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AN EXAMPLE OF INFINITE DIMENSIONAL REFLEXIVE SPACE NON-ISOMORPHIC TO ITS CARTESIAN SQUARE

by

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The problem whether every infinite dimensional Banach space X is isomorphic to its square X^2 was raised in [2] and remained unsolved until 1959. All known counterexamples, however, (cf. [4], [8]), were non-reflexive. Several authors (cf. [1], [4]) asked if there exist reflexive spaces with these properties.

The positive answer to this question is given in the following theorem.

THEOREM.- Let $(p_i)_{i=1}^{\infty}$ be a strictly decreasing sequence of real numbers greater than 2, and let $1<p\leqslant \lim\limits_{i\to\infty} p_i$. Then there exists a sequence $(n_i)_{i=1}^{\infty}$ of positive integers such that the space

$$X = \left(\sum_{i=1}^{\infty} \oplus \ell_{p_i}^{n_i} \right)_p$$

is not isomorphic to X^2 .

(Given a sequence $(X_i)_{i=1}^{\infty}$ of Banach spaces and $p>1$, we denote by $\left(\sum_{i=1}^{\infty} \oplus X_i \right)_p$ the space of all sequences $x = (x_i)_{i=1}^{\infty}$, with $x_i \in X_i$ for $i = 1,2,\ldots$, and $\|x\| = \left(\sum_{i=1}^{\infty} \|x_i\|_i^p \right)^{1/p} < \infty$. ℓ_p^n denotes, as usual, the space $\left(\sum_{i=1}^{\infty} \oplus X_i\right)_p$, where $\dim X_i = 1$ for $i\leqslant n$, and $\dim X_i = 0$ for $i>n$) .

The proof of this theorem and, in fact, of a more general result will appear in Studia Mathematica [6]. We shall not go here into detail, we would like, however, to formulate two facts established and used in this proof and seeming to be useful also in other problems.

PROPOSITION 1.- Let $p>2$, $0<c<1$, and $\varepsilon>0$. Then there exists such an N that in every linear subspace $Z\subset \ell_p^n$ with $\dim Z>cn>cN$ there exists a vector $x\in Z$ such that $\|x\| = 1$ and

$$\mathrm{Card}\,(\{j\in \{1,\ldots, n\} : |x_j| > \varepsilon\, n^{-1/p}\}) < \varepsilon\, n \quad .$$

One can take $N = \left(\frac{\varepsilon^6 c}{2\pi}\right)^{p/(2-p)}$.

COROLLARY.- Let $p>2$, $0<c<1$, $\beta>1$. Then there exist positive constants M, α such that every linear subspace $Z\subset \ell_p^n$ with $\dim Z>cn>cM$ contains a subspace Y such that

$$m = \dim Y > \alpha\, n^{(p-2)/6p} ,$$

$$d(Y , \ell_p^m) < \beta .$$

(For any isomorphic Banach spaces X_1 , X_2 we denote by $d(X_1 , X_2)$ the greatest lower bound of numbers $\|T\|\,\|T^{-1}\|$, where T is an isomorphism of X_1 onto X_2) .

PROPOSITION 2.- Let $p\geqslant 2$, $p>q>1$, $K>0$. Let $(X_i)_{i=1}^{\infty}$ be a sequence of Banach spaces, whose modulus of convexity $\delta_{X_i}(\varepsilon)$ admits the estimate $\delta_{X_i}(\varepsilon) > K\varepsilon^p$ for $0<\varepsilon\leqslant 2$, $i = 1,2,\ldots$ Then the modulus of convexity of the space $X = (\sum_{i=1}^{\infty} \oplus X_i)_q$ can be estimated from below by $L\varepsilon^p$, where L is a positive constant depending only on p,q,K .

Proposition 2 is a partial improvement of a known result of Day [5] .

The proof of Proposition 1 uses some ideas of Kadec and Pelczynski[7] and also certain properties of projection constants. Corollary may be deduced from it by a standard "gliding hump" procedure (cf. [3], [7]) .

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