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THE MILNORRING OF A LOCAL RING

par

E. A. M. HORNIX

Let F be a field. Milnor defined a ring $k_*(F)$, and in the case that characteristic $(F) \neq 2$ he studied maps between $k_*(F)$, and groups or rings which play a role in the theory of quadratic forms. The aim of this talk is to extend some of his definitions and results to local rings. We do not suppose thaht 2 is a unit of the local ring. The only restriction for the local rings is, that the residue field has more than 3 elements.
Sections 1,2,3 give a survey of [3], though the definitions of [3] are a bit generalized. In section 4, the analogue of Milnor's map s_* is given, and section 5 covers the example of a field of characteristic 2.

1. We repeat some of the definitions given by Milnor [6]. Let F be a field, denote $U(F) = \{x \in F \mid x$ is invertible$\}$. Let M be the $\mathbb{Z}$-module $U(F)$, and denote $T(M)$ for the tensoralgebra of M. We write $\ell : M \to T(M)$ for the imbedding of M in $T(M)$. $K_*(F)$ is defined as $T(M)$ mod I, and I is the two-sided ideal of M, generated by $\{\ell(a)\,\ell(1-a) \mid a, 1-a \in U(F)\}$. Remark thaht $\langle -a,1\rangle \otimes \langle -(1-a),1\rangle \cong 2H$, as soon as $a, 1-a \in U(F)$ and $2 \neq 0 \in F$. $K_*(F) = Z \oplus K_1(F) \oplus K_2(F) \oplus \dots$, and here $K_n(F) = \ell(M) \otimes \dots \otimes \ell(M) \bmod \ell(M) \otimes \dots \otimes \ell(M) \cap I$.
The elements of $K_n(F)$ are again denoted as sums of terms $\ell(a_1) \dots \ell(a_n)$. Finally, $k_*(F)$ is defined as $Z \oplus K_1(F) / 2K_1(F) \oplus K_2(F) / 2K_2(F) \oplus \dots$ (1). We remark that for $a \in U(A)$ and $x \in k_*(F)$, the element $\overline{\ell}(a^2)x = 2\overline{\ell}(a)x = 0 \in k_*(F)$. In fact, the defining relations for $k_*(F)$ are :

$$\overline{\ell}(ab) = \overline{\ell}(a) + \overline{\ell}(b) \qquad a \in U(A),\ b \in U(A)$$
$$\overline{\ell}(a)\,\overline{\ell}(1-a) = 0 \qquad a, 1-a \in U(A)$$
$$2\overline{\ell}(a) = 0 \qquad a \in U(A)$$

Suppose now that $\mathrm{char}(F) \neq 2$. We write Quad(F) for the Grothendieck monoïde of finite-dimensional quadratic spaces over F. Milnor proved, that there exists a well-defined map

$$SW : \mathrm{Quad}(F) \to k_*(F) \text{ such that}$$
$$SW\,\langle a_1,\dots,a_n\rangle = (1 + \overline{\ell}(a_1)) \dots (1 + \overline{\ell}(a_n)) .$$

(1) Write $\overline{\ell}(a)$ for the class of $\ell(a)$ in $K_1(F)$, etc.

We denote the Grothendieck-Writting of finite-dimensional quadratic spaces over F by $W(F)$, and we write $I(F) \subset W(F)$ for the kernel of the dimension map $W(F) \to \mathbb{Z}/2\mathbb{Z}$. Milnor proved also, that there exists a homomorphism

$$s_* : k_*(F) \to \bigoplus_{n \geqslant 1} I^n(F) / I^{n+1}(F) \quad \text{such that}$$

$$s_n : K_n(F) / 2K_n(F) \to I^n(F) / I^{n+1}(F) \quad \text{and}$$

$$s_n : \overline{\ell}(a_1) \ldots \overline{\ell}(a_n) = (\langle a_1\rangle - 1) \ldots (\langle a_n\rangle - 1) + I^{n+1}(F) \quad .$$

2. Let A be a local ring with maximal ideal $\underline{m}$. Denote $U(A) = \{a \in A \mid a$ has inverse in $A\}$. If $2 \in \underline{m}$, then every nondegenerate quadratic form on A of finite dimension has even dimension.

