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ON THE SIMILARITY BETWEEN THE IWASAWA
PROJECTION AND THE DIAGONAL PART

by

J.J. Duistermaat

1. Statement of the result.

Let G be a real connected semisimple Lie group with finite center and $G = KAN$ its Iwasawa decomposition. Via the adjoint representation, and with respect to a suitable basis in \mathfrak{g} , K , resp. A , resp. N are the set of matrices in G which are orthogonal, resp. diagonal with positive entries, resp. upper triangular.

The Iwasawa projection H from G onto the Lie algebra \mathfrak{a} of A is defined by

$$(1.1) \quad x \in K \cdot \exp H(x) \cdot N, \quad x \in G.$$

Obviously H factorizes through the projection from G onto the (non-compact Riemannian) symmetric space $K \backslash G$. If \mathfrak{s} (called \mathfrak{p} by everybody else) denotes the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to the killing form, then the Cartan decomposition $G = K \cdot \exp \mathfrak{s}$ yields that

$$(1.2) \quad \mathfrak{s} \xrightarrow{\exp} G \rightarrow K \backslash G$$

is a diffeomorphism from \mathfrak{s} onto $K \backslash G$. So the Iwasawa projection can be studied by looking at the mapping

$$(1.3) \quad \gamma = H \circ \exp : \mathfrak{s} \rightarrow \mathfrak{a}.$$

On the other hand we have the orthogonal projection

$$(1.4) \quad \pi : \mathfrak{s} \rightarrow \mathfrak{a}$$

with respect to the Killing form. In the above matrix terminology, \mathfrak{s} is the space of symmetric matrices in \mathfrak{g} and π is the operation of taking the diagonal part of the symmetric matrix. So this projection has a very simple minded interpretation, whereas the Iwasawa projection is a rather more mysterious object.

Theorem 1.1. There is a real analytic map $\Psi : \mathfrak{s} \rightarrow K$ such that

- i) $\Phi_X : k \rightarrow k \cdot \Psi(\text{Ad } k^{-1}(X))$ is a diffeomorphism from K onto K , for each $X \in \mathfrak{s}$.
- ii) $\gamma(\text{Ad } \Psi(X)^{-1}(X)) = \pi(X)$ for all $X \in \mathfrak{s}$.

That is, we can turn the Iwasawa projection into the orthogonal projection by an action of $\text{Ad } K$, the element of K depending analytically on $X \in \mathfrak{s}$.

It also follows from the theorem that the images of an $\text{Ad } K$ -orbit in \mathfrak{s} under γ and π are the same. This was obtained before by Kostant [4] who showed separately that both images are equal to the convex hull of the intersection of the $\text{Ad } K$ -orbit in \mathfrak{s} with \mathfrak{a} . Since this intersection is equal to a Weyl group orbit in \mathfrak{a} , which is finite, this image is a convex polytope. Very remarkable because an $\text{Ad } K$ -orbit is such a roundish object!

Later Heckman [3] reduced the convexity theorem for the Iwasawa projection to the convexity theorem for the diagonal part, for which the proof is much simpler, using a homotopy argument. This homotopy argument actually is one of the elements in the proof of Theorem 1.1.

For me the major motivation for wanting the theorem was the study in [2], together with Kolk and Varadarajan, of the asymptotic behaviour of integrals of the form

$$(1.5) \quad I_{\mathfrak{a}}(X, \xi) = \int_K e^{i\langle \gamma(\text{Ad } k^{-1}(X)), \xi \rangle} \cdot a(X, k) dk$$

as $\|\xi\| \rightarrow \infty$, $\xi \in \mathfrak{a}^*$. The matrix coefficients of the principal series representations of G are given by such integrals, the simplest case being the elementary spherical functions where

$$(1.6) \quad a(X, k) = e^{-\langle \gamma(\text{Ad } k^{-1}(X)), \rho \rangle}.$$

