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APPROXIMATION THEOREMS AND NASH CONJECTURE

by Alberto TOGNOLI

Summary:

The purpose of this lecture is to illustrate some applications of Weierstrass' and Whitney's approximation theorems in their relative form.

In particular it will be mentioned how from these descend a theorem which affirms that the classification of the analytic fiber bundle on a coherent real analytic space coincides with the topological one.

Then, using Weierstrass' relative approximation theorem, an outline of the proof of the following fact will be given: every compact differentiable variety admits a structure of regular algebraic variety.

§ 1. THE RELATIVE APPROXIMATION THEOREMS

a) Some definitions.

In this article we shall study only entities defined on the real field. Let $U$ be an open set of $\mathbb{R}^n$, $\mathcal{O}_U$ denotes the sheaf of germs of the real analytic functions on $U$ and $\Gamma(\mathcal{O}_U)$ the ring of (global) sections of $\mathcal{O}_U$.

A function $f \in \Gamma(\mathcal{O}_U)$ is said algebraic if for any point $x \in U$ there exists a neighbourhood $U_{x_0}$ and some polynomials $a_i : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{i=0}^n (f(x))^i a_i(x) = 0, \forall x \in U_{x_0}$$

Let $\mathcal{A}_U$ denote the sheaf of germs of algebraic functions.

Let $V$ be a closed subset of $U$, $V$ is said an analytic subset of $U$ if the following condition is satisfied: for every $a \in V$ there exists an open neighbourhood $U_a$ such that:

$$V \cap U_a = \{ x \in U_a \mid f_1(x) = \ldots = f_q(x) = 0, f_i \in \Gamma(\mathcal{O}_{U_a}) \}.$$ 

Let $V$ be an analytic subset of $U$ and $\mathcal{I}_V$ denote the ideal subsheaf of $\mathcal{O}_U$ of germs of the analytic functions that are identically zero on $V$. 
Finally we denote $O^\circ = O_U/\mathfrak{J}_V$, the sheaf $O^\circ$ is said the sheaf of germs of analytic functions on $V$.

In such a way, to any analytic set $V$ of $U$, is associated a local ringed space.

Then a local ringed space $(X, O_X)$ is said a real analytic space if:

I) $X$ is paracompact.

II) $(X, O_X)$ is locally isomorphic to a ringed space associated to an analytic subset of an open set of $\mathbb{R}^n$.

In a similar way we define algebraic set of $U$ any closed set that, locally, is the set of zeros of algebraic functions, and we associate to any algebraic set $V$ the sheaf $\mathcal{A}_V = \mathcal{A}_U/\mathfrak{J}_V$ of germs of algebraic functions restricted to $V$.

Finally a local ringed space $(X, O_X)$ is said an algebraic space if it is paracompact and locally isomorphic to a ringed space associated to an algebraic set.

A closed set $V$ of $\mathbb{R}^n$ is said an affine variety if there exist some polynomials $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, q$ such that $V = \{ x \in \mathbb{R}^n | f_1(x) = \ldots = f_q(x) = 0 \}$.

Let $V$ be an affine variety, we shall denote $\mathcal{R}_V$ the sheaf of germs of regular functions on $V$. Using affine varieties $(V, \mathcal{R}_V)$ as local models one defines algebraic varieties (see [1]).

If $X, Y$ are real analytic spaces or algebraic spaces or algebraic varieties we shall use the term morphism (and isomorphism) of $X$ into $Y$ instead of morphism (and isomorphism) of ringed spaces. If $X, Y$ are analytic spaces a morphism is usually said an analytic map.

Let $U$ be an open set of $\mathbb{R}^n$, $V$ an analytic set, $x_0 \in V$ and $V_{x_0}$ the germ of $V$ at $x_0$.

We shall say that $V$ is regular in the point $x_0$ if it is possible to find $q = n - \dim V$ analytic functions $f_1, \ldots, f_q$, defined on a neighbourhood $U_{x_0}$ of $x_0$, such that:

I) $V \cap U_{x_0} = \{ x \in U_{x_0} | f_1(x) = \ldots = f_q(x) = 0 \}$

II) $(df_1)(x_0), \ldots, (df_q)(x_0)$ are linearly independent.

Let $(X, O_X)$ be a real analytic space, we shall say that $x_0 \in X$ is a regular point if there exists a neighbourhood $U_{x_0}$ of $x_0$ that is isomorphic to an analytic set containing only regular points. A point that is not regular is called singular. A similar definition of regular point is given for algebraic spaces and algebraic varieties.
Let \((X, \mathcal{O}_X)\) be a real analytic space (real algebraic variety) containing only regular points then \(X\) is called an analytic (algebraic) real manifold. An algebraic space that contains only regular points is called a regular algebraic space.

Let \(U\) be an open set of \(\mathbb{R}^n\) and \(V\) an analytic (algebraic) subset of \(U\); it is a well known fact, (see [2],[3]), that in general the sheaf \(\mathcal{J}_V (\mathcal{F}_V)\) is not coherent considered as \(\mathcal{O}_U\) - module (\(\alpha_U\) - module).

