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Some functors related to polynomial theory. II

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SOME FUNCTORS RELATED TO POLYNOMIAL THEORY, II

by

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1. Introduction. We consider the following natural transformation:

$$T^m : \mathcal{P}_R^m(X,Y) \to \text{Map}(X,Y), \quad T^m(f) = f^R$$

where $R$ denotes a commutative ring with 1, $X,Y$ – $R$-modules, and $\mathcal{P}_R^m(X,Y)$ is the $R$-module of all forms of degree $m$ on the pair $(X,Y)$ (in the sense of N.Roby [2]).

An element of $\mathcal{P}_R^m(X,Y)$ is a system $f=(f_A)$ indexed by all commutative $R$-algebras $A$, where $f_A : X \otimes A \to Y \otimes A$ are mappings satisfying the following conditions:

(i) $(1 \otimes u)f_A = f_B (1 \otimes u)$ for any $R$-algebra homomorphism $u : A \to B$,

(ii) $f_A(xa) = f_A(x)a^m$ for any $R$-algebra $A$, any $x \in X \otimes A$ and $a \in A$.

It is proved in [1] that in the case $X = R^n$, $Y = R$ we obtain:

$$T^m : R[T_1, \ldots, T_n] \to \text{Map}(R^n, R), \quad T^m(f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n).$$

It is well-known that the above homomorphism is not always injective; this is the starting point and the motivation of the following considerations.

It is known from [2] that the functor $\mathcal{P}_R^m(x, -)$ is represented by the $m$-th divided power $\mathcal{P}_R^m(x)$ of the module $X$. Similarly, it is proved in [1] that $\mathcal{P}_R^m(x, -) = \text{Ker} \ T^m$ is represented by $\mathcal{P}_R^m(x)$ where:

$$\mathcal{P}_R^m(x) = \mathcal{P}_R^m(x) / R\{x(m); x \in x\}.$$  

The above module is generated by the classes of elements:

$$\mathcal{V}_{m_1, \ldots, m_k}(x_1, \ldots, x_k) = x_1^{m_1} \ldots x_k^{m_k}, \quad m_i \geq 0, \quad m_1 + \ldots + m_k = m, \quad x_1, \ldots, x_k \in x,$$

which are denoted by $\mathcal{V}_{m_1, \ldots, m_k}(x_1, \ldots, x_k)$.

It is easy to see that $\mathcal{P}_R^m$ is an endo-functor of the category $R$-Mod.

We recall the following results contained in [1]:

Lemma 1.1. $\mathcal{P}_R^m$ commutes with direct limits.

Lemma 1.2. $\mathcal{P}_R^m(x)$ is finitely generated if so is $x$.

Theorem 1.3. There exist the natural isomorphisms:
Theorem 1.4. For a finitely generated $R$-module $X$, the following conditions are equivalent:

(i) $\hat{F}_R^m(X)_P = 0$

(ii) $\hat{F}_R^m(X)_P = 0$ for any $P \in \text{Max}(R)$

(iii) For any $P \in \text{Max}(R)$: either $\dim_{R/P}(X/PX) \leq 1$ or $m \leq |R/P|$. In particular, $\hat{F}_R^m = 0$ iff $m \leq \inf \{|R/P| : P \in \text{Max}(R)\}$.

2. The structure of $\hat{F}_R^m(X)$. We shall give some structural informations on $\hat{F}_R^m(X)$ which generalize results contained in [1]. The first step is the following

Lemma 2.1. If $P \in \text{Spec}(R) - \text{Max}(R)$ then $\hat{F}_R^m(X) = 0$ for any $R$-module $X$.
Moreover, if $X$ is finitely generated then $\text{Ann}(\hat{F}_R^m(X)) \neq P$.

Proof: Observe that $R/P$ is an infinite domain (it is not a field!) and hence $d(R/P) = \infty$. It follows from Theorem 1.3 and 1.4 that $\hat{F}_R^m(X)_P = \hat{F}_R^m(X)_P = 0$. Then the second part of the lemma follows from Lemma 1.2.

Corollary 2.2. If $\dim(R) > 0$ then:

(1) $\hat{F}_R^m(X)$ are torsion modules.

(2) $\hat{F}_R^m(X)$ is free iff it is zero.

