)$ *is the solution of the cell problem*

$$\begin{cases} -\operatorname{div}\big(A(y)\nabla(\chi_1^j+y_j)\big)=0 & \textit{in } Y_1,\\ -\operatorname{div}\big(A(y)\nabla(\chi_2^j+y_j)\big)=0 & \textit{in } Y_2,\\ A(y)\nabla(\chi_1^j+y_j)\cdot n_1=-A(y)\nabla(\chi_2^j+y_j)\cdot n_2 & \textit{on } \Gamma,\\ A(y)\nabla(\chi_1^j+y_j)\cdot n_1=-h(\chi_1^j-\chi_2^j) & \textit{on } \Gamma,\\ \mathcal{M}_{Y_1}(\chi_1^j)=0, \quad \chi_1^j \textit{ is } Y\textit{-periodic.} \end{cases} \tag{3.25}$$

And u_1 *is the unique solution of the homogenized problem* (3.12) *with* $A^0_\gamma=(a^0_{ij})_{1\le i,j\le n}$ *defined by*

$$a^0_{ij}=\theta_1\mathcal{M}_{Y_1}\left(a_{ij}+\sum_{k=1}^n a_{ik}\frac{\partial\chi_1^j}{\partial y_k}\right)+\theta_2\mathcal{M}_{Y_2}\left(a_{ij}+\sum_{k=1}^n a_{ik}\frac{\partial\chi_2^j}{\partial y_k}\right). \tag{3.26}$$

We also have the following convergences:

$$\begin{aligned} &A^\varepsilon\widetilde{\nabla u_{1\varepsilon}}\rightharpoonup A^1_\gamma\nabla u_1 \quad \textit{weakly in } L^2(0,T;L^2(\Omega)),\\ &A^\varepsilon\widetilde{\nabla u_{2\varepsilon}}\rightharpoonup A^2_\gamma\nabla u_1 \quad \textit{weakly in } L^2(0,T;L^2(\Omega)), \end{aligned} \tag{3.27}$$

where $A^l_\gamma=(a^l_{ij})_{n\times n}$ $(l=1,2)$ *is defined by*

$$a^l_{ij}=\theta_l\mathcal{M}_{Y_l}\left(a_{ij}+\sum_{k=1}^n a_{ik}\frac{\partial\chi_l^j}{\partial y_k}\right). \tag{3.28}$$

Proof. The proof of Theorem 3.2 follows from a similar argument as that of Theorem 3.1. The only difference is that we need to handle the interface term now.

Let $\varphi \in \mathcal{D}(0,T)$ and $v_{i\varepsilon}$ $(i=1,2)$ be given by (3.17). For the interface term, Proposition 2.12 shows that

$$\begin{aligned}
&\varepsilon^{-1}\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(v_{1\varepsilon}-v_{2\varepsilon})\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t\\
=&\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(\phi_1\psi_1^\varepsilon-\phi_2\psi_2^\varepsilon)\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t\\
=&\frac{1}{\varepsilon|Y|}\int_0^T\int_{\Omega\times\Gamma} h(y)\big[\mathcal{T}_1^\varepsilon(u_{1\varepsilon})-\mathcal{T}_2^\varepsilon(u_{2\varepsilon})\big]\big[\psi_1\mathcal{T}_1^\varepsilon(\phi_1)-\psi_2\mathcal{T}_2^\varepsilon(\phi_2)\big]\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t.
\end{aligned}$$

On the other hand, Theorem 2.13 gives that there exists $\zeta \in L^2(0,T;L^2(\Omega))$ such that

$$\varepsilon^{-1}\big[\mathcal{T}_1^\varepsilon(u_{1\varepsilon})-\mathcal{T}_2^\varepsilon(u_{2\varepsilon})\big] \rightharpoonup \widehat{u}_1-\widehat{u}_2+y_\Gamma\nabla u_1+\zeta \text{ weakly in } L^2(0,T;L^2(\Omega\times\Gamma)) \tag{3.29}$$

for the above subsequence. From (3.18) and (3.29), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(v_{1\varepsilon}-v_{2\varepsilon})\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t\\
=&\frac{1}{|Y|}\int_0^T\int_{\Omega\times\Gamma} h(y)\big(\widehat{u}_1-\widehat{u}_2+y_\Gamma\nabla u_1+\zeta\big)(\Phi_1-\Phi_2)\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t.
\end{aligned}\tag{3.30}$$

We also notice that to prove u_1 satisfies (3.12), we need the following identity

$$\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega\times Y_1} A(\nabla u_1+\nabla_y\widehat{u}_1)(\nabla\Psi+\nabla_y\Phi_1)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{1}{|Y|}\int_{\Omega\times Y_2} A(\nabla u_1+\nabla_y\check{u}_2)(\nabla\Psi+\nabla_y\Phi_2)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\frac{1}{|Y|}\int_{\Omega\times\Gamma} h(y)(\widehat{u}_1-\check{u}_2)(\Phi_1-\Phi_2)\,\mathrm{d}x\,\mathrm{d}\sigma_y\\
&=A_\gamma^0\nabla u_1\nabla\Psi,
\end{aligned}\tag{3.31}$$

where $A_\gamma^0=(a_{ij}^0)_{1\le i,j\le n}$ is defined by (3.26).

The other parts of the proof can be done by a similar argument as that in Theorem 3.1. □

Remark 3.3. For $\gamma\le -1$, Jose [18] proved that

$$A^\varepsilon\widetilde{\nabla u_{1\varepsilon}}+A^\varepsilon\widetilde{\nabla u_{2\varepsilon}} \rightharpoonup (A_\gamma^1+A_\gamma^2)\nabla u_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega)),$$

where $A_\gamma^l (l=1,2)$ is defined by (3.15) and (3.28) for $\gamma<-1$ and $\gamma=-1$, respectively. Here, we obtain separately the convergences of $A^\varepsilon\widetilde{\nabla u_{1\varepsilon}}$ and $A^\varepsilon\widetilde{\nabla u_{2\varepsilon}}$, as presented in Theorems 3.1 and 3.2.