The idea in [2] was to consider (1.5) as an oscillatory integral, for which the asymptotics is concentrated at the stationary points of the "phase function".

$$(1.7) \quad F_{X, \xi} : k \rightarrow \langle \gamma(\text{Ad } k^{-1}(X)), \xi \rangle$$

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on K . We then observed that $F_{X,\xi}$ had exactly the same critical points and critical values as its "infinitesimal counterpart"

$$(1.8) \quad f_{X,\xi} = \lim_{t \rightarrow 0} \frac{1}{t} F_{tX,\xi} : k \rightarrow \pi(\text{Ad } k^{-1}(X)), \xi \rangle.$$

These critical points in turn had such a special, rigid structure that the asymptotics of (1.5) could be obtained by a repeated application of the method of stationary phase.

It had already been observed in [2] that the equality of critical points and critical values of $F_{X,\xi}$ and $f_{X,\xi}$ leads to the existence of a diffeomorphism $\phi_{X,\xi} : K \rightarrow K$ such that $F_{X,\xi} \circ \phi_{X,\xi} = f_{X,\xi}$.

However, the diffeomorphism is not unique and at that time I could not find $\phi_{X,\xi}$ depending smoothly on X and ξ . Already continuous dependence on ξ would imply, replacing ξ by $t\xi$, dividing by t , and letting $t \rightarrow 0$, that $F_{X,\xi} \circ \phi_{X,0} = f_{X,\xi}$. That is, one could find a diffeomorphism ϕ_X not depending on ξ . Then, using the substitution of variables

$$(1.9) \quad k = \phi_X(l), \quad l \in K,$$

the integral (1.5) can be rewritten as ($X \in \mathfrak{s}$, $\xi \in \mathfrak{a}^*$)

$$(1.10) \quad I_{\mathfrak{a}}(X,\xi) = \int_K e^{i\langle \pi(\text{Ad } k^{-1}(X)), \xi \rangle} \int_{\mathfrak{a}} \phi_X(k) \cdot \left| \det \frac{\partial \phi_X}{\partial k}(k) \right| dk.$$

In this way the study of the asymptotic behaviour would be reduced to doing stationary phase with the simpler $f_{X,\xi}$ as the phase function, rather than $F_{X,\xi}$. (Such asymptotics has been done before by Clerc and Barlet [1].)

It is one of the applications of Theorem 1.1, that the integral representation (1.10) actually holds with a $\phi_X(k)$ which depends analytically on X and k simultaneously. For instance, for the elementary spherical functions this leads to an integral formula of the form

$$(1.11) \quad \phi_{\xi}(\exp X) = \int_K e^{i\langle \pi(\text{Ad } k^{-1}(X)), \xi \rangle} b(\text{Ad } k^{-1}(X)) dk,$$

for some analytic function $b : \mathfrak{s} \rightarrow \mathbb{R}$. As an application of the analyticity of b , one can note that replacing ξ , resp. X by $i\xi$, resp. iX , one obtains the elementary spherical functions for the compact symmetric space which is dual to $K \backslash G$. (In this case ξ has to be taken in a weight lattice.) So also for these functions an integral formula like (1.10) holds, at least for small $\|X\|$. I owe

this observation to Richard van den Dries (T.H. Delft), who is using this integral formula in his characterization of invariant pseudo-differential operators on compact symmetric spaces in terms of their eigenvalues.

2. SL(2, R).

For $G = \text{SL}(2, \mathbb{R})$, $\dim K = \dim \mathfrak{a} (= 1)$, so the substitution of variables is unique up to a flip. In order to determine it explicitly, write the elements of K as

$$(2.1) \quad k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \theta \in \mathbb{R}/2\pi\mathbb{Z},$$

and the elements of \mathfrak{a} as

$$(2.2) \quad X = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Then $Y = \text{Ad } k^{-1}(X) = k^{-1}Xk$ is the general element of \mathfrak{s} , and $\phi_X(k)$ is the element of K with the coordinate μ given implicitly by

$$(2.3) \quad e^{2t} \cos^2 \mu + e^{-2t} \sin^2 \mu = e^{2t} \cos 2\theta.$$

From this one can determine $\Psi(Y) = k^{-1} \cdot \phi_X(k)$. It is not entirely trivial to verify that this defines a real analytic mapping $\Psi : \mathfrak{s} \rightarrow \mathfrak{a}$.