If $\dim(R) > 0$ then:

(3) $\hat{F}_R^m(X)$ is projective iff it is zero.

Now we explain the structure of $\hat{F}_R^m(X)$ over Noetherian rings.

Theorem 2.3. Let $R$ be a Noetherian ring and let $X$ be a finitely generated $R$-module. Then there exists a natural $R$-isomorphism:

$$\hat{F}_R^m(X) \cong \bigoplus_{P \in \text{Max}(R)} \hat{F}_R^m(R/P)_P^k(P_{k}(X/PX)),$$

induced by $X \rightarrow X/P_{k}(X/PX)$, for all sufficiently large $k_P$.

Proof: We can assume that $\text{Ann}(\hat{F}_R^m(X)) \neq R$. Let $\text{Ann}(\hat{F}_R^m(X)) = \cap_{i=1}^s Q_i$ be a primary decomposition, and let $P_i = \text{rad}(Q_i)$. Observe that $P_i \subseteq Q_i$ for all sufficiently large $k_i$. Denote $I = P_1 \ldots P_s \subseteq \text{Ann}(\hat{F}_R^m(X))$. Since $P_1, \ldots, P_s \in \text{Max}(R)$ by Lemma 2.1, it follows that $R/I \cong R/P_{k_i}^i$ and hence:
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\[ \mathcal{P}_R^m(X) = \mathcal{P}_R^m(X) / \mathcal{P}_R^m(\text{X/IX}) = s \mathcal{P}_R^m(\text{X/F}_1^k \text{X}) \]

If \( P \in \text{Max}(R) - \{P_1, \ldots, P_s\} \) then \( I + P^k = R \) for each natural \( k \), and hence:

\[ \mathcal{P}_R^m(\text{X/F}_1^k \text{X}) = \mathcal{P}_R^m(\text{X}) / P^k \mathcal{P}_R^m(\text{X}) = 0. \]

This completes the proof.

**Corollary 2.4.** If \( R \) is a Noetherian ring then there exists a natural \( R \)-isomorphism

\[ \mathcal{P}_R^m(\text{X}) \cong \bigoplus_{P \in \text{Max}(R)} \mathcal{P}_P^m(\text{X}_P) \]

induced by \( X \to X_P \).

**Proof:** Compare the decompositions from Theorem 2.3 for \( X \) and \( X_P \) in the case if \( X \) is finitely generated. Next apply Lemma 1.1.

The same argument prove the following

**Corollary 2.5.** If \( R \) is a local Noetherian ring then there exists a natural \( R \)-isomorphism:

\[ \mathcal{P}_R^m(\text{X}) \cong \mathcal{P}_R(\text{X} \otimes \hat{R}) \]

induced by \( X \to X \otimes \hat{R} \).

Observe that the above two corollaries reduce the computation of \( \mathcal{P}_R^m(\text{X}) \) for Noetherian \( R \) to the case when \( R \) is local and complete. Theorem 2.3 reduces this problem (for finitely generated \( X \)) to the case when \( R \) is local Artinian. This case will be studied in the next section.

3. The Artinian case. Let \((R, \mathfrak{p})\) be an Artinian local ring. Then \( \mathfrak{p}^k = 0 \) for some natural \( k \). Observe that \( r^2 = 0 \) for any \( r \in \mathfrak{p}^{k-1} \) (if \( k > 1 \)). This is the motivation of the following.

**Proposition 3.1.** If \( r^2 = 0 \) in \( R \) and \( m \leq 5 \) then \( \mathcal{P}_R^m(\text{X}) = 0 \) for any \( R \)-module \( \text{X} \).

**Proof:** To start with, we give some general formulas. It follows from [1] that:

\[ \sum_{\substack{m_1 > 0 \ \text{m}\_1 \ldots, \text{m}\_n \ (x_1, \ldots, x_n) = 0 \ \text{for any} \ x_1, \ldots, x_n \in \text{X}.}} \]

Denote \( /\text{m}\_1, \ldots, \text{m}\_n/ = \sum_{\substack{m_1 > 0 \ \text{m}\_1 \ldots, \text{m}\_n \ (x_1, \ldots, x_n) \ \text{for} \ m_1 > 0, \ \sum \text{m}\_1 = \text{m}} \). We must prove that \( r \) annihilates all this generators. We have:

1. \( \Sigma /\text{m}\_1, \ldots, \text{m}\_n/ = 0. \)

Replacing \( x_1 \) by \( rx_1 \) and \((1+r)x_1\) we get:

2. \( r\Sigma /\text{m}\_1, \ldots, \text{m}\_n/ = 0 \)

(2') \( \Sigma (1+rm_1)/\text{m}\_1, \ldots, \text{m}\_n/ = 0 \)
since \( r^2 = 0 \) and \((1+r)^k = 1+kr\). In view of (1) and (2) we get from (2') :

\[
(3) \quad r \in \bigcap_{k=3}^{m-n+1} \frac{k}{k-2} \Gamma /k,m_1,m_2,\ldots, m_n/ = 0.
\]

In particular, it follows that :

(a) \( r/1,\ldots, 1/= 0 \) by (1) (n=m)
(b) \( r/2,1,\ldots, 1/= 0 \) by (1) and (2) (n=m-1)
(c) \( r/3,1,\ldots, 1/= 0 \) by (2) (n=m-2)
(d) \( r/1,m-1/= 0 \) by (2) (n=2).

For \( m \leq 2 \) there is nothing to prove. For \( m=3 \) we utilize (a),(b). For \( m=4 \) we get
\[
r/3,1/= r/1,3/= r/2,1,1/= r/1,2,1/= r/1,1,2/= r/1,1,1,1/= 0.
\]
Hence also
\[
r/2,2/= 0 \quad \text{by (1)}.
\]
For \( m=5 \) we have
\[
r/1,4/= r/3,1,1/= r/2,1,1,1/= r/1,1,1,1,1/= 0 \quad \text{and analogously for any permutation. Then (2) and (3) get us}
\[
r/1,2,2/= r/3,2/= 0.
\]
This completes the proof.

Remark 3.2. Using the same formulas (when we also replace \( x_1 \) by \(-x_1\)) we can prove the above proposition for \( m \leq 7 \) with the assumption that \( 2 \) is invertible in \( R \).

Corollary 3.3. Let \( R \) be a Noetherian ring and \( m \geq 5 \) (or \( m \geq 7 \) and \( 2 \) is invertible in \( R \)). Then there exists a natural \( R \)-isomorphism :

\[
\mathcal{C}^m_R(X) \cong \bigoplus_{P \in \text{Max}(R)} \mathcal{C}^m_{R/P}(X/PX)
\]

induced by \( X \to X/PX \).

Proof : It can be assumed that \( X \) is finitely generated. In view of Theorem 2.3, it suffices to prove that \( \mathcal{C}^m_R(X) \cong \mathcal{C}^m_{R/P}(X/PX) \) for any Artinian local \((R,P)\).

If \( P^k = 0, P^{k-1} \neq 0 \) and \( k > 1 \) (i.e. \( R \) is not a field) then :

\[
\mathcal{C}^m_R(X) = \mathcal{C}^m_R(X) / P^{k-1}\mathcal{C}^m_R(X) \cong \mathcal{C}^m_{R/P}(X/P^{k-1}X)
\]

by Proposition 3.1 and Remark 3.2. Induction on \( k \) completes the proof.

Remark 3.4. The assumptions of the above corollary are necessary. In fact, it can be computed that :

\[
\mathcal{C}^6_R(z_4^2) = Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_4, \quad \mathcal{C}^6_R(z_9^2) = Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_9.
\]

Remark 3.5. Since the dimensions of \( \mathcal{C}^m_R(X) \) over fields are known (see [1]),

Corollary 3.3 solves the problem of computation of \( \mathcal{C}^m_R(X) \) over Noetherian rings for small \( m \). For example, it can be proved that :

\[
\mathcal{C}^2_R(z^n) = \binom{n}{2} Z_2
\]
\[
\mathcal{C}^3_R(z^n) = 2^{\binom{n+1}{3}} Z_2 \oplus \binom{n}{2} Z_3
\]
\[ \mathcal{F}_n^5(z^n) = (3 \binom{n}{2} + 5 \binom{n}{3} + 3 \binom{n}{4})z_2 \otimes 2^{\binom{n+1}{3}}z_3 \]

where \( \binom{n}{k} = 0 \) for \( n < k \). However, the problem is open for large \( m \).

REFERENCES
