GORO SHIMURA

The periods of abelian varieties with complex multiplication and the spectral values of certain zeta functions

Mémoires de la S. M. F. 2e série, tome 2 (1980), p. 103-106
<http://www.numdam.org/item?id=MSMF_1980_2_2__103_0>

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
2e série, Mémoire n° 2, 1980, p. 103-106

THE PERIODS OF ABELIAN VARIETIES WITH COMPLEX MULTIPLICATION AND THE SPECIAL VALUES OF CERTAIN ZETA FUNCTIONS

by

Goro SHIMURA

Let K be a CM-field of degree 2n and I_K the free $\mathbb{Z}$-module generated by all embeddings of K into $\mathbb{C}$. Given a CM-type $\varphi = \sum_{i=1}^{n} \tau_i$ of K, take a $\overline{\mathbb{Q}}$-rational abelian variety of type (K,φ) and a $\overline{\mathbb{Q}}$-rational holomorphic 1-form ω_i on A such that $\omega_i \circ a = a^{\tau_i}\omega_i$ for all $a \in K$. As shown in [2, p.383], there is a non-zero complex number $p_K(\tau_i,\varphi)$ depending only on K,φ, and τ_i such that

$$[\pi . p_K(\tau_i,\varphi)]^{-1} \int_c \omega_i \in \overline{\mathbb{Q}}$$

for every $c \in H_1(A,\mathbb{Z})$. The quantity $p_K(\tau_i,\varphi)$ can actually be chosen to be a positive real number; it is also given as the value of a certain $\overline{\mathbb{Q}}$-rational (meromorphic) Hilbert modular form at a CM-point (see [2]). Now denote by ρ the complex conjugation, and put $p_K(\tau_i\rho,\varphi) = p_K(\tau_i,\varphi)^{-1}$. Then we have

Theorem 1 : *If* $\varphi_1, \ldots, \varphi_m$ *are CM-types of* K *and* τ *is an embedding of* K *into* $\mathbb{C}$, *the product* $\prod_{i=1}^{m} p_K(\tau, \varphi_i)^{s_i}$ *with* $s_i \in \mathbb{Z}$, *up to algebraic factors, depends only on* τ *and* $\sum_{i=1}^{m} s_i \varphi_i$. *Moreover, if* L *is a CM-field containing* K *and* ψ *is a CM-type of* L *whose restriction to* K *is* $\sum_{i=1}^{m} s_i \varphi_i$, *then the above product equals, up to algebraic factors, to* $\prod_{\sigma} p_L(\sigma, \psi)$, *where* σ *runs over all embeddings of* L *into* $\mathbb{C}$, *which coincide with* τ *on* K.

The proof is given in [3] . To express this theorem in a different way, we consider two linear maps

$$\mathrm{Res}_{L/K} : I_L \longrightarrow I_K \quad , \quad \mathrm{Inf}_{L/K} : I_K \longrightarrow I_L .$$

Here $\mathrm{Res}_{L/K}(\sigma)$ is the sum of all restrictions of σ to K; $\mathrm{Inf}_{L/K}(\tau)$ is the sum of all extensions of τ to L.

Theorem 2 : *The above* p_K *can be extended to a bilinear map of* $I_K \times I_K$ *into* $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ *with the following properties* :

1) $p_K(\alpha\rho, \beta) = p_K(\alpha, \beta\rho) = p_K(\alpha, \beta)^{-1}$ *for* $\alpha, \beta \in I_K$;

2) $p_K(\alpha, \mathrm{Res}_{L/K}\beta) = p_L(\mathrm{Inf}_{L/K}\alpha, \beta)$, $p_K(\mathrm{Res}_{L/K}\beta, \alpha) = p_L(\beta, \mathrm{Inf}_{L/K}\alpha)$ *for* $\alpha \in I_K$, $\beta \in I_L$, *and* $K \subset L$;

3) $p_M(\gamma\alpha, \gamma\beta) = p_K(\alpha, \beta)$ *if* γ *is an isomorphism of* M *onto* K.

Theorem 3 : *If* (L, ψ) *is the reflex of* (K, φ), *we have* $p_K(\sigma, \varphi) = p_L(\psi\sigma, id_L)$ *for every embedding* σ *of* K *into* $\mathbb{C}$.

These theorems imply various algebraic relations among the periods. For example, we have :

Theorem 4 : For $\alpha \in I_K$, let $t(\alpha)$ denote the rank of the module $\sum_{\gamma \in G} \mathbb{Z}\alpha\gamma$, where G is the Galois group over $\mathbb{Q}$ of the Galois closure of K. If $\sum_{i=1}^{n} \tau_i$ is a CM-type of K, then for every $\beta \in I_K$, the module

$$\{(e_1,\ldots,e_n) \in \mathbb{Z}^n \mid \prod_{i=1}^{n} p_K(\tau_i,\beta)^{e_i} = 1\}$$

has rank at least $n-t(\beta-\beta\rho)$.

If β is a CM-type, we have $t(\beta-\beta\rho) = t(\beta) - 1$. Theorems 2, 3 and 4 will be proved in [4].

The quantities p_K occur as the values of an L-function of a CM-field with an algebraic valued Hecke character of infinite order (see [1, Theorem 2]). As a new example of a zeta function whose values are given by p_K, we consider

$$D(s) = \sum_{0 \neq x \equiv a(\Lambda)} \mu\,(Tr_{K/\mathbb{Q}}(yxx^{\rho}))\,x^{\Phi}(x^{\tau})^{-k}|x^{\tau}|^{-2s} \qquad (s \in \mathbb{C}).$$

Here Λ is a lattice in K and $a \in K$; $0 < k \in \mathbb{Z}$; τ is an embedding of K into $\mathbb{C}$; μ denotes the Fourier coefficients of an elliptic modular form $g(z) = \sum \mu(b)e^{2\pi ibz}$; y is a real element of K such that y^{τ} is its only positive conjugate; Φ is an element of I_K with non-negative coefficients.

Theorem 5 : The series D is convergent for sufficiently large $Re(s)$ and can be continued to a meromorphic function on the whole plane.

Theorem 6 : Suppose that g is a cusp form of weight ℓ, $\mu(b)$ are all algebraic, and τ and $\tau\rho$ occur in Φ with the same multiplicity, say q. Let m be an integer such that

$$(2n - 1 - k + \ell + \deg(\Phi))/2 < m \leq q .$$

Then $D(m)$ is $\pi^k p_K(k\tau - \Phi, 2\tau)$ times an algebraic number.

A more general result holds for a series of a similar type with a Hilbert modular form (which is not necessarily a cusp form) in place of g. The details will be given in [4].

References

[1] G. Shimura, On some arithmetic properties of modular forms of one and several variables, Ann. of Math. 102 (1975), 491-515.

[2] G. Shimura, On the derivatives of theta functions and modular forms, Duke Math. J. 44 (1977), 365-387.

[3] G. Shimura, Automorphic forms and the periods of abelian varieties, J. Math. Soc. Japan, 31 (1979), 561-592.

[4] G. Shimura, The arithmetic of certain zeta functions and automorphic forms on orthogonal groups, to appear in Ann. of Math., 110 (1980) .

Princeton University
Department of Mathematics
Princeton, N. J. 08540
(U. S. A.)