We denote (a,b,c) for the form q which has a basis e,f satisfying $q(e) = a$, $q(f) = b$, $(e,f) = c$. The form (a,b,c) is nondegenerate if and only if $4ab - c^2 \in U(A)$. If $|A \bmod \underline{m}| > 3$ then we may choose $a,b,1$ such that $a,b \in U(A)$. In that case $(a,b,1) \cong a(1,ab,1)$ and ab determines an invariant of $(a,b,1)$ which we will describe now.

The following notions can be found in the notes of the 1968 Montpellier conference, Micali, Villamayor [4].

Let A be an arbitrary ring, define $A^o = \{a \in A \mid 1-4a \in U(A)$. A^o is a group under $o : A^o \times A^o \to A^o$, $a \circ b = a+b-4ab$.

The inverse of a in A^o is the element $\frac{-a}{1-4a}$. Define $J(A) = \{x - x^2 \mid 1-2x \in U(A)\}$.

If $a \in A^o$, then $a \circ a \in J(A)$. $J(A)$ is a subgroup of A^o, we denote $G(A) = A^o \bmod J(A)$. There exist homomorphisme $\sigma : A^o \to U(A)$, $\sigma(a) = 1-4a$, $\overline{\sigma} : G(A) \to U(A) \bmod U(A)^2$, $\overline{\sigma}(a \circ J) = (1-4a)\, U(A)^2$.

Examples. (1) If $2 \in U(A)$ then $\overline{\sigma}$ is an isomorphism..

(2) If $2 = 0$ then $A = A^o$ and $a \circ b = a+b$, $G(A) = A^+ \bmod \wp(A)$.

Let A be a local ring. The quadratic form $a(1,d,1)$ is nondegenerate if and only if $a \in U(A)$, $d \in A^o$. The class $d \circ J(A)$ is an invariant for the isometry class of $a(1,d,1)$, for the proof see [3].

In general, we have the following result : Suppose that q is a nondegenerate quadratic form of dimension $2n$. Then

$$q \cong \bigoplus_{i=1}^{n} a_i(1,d_i,1)$$

$a_i \in U(A)$, $d_i \in A^o$, $1 \leqslant i \leqslant n$ and $d_1 \circ d_2 \circ \ldots \circ d_n \circ J(A)$ is an invariant for the isometry class of q.

Examples. (1) F is a field of characteristic $\neq 2$. $q \cong \bigoplus_{i=1}^{n} a_i(1,d_i,1)$, then $\overline{\sigma}(d_1 \circ \ldots \circ d_n \circ J(A))$ is the discriminant of q.

(2) F is a field of characteristic 2, q as before. Then $d_1 \circ \ldots \circ d_n \circ J(A)$ is the Arf invariant of q.

3. Suppose again that A is a local ring. $|A \bmod \underline{m}| > 3$. It is clear, that for the determination of the isometry class of $a(1,d,1)$ a role is played by $d_{\,0}\, J(A)$ and by $a \in U(A)$. So in the definition of the Milnorring of A, $G(A)$ and $U(A)$ should play a role. In the case of a fields of char $\neq 2$ it seemed important to remark that

$$\langle -a,1\rangle \otimes \langle -(1-a),1\rangle \cong 2\,\mathbb{H} \quad , \qquad a\ ,\ 1-a \in U(A).$$

We translate that remark :

if $a \in U(A) \cap A^{o}$ then $\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} \otimes (1,a,1) \cong 2\,\mathbb{H}$. Here $\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}$ denotes a symmetric bilinear form, and we use the tensorproduct which is defined for symmetric bilinear forms and quadratic forms by H. Bass [1].

We now give a construction of the ring $g_*(A)$, which is almost equivalent to the construction of $k_*(A)$.