4. Asymptotic behavior of the energy

In this section, we study the asymptotic behavior of the energy which plays a key role in the study of the corrector results, as evidenced in [1, 11], to name a few. To do that, we need some stronger assumptions than those of the convergence results.

Still let $\gamma\le -1$. We suppose that for the data $f_{i\varepsilon}\in L^2(0,T;L^2(\Omega_{i\varepsilon}))$, there exists $f_i\in L^2(0,T;L^2(\Omega))$ such that

$$\|f_{i\varepsilon}-f_i\|_{L^2(0,T;L^2(\Omega_{i\varepsilon}))}\to 0,\quad i=1,2. \tag{4.1}$$

Remark 4.1.

(i) According to Proposition 2.7, assumption (4.1) is equivalent to

$$\begin{aligned}&\int_0^T\int_{\Lambda_{i\varepsilon}}|f_{i\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}t\to 0,\\ &\mathcal{T}_i^\varepsilon(f_{i\varepsilon})\to f_i \ \text{ strongly in } \ L^2(0,T;L^2(\Omega\times Y_i)).\end{aligned}\tag{4.2}$$

This implies that

$$(\widetilde{f}_{1\varepsilon},\widetilde{f}_{2\varepsilon})\rightharpoonup(\theta_1 f_1,\theta_2 f_2)\text{ weakly in } \ L^2(0,T;L^2(\Omega))\times L^2(0,T;L^2(\Omega)).$$

(ii) Donato and Jose [11] introduced the following assumption

$$\begin{cases}f_{i\varepsilon}\in L^2(0,T;L^2(\Omega)),\\ f_{i\varepsilon}\to f_i \ \text{ strongly in } \ L^2(0,T,L^2(\Omega)),\end{cases}\tag{4.3}$$

which implies assumption (4.1).

Next, we introduce some assumptions on the initial data $(U^0_{1\varepsilon},U^0_{2\varepsilon})$.

Concerning $U^0_{i\varepsilon}\in L^2(\Omega_{i\varepsilon})(i=1,2)$, we make the following assumption (see [11]): there exists $U^0\in L^2(\Omega)$ such that

$$\widetilde{U^0_{1\varepsilon}}+\widetilde{U^0_{2\varepsilon}}\to U^0\text{ strongly in } \ L^2(\Omega),\tag{4.4}$$

which is equivalent to

$$\begin{cases}\widetilde{U^0_{i\varepsilon}}\rightharpoonup\theta_i U^0\text{ weakly in } \ L^2(\Omega),\quad i=1,2,\\ \|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})}+\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})}\to\|U^0\|^2_{L^2(\Omega)}.\end{cases}\tag{4.5}$$

Remark 4.2. Assumption (4.4) is also equivalent to

$$\|U^0_{i\varepsilon}-U^0\|_{L^2(\Omega_{i\varepsilon})}\to 0.$$

This is easily obtained from the fact that $\Omega_{1\varepsilon}$ and $\Omega_{2\varepsilon}$ are disjoint.

Now, we consider the asymptotic behavior of the energy. For each ε, the energy $E^\varepsilon(t)$ associated to problem (1.1) is defined by

$$\begin{aligned}E^\varepsilon(t):=&\frac12\int_{\Omega_{1\varepsilon}}|u_{1\varepsilon}(t)|^2\mathrm{d}x+\frac12\int_{\Omega_{2\varepsilon}}|u_{2\varepsilon}(t)|^2\mathrm{d}x+\int_0^t\int_{\Omega_{1\varepsilon}}A^\varepsilon\nabla u_{1\varepsilon}\nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s\\&+\int_0^t\int_{\Omega_{2\varepsilon}}A^\varepsilon\nabla u_{2\varepsilon}\nabla u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s+\varepsilon^\gamma\int_0^t\int_{\Gamma^\varepsilon}h^\varepsilon|u_{1\varepsilon}-u_{2\varepsilon}|^2\,\mathrm{d}\sigma_x\,\mathrm{d}s.\end{aligned}\tag{4.6}$$

Choosing $(u_{1\varepsilon},u_{2\varepsilon})$ as test function in (3.4) and integrating by parts, $E^\varepsilon(t)$ can be rewritten as

$$E^\varepsilon(t):=\frac12\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})}+\frac12\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})}+\int_0^t\int_{\Omega_{1\varepsilon}}f_{1\varepsilon}u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\Omega_{2\varepsilon}}f_{2\varepsilon}u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s.\tag{4.7}$$

Theorem 4.3 (convergence of energy for $\gamma\le -1$)**.** *Let A^ε and h^ε be defined by* (3.1) *and* (3.2)*, respectively. Suppose that* (4.1) *and* (4.4) *hold. If u_ε is the solution of problem* (1.1) *with $\gamma\le -1$, then*

$$E^\varepsilon\to E\ \textit{strongly in}\ \mathcal{C}^0([0,T]),$$

where E is the energy associated to the corresponding homogenized problem, defined by

$$E(t):=\frac12\int_\Omega|u_1|^2\mathrm{d}x+\int_0^t\int_\Omega A^0_\gamma\nabla u_1\nabla u_1\,\mathrm{d}x\,\mathrm{d}s$$

with A^0_γ being the corresponding homogenized matrix.

Proof. From the homogenization results in Section 3, we have

$$\mathcal{T}_i^\varepsilon(u_{i\varepsilon}) \rightharpoonup u_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega,H^1(Y_i))), \quad i=1,2.$$

By (3.6) and (4.2), we use Proposition 2.5 to obtain that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^t\int_{\Omega_{i\varepsilon}} f_{i\varepsilon}u_{i\varepsilon}\,\mathrm{d}x\,\mathrm{d}s &= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^t\int_{\Omega\times Y_i}\mathcal{T}_i^\varepsilon(f_{i\varepsilon})\mathcal{T}_i^\varepsilon(u_{i\varepsilon})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\ &= \theta_i\int_0^t\int_\Omega f_iu_1\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{4.8}$$

Notice that a direct computation gives

$$E(t)=\frac{1}{2}\|U^0\|^2_{L^2(\Omega)}+\int_0^t\int_\Omega(\theta_1f_1+\theta_2f_2)u_1\,\mathrm{d}x\,\mathrm{d}s.$$

Combining this with (4.5), (4.7) and (4.8), we conclude that $E^\varepsilon \to E$, $\forall t\in[0,T]$. Following the standard framework of argument, we use the Ascoli−Arzelà theorem to get the proof of Theorem 4.3. □

5. Corrector results

In this section, we are devoted to the corrector results for problem (1.1) with $\gamma\le -1$, which are new. The proofs mainly rely on the unfolding method. Our method is quite different from that in [11], which is used to prove the correct results for the case $-1<\gamma\le 1$.