We shall say that an analytic (algebraic) subset of \(U\) is coherent if the sheaf \(\mathcal{J}_V (\mathcal{F}_V)\) is a coherent \(\mathcal{O}_U\) - module (\(\alpha_U\) - module).

An analytic (algebraic) space is called coherent if any point \(x_0 \in X\) has a neighbourhood isomorphic to an analytic (algebraic) coherent subset of some open set of \(\mathbb{R}^n\).

It is known that an algebraic space is coherent if and only if the associated real analytic space is coherent (see [3]). Finally we remember that any real analytic manifold and any regular algebraic space is coherent.

Let \(V\) be an affine variety of \(\mathbb{R}^n\), \(x_0 \in V\) and \(\mathcal{J}(V, x_0)\), \((I(V, x_0))\) the rings of germs of analytic functions (and of polynomials) that are zero on the germ \(V, x_0\) of \(V\) at \(x_0\) (on \(V\)).

Let \(\mathcal{O}_{x_0}\) be the ring of germs of analytic functions defined in some neighbourhoods of \(x_0\) in \(\mathbb{R}^n\).

We shall say that the point \(x_0\) is an almost regular point of \(V\) if \(\mathcal{J}(V, x_0)\) is generated, as \(\mathcal{O}_{x_0}\) - module, by \(I(V, x_0)\).

An affine variety \(V\) is said almost regular if \(V\) is almost regular in any point.

It is easy to prove that \(x_0\) is an almost regular point of \(V\) if, and only if, the intersection of all the germs of complex analytic sets of \(\mathbb{C}^n\) that contains \(V, x_0\) is the germ of a complex affine variety that contains \(V\) (see [4]). As a consequence we have that any regular point of \(V\) (considered as affine variety) is almost regular.

b) The approximation theorems.

In the suite we will give some applications of the following theorems:

**Theorem 1.** Let \(U\) be open in \(\mathbb{R}^n\), \(V\) a coherent analytic subset of \(U\) and \(\mathcal{J} \in \Gamma(\mathcal{O}_V)\) an analytic function on \(V\). Let \(\{K_n\}_{n \in \mathbb{N}}\) be a sequence of compact
sets in $U$ such that:

$$K_n \subseteq K_{n+1}, \quad \bigcup_{n \in \mathbb{N}} K_n = U.$$

Let $\{n_\ell\}_{\ell \in \mathbb{N}}$ be a sequence of positive integers.

Finally, let $\{c_\ell\}_{\ell \in \mathbb{N}}$ be a sequence of positive numbers.

Then for any function $f : U \rightarrow \mathbb{R}$ of class $C^\infty$ such that $f|_V = c|_V$ there exists an analytic function $h : U \rightarrow \mathbb{R}$ with the following properties:

I) $\left| \frac{\partial^\alpha (f - h)(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right| < \epsilon_\ell$ for any $x \in K_{p+1} - K_p$ and $0 \leq \alpha < n_p$

II) $f|_V = h|_V$

**THEOREM 2.** Let $U$ be an open set of $\mathbb{R}^n$, $V$ a compact affine almost regular variety contained in $U$. Suppose that $V$, considered as analytic set, is coherent and denote by $p : \mathbb{R}^n \rightarrow \mathbb{R}$ a polynomial function.

Let $f : U \rightarrow \mathbb{R}$ be a function of class $C^\infty$ such that $f|_V = p|_V$, $H$ a compact set of $U$ and $\epsilon$ a positive number.

Then, for every positive integer $q$, there exists a polynomial $h : \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

I) $\left| \frac{\partial^\alpha (f - h)(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right| < \epsilon$, for any $x \in H$, $0 \leq \alpha < q$

II) $f|_V = h|_V$

**THEOREM 3.** Let $U$ be an open set of $\mathbb{R}^n$, $V$ a compact, coherent affine almost regular variety contained in $U$ and $p : U \rightarrow \mathbb{R}$ an algebraic function.

Let $f : U \rightarrow \mathbb{R}$ be a function of class $C^\infty$ such that $f|_V = p|_V$, $H$ a compact set of $U$ and $\epsilon$ a positive number.

Then, for every positive integer $q$, there exists an algebraic function $h : U \rightarrow \mathbb{R}$ such that conditions I) and II) of theorem 2 are satisfied.

We shall give a sketch of the proof of theorem 2.

Let $\mathbb{R}[X_1, \ldots, X_n]$ and $\mathbb{R}[[X_1, \ldots, X_n]]$ be the ring of convergent power series and formal power series.

In the following on local rings we shall consider the $\mathfrak{p}$-adic topology and we shall denote by $\hat{A}$ the completion of $A$. 
A ring \( A \) is said analytic (or formal) if \( A = \mathbb{R}[X_1, \ldots, X_n]/\mathfrak{J} \) where \( \mathfrak{J} \) is an ideal.