We start with the $\mathbb{Z}$-module $M = A^{o} \oplus U(A)$, and we denote $\omega(a) = (a,0)$ for $a \in A^{o}$ and $\gamma(a) = (0,a)$ for $a \in U(A)$. $T(M)$ is again the tensor algebra of M. $\mathcal{J}$ is the two-sided ideal of $T(M)$ generated by

$$\{\omega(a)\ \gamma(a) \mid a \in A^{o} \cap U(A)\} \cup \{\gamma(a)\ \omega(a) \mid a \in A^{o} \cap U(A)\} \cup$$
$$\cup\ \{\omega(a) \mid a \in J(A)\}\ .$$

$g_*(A) = T(M) \bmod \mathcal{J}$, $g_*(A)$ is isomorphic with $Z \oplus g_1(A) \oplus g_2(A) \oplus \ldots$,

$g_i(A) = M \otimes \ldots \otimes M / M \otimes \ldots \otimes M \cap \mathcal{J}$.

We denote $\overline{g}(a)$ for the image of $\gamma(a)$ $(a \in U(A))$ in $g_*(A)$. We write $\overline{O}(A)$ for the image of $\omega(a)$ $(a \in A^{o})$ in $g_*(A)$. In fact, $g_*(A)$ satisfies the following defining relations :

$$\overline{g}(ab) = \overline{g}(a) + \overline{g}(b) \quad , \qquad a \in U(A),\ b \in U(A)$$

$$\overline{O}(a_{\,0}\, b) = \overline{O}(a) + \overline{O}(b) \quad , \qquad a,b \in A^{o}$$

$$\overline{g}(a)\ \overline{O}(a) = \overline{O}(a)\ \overline{g}(a) = 0 \quad , \qquad a \in A^{o} \cap U(A)$$

$$\overline{O}(a) = 0 \quad , \qquad a \in J(A) \quad .$$

We would like to define a map

$$SW : Quad(A) \to g_*(A).$$

The analogue of Milnor's definition is for <u>even dimensional</u> forms :

(DEF) : $SW(a_1(1,d_1,1) \oplus a_2(1,d_2,1) \oplus \ldots \oplus a_n(1,d_n,1)) =$

$$= (1 + \overline{g}(-1) + \overline{O}(d_1) + \overline{g}(a_1)\ \overline{O}(d_1)) \ldots (1 + \overline{g}(-1) + \overline{O}(d_n) + \overline{g}(a_n)\ \overline{O}(d_n))\ .$$

This definition works for $n = 1$: if $a(1,d,1) \cong a_1(1,d_1,1)$ then $d_{\,0}\, J(A) = d_{1\,0}\, J(A)$, so $\overline{O}(d) = \overline{O}(d_1)$, and it can easily be proved that $\overline{g}(a)\ \overline{O}(d) = \overline{g}(a_1)\ \overline{O}(d_1)$.

For the proof that the definition works for $n = 2$, we have to impose some extra conditions. Some of these come from the commutativity of Quad(A). The more important conditions are :

W_2 : $\bar{g}(1-4a)\,\bar{O}(b) - \bar{O}(a)\,\bar{O}(b)$ should be equal to 0, as soon as $a \in A^o$, $b \in U(A) \cap A^o$.

W_7 : $\bar{g}(a)\,\bar{g}(a)\,\bar{O}(b)\,\bar{O}(d) - \bar{g}(a)\,\bar{O}(b)\,\bar{O}(d)\,\bar{O}(d)$ should be equal to 0 for $a \in U(A)$, $b \in A^o \cap U(A)$, $d \in A^o$.

So we consider the ring $g_*(A)$ mod $Cg_*(A)$, $Cg_*(A)$ being the ideal in $g_*(A)$ generated by the elements mentionel in W_2, W_7 and by some more elements. For an explicit and precise definition see [3].

Let us denote $\bar{g}(a)$ for $\bar{g}(a) + Cg_*(A)$, $\bar{O}(a)$ for $\bar{O}(a) + Cg_*(A)$.

Suppose that $2 \in \underline{m}$. Then one can prove that the map SW : Quad(A) $\to g_*(A)$ mod $Cg_*(A)$ as proposed in (DEF), is well-defined.

