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AN AVERAGE FORM OF ARTIN'S CONJECTURE

by

P. J. STEPHENS

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A more detailed account of the theorems described in this lecture is published in [1].

The order, $e_p(a)$, of a modulo p is defined to be the least positive integer n for which $a^n \equiv 1 \pmod{p}$. It is well-known that there are $\phi(d)$ numbers of order d in any interval of length p . In particular, there are $\phi(p-1)$ primitive roots in any interval of length p . We define

$$M_p(N) = \sum_{\substack{a \leq N \\ e_p(a) = p-1}} 1, \quad (1)$$

and deduce from the above remark that

$$M_p(N) = \phi(p-1) \left\{ \frac{N}{p} + O(1) \right\}. \quad (2)$$

Using the notation

$$N_a(x) = \sum_{\substack{p \leq x \\ e_p(a) = p-1}} 1. \quad (3)$$

Artin conjectured that

$$N_a(x) \sim A(a) x / \log x.$$

Forty years elapsed before Hooley proved that

$$N_a(x) = \frac{A(a) x}{\log x} + O\left(\frac{x \log \log x}{\log^2 x}\right),$$

is a consequence of the Riemann hypothesis holding for certain algebraic number fields.

A year later Goldfeld showed, without any hypothesis, that

$$N_a(x) = A \operatorname{li}(x) + O(x / (\log x)^D) \quad (4)$$

where $A = \prod_p \left(1 - \frac{1}{p(p-1)}\right)$ is true for almost all $a \leq N$ where N satisfied

$$x \geq N > \exp \{c_1 (\log x \log \log x)^\delta\}.$$

He also found a bound for the number of exceptions.

In this note we shall outline the proof of the following theorem.

THEOREM A . If $N > \exp \{4(\log x \log \log x)^{1/2}\}$, then

$$\frac{1}{N} \sum_{a \leq N} N_a(x) = A \operatorname{li}(x) + O\left(\frac{x}{(\log x)^D}\right) \quad (5)$$

where D is an arbitrary constant greater than 1 .

By proving the following theorem it is possible to deduce a bound for the number of exceptions to (4) . We only state it here.

THEOREM B . If $N > \exp \{6(\log x \log \log x)^{1/2}\}$, then

$$\frac{1}{N} \sum_{a \leq N} \{N_a(x) - A \operatorname{li}(x)\}^2 \ll x^2 / (\log x)^E ,$$

where E is an arbitrary constant greater than 2 .

The following lemma may be proved using the Siegel-Walfisz theorem.

LEMMA . For any choice of the constant D we have

$$\sum_{p \leq x} \frac{\phi(p-1)}{p} = A \operatorname{li}(x) + O\left(\frac{x}{(\log x)^D}\right) . \quad (6)$$

We define

$$c(\chi) = \frac{1}{(p-1)} \sum_a' \chi(a)$$

where \sum_a' runs over all the primitive roots in an interval of length p . Then

$$\sum_{\chi \pmod{p}} c(\chi) \chi(a) = \begin{cases} 1 & \text{if } a \text{ is a primitive root mod } p , \\ 0 & \text{else.} \end{cases}$$

It is clear that

$$c(\chi_0) = \phi(p-1) / (p-1) .$$

If $\chi \neq \chi_0$, then we may express the sum form $c(\chi)$ in terms of Ramanujan sums to obtain

$$|c(\chi)| \leq 1 / \operatorname{ord} \chi ,$$

where $\operatorname{ord} \chi$ is the least integer d for which $\chi^d = \chi_0$.

Then we have

$$\begin{aligned}
 \frac{1}{N} \sum_{a \leq N} N_a(x) &= \frac{1}{N} \sum_{p \leq x} M_p(N) \\
 &= \frac{1}{N} \sum_{p \leq x} \sum_{a \leq N} \sum_{\chi(\bmod p)} c(\chi) \chi(a) \\
 &= \frac{1}{N} \sum_{p \leq x} \sum_{\chi(\bmod p)} c(\chi) \sum_{a \leq N} \chi(a) \\
 &= \frac{1}{N} \sum_{p \leq x} \frac{\phi(p-1)}{(p-1)} \{[N] - [N/p]\} \\
 &+ O\left(\frac{1}{N} \sum_{p \leq x} \sum_{\substack{\chi(\bmod p) \\ \chi \neq \chi_0}} \frac{1}{\text{ord } \chi} \left| \sum_{a \leq N} \chi(a) \right|\right) \\
 &= \sum_{p \leq x} \frac{\phi(p-1)}{p} + O\left(\frac{x}{N \log x}\right) + O\left(\frac{S_1}{N}\right), \quad (7)
 \end{aligned}$$

where

$$S_1 = \sum_{p \leq x} \sum_{\substack{\chi(\bmod p) \\ \chi \neq \chi_0}} \frac{1}{\text{ord } \chi} \left| \sum_{a \leq N} \chi(a) \right|.$$

Since any non-principal character mod p is primitive, we have

$$S_1 = \sum_{p \leq x} \sum_{\chi(\bmod p)}^* \frac{1}{\text{ord } \chi} \left| \sum_{a \leq N} \chi(a) \right|$$

where \sum^* denotes summation over primitive characters. We now use Holder's inequality to obtain

$$S_1^{2r} \leq \left\{ \sum_{p \leq x} \sum_{\chi(\bmod p)}^* \left(\frac{1}{\text{ord } \chi}\right)^{2r/(2r-1)} \right\}^{2r-1} \sum_{p \leq x} \sum_{\chi(\bmod p)}^* \left| \sum_{a \leq N} \chi(a) \right|^{2r}. \quad (8)$$

The first sum on the right hand side of this inequality is easily estimated to be

$$\ll (c_2 x)^{2r-1}. \quad (9)$$

The second sum is

$$\begin{aligned}
 S_2 &= \sum_{p \leq x} \sum_{\chi(\bmod p)}^* \left| \sum_{a \leq N} \chi(a) \right|^{2r} \\
 &= \sum_{p \leq x} \sum_{\chi(\bmod p)} \left| \sum_{a \leq N^r} \tau'_r(a) \chi(a) \right|^2
 \end{aligned}$$

where $\tau'_r(a)$ is the number of ways of writing a as the product of r factors each of which is less than N . If we define $\tau_r(a)$ to be the number of ways of

writing a as the product of r factors, then $\tau'_r(a) \leq \tau_r(a)$. Then large sieve inequality (see Gallagher [2]) gives

$$\begin{aligned} S_2 &\ll (x^2 + N^r) \sum_{a \leq N^r} \{\tau'_r(a)\}^2 \\ &\ll (x^2 + N^r) \sum_{a \leq N^r} \tau_r^2(a). \end{aligned} \quad (10)$$

Mardjanichvili [3] estimated the sum on the right hand side of this inequality to be

$$\ll N^r \{\log(e N^r)\}^{r^2-1}. \quad (11)$$

We thus have from (8), (9), (10) and (11) that

$$S_1^{2r} \ll (c_2 x)^{2r-1} (x^2 + N^r) N^r \{\log(e N^r)\}^{r^2-1}.$$

By choosing $r = \lceil \frac{2 \log x}{\log N} \rceil + 1$, we deduce

$$S_1 \ll x^{1-1/2r} N \{\log(e x^2)\}^{(r^2-1)/2r}$$

or,

$$S_1/N \ll x / (\log x)^D.$$

This together with (6) and (7) yields (5).

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