It is known that analytic and formal rings are local noetherian rings and Hausdorff spaces (with respect to \( \mathfrak{m} \)-adic topology).

From the last assertion the following equality is clear: for any ideal \( \mathfrak{J} \) of an analytic or formal ring \( A \) we have

\[
\hat{\mathfrak{J}} = \hat{A} \cdot \mathfrak{J} = \{ x \in A | x = \sum_{i=1}^{q} a_i g_i, a_i \in \hat{A}, g_i \in \mathfrak{J} \}.
\]

(\( \hat{A} \cdot \mathfrak{J} \) is dense in \( \hat{J} \), but \( \hat{A} \cdot \mathfrak{J} \) is an ideal, then closed, and we have \( \hat{J} = \hat{A} \cdot \mathfrak{J} \)).

Let \( U \) be an open set of \( \mathbb{R}^n \), \( 0 \) the origin and suppose \( 0 \in U \). Let \( E \) be a set contained in \( U \) and \( g \) a function of class \( \mathcal{C}^\infty \) defined in a neighbourhood of \( 0 \); we shall say that \( g \) has on \( E \), in \( 0 \), a zero of infinite order if for any \( p \in \mathbb{N} \) there exists a positive number \( C_p \) and a neighbourhood \( B_p \) of \( 0 \) such that on \( B_p \cap E \) we have:

\[
|g(x)| < C_p \|x\|^p
\]

where

\[
x = (x_1, \ldots, x_n), \quad \|x\| = \sum_{i=1}^{n} x_i^2.
\]

We remark that if \( g \) has a zero of infinite order on \( E \) in \( 0 \) then any function \( h \) having the same formal development has the same property.

Finally we shall denote by \( \mathcal{J}(E_0) \) the subset of \( \mathbb{R}[[X_1, \ldots, X_n]] \) of the elements associated to a germ of a \( \mathcal{C}^\infty \) function having a zero of infinite order on \( E \) in \( 0 \).

If \( E_0 \) is a germ of analytic set (algebraic variety) we shall denote by \( \mathcal{J}(E_0) \) (\( \mathcal{P}(E_0) \)) the ring of germs of analytic functions (polynomials) that are zero on \( E_0 \).

It is clear that in the above definitions the choice of the origin as fixed point is inessential.

Using the above notation we have the following

**Lemma 1.** Let \( V \) be an affine variety of \( \mathbb{R}^n \) and \( x \in V \) be an almost regular point, then we have

\[
\hat{P}(V_x) = \hat{\mathcal{J}}(V_x) = \mathcal{J}(V_x)
\]

*Proof:* The first equality is a consequence of the definition of almost regular point, the second is proved in [6].
LEMMA 2. - Let $V$ be an affine variety of $\mathbb{R}^n$, $x \in V$ be an almost regular point and suppose that $V$, considered as an analytic space, is coherent in $x$.

Let $f : U(x) \to \mathbb{R}$ be a function of $C^\infty$ class defined on a neighbourhood $U(x)$ of $x$ in $\mathbb{R}^n$.

If $f\big|_{U(x) \cap V} = 0$ there exist some polynomials $\xi_1, \ldots, \xi_q$ and some $C^\infty$ functions $\alpha_1, \ldots, \alpha_q$, defined on a neighbourhood $U'(x)$ of $x$, such that:

$$f(y) = \sum_{i=1}^q \alpha_i(y) \xi_i(y), \quad \forall y \in U'(x) \quad \text{and} \quad \xi_i\big|_V = 0.$$ 

Proof: By hypothesis there exists a neighbourhood $D(x)$ of $x$ in $V$ and some polynomials $\xi_1, \ldots, \xi_q$ such that: $\xi_i\big|_V = 0$, $i = 1, \ldots, q$, for any $y \in D(x)$ the ring $\mathcal{O}(V)$ is generated by $\xi_1, \ldots, \xi_q$.

For any $y \in D(x)$ the germ $f_y$ of $f$ is, in virtue of lemma 1, of the form

$$(1) \quad f_y = \sum_{i=1}^q (\alpha_i)_y \xi_i(y) \quad \text{where} \quad (\alpha_i)_y \in \mathbb{R}[[X_1, \ldots, X_n]].$$

By a result of Malgrange (see [8]) from (1) we deduce that $f_x$ is a linear combination of $(\xi_i)$ with $C^\infty$ coefficients and the lemma is proved.

LEMMA 3. - Let $V$ be an affine, compact, almost regular subvariety of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a function of class $C^\infty$ defined on a neighbourhood $U$ of $V$.