Now we present two necessary results. The first one is the compactness result of $\widetilde{u}_{1\varepsilon}+\widetilde{u}_{2\varepsilon}$ in $\mathcal{C}^0([0,T];H^{-1}(\Omega))$, which can be proved by repeating the proof of Theorem 4.8 in [11], step by step.

Proposition 5.1. *Let $\gamma\le -1$. Suppose that (3.3) and (3.5) hold and u_ε is the solution of problem (1.1). Then, we have*

$$\widetilde{u}_{1\varepsilon}+\widetilde{u}_{2\varepsilon}\to\theta_1u_1+\theta_2u_2 \quad \textit{in} \quad \mathcal{C}^0([0,T];H^{-1}(\Omega)).$$

Moreover, we have

$$\widetilde{u}_{1\varepsilon}+\widetilde{u}_{2\varepsilon}\to u_1 \quad \textit{in} \quad \mathcal{C}^0([0,T];H^{-1}(\Omega)).$$

The second one is a classical result due to Cioranescu *et al.* [3].

Proposition 5.2. *Let $\{D_\varepsilon\}$ be a sequence of $n\times n$ matrices in $M(\alpha,\beta,\mathcal{O})$ for some open set $\mathcal{O}$, such that $D_\varepsilon\to D$ a.e. on $\mathcal{O}$ (or more generally, in measure in $\mathcal{O}$). If $\zeta_\varepsilon\rightharpoonup\zeta$ weakly in $L^2(\mathcal{O})$, then*

$$\int_{\mathcal{O}}D\zeta\zeta\,\mathrm{d}x\le\liminf_{\varepsilon\to 0}\int_{\mathcal{O}}D_\varepsilon\zeta_\varepsilon\zeta_\varepsilon\,\mathrm{d}x.$$

Furthermore, if

$$\limsup_{\varepsilon\to 0}\int_{\mathcal{O}}D_\varepsilon\zeta_\varepsilon\zeta_\varepsilon\,\mathrm{d}x\le\int_{\mathcal{O}}D\zeta\zeta\,\mathrm{d}x.$$

then

$$\int_{\mathcal{O}}D\zeta\zeta\,\mathrm{d}x=\lim_{\varepsilon\to 0}\int_{\mathcal{O}}D_\varepsilon\zeta_\varepsilon\zeta_\varepsilon\,\mathrm{d}x \;\textit{ and }\; \zeta_\varepsilon\to\zeta \;\textit{ strongly in }\; L^2(\mathcal{O}).$$

5.1. The case $\gamma < -1$

Theorem 5.3. *Let A^ε and h^ε be defined by (3.1) and (3.2), respectively. For $\gamma < -1$, suppose that u_ε is the solution of problem (1.1) with (4.1) and (4.4). If u_1 is the solution of the homogenized problem (3.12) with A^0_γ defined by (3.13), then we have*

$$\begin{aligned} &\|\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - u_1\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \to 0, \\ &\|\nabla u_{1\varepsilon} - C^\varepsilon \nabla u_1\|_{L^2([0,T];L^1(\Omega_{1\varepsilon}))} \to 0, \\ &\|\nabla u_{2\varepsilon} - C^\varepsilon \nabla u_1\|_{L^2([0,T];L^1(\Omega_{2\varepsilon}))} \to 0, \end{aligned} \tag{5.1}$$

where the corrector matrix is defined by

$$\begin{cases} C^\varepsilon(x) = C\left(\dfrac{x}{\varepsilon}\right) & a.e. \text{ on } \Omega, \\ C_{ij}(y) = \delta_{ij} + \dfrac{\partial \widehat{\chi}_j}{\partial y_i}(y) & a.e. \text{ on } Y, \end{cases} \tag{5.2}$$

with $\widehat{\chi}_j$ being the solution of the cell problem (3.11).

The proof is based on the following lemma.

Lemma 5.4. *Keep the notations and assumptions in Theorem 5.3. For any $\Phi \in \mathcal{C}^\infty([0,T], \mathcal{D}(\Omega))$, set*

$$\begin{aligned} \rho_\varepsilon(t) :=& \frac{1}{2}\int_\Omega |\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - \Phi|^2 \mathrm{d}x + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon(\nabla u_{1\varepsilon} - C^\varepsilon \nabla\Phi)(\nabla u_{1\varepsilon} - C^\varepsilon \nabla\Phi)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon(\nabla u_{2\varepsilon} - C^\varepsilon \nabla\Phi)(\nabla u_{2\varepsilon} - C^\varepsilon \nabla\Phi)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Then we have

$$\limsup_{\varepsilon\to 0} \|\rho_\varepsilon\|_{\mathcal{C}^0([0,T])} \le \|\rho\|_{\mathcal{C}^0([0,T])}, \tag{5.3}$$

where

$$\rho(t) = \frac{1}{2}\|u_1 - \Phi\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^0_\gamma(\nabla u_1 - \nabla\Phi)(\nabla u_1 - \nabla\Phi)\,\mathrm{d}x\,\mathrm{d}s. \tag{5.4}$$