Suppose $2 \notin \underline{m}$. If A is a field, then $g_*(A)$ and $k_*(A)$ are not isomorphic. We should have identified A^o and $U(A)$. More precisely, choose $M = U(A) \oplus A^o \bmod \{\gamma(1-4a) - \omega(a) \mid a \in A^o\}$ and repeat the definition of $T(M)$ mod $\mathcal{J}$, hence the defining relations for $T(M)$ mod $\mathcal{J}$ are

$$\bar{g}(1-4a) = \bar{O}(a) \quad , \qquad a \in A^o$$

$$\bar{g}(ab) = \bar{g}(a) + \bar{g}(b) \quad , \qquad a,b \in U(A)$$

$$\bar{g}(a)\,\bar{O}(a) = \bar{O}(a)\,\bar{g}(a) \quad , \qquad a \in A^o \cap U(A)$$

$$\bar{g}(a) = 0 \quad , \qquad a \in U(A)^2$$

In fact, this was the definition, proposed in [3] for any local ring A with 2 unit in A.

It is then easily proved that $Cg_*(A) = 0$, and that SW is defined on all of Quad(A), such that

(*) : $SW\,\langle a_1,\ldots,a_n\rangle = (1 + \bar{g}(a_1))\ \ldots\ (1 + \bar{g}(a_n))$.

For isometry classes of even dimension, the definitions (*) and (DEF) coïncide.

There are situations in which we have that $2 \in U(A)$ and that we want to restrict ourselves to isometry classes of even-dimensional forms. It is possible to define $g_*(A)$ based on $M = U(A) \oplus A^o$. The map SW can be defined as proposed in (DEF). For proving this, the proofs in [3] can completely be repeated. The map SW as proposed in (*) cannot be defined, since $2\,\bar{g}(a)$ $(a \in U(A))$ is not necessarily equal to 0.

4. We give now the analogue for the map s_*. For convenience, <u>we work with rings</u> $g_*(A)$, <u>based on</u> $M = U(A) \oplus A^o$.

$W_q(A)$ is the Witt-group of free finite-dimensional nondegenerate quadratic forms on A. $W(A)$ is the Wittring of free finite-dimensional nondegenerate symmetric bilinear forms on A, $I(A) \subset W(A)$ is the ideal of forms of even dimension. We denote the class of a form in $W_q(A)$, $W(A)$ by square brackets. $W_q^o(A) \subset W_q(A)$ is the Witt-group of forms of even dimension.

It is well known that $W_q(A)$ can be considered as an $W(A)$-module. According to definitions given by Micali + Villamayor [5], we give $W_q(A)$ a structure of ring by defining :

$$q_1 \cdot q_2 = (\, , \,)_{q_1} \otimes q_2 \quad .$$

This definition induces a structure of ring on $\bigoplus_{n \geqslant 0} I^n(A)\, W_q^o(A) \bmod I^{n+1}(A)\, W_q^o(A)$.

In analogy with Milnor's definition, we would like to define a homomorphism of rings $s_* : g_*(A) \bmod Cg_*(A) \to \bigoplus_{i \geqslant 0} I^n(A)\, W_q^o(A) \bmod I^{n+1}(A)\, W_q^o(A) \oplus \bigoplus_{n \geqslant 1} I^n(A) \bmod I^{n+1}(A)$

For $a \in A^o$, we propose to define $s_1\, \overline{o}(a) = [-1,-a,1] + I(A)\, W_q^o(A)$.

If $a \in U(A)$ we would like to define

$$s_1\, \overline{g}(a) = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} + I^2(A) \; .$$

The map s_1 can be extended to a homomorphism of rings, if the image of s_1 satisfies the defining relations of $g_*(A) \bmod Cg_*(A)$. It is clear that the following results hold :

4.1. $$\begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & b \end{bmatrix} \in \begin{bmatrix} -1 & 0 \\ 0 & ab \end{bmatrix} + I^2(A) \quad , \qquad a,b \in U(A)$$

4.2. $$[-1,-a,1] + [-1,-b,1] \in [-1,-a\circ b,1] + I(A)W_q^o(A) \quad , \qquad a,b \in A^o$$

4.3. $$\begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} \cdot [-1,-a,1] = 0 \quad , \qquad a \in U(A) \cap A^o \; .$$

For proving the other relations, we derive some formulas.