Let $K$ be a compact set of $U$, and suppose $f|_K = 0$, then there exist some polynomials $\xi_1, \ldots, \xi_q$ and some functions $\alpha_1, \ldots, \alpha_q$ of class $C^\infty$ defined on a neighbourhood $U_K$ of $K$ such that:

$$f(x) = \sum_{i=1}^q \alpha_i(x) \xi_i(x), \quad \forall x \in U_K \quad \text{and} \quad \xi_i\big|_V = 0, \quad i = 1, \ldots, q.$$ 

Proof: $V$ is almost regular and compact then there exist some polynomials $\xi_1, \ldots, \xi_q$ such that: $\xi_i\big|_V = 0$, $(\xi_i)_{i=1, \ldots, q}$ generate $\mathcal{O}(V)$ for any $x \in V$ and if $x \notin V$ then there exists $i$ such that $\xi_i(x) \neq 0$.

For any $x \in U$ there exists a neighbourhood $U_x$ and some functions of class $C^\infty$: $(\xi_i)_{i=1, \ldots, q}$ such that

$$(1) \quad f(y) = \sum_{j=1}^q \alpha_j(y) \xi_j(y), \quad \forall y \in U_x.$$
In fact, if \( x \in V \), (1) is a consequence of lemma 2, if \( x \notin V \) then there exists \( \varepsilon_i \) such that \( \varepsilon_i(x) \neq 0 \) and we can write \( f(y) = f(y)/\varepsilon_i(y) \cdot \varepsilon_i(y) \).

So we have proved that there exists a finite open, \((in \ R^n)\), covering \( \{U_i\} \) of \( K \) and functions \( \{a_j\}_{j=1}^{q} \) of class \( C^\infty \), such that we have:

\[
f(y) = \sum_{j=1}^{q} a_j(y) \varepsilon_j(y) , \forall y \in U_i .
\]

Let \( \{\rho_i\}_{i=1}^{s} \) be a partition of unity of class \( C^\infty \) relative to the covering \( \{U_i\} \) \( i = 1, \ldots, s \).

The we have:

\[
f(x) = f(x) \cdot \sum_{i=1}^{s} \rho_i(x) = \sum_{j=1}^{q} \sum_{i=1}^{s} a_j(x) \varepsilon_j(x) = \sum_{i,j}^{s} \rho_i(x) a_j(x) \varepsilon_j(x) = \sum_{i}^{s} \rho_i(x) \sum_{j}^{q} a_j(x) \varepsilon_j(x)
\]

where \( a_j = \sum_{i=1}^{s} a_j \rho_i \).

The functions \( a_j \) are of class \( C^\infty \) and the lemma is proved.

**Proof of theorem 2.** We have \( f - p \mid_V \equiv 0 \) then it is enough to prove the theorem for the function \( g = f - p \) such that \( g \mid_V \equiv 0 \).

Lemma 3 affirms that there exist some polynomials \( \varepsilon_1, \ldots, \varepsilon_q \) and \( C^\infty \) functions \( a_1, \ldots, a_q \) defined on a neighbourhood \( U_K \) of \( K \) such that:

\[
\varepsilon_j(x) = \sum_{j=1}^{q} a_j(x) \varepsilon_j(x) , \ x \in U_K \text{ and } \varepsilon_j \mid_V \equiv 0 , \ j = 1, \ldots, q .
\]

It is now possible, by the classical Weierstrass approximation theorem, to choose polynomials \( a_j \) such that the polynome \( \sum_{j=1}^{q} a_j \varepsilon_j + p \) satisfies the conditions of theorem 2.

**Remark:** The proof of theorem 3 is quite similar.

The proof of theorem 1 is of the same type but more difficult because in general we need infinitely many elements of \( \Gamma \mid \chi \) to generate \( \mathcal{F} \mid \chi \), \( x \in V \).
§ 2. APPROXIMATION THEOREMS IN THE CASE OF MANIFOLDS

It is a natural problem to see if it is possible to deduce from theorems 1, 2, 3 some results of the following type:

1') let $X, Y$ be two real analytic spaces and $f : X \to Y$ a continuous map, then $f$ can be approached by analytic maps $f_i : X \to Y$ such that any $f_i$ is in the same homotopy class of $f$.

2') let $X, Y$ be two affine, compact varieties and $f : X \to Y$ a continuous map, then $f$ can be approached by a sequence of morphisms.

3') let $X, Y$ be two compact algebraic spaces and $f : X \to Y$ a continuous map, then $f$ can be approached by a sequence of morphisms $f_n : X \to Y$ such that any $f_n$ is in the same homotopy class of $f$.

It is also possible to see for "relative problem" of type 1'), 2'), 3').

In the next proposition we shall give a partial solution to problem 1').

PROPOSITION 1. - Let $X$ be a coherent real analytic space and suppose that for any connected component $X_i$ of $X$ we have $\dim X_i < +\infty$.

Let $Y$ be a real analytic manifold, $d : Y \times Y \to \mathbb{R}$ a continuous metric and $f : X \to Y$ a continuous map.

Then, for any $\varepsilon > 0$, there exists an analytic map $h : X \to Y$ such that:

$$d(f(x), h(x)) < \varepsilon, \quad \forall x \in X \text{ and } h \text{ is homotopic to } f.$$  

Proof: We may suppose $X$ connected.