The Jacobian of ϕ_X is equal to

$$(2.4) \quad \frac{2|t \sin 2\theta| \cdot e^{t \cos 2\theta}}{\sqrt{2 \cosh(2t) - \cosh(2t \cos 2\theta)}},$$

leading to the following formula for the elementary spherical function:

$$(2.5) \quad \phi_\xi(\exp X) = \frac{1}{2\pi} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{it\tau \cos 2\theta} \frac{|t \sin 2\theta|}{\sqrt{\frac{\cosh(2t) - \cosh(2t \cos 2\theta)}{2}}} d\theta.$$

Here we have written $\langle X, \xi \rangle = t\tau$. This can also be written as

$$(2.6) \quad \phi_\xi(\exp X) = \frac{4}{\pi} \int_0^t \cos \tau s \cdot \frac{ds}{\sqrt{4(\cosh(2t) - \cosh(2s))}}.$$

A similar formula for all rank one symmetric spaces can be found in Koornwinder [8], formula (2.16) and (2.18).

I prefer (2.5) over (2.6), because there are no boundary points nor singularities for the integrand as in (2.6). To see the analyticity of the integrand in (2.5) we write

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$$(2.7) \quad \cosh(2t) - \cos(2t \cos 2\theta) = (2t \sin 2\theta)^2 \sum_{n=1}^{\infty} \frac{(2t)^{2n-2}}{(2n)!} \sum_{k=0}^{n-1} (\cos 2\theta)^{2k},$$

from which

$$(2.8) \quad \phi_{\xi}(\exp X) = \frac{1}{2\pi} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{it\tau \cos 2\theta} \left[\sum_{n=0}^{\infty} \sum_{k=0}^n \frac{2 \cdot (2t)^{2(n-k)}}{(2n+2)!} (t \cos 2\theta)^{2k} \right]^{-1/2} d\theta.$$

In turn this allows us to write

$$(2.9) \quad \phi_{\xi}(\exp X) = \sum_{k=0}^{\infty} c_k(t^2) \cdot \left(\frac{1}{i} \frac{\partial}{\partial \tau}\right)^{2k} \frac{1}{2\pi} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{it\tau \cos 2\theta} d\theta,$$

where the c_k are suitable power series in t^2 with some positive radius of convergence. So the elementary spherical function, which is a hypergeometric function, can be obtained from the Bessel function

$$(2.10) \quad \psi_{\xi}(\exp X) = \frac{1}{2\pi} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{it\tau \cos 2\theta} d\theta,$$

which is the elementary spherical function for the Cartan motion group, by applying an infinite order differential operator with respect to the eigenvalue (= character) parameter τ , with coefficients which are Ad K -invariant functions on \mathfrak{s} . This is the strategy in Stanton and Tomas [7]. That such a description is possible for all real rank one spaces can be derived from the previously mentioned explicit formulae of Koornwinder [8], but can also be read of from (1.11).

This description would generalize to arbitrary symmetric spaces if the amplitude $b(\text{Ad } k^{-1}(X))$ in (1.11) could be written as

$$(2.11) \quad b(\text{Ad } k^{-1}(X)) = \sum_{\mathbf{m}} c_{\mathbf{m}}(X) \cdot \pi(\text{Ad } k^{-1}(X))^{\mathbf{m}}$$

($\mathbf{m} = (m_1, \dots, m_{\dim \mathfrak{a}})$ a multi-index), where the $c_{\mathbf{m}}$ are Ad k -invariant functions on \mathfrak{