4.4. <u>Suppose</u> $2 \in \underline{m}$. Let $1-pq \in U(A)$, $d \in U(A) \cap A^o$. Then

$$\begin{bmatrix} p & 1 \\ 1 & q \end{bmatrix} \cdot [-1,-d,1] = \begin{bmatrix} d(pq-1) & 0 \\ 0 & 1 \end{bmatrix} \cdot \left[\frac{pd}{1-4d}, \frac{q}{1-pq}, 1\right] \; .$$

<u>Proof.</u> Let e,f be a basis of V, let $(\, , \,)$ be a symmetric bilinear form on V such that $(e,e) = p$, $(f,f) = q$, $(e,f) = 1$. Let x,y be a basis of W, $q : W \to A$ a quadratic form and $q(x) = -1$, $q(y) = -d$, $(x,y) = 1$.

The bilinear form and the quadratic form are nondegenerate. Choose $X = e \otimes (2dx+y)$, $Y = (-qe+f) \otimes x$, $S = f \otimes y$, $T = (-e+pf) \otimes (x+2y)$. Since $2 \in \underline{m}$, we have that X, Y, S, T is a basis of $V \otimes W$. Moreover, $\langle X\rangle + \langle Y\rangle \perp \langle S\rangle + \langle T\rangle$. It is clear that $\langle X\rangle + \langle Y\rangle \cong (\frac{pd}{1-4d}, \frac{q}{1-pq}, 1)$ and that

$\langle S\rangle + \langle T\rangle \cong (-qd, \frac{-p}{(1-4d)(1-pq)}, 1)$.

4.5. Lemma. $I(A)\ W_q^o(A)$ is generated as an additive group by elements of the form

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} [1,d,1] \quad , a \in A^o,\ d \in A^o \cap U(A).$$

4.6. Let $a \in A^o$, $d \in U(A) \cap A^o$. Then we have that

$$\begin{bmatrix} 1-4a & 0 \\ 0 & -1 \end{bmatrix} . [-1,-d,1] - \begin{bmatrix} -2 & 1 \\ 1 & -2a \end{bmatrix} . [-1,-d,1] \in I^2(A)\ W_q^o(A) .$$

Proof. If $2 \notin \underline{m}$ then this statement is easily proved. So suppose $2 \in \underline{m}$. Applying (4.4.) we find that

$$\begin{bmatrix} -2 & 1 \\ 1 & -2a \end{bmatrix} . [-1,-d,1] = \begin{bmatrix} d(4a-1) & 0 \\ 0 & 1 \end{bmatrix} . \left[\frac{-2d}{1-4d}, \frac{-2a}{1-4a}, 1\right] =$$

$$= [\rho] . \begin{bmatrix} d(4a-1) & 0 \\ 0 & 1 \end{bmatrix} . \left[-1, \frac{-4ad}{(1-4d)(1-4a)}, 1\right] \text{ for certain } \rho \in U(A)$$

Now we consider the form $\begin{bmatrix} 1-4a & 0 \\ 0 & -1 \end{bmatrix} . [-1,-d,1]$.

Let e,f be a basis of V, and let $(\ , \ )$ be a symmetric bilinear form satisfying $(e,e) = 1-4a$, $(f,f) = -1$, $(e,f) = 0$.

Let x,y be a basis of W, and let q be a quadratic form such that $q(x) = -1$, $q(y) = -d$, $(x,y) = 1$. Denote $A = e \otimes y$, $B = (e+f) \otimes (x+2y)$, $C = f \otimes x$, $D = (e + (1-4a)f) \otimes (2dx+y)$.

A, B, C, D is a basis for $V \otimes W$ and $\langle A\rangle + \langle B\rangle \perp \langle C\rangle + \langle D\rangle$.

$$\langle A\rangle + \langle B\rangle \cong (\frac{-d}{1-4a}, \frac{-4a}{1-4d}, 1) \cong -d(4a-1)(-1, \frac{-4ad}{(1-4a)(1-4d)}, 1)$$

$$\langle C\rangle + \langle D\rangle \cong (1, \frac{4ad}{(1-4a)(1-4d)}, 1) \cong -(-1, \frac{-4ad}{(1-4a)(1-4d)}, 1)$$

Hence $V \otimes W \cong (-1) \begin{pmatrix} d(4a-1) & 0 \\ 0 & 1 \end{pmatrix} . (-1, \frac{-4ad}{(1-4a)(1-4d)}, 1)$.