There exists an analytic proper injective map $j : X \to \mathbb{R}^n$, $n = 2 \dim X + 1$, such that $j : X \to j(X)$ is homeomorphism and $j(X)$ is a coherent real analytic space (see [9]).

It is then clear that it is enough to solve the problem for the analytic subspace $j(X)$ of $\mathbb{R}^n$ and the function $f' = f \circ j^{-1}$, so in the following we shall suppose $X$ subspace of $\mathbb{R}^n$.

It is known that $Y$ may be considered as a submanifold of $\mathbb{R}^m$, $m > 2 \dim Y + 1$ and there exists a tubular neighbourhood $U$ of $Y$ in $\mathbb{R}^m$. 
By definition of tubular neighbourhood there exists an analytic map \( p : U \rightarrow Y \) such that: \( p(x) = x \), if \( x \in Y \), and \( p \) is homotopic to the identity map \( i : U \rightarrow U \).

Any continuous map \( f : X \rightarrow Y \) may be approached by \( C^\infty \) maps \( f'_1 : X \rightarrow U \subset \mathbb{R}^m \) (see [10]); theorem 1 asserts that we can approach \( f'_1 \) by analytic maps \( f''_1 : X \rightarrow U \subset \mathbb{R}^m \).

If \( f'_1 \) is close enough to \( f \) and \( f''_1 \) to \( f'_1 \) the analytic map

\[
    f_1 = p \circ f''_1 : X \rightarrow Y
\]

approaches \( f \) in the required sense.

Finally it is easy to verify that if \( f''_1 \) approaches \( f \) then \( f_1 \) is homotopic to \( f \).

The proposition is now proved.

The demonstration of proposition 1 points out that we obtain results of type 1'), 2'), 3') if the following conditions are satisfied:

a) \( X \) and \( Y \) are imbedded in some euclidian space;

b) \( Y \) has a tubular neighbourhood.

So we can affirm that (at least following this way) we cannot solve the problem 1') if \( Y \) is singular (it is known that, if \( Y \) has at least a singular point, it is impossible to find a tubular neighbourhood).

Analogously we cannot solve problem 2') and we can solve problem 3') only if \( X \) and \( Y \) are isomorphic to algebraic subspaces of some euclidian space\(^{(*)}\) and \( Y \) is regular at any point (the existence of tubular neighbourhoods for algebraic regular subspaces of \( \mathbb{R}^n \) is proved in [3]).

It is not difficult to convince ourself that result 1'), if \( Y \) is singular, result 2'), result 3') if \( X \) or \( Y \) are not imbedded are false (at least in general).

For example let:

\[
    X = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 - 9 = 0 \}
\]

\[
    Y_1 = \{(x,y) \in \mathbb{R}^2 | x^2 + (y-1)^2 - 1 = 0 \}
\]

\[
    Y_2 = \{(x,y) \in \mathbb{R}^2 | x^2 + (y+1)^2 - 1 = 0 \}
\]

\( Y = Y_1 \cup Y_2 \) and \( f : X \rightarrow Y \) the projection of \( X \) into \( Y \) from the origin \( O \) of \( \mathbb{R}^2 \).

It is easy to verify that:

\( f \) is continuous but for any analytic map \( f' : X \rightarrow Y \) we have \( f'(X) \subset Y_1 \) or \( f'(X) \subset Y_2 \).

\(^{(\ddagger)}\) In general a regular compact algebraic space is not isomorphic to a subspace of an euclidian space (see [3]).
So we conclude that \( f \) cannot be approximated by analytic map and any analytic map \( f':X \to Y \) is not homotopic to \( f \).

About the problem 2') we remark the following: if it should be possible to obtain results of type 2') then we shall also have that two compact regular affine varieties are isomorphic if and only if they are \( C^\infty \) isomorphic and this is false (in fact for proving this last result we need a stronger version of 2') involving approximation of derivatives).

About the problem 3') we remark the circle \( S^1 \) may be considered as a real algebraic subspace of \( \mathbb{R}^2 \), and also with the algebraic structure induced by \( \mathbb{R} \) identifying \( S^1 \) with \( \mathbb{R}/\mathbb{Z} \). It is easy to verify that \( S^1 \), endowed with the last structure, has no global algebraic function; we shall denote \( S^1 \) the circle with this last structure.

It is now clear that the identity map \( i : S^1 \to S^1 \) cannot be approximated by morphisms of algebraic structures and any morphism is not homotopic to \( i \).

Using theorem 1 in the relative form we can strenghten proposition 1 and we obtain:

**THEOREM 4.** Let \( X \) be a real coherent analytic space, \( X' \) a coherent analytic subspace of \( X \) such that \( \dim X' < +\infty \).

Let \( Y \) be a real analytic manifold, \( d : Y \times Y \to \mathbb{R} \) a continuous metric and \( f : X \to Y \) a continuous map such that \( f|_{X'} \) is analytic.