Proof. We first decompose ρ_ε into three terms:

$$\rho_\varepsilon = \rho_{1\varepsilon} + \rho_{2\varepsilon} - \rho_{3\varepsilon}, \tag{5.5}$$

where

$$\begin{aligned} \rho_{1\varepsilon} =& \frac{1}{2}\int_{\Omega_{1\varepsilon}} |u_{1\varepsilon}|^2 \mathrm{d}x + \frac{1}{2}\int_{\Omega_{2\varepsilon}} |u_{2\varepsilon}|^2 \mathrm{d}x \\ &+ \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s, \\ \rho_{2\varepsilon} =& \frac{1}{2}\int_\Omega |\Phi|^2 \mathrm{d}x + + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon \nabla\Phi C^\varepsilon \nabla\Phi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon C^\varepsilon \nabla\Phi C^\varepsilon \nabla\Phi\,\mathrm{d}x\,\mathrm{d}s, \\ \rho_{3\varepsilon} =& \int_\Omega (\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon})\Phi \mathrm{d}x + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon \nabla\Phi \nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon C^\varepsilon \nabla\Phi \nabla u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} C^\varepsilon \nabla\Phi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} C^\varepsilon \nabla\Phi\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{5.6}$$

Step 1. In this step, we consider the term $\rho_{3\varepsilon}$ which is more complicated than the other two terms. Write $\rho_{3\varepsilon}$ in the form:

$$\rho_{3\varepsilon} = \rho_{3\varepsilon}^1 + \rho_{3\varepsilon}^2 + \rho_{3\varepsilon}^3,$$

where

$$\begin{aligned}
\rho_{3\varepsilon}^1 &= \int_\Omega (\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon})\varPhi \mathrm{d}x, \\
\rho_{3\varepsilon}^2 &= \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon \nabla\varPhi \nabla u_{1\varepsilon}\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon C^\varepsilon \nabla\varPhi \nabla u_{2\varepsilon}\, \mathrm{d}x\, \mathrm{d}s, \\
\rho_{3\varepsilon}^3 &= \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} C^\varepsilon \nabla\varPhi\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} C^\varepsilon \nabla\varPhi\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned}$$

For the term $\rho_{3\varepsilon}^1$, we have

$$\max_{t\in[0,T]} \Big| \int_\Omega [(\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon})\varPhi - u_1\varPhi] \mathrm{d}x \Big| \le \|\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - u_1\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))} \|\varPhi\|_{\mathcal{C}^0([0,T];H_0^1(\Omega))}.$$

Thanks to Proposition 5.1, we obtain

$$\rho_{3\varepsilon}^1 \to \int_\Omega u_1 \varPhi \mathrm{d}x \quad \text{in } \ \mathcal{C}^0([0,T]).$$

For the term $\rho_{3\varepsilon}^2$, by (3.7) and (5.2), we use Proposition 2.5 to get

$$\begin{aligned}
\rho_{3\varepsilon}^2(t) \to& \frac{1}{|Y|}\int_0^t \int_{\Omega\times Y_1} A(y)[\nabla\varPhi + \nabla_y \widehat{\varPhi}][\nabla u_1 + \nabla_y \widehat{u}_1]\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s \\
&+ \frac{1}{|Y|}\int_0^t \int_{\Omega\times Y_2} A(y)[\nabla\varPhi + \nabla_y \widehat{\varPhi}][\nabla u_1 + \nabla_y \widehat{u}_2]\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s \\
=& \frac{1}{|Y|}\int_0^t \int_{\Omega\times Y} A(y)[\nabla\varPhi + \nabla_y \widehat{\varPhi}][\nabla u_1 + \nabla_y \widehat{u}]\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s, \quad \forall t\in[0,T],
\end{aligned} \tag{5.7}$$

where $\widehat{u}$ is given by Theorem 3.1 and $\widehat{\varPhi}$ is defined by

$$\widehat{\varPhi} = \sum_{i=1}^n \frac{\partial\varPhi}{\partial x_i}\widehat{\chi}_i. \tag{5.8}$$

Furthermore, as we did for getting (3.21), we obtain

$$\rho_{3\varepsilon}^2(t) \to \int_0^t \int_\Omega A_\gamma^0 \nabla\varPhi\nabla u_1\, \mathrm{d}x\, \mathrm{d}s, \quad \forall t\in[0,T]. \tag{5.9}$$

The Ascoli−Arzelà theorem shows that convergence (5.9) still holds in $\mathcal{C}^0([0,T])$. In fact, by (3.1) and the assumption on $\varPhi$, the Hölder inequality gives

$$\begin{aligned}
|\rho_{3\varepsilon}^2| \le& \|A\|_{L^\infty(Y)} \|C^\varepsilon\|_{L^2(\Omega)} \|\nabla\varPhi\|_{L^2(0,T;L^\infty(\Omega))} \\
&\cdot \big[\|\nabla u_{1\varepsilon}\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}))} + \|\nabla u_{2\varepsilon}\|_{L^2(0,T;L^2(\Omega_{2\varepsilon}))}\big].
\end{aligned}$$

From Proposition 8.5, 2, we know there exists a constant C_1 (independent of ε) such that

$$\|C^\varepsilon\|_{L^2(\Omega)} \le C_1. \tag{5.10}$$

Together with (3.1), (3.6) and the assumption on Φ, we have the following estimate:

$$|\rho_{3\varepsilon}^2(t)| \le c, \qquad \forall t \in [0,T],$$

where c is independent of t and ε. Moreover, as $s \to 0^+$,

$$\begin{aligned} |\rho_{3\varepsilon}^2(t+s) - \rho_{3\varepsilon}^2(t)| &\le s^{\frac{1}{2}} \|A\|_{L^\infty(Y)} \|C^\varepsilon\|_{L^2(\Omega)} \|\nabla \Phi\|_{L^\infty(0,T;L^\infty(\Omega))} \\ &\quad \cdot \left[\|\nabla u_{1\varepsilon}\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}))} + \|\nabla u_{2\varepsilon}\|_{L^2(0,T;L^2(\Omega_{2\varepsilon}))} \right] \\ &\le c s^{\frac{1}{2}} \to 0, \quad \text{uniformly with respect to } \varepsilon. \end{aligned} \tag{5.11}$$

Hence we conclude that

$$\rho_{3\varepsilon}^2 \to \int_0^t \int_\Omega A_\gamma^0 \nabla \Phi \nabla u_1 \, \mathrm{d}x \, \mathrm{d}s \quad \text{in } \ \mathcal{C}^0([0,T]).$$