Now it is easily proved that

$$\begin{bmatrix} 1-4a & 0 \\ 0 & -1 \end{bmatrix} . [-1,-d,1] - \begin{bmatrix} -2 & 1 \\ 1 & -2a \end{bmatrix} . [-1,-d,1] =$$

$$= \begin{bmatrix} \rho & 0 \\ 0 & -1 \end{bmatrix} . \begin{bmatrix} d(4a-1) & 0 \\ 0 & 1 \end{bmatrix} . \left[-1, \frac{-4ad}{(1-4a)(1-4d)}, 1\right] .$$

4.7. $$\begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot [-1,-c,1] - \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot [-1,-c,1] \in I(A)^4\, W_q^o(A),\ a \in U(A),\ b \in U(A) \cap A^o,$$
$c \in A^o$.

Proof. Since $b \in U(A) \cap A^o$ we have that
$$\begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot [-1,-c,1] = \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot [-1,-b,1] \in \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1,-b,1] +$$
$$+ I^2(A)\, W_q^o(A) .$$

It is clear that $\begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Now we calculate : $$\begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot [-1,-c,1] -$$
$$- \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2b \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot [-1,-c,1] \in$$
$$\begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot [-1,-b,1] - \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1 & -2c \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1,-b,1] + I^4(A)\, W_q^o(A) = \text{(applying (4.4.))} =$$
$$= \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1,-b,1] -$$
$$- \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1-4c & 0 \\ 0 & -1 \end{bmatrix} \cdot [-1,-b,1] + I^4(A)\, W_q^o(A) =$$
$$= I^4(A)\, W_q^o(A).$$

The relations (4.1), (4.2), (4.3), (4.6), (4.7) are translations of relations, which have been mentioned explicity in the definition of $g_*(A) \bmod Cg_*(A)$. The other relations have to do with commutativity. Now, $\bigoplus_{n \geq 0} I^n(A)\, W_q^o(A) \bmod I^{n+1}(A)\, W_q^o(A) \oplus \bigoplus_{n \geq 1} I^n(A) \bmod I^{n+1}(A)$ is commutative with respect to multiplication. So we have verified that the defining relations for $g_*(A) \bmod Cg_*(A)$ also hold for the image of s_1. Hence the following theorem is proved :

4.8. Theorem. There exists a well-defined homomorphism of rings
$$s_* : g_*(A) \bmod Cg_*(A) \to \bigoplus_{n \geq 0} I^n(A)\, W_q^o(A) \bmod I^{n+1}(A)\, W_q^o(A) \oplus$$

$\bigoplus_{n\geqslant 1} I^n(A) \bmod I^{n+1}(A)$, such that

$$s_1\,\bar{g}(a) = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} + I^2(A) \quad , \qquad a \in U(A)$$

$$s_1\,\bar{O}(a) = [-1,-a,1] + I(A)\, W_q^o(A) \quad , \qquad a \in A^o$$

We denote s_n for the restriction of s_* to $g_n(A) \bmod Cg_*(A) \cap g_n(A)$.

We denote $\mathcal{O}(A) \subset g_*(A) \bmod Cg_*(A)$ for the two-sided ideal, generated by $\{\bar{O}(a) \mid a \in A^o\}$. Let us write $\mathcal{O}_n(A)$ for the intersection of $\mathcal{O}(A)$ with $g_n(A) \bmod Cg_*(A) \cap g_n(A)$.

Denote s_n for the restriction of s_* to $\mathcal{O}_n(A)$, and denote the restriction of s_* to $\mathcal{O}(A)$ by s_*.