Then for any \( \varepsilon > 0 \) there exists an analytic map \( h : X \to Y \) such that:

\[
|f(x) - h(x)| < \varepsilon, \quad \forall x \in X \quad \text{and} \quad f \text{ is homotopic to } h.
\]

The idea for proving theorem 4 is the following: let \( X = \bigcup X_n \) the decomposition of \( X \) into irreducible components; then one, using proposition 1, approximate \( f|_{X_1} \) by \( f^1 : X_1 \to Y \), after, without changing \( f^1|_{X_1 \cap X_2} \), one approximate \( f|_{X_1 \cup X_2} \) ....

The family \( \{X_n\}_{n \in \mathbb{N}} \) is locally finite so we can construct an analytic approximation of \( f \).

Theorem 4 is proved in [11].

A problem tied to problem 1') is the following

1") Let \( X \) be a coherent real analytic space and \( (B \to X, G, F) \) an analytic fiber bundle with structural Lie group \( G \) and fiber \( F \). Suppose \( F \) is an analytic

(*) In fact one proves that if two affine varieties \( X, X' \) are isomorphic then their complexifications are birationally equivalent.
tic manifold and \( \gamma : X \to B \) be a continuous cross section.

We ask if it is possible to approach \( \gamma \) by analytic cross sections. A partial affirmative answer is given by

**Proposition 2.** Let \( X \) be an analytic manifold and \( B \to X \) an analytic fiber bundle the fiber of which is a manifold. Let \( d : B \times B \to \mathbb{R} \) be a continuous metric on \( B \), \( X' \) a coherent analytic subspace of \( X \) and \( \gamma : X \to B \) a continuous cross section such that \( \gamma|_{X'} \) is analytic.

Then for any \( \varepsilon > 0 \) there exists an analytic cross section \( \gamma_a : X \to B \) such that:

\[
\gamma_a|_{X'} = \gamma|_{X'}, \quad d(\gamma(x), \gamma_a(x)) < \varepsilon, \quad \forall x \in X \quad \text{and} \quad \gamma_a \text{ is homotopic to } \gamma.
\]

**Proof:** \( B \) is an analytic manifold then, by Proposition 1, the map \( \gamma : X \to B \) may be approached by analytic maps \( \gamma_i : X \to B \) such that \( \gamma|_{X'} = \gamma_i|_{X'} \).

In general the maps \( a_i = \pi \circ \gamma_i : X \to X \) are not the identity but, if \( \gamma_i \) is close enough to \( \gamma \), we know that \( a_i \) is an isomorphism of analytic manifolds.

It is now clear that \( \gamma_i = \gamma_i \circ a_i^{-1} : X \to B \) is an analytic cross section of \( B \) and, if \( \gamma_i \) is close enough to \( \gamma \), then \( \gamma_i \) satisfies the condition \( d(\gamma_i(x), \gamma(x)) < \varepsilon, \quad \forall x \in X \).

If \( x \in X' \) we have \( a_i(x) = x \) then \( \gamma_i(x) = \gamma_i(x) = \gamma(x) \). The Proposition 1 asserts that, if \( \gamma_i \) is close enough to \( \gamma \), there exists a homotopy \( t \to \gamma_i \) tying \( \gamma_i \) to \( \gamma \); it is clear that \( \gamma_i \) ties \( \gamma_i \) to \( \gamma \).

The proof is acquired.

As a consequence of the theorem 4 and the proposition 2 we can prove the following

**Proposition 3.** Let \( X \), \( \dim X < +\infty \), be a real coherent analytic space and \( B \to X \) a topological principal fibre bundle of structural group \( G \). If \( G \) is a connected (or a compact) Lie group then there exists an analytic fiber bundle \( B \to X \) that is topologically equivalent to \( B \).

Let \( X \) be a real analytic manifold and \( G \) a Lie group.

Let \( B_i \to X \), \( i = 1,2 \), be two analytic principal fiber bundles with structural group \( G \), then \( B_i \) is analytically isomorphic to \( B_2 \) if and only if \( B_i \) is topologically isomorphic to \( B_2 \).

(*) Here we need that \( \gamma_i \) and their "first derivative" approach \( \gamma \) and its first derivative and this is possible by theorem 1.
Proof: It is known (see [10]), that if the Lie group $G$ is connected then, in the bundle $\mathcal{B} \rightarrow X$, the structural group may be reduced to a compact subgroup $G'$.

For any $n \in \mathbb{N}$ there exists a universal bundle $U(G',n) \rightarrow D(G',n)$ relative to the group $G'$; it is known (see [10]) that the universal bundle $U(G',n) \rightarrow D(G',n)$ may be endowed of real analytic structure.

To prove the first part of the proposition it is enough to show that any continuous map $\varphi : X \rightarrow D(G',n)$, $n = \dim X$, is homotopic to an analytic map $\varphi_a : X \rightarrow D(G',n)$ and this is proved in proposition 1.