For the term $\rho_{3\varepsilon}^3$, arguing as we treated $\rho_{3\varepsilon}^2$, we get

$$\rho_{3\varepsilon}^3 \to \int_0^t \int_\Omega A_\gamma^0 \nabla u_1 \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s \quad \text{in } \ \mathcal{C}^0([0,T]).$$

Combining this with the convergence of $\rho_{3\varepsilon}^1$ and $\rho_{3\varepsilon}^2$, we have

$$\rho_{3\varepsilon} \longrightarrow \int_\Omega u_1 \Phi \mathrm{d}x + \int_0^t \int_\Omega A_\gamma^0 \nabla \Phi \nabla u_1 \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega A_\gamma^0 \nabla u_1 \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s \quad \text{in } \ \mathcal{C}^0([0,T]). \tag{5.12}$$

Step 2. For $\rho_{2\varepsilon}$, we can easily verify that it is bounded in $L^\infty(0,T)$. Similar property holds for its time derivative due to the smoothness of Φ. Following the computation in (5.7) and (5.9), we have

$$\rho_{2\varepsilon} \to \frac{1}{2} \int_\Omega |\Phi|^2 \mathrm{d}x + \int_0^t \int_\Omega A_\gamma^0 \nabla \Phi \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s, \qquad \forall t \in [0,T], \tag{5.13}$$

where $\widehat{\Phi}$ is defined by (5.8). Thus it follows that

$$\rho_{2\varepsilon} \to \frac{1}{2} \int_\Omega |\Phi|^2 \mathrm{d}x + \int_0^t \int_\Omega A_\gamma^0 \nabla \Phi \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s \quad \text{in } \ \mathcal{C}^0([0,T]). \tag{5.14}$$

Step 3. For $\rho_{1\varepsilon}$, it follows from (4.6) that

$$\rho_{1\varepsilon}(t) \le E^\varepsilon(t), \quad \forall t \in [0,T].$$

This yields

$$0 \le \rho_\varepsilon(t) = \rho_{1\varepsilon}(t) + \rho_{2\varepsilon}(t) - \rho_{3\varepsilon}(t) \le E^\varepsilon(t) + \rho_{2\varepsilon}(t) - \rho_{3\varepsilon}(t), \quad \forall t \in [0,T]. \tag{5.15}$$

By Theorems 4.3, (5.12) and (5.14), we have

$$E^\varepsilon(t) + \rho_{2\varepsilon}(t) - \rho_{3\varepsilon}(t) \to \rho(t) \ \text{ in } \mathcal{C}^0([0,T]).$$

This, together with (5.15), implies (5.3). The proof of Lemma 5.4 is completed. □

Proof of Theorem 5.3. By Lemma 5.4 and the classical density result, we can easily prove Theorem 5.3 by standard arguments (see also [11]). For the reader's convenience and the completeness, we include the following details.

In view of $u_1 \in L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}^0([0,T];L^2(\Omega))$, we know that for $\delta > 0$, there exists $\Phi \in \mathcal{C}^\infty([0,T];\mathcal{D}(\Omega))$ such that

$$\begin{cases} \|u_1 - \Phi\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \le \delta, \\ \|\nabla u_1 - \nabla \Phi\|_{L^2(0,T;L^2(\Omega))} \le \delta. \end{cases} \tag{5.16}$$

Combining this with Lemma 5.4, we have

$$\limsup_{\varepsilon\to 0} \|\rho_\varepsilon\|_{\mathcal{C}^0([0,T])} \le \|\rho\|_{\mathcal{C}^0([0,T])} \le c\delta^2, \tag{5.17}$$

where c is independent of ε and δ.

The ellipticity of A^ε implies that $\frac{1}{2}\int_\Omega |\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - \Phi|^2 \mathrm{d}x \le \rho_\varepsilon(t)$. Together with the triangle inequality and (5.16), it follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \|\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - u_1\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))} &\le \limsup_{\varepsilon\to 0} \big\{2\|\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - \Phi\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))} + 2\delta^2\big\}\\ &\le c\big(\limsup_{\varepsilon\to 0}\|\rho_\varepsilon\|_{\mathcal{C}^0([0,T])} + \delta^2\big)\\ &\le c\delta^2.\end{aligned} \tag{5.18}$$

On the other hand, by the triangle inequality, (5.10), (5.16) and the Hölder inequality, we deduce that

$$\begin{aligned}&\int_0^T \|\nabla u_{1\varepsilon} - C^\varepsilon \nabla u_1\|^2_{L^1(\Omega_{1\varepsilon})}\mathrm{d}t + \int_0^T \|\nabla u_{2\varepsilon} - C^\varepsilon \nabla u_1\|^2_{L^1(\Omega_{2\varepsilon})}\mathrm{d}t\\ &\quad\le 2\int_0^T \|\nabla u_{1\varepsilon} - C^\varepsilon\nabla\Phi\|^2_{L^1(\Omega_{1\varepsilon})}\mathrm{d}t + 2\|C^\varepsilon\|^2_{L^2(\Omega_{1\varepsilon})}\int_0^T \|\nabla u_1 - \nabla\Phi\|^2_{L^2(\Omega_{1\varepsilon})}\mathrm{d}t\\ &\qquad + 2\int_0^T \|\nabla u_{2\varepsilon} - C^\varepsilon\nabla\Phi\|^2_{L^1(\Omega_{2\varepsilon})}\mathrm{d}t + 2\|C^\varepsilon\|^2_{L^2(\Omega_{2\varepsilon})}\int_0^T \|\nabla u_1 - \nabla\Phi\|^2_{L^2(\Omega_{2\varepsilon})}\mathrm{d}t\\ &\quad\le c\int_0^T \|\nabla u_{1\varepsilon} - C^\varepsilon\nabla\Phi\|^2_{L^2(\Omega_{1\varepsilon})}\mathrm{d}t + c\int_0^T \|\nabla u_{2\varepsilon} - C^\varepsilon\nabla\Phi\|^2_{L^2(\Omega_{2\varepsilon})}\mathrm{d}t + c\delta^2.\end{aligned}$$