4.9. Theorem. $s_* : \mathcal{O}(A) \to \bigoplus_{n\geqslant 0} I^n(A)\, W_q^o(A) \bmod I^{n+1}(A)\, W_q^o(A)$ is a surjective homomorphism of rings.

Proof. The elements of $\mathcal{O}(A)$ are of the form $\sum_{i=1}^{n} x_i\, \bar{O}(a_i) y_i$, with $a_i \in A^o$, $x_i, y_i \in g_*(A) \bmod Cg_*(A)$.

So $s_*\, \mathcal{O}(A) \subset \bigoplus_{n\geqslant 0} I^n(A)\, W_q^o(A) \bmod I^{n+1}(A)\, W_q^o(A)$.

Lemma (4.5) proves that s_* maps $\mathcal{O}(A)$ surjectively on

$$\bigoplus_{n\geqslant 0} I^n(A)\, W_q^o(A) \bmod I^{n+1}(A)\, W_q^o(A)$$

We will now prove, that s_1 is an injective map on $\mathcal{O}_1(A)$.

4.10. There exists a homomorphism of groups

$$\mathrm{discr} : W_q^o(A) \to G(A) \quad , \quad \text{satisfying}$$

$$\mathrm{discr}\,[a]\,[1,d,1] = d \circ J(A) \quad , \qquad a \in U(A),\ d \in A^o.$$

The following sequence is exact :

$$1 \to I(A)\, W_q^o(A) \to W_q^o(A) \xrightarrow{\mathrm{discr}} G(A) \to 1 \ .$$

Proof. The existence of the homomorphism discr follows from what is said in section 2. The map discr is surjective since $\mathrm{discr}\,[1,d,1] = d \circ J(A)$, $d \in A^o$. Lemma (4.5) shows that $I(A)\, W_q^o(A)$ is generated by elements of the form $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \cdot [1,d,1]$, $a \in U(A)$, $d \in A^o$. Hence $I(A)\, W_q^o(A) \subset \ker(\mathrm{discr})$. Suppose that $\bigoplus_{i=1}^{n} [a_i]\,[1,d_i,1] \in W_q^o(A)$ and that $d_1 \circ \cdots \circ d_n \circ J(A) = J(A)$. Then we have that $\bigoplus_{i=1}^{n} [a_i]\,[1,d_i,1] = \bigoplus_{i=1}^{n-1} [a_i]\,[1,d_i,1] \oplus [a_n]\,[1,d_1 \circ \cdots \circ d_{n-1}]$. Applying (4.2) we find that $\bigoplus_{i=1}^{n} [a_i][1,d_i,1] \in \bigoplus_{i=1}^{n-1} [a_i][1,d_i,1] \oplus \bigoplus_{i=1}^{n-1} [a_n][1,d_i,1] + I(A)\, W_q^o(A) =$

$$= I(A) W_q^o(A) \ .$$

Remark. Compare Knebusch [2], (7.10).

4.11. Theorem. $s_1 : \mathcal{O}_1(A) \to W_q^o(A) \bmod I(A)W_q^o(A)$ is an isomorphism of additive groups.

Remark. We cannot repeat Milnor's proof for the injectivity of s_2, since we do not work with 1-dimensional quadratic forms.

5. Example. F is a field of characteristic 2, $F \neq \mathbb{F}_2$.
We have $U(F) = \{a \in F \mid a \neq 0\}$, $F^o = F$.

The most important defining relations for $g_*(F) \bmod Cg_*(F)$ are

$$\bar{g}(ab) = \bar{g}(a) + \bar{g}(b) \quad , \qquad a,b \neq 0$$

$$\bar{O}(a+b) = \bar{O}(a) + \bar{O}(b) \quad ,$$

$$\bar{g}(a)\,\bar{O}(a) = 0 \quad , \qquad a \neq 0$$

$$\bar{O}(a)\,\bar{O}(b) = 0 \quad .$$

The elements of $g_n(F) \bmod Cg_*(F) \cap g_n(F)$ can be written as sums of elements of the type

$$\bar{g}(a_1)\ \ldots\ \bar{g}(a_n) \quad , \quad \bar{g}(a_1)\ \ldots\ \bar{g}(a_{n-1})\ \bar{O}(b) \quad .$$

The elements of $\mathcal{O}_n(F) \bmod Cg_*(F) \cap \mathcal{O}_n(F)$ are sums of terms $\bar{g}(a_1)\ldots\bar{g}(a_{n-1})\bar{O}(b)$.