To prove the second part of proposition we recall that, given the fiber bundles $\mathcal{B}_1, \mathcal{B}_2$, there exists another fiber bundle $\mathcal{B}_{1,2} \rightarrow X$ such that $\mathcal{B}_1$ is topologically (analytically) isomorphic to $\mathcal{B}_2$ if and only if $\mathcal{B}_{1,2}$ has at least one continuous (analytic) section (for the construction of $\mathcal{B}_{1,2}$ see [13]).

It is now clear that the proposition 2 proves the second part of this proposition.

Proposition 2 is a particular case of the following

THEOREM 5. Let $X$ be a real coherent analytic space, $\dim X < \infty$ and $X'$ a coherent analytic subspace.

Let $\mathcal{B} \rightarrow X$ be a real analytic fiber bundle of structural Lie group $G$ and fiber the analytic manifold $F$.

Let $d : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}$ a continuous metric, $\gamma : X \rightarrow \mathcal{B}$ a continuous cross section such that $\gamma|_X$ is analytic.

Then, if $G$ is connected, for any $\varepsilon > 0$ there exists an analytic section $\gamma_a : X \rightarrow \mathcal{B}$ such that:

$\gamma|_{X'} = \gamma_a|_{X'}$, $d(\gamma(x), \gamma_a(x)) < \varepsilon$, $\forall x \in X$ and $\gamma$ is homotopic to $\gamma_a$.

Remark: It is possible to prove a version of proposition 1 and 2 for compact regular algebraic sets of $\mathbb{R}^n$ (the proofs are formally the same).

Also a weak form of proposition 3 may be proved for the compact algebraic subsets of $\mathbb{R}^n$.

§ 3. AN APPLICATION OF THEOREM 2

Let $V$ be a compact differentiable submanifold of $\mathbb{R}^n$; J. Nash in [14], has put the following problems:
I) does it exist an affine regular variety $V_a$ isomorphic (as differentiable manifold) to $V$?

II) if there exists $V_a$, is it possible to realize $V_a$ as a submanifold of $\mathbb{R}^n$ close to $V$?

Nash has proved that there exists an affine variety $V'$ such that $V'$ has an analytic component $V_a$ that solves problems I) and II). In the terminology we have introduced we can say that Nash has solved problems I) and II) with a regular compact algebraic set $V_a$. Using theorem 2 we can prove that the problem I) has an affirmative resolution and problem II) can be solved if $n > 2 \dim V$ (\(*\)). We now shall give some definitions to explain problem II). Let $L, L'$ be two linear $r$-dimensional subspaces of $\mathbb{R}^n$ and $x_1, \ldots, x_r, y_1, \ldots, y_{n-r}$ a system of orthogonal coordinates of $\mathbb{R}^n$ such that: $L = \{(x_1, \ldots, x_r, y_1, \ldots, y_{n-r}) | y_1 = \ldots = y_{n-r} = 0\}$. We shall say that $L'$ is an $\varepsilon$-approximation of $L$ if $L'$ has equations of the form

$$
y_i = \sum_{j=1}^{r} a_{ij} x_j + c_i, \quad i = 1, \ldots, n-r
$$

with the condition $\sum_{i,j} |a_{ij}|^2 + \sum c_i^2 < \varepsilon$

Let $V$ be a compact differentiable manifold of dimension $r$ differentiably embedded in $\mathbb{R}^n$. At each point $x \in V$ take the disc $D_x$ of radius $\delta$ contained in the $n-r$ dimensional linear space orthogonal to $V$.

If $\delta$ is small enough it is known that the union of all these discs has the structure of a fibre bundle over $V$.

This bundle is called the normal bundle of radius $\delta$ (\(**\)) and it is denoted by $B(\delta)$.

The set $B(\delta)$ is an open neighbourhood of $V$ in $\mathbb{R}^n$ and the projection $p : B(\delta) \to V$ defined by: $p(y) = x$ if $y \in D_x$ is a differentiable map.

Let $V'$ be a differentiable manifold of $\mathbb{R}^n$, we shall say that $V'$ is an $\varepsilon$-approximation of $V$ if:

1°/ $V'$ is contained in the tubular neighbourhood $B(\varepsilon)$ of $V$

2°/ $p : V' \to V$ is an isomorphism of the differentiable structures

3°/ for any $x \in V'$ the tangent linear variety to $V'$ at $x$ is an $\varepsilon$-approximation of the tangent linear variety to $V$ at $p(x)$.

\(\text{\textbf{(*)}}\) The author conjectures that problem II) can be solved without any restriction on the codimension of $V$.

\(\text{\textbf{(**)}}\) $B(\delta)$ is also called the tubular neighbourhood of radius $\delta$.\n
Let $V$ be a differentiable submanifold of $\mathbb{R}^n$ we shall say that $V$ has, (in $\mathbb{R}^n$) an algebraic $\varepsilon$-approximation if there exists an affine regular subvariety $V'$ of $\mathbb{R}^n$ that is an $\varepsilon$-approximation of $V$.

We shall say that $V$ admits algebraic approximation if, for any $\varepsilon > 0$, $V$ has an algebraic $\varepsilon$-approximation.