Thanks to the ellipticity of A^ε, we have

$$\int_0^T \|\nabla u_{1\varepsilon} - C^\varepsilon \nabla u_1\|^2_{L^1(\Omega_{1\varepsilon})}\mathrm{d}t + \int_0^T \|\nabla u_{2\varepsilon} - C^\varepsilon \nabla u_1\|^2_{L^1(\Omega_{2\varepsilon})}\mathrm{d}t \le c(\rho_\varepsilon(T) + \delta^2). \tag{5.19}$$

This, together with (5.17) and (5.18), shows that

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \|\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - u_1\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))}\\ &\quad + \limsup_{\varepsilon\to 0}\Big\{\|\nabla u_{1\varepsilon} - C^\varepsilon\nabla u_1\|^2_{L^2(0,T;L^1(\Omega_{1\varepsilon}))} + \|\nabla u_{2\varepsilon} - C^\varepsilon\nabla u_1\|^2_{L^2(0,T;L^1(\Omega_{2\varepsilon}))}\Big\}\\ &\le c\limsup_{\varepsilon\to 0}\|\rho_\varepsilon\|_{\mathcal{C}^0([0,T])} + c\delta^2 \le c\delta^2,\end{aligned} \tag{5.20}$$

which implies (5.1) owing to δ being arbitrary. □

5.2. The case $\gamma = -1$

For the case $\gamma = -1$, because of the presence of the integral of the jump between $\widehat{u}_1$ and $\check{u}_2$ on the interface Γ in the limit problem (3.22), the proofs of the corrector results are quite different from those in the case $\gamma < -1$.

Theorem 5.5. *Let A^ε and h^ε be defined by (3.1) and (3.2), respectively. For $\gamma = -1$, suppose that u_ε is the solution of problem (1.1) with (4.1) and (4.4). If u_1 is the solution of the homogenized problem (3.12) with A^0_γ defined by (3.26), then*

$$\begin{aligned}&\|\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - u_1\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \to 0,\\ &\|\nabla u_{1\varepsilon} - C^\varepsilon\nabla u_1\|_{L^2([0,T];L^1(\Omega_{1\varepsilon}))} \to 0\\ &\|\nabla u_{2\varepsilon} - D^\varepsilon\nabla u_1\|_{L^2([0,T];L^1(\Omega_{2\varepsilon}))} \to 0,\end{aligned} \tag{5.21}$$

where the corrector matrices C^ε and D^ε are defined by

$$\begin{cases} C^\varepsilon(x) = C\left(\dfrac{x}{\varepsilon}\right) & a.e.\ on\ \Omega_{1\varepsilon}, \\ C_{ij}(y) = \delta_{ij} + \dfrac{\partial \chi_1^j}{\partial y_i}(y) & a.e.\ on\ Y_1, \\ D^\varepsilon(x) = D\left(\dfrac{x}{\varepsilon}\right) & a.e.\ on\ \Omega_{2\varepsilon}, \\ D_{ij}(y) = \delta_{ij} + \dfrac{\partial \chi_2^j}{\partial y_i}(y) & a.e.\ on\ Y_2. \end{cases}$$

Here $\chi_2^j \in H^1_{\text{per}}(Y_2)$ (still denoted by χ_2^j) is the extension by periodicity of χ_2^j, and (χ_1^j, χ_2^j) is the solution of the cell problem (3.25).

To prove this result, we need the following result corresponding to Lemma 5.4.

Lemma 5.6. *Keep the notations and assumptions in Theorem 5.5. For any $\Phi \in \mathcal{C}^\infty([0,T], \mathcal{D}(\Omega))$, write*

$$g_\varepsilon(t) := \frac{1}{2}\int_\Omega |\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon} - \Phi|^2 \mathrm{d}x + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon(\nabla u_{1\varepsilon} - C^\varepsilon\nabla\Phi)(\nabla u_{1\varepsilon} - C^\varepsilon\nabla\Phi)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon(\nabla u_{2\varepsilon} - D^\varepsilon\nabla\Phi)(\nabla u_{2\varepsilon} - D^\varepsilon\nabla\Phi)\,\mathrm{d}x\,\mathrm{d}s.$$

Then we have

$$\limsup_{\varepsilon\to 0} \|g_\varepsilon\|_{\mathcal{C}^0([0,T])} \le \|\rho(t)\|_{\mathcal{C}^0([0,T])}, \tag{5.22}$$

where $\rho(t)$ is given by (5.4) with A^0_γ defined by (3.26).

Proof. The proof of Lemma 5.6 follows from a similar argument as that of Lemma 5.4. Here, we only indicate the different parts. We first decompose g_ε into three terms:

$$g_\varepsilon = g_{1\varepsilon} + g_{2\varepsilon} - g_{3\varepsilon}, \tag{5.23}$$

where

$$g_{1\varepsilon} = \frac{1}{2}\int_{\Omega_{1\varepsilon}} |u_{1\varepsilon}|^2 \mathrm{d}x + \frac{1}{2}\int_{\Omega_{2\varepsilon}} |u_{2\varepsilon}|^2 \mathrm{d}x$$
$$+ \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon}\nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon}\nabla u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s,$$
$$g_{2\varepsilon} = \frac{1}{2}\int_\Omega |\Phi|^2 \mathrm{d}x + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon\nabla\Phi C^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon D^\varepsilon\nabla\Phi D^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s,$$
$$g_{3\varepsilon} = \int_\Omega (\widetilde{u}_{1\varepsilon} + \widetilde{u}_{2\varepsilon})\Phi \mathrm{d}x + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon\nabla\Phi\nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon D^\varepsilon\nabla\Phi\nabla u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} C^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} D^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s. \tag{5.24}$$

Step 1. In this step, we study the term $g_{1\varepsilon}$. It is different from the corresponding one in the case $\gamma < -1$, due to the consideration of interface term.