Let $\bigoplus_{i=1}^{n} a_i(1,d_i,1)$ be a quadratic form.

$$SW(\bigoplus_{i=1}^{n} a_i(1,d_i,1)) = 1 + \bar{O}(d_1 \circ \ldots \circ d_n) + \sum_{i=1}^{n} \bar{g}(a_i)\,\bar{O}(d_i) \quad .$$

Hence, $SW(\mathbb{H}) = 0$, and we can extend SW to a map

$$SW : W_q(F) \to \mathcal{O}(F) \bmod Cg_*(F) \cap \mathcal{O}(F) \quad .$$

We calculate the action of SW on $I^n(F)\,W_q(F)$.

$$SW\,[1,d,1] = 1 + \bar{O}(d)$$

$$SW \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} [1,d,1] = 1 + \bar{g}(a)\,\bar{O}(d) \quad .$$

$$SW \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} [1,d,1] = 0$$

Hence, SW acts trivially on $I^2(F)W_q(F)$.

We calculate $s_* : \mathcal{O}(F) \to \bigoplus_{n \geq 0} I^n(F)W_q(F) \bmod I^{n+1}(F)W_q(F)$.

$$s_1\bar{O}(a) = [1,a,1] + I(F)W_q(F) \quad .$$

$$s_2(\bigoplus_{i=1}^{n} \bar{g}(a_i)\,\bar{O}(d_i)) = \bigoplus_{i=1}^{n} \begin{bmatrix} a_i & 0 \\ 0 & 1 \end{bmatrix} [1,d_i,1] + I^2(F)W_q(F).$$

It is easy to see, that :

$$SW \circ s_2(x) = 1 + x \quad , \quad x \in \mathcal{O}_2(F)/Cg_*(F) \cap \mathcal{O}_2(F).$$

This proves that s_2 is a monomorphism.

There are no results about the injectivity of s_i , $i \geqslant 3$.

$$s_2 : \mathcal{O}_2(F) \bmod \mathcal{O}_2(F) \cap Cg_*(F) \to I(F)W_q(F) \bmod I^2(F)W_q(F)$$

is an isomorphism of additive groups.

We refer to another description of $I(F)W_q(F) \bmod I^2(F)W_q(F)$ by C.H. Sah, [7].

Let $Cl[M,q]$ denote the class of the Clifford algebra of (M,q) in the ungraded Brauergroup of F. $Cl[M,q]$ is an element of ${}_2Br(F)$, the subgroup generated by the elements of order 2 of $Br(F)$. Cl induces a split exact sequence :

$$0 \to I^2(F)W_q(F) \to I(F)W_q(F) \xrightarrow{Cl} {}_2Br(F) \to 0$$

Hence, Cl induces an isomorphism

$$\overline{Cl} : I(F)W_q(F) \bmod I^2(F)W_q(F) \to {}_2Br(F) \quad .$$

In proving this theorem, C.H. Sah uses the following result :

Denote $(a,d]$ for the F-algebra H with F-basis 1, u, v, uv and with relations $u^2 = a \neq 0$, $v^2 + v = d$, $uv + vu = 1$.

H is a quaternion algebra with norm form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. (1,d,1). The class of the Clifford algebra of $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. (1,d,1) is equal to the class $[H]$ of H in the Brauer group.

Combining these results, we find that

$$\overline{Cl} \circ s_2 : \mathcal{O}_2(F) \bmod \mathcal{O}_2(F) \cap Cg_*(F) \to {}_2Br(F)$$

is an isomorphism of groups.

$\overline{Cl} \circ s_2(\bigoplus_{I=1}^{n} \overline{g}(a_i) \overline{O}(d_i)) = \bigotimes_{i=1}^{n} [(a_i,d_i]]$, since tensor product induces multiplication in ${}_2Br(F)$.

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