A formulation of problem II is the following:

Any compact differentiable submanifold of $\mathbb{R}^n$ admits algebraic approximation?

It is possible to prove the following

THEOREM 6. - Let $V$ be a compact differentiable submanifold of $\mathbb{R}^n$, $n > 2\dim V$, then $V$ admits algebraic approximation.

COROLLARY. - Any compact differentiable manifold is isomorphic to a regular affine variety.

Theorem 6 is proved in [4] we shall give here an idea of the proof. We need the following

LEMMA. - Any compact differentiable manifold is in the same cobordism class of a compact, regular affine variety.

Proof: Let $P_n(\mathbb{R})$ be the $n$-projective space on the real numbers. We denote by $z_0, \ldots, z_n, w_0, \ldots, w_m$, $m < n$ two systems of coordinates of $P_n(\mathbb{R})$ and $P_m(\mathbb{R})$.

We put:

$$H_{n,m}(\mathbb{R}) = \{ [z_j^*] [z_j^*] \in P_n(\mathbb{R}) \times P_m(\mathbb{R}) | w_0^* z_0 + w_1^* z_1 + \ldots + w_m^* z_m = 0 \}$$

It is known (see [16]) that the manifolds $P_n(\mathbb{R})$, $H_{n,m}(\mathbb{R})$ are generators of cobordism ring.

Then to prove the lemma it is enough to show that $P_n(\mathbb{R})$ has a structure of regular affine variety.

Let us consider the map $\chi_{ik} : P_n(\mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\chi_{ik}(x_j) = \frac{1}{\sum_{j=0}^{n} x_1^2}$$

It is easy to verify that the map $X : P_n(\mathbb{R}) \rightarrow \mathbb{R}^{(n+1)^2}$ defined by

$$X(x) = \{ \chi_{ik}(x) \}_{i,k=0,\ldots,n}$$

is injective, of maximum rank at any point and the set $X(P_n(\mathbb{R}))$ is the regular affine subvariety $W$ of $\mathbb{R}^{(n+1)}$ defined by the equations:
\[ \sum_{i=0}^{n} \chi_{ii} = 1 \]
\[ \chi_{ik} \chi_{lr} = \chi_{il} \chi_{kr} \]
\[ \chi_{ik} = \chi_{ki} \quad i, k, l, r = 0, \ldots, n. \]

So we have proved that \( P_n(\mathbb{R}) \) is isomorphic to \( W \); it is now easy to verify that \( W \) is a regular affine subvariety of \( \mathbb{R}^{(n+1)^2} \).

Let \( V_1, V_2 \) be two differentiable manifolds and suppose that \( V_1 \) is in the same cobordism class of \( V_2 \).

By Whitney's embedding theorem (see [17]), we may suppose that there exists a differential submanifold, with boundary \( W \) of \( \mathbb{R}^{n+1} \) such that, if \( x_1, \ldots, x_{n+1} \) are coordinates in \( M \), we have:

1°/ \( W \subset \{ ||x_1|| x_{n+1} > 0 \} \), the boundary \( \partial W = V_1 \cup V_2 \) of \( W \) is equal to \( W \cap \{ ||x_1|| x_{n+1} = 0 \} \).

2°/ the set \( \hat{W} = W \cup \{ (x_1, \ldots, x_{n+1}) | (x_1, \ldots, -x_{n+1}) \in W \} \) is a differentiable submanifold of \( \mathbb{R}^{n+1} \).

3°/ the hyperplane \( x_{n+1} = 0 \) cuts transversally \( \hat{W} \).

Furthermore if \( V_1 \) is an affine regular variety we may suppose that \( W \) is the disjoint union of a regular affine subvariety \( V'_1 \) of \( \mathbb{R}^{n+1} \), isomorphic to \( V_1 \), and of a differentiable submanifold \( V'_2 \) isomorphic to \( V_2 \).

The manifold \( W \) shall be said the torus constructed on \( V_1 \) and \( V_2 \).

The idea of the proof of theorem 6 is the following: let \( V_2 \) be a compact differentiable manifold and \( V_1 \) a regular compact affine variety in the same cobordism class. Let \( \hat{W} \) be the torus constructed on \( V_1 \) and \( V_2 \). Then we approach \( \hat{W} \) by an affine regular variety \( \hat{W}' \) in such a way that the intersection of \( \hat{W}' \) with the hyperplane \( x_{n+1} = 0 \) is composed by two analytic compact manifolds \( V'_1, V'_2 \) that are \( \varepsilon \)-approximation of \( V_1, V_2 \) for some \( \varepsilon \).

But if in the approximation process we use theorem 2 instead of the classical Weierstrass theorem we can obtain \( V_1 = V'_1 \). So we have that \( V'_1 \cup V'_2 \) is a regular affine subvariety of \( \mathbb{R}^n \), \( V'_1 = V_1 \) is an affine regular subvariety and we can conclude that \( V'_2 \) is affine and an \( \varepsilon \)-approximation of \( V_2 \).
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