By (4.6), we have

$$g_{1\varepsilon} = E^\varepsilon - \varepsilon^{-1}\int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2\,\mathrm{d}\sigma_x\,\mathrm{d}s, \quad \forall t \in [0,T].$$

From Theorem 4.3, we know that

$$E^\varepsilon \to E \quad \text{strongly in } \mathcal{C}^0([0,T]). \tag{5.25}$$

For the interface term $\varepsilon^{-1}\int_0^t\int_{\Gamma^\varepsilon} h^\varepsilon|u_{1\varepsilon}-u_{2\varepsilon}|^2 \mathrm{d}\sigma_x ds$, we know that it is bounded in $H^1(0,T)$ due to (3.2) and (3.6). By the compactness of the injection $H^1(0,T)\subset\mathcal{C}^0([0,T])$, we get that

$$g_{1\varepsilon} \text{ is compact in } \mathcal{C}^0([0,T]). \tag{5.26}$$

On the other hand, for any $t\in[0,T]$, we use Proposition 2.10 to obtain

$$\begin{aligned}\varepsilon^{-1}\int_0^t\int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon}-u_{2\varepsilon})^2\,\mathrm{d}\sigma_x\,\mathrm{d}s &\geq \varepsilon^{-1}\int_0^t\int_{\widehat{\Gamma}^\varepsilon} h^\varepsilon (u_{1\varepsilon}-u_{2\varepsilon})^2\,\mathrm{d}\sigma_x\,\mathrm{d}s\\ &= \frac{1}{|Y|}\int_0^t\int_{\Omega\times\Gamma} h(y)\left(\frac{\mathcal{T}_1^\varepsilon(u_{1\varepsilon})-\mathcal{T}_2^\varepsilon(u_{2\varepsilon})}{\varepsilon}\right)^2\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}s.\end{aligned} \tag{5.27}$$

Notice that (3.23) and (3.29) imply $\varepsilon^{-1}\big[\mathcal{T}_1^\varepsilon(u_{1\varepsilon})-\mathcal{T}_2^\varepsilon(u_{2\varepsilon})\big]\rightharpoonup \widehat{u}_1-\check{u}_2$ weakly in $L^2(0,T;L^2(\Omega\times\Gamma))$. Combining this with (5.27), then making use of Proposition 5.2 with $D=h$, we obtain

$$\liminf_{\varepsilon\to 0}\left[\varepsilon^{-1}\int_0^t\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})^2\,\mathrm{d}\sigma_x\,\mathrm{d}s\right] \geq \frac{1}{|Y|}\int_0^t\int_{\Omega\times\Gamma} h(y)(\widehat{u}_1-\check{u}_2)^2\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}s, \tag{5.28}$$

where $\widehat{u}_1$ and $\check{u}_2$ are given by Theorem 3.2. Hence

$$\limsup_{\varepsilon\to 0} g_{1\varepsilon} \leq E - \frac{1}{|Y|}\int_0^t\int_{\Omega\times\Gamma} h(y)(\widehat{u}_1-\check{u}_2)^2\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}s, \quad \forall t\in[0,T]. \tag{5.29}$$

Step 2. This step is devoted to the study of $g_{3\varepsilon}$. Decompose $g_{3\varepsilon}$ into three terms:

$$g_{3\varepsilon} = g_{3\varepsilon}^1 + g_{3\varepsilon}^2 + g_{3\varepsilon}^3,$$

where

$$\begin{aligned} g_{3\varepsilon}^1 &= \int_\Omega (\widetilde{u}_{1\varepsilon}+\widetilde{u}_{2\varepsilon})\Phi \mathrm{d}x,\\ g_{3\varepsilon}^2 &= \int_0^t\int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon \nabla\Phi\nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Omega_{2\varepsilon}} A^\varepsilon D^\varepsilon\nabla\Phi\nabla u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s,\\ g_{3\varepsilon}^3 &= \int_0^t\int_{\Omega_{1\varepsilon}} A^\varepsilon\nabla u_{1\varepsilon} C^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Omega_{2\varepsilon}} A^\varepsilon\nabla u_{2\varepsilon} D^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Repeating the arguments about $\rho_{3\varepsilon}^1$ in Lemma 5.4, we have

$$g_{3\varepsilon}^1 \to \int_\Omega u_1\Phi\,\mathrm{d}x \;\text{ in } \mathcal{C}^0([0,T]). \tag{5.30}$$

For the term $g_{3\varepsilon}^2$, by Proposition 2.5 and Theorem 3.2, arguing as we did for getting (5.7), we obtain

$$\begin{aligned} g_{3\varepsilon}^2(t) \to &\frac{1}{|Y|}\int_0^t\int_{\Omega\times Y_1} A(y)[\nabla\Phi+\nabla_y\widehat{\Phi}_1][\nabla u_1+\nabla_y\widehat{u}_1]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\ &+\frac{1}{|Y|}\int_0^t\int_{\Omega\times Y_2} A(y)[\nabla\Phi+\nabla_y\widehat{\Phi}_2][\nabla u_1+\nabla_y\check{u}_2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \;\text{ for any } t\in[0,T],\end{aligned}$$

where $\widehat{\Phi}_i$ is defined by

$$\widehat{\Phi}_i = \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j} \chi_i^j, \quad i = 1, 2.$$

Moreover, (3.31) allows us to deduce

$$g_{3\varepsilon}^2(t) \to \int_0^t \int_\Omega A_\gamma^0 \nabla \Phi \nabla u_1 \, \mathrm{d}x \, \mathrm{d}s - \frac{1}{|Y|} \int_0^t \int_{\Omega \times \Gamma} h(y)(\widehat{\Phi}_1 - \widehat{\Phi}_2)(\widehat{u}_1 - \check{u}_2) \, \mathrm{d}x \, \mathrm{d}\sigma_y \, \mathrm{d}s, \quad \forall t \in [0, T]. \tag{5.31}$$

This convergence still holds in $\mathcal{C}^0([0, T])$ due to the Ascoli−Arzelà theorem. In fact, it is easily obtained from the following estimate corresponding to that of $\rho_{3\varepsilon}^2$:

$$\begin{aligned} |g_{3\varepsilon}^2(t)| \leq \|A\|_{L^\infty(Y)} \big\{ & \|C^\varepsilon\|_{L^2(\Omega_{1\varepsilon})} \|\nabla \Phi\|_{L^2(0,T;L^\infty(\Omega))} \|\nabla u_{1\varepsilon}\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}))} \\ & + \|D^\varepsilon\|_{L^2(\Omega_{2\varepsilon})} \|\nabla \Phi\|_{L^2(0,T;L^\infty(\Omega))} \|\nabla u_{2\varepsilon}\|_{L^2(0,T;L^2(\Omega_{2\varepsilon}))} \big\}. \end{aligned}$$

In [19], Monsurrò proved that there exist two constants C_1 and C_2 (independent of ε) such that

$$\|C^\varepsilon\|_{L^2(\Omega_{1\varepsilon})} \leq C_1 \quad \text{and} \quad \|D^\varepsilon\|_{L^2(\Omega_{2\varepsilon})} \leq C_2. \tag{5.32}$$

Together with (3.1), (3.6) and the assumption on Φ, we have

$$|g_{3\varepsilon}^2(t)| \leq c, \qquad \forall t \in [0, T],$$

where the constant c is independent of t and ε. Moreover, as $s \to 0^+$, we get the following estimate corresponding to (5.11):

$$\begin{aligned} |g_{3\varepsilon}^2(t+s) - g_{3\varepsilon}^2(t)| \leq & \; s^{\frac{1}{2}} \|A\|_{L^\infty(Y)} \|\nabla \Phi\|_{L^\infty(0,T;L^\infty(\Omega))} \\ & \cdot \big\{ \|C^\varepsilon\|_{L^2(\Omega_{1\varepsilon})} \|\nabla u_{1\varepsilon}\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}))} + \|D^\varepsilon\|_{L^2(\Omega_{2\varepsilon})} \|\nabla u_{2\varepsilon}\|_{L^2(0,T;L^2(\Omega_{2\varepsilon}))} \big\} \\ \leq & \; c s^{\frac{1}{2}} \to 0, \quad \text{uniformly with respect to } \varepsilon. \end{aligned}$$

Hence we have

$$g_{3\varepsilon}^2 \to \int_0^t \int_\Omega A_\gamma^0 \nabla \Phi \nabla u_1 \, \mathrm{d}x \, \mathrm{d}s - \frac{1}{|Y|} \int_0^t \int_{\Omega \times \Gamma} h(y)(\widehat{\Phi}_1 - \widehat{\Phi}_2)(\widehat{u}_1 - \check{u}_2) \, \mathrm{d}x \, \mathrm{d}\sigma_y \, \mathrm{d}s \;\; \text{in } \mathcal{C}^0([0, T]). \tag{5.33}$$

Similarly, we also have

$$g_{3\varepsilon}^3 \to \int_0^t \int_\Omega A_\gamma^0 \nabla u_1 \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s - \frac{1}{|Y|} \int_0^t \int_{\Omega \times \Gamma} h(y)(\widehat{u}_1 - \check{u}_2)(\widehat{\Phi}_1 - \widehat{\Phi}_2) \, \mathrm{d}x \, \mathrm{d}\sigma_y \, \mathrm{d}s \;\; \text{in } \mathcal{C}^0([0, T]). \tag{5.34}$$

Step 3. For any $t \in [0, T]$, following the computation in (5.31), we obtain the following pointwise limit

$$\begin{aligned} g_{2\varepsilon} \to & \frac{1}{2} \int_\Omega |\Phi|^2 \mathrm{d}x + \frac{1}{|Y|} \int_0^t \int_{\Omega \times Y_1} A(y)[\nabla \Phi + \nabla_y \widehat{\Phi}_1][\nabla \Phi + \nabla_y \widehat{\Phi}_1] \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \\ & + \frac{1}{|Y|} \int_0^t \int_{\Omega \times Y_2} A(y)[\nabla \Phi + \nabla_y \widehat{\Phi}_2][\nabla \Phi + \nabla_y \widehat{\Phi}_2] \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \\ = & \frac{1}{2} \int_\Omega |\Phi|^2 \mathrm{d}x + \int_0^t \int_\Omega A_\gamma^0 \nabla \Phi \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s - \frac{1}{|Y|} \int_0^t \int_{\Omega \times \Gamma} h(y)(\widehat{\Phi}_1 - \widehat{\Phi}_2)^2 \, \mathrm{d}x \, \mathrm{d}\sigma_y \, \mathrm{d}s. \end{aligned}$$

Moreover, the same arguments as those of $\rho_{2\varepsilon}$ in Lemma 5.4 show that

$$\begin{aligned} g_{2\varepsilon} \to & \frac{1}{2} \int_\Omega |\Phi|^2 \mathrm{d}x + \int_0^t \int_\Omega A_\gamma^0 \nabla \Phi \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s \\ & - \frac{1}{|Y|} \int_0^t \int_{\Omega \times \Gamma} h(y)(\widehat{\Phi}_1 - \widehat{\Phi}_2)^2 \, \mathrm{d}x \, \mathrm{d}\sigma_y \, \mathrm{d}s \;\; \text{in } \mathcal{C}^0([0, T]). \end{aligned} \tag{5.35}$$

Now we focus on g_ε. Making use of (5.26)−(5.30) and (5.33)–(5.35), we have

$$g_\varepsilon \text{ is compact in } \mathcal{C}^0([0,T])$$

and

$$0 \leq \limsup_{\varepsilon\to 0} g_\varepsilon \leq \rho - \frac{1}{|Y|}\int_0^t \int_{\Omega\times\Gamma} h(y)\left[(\widehat{u}_1 - \breve{u}_2) - (\widehat{\Phi}_1 - \widehat{\Phi}_2)\right]^2 \mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}s \leq \rho, \quad \forall t \in [0,T],$$

where we used the assumption on h in the last inequality. This implies the desired result. □

Proof of Theorem 5.5. With Lemma 5.6 at our disposal, the Proof of Theorem 5.5 is completed by repeating the details in the Proof of Theorem 5.3. □

Remark 5.7. For the case $\gamma \in (-1, 1]$, the corrector results were proved by the oscillating test functions method in [11]. In fact, these results can be also proved by the periodic unfolding method. The argument is similar to that of Theorem 5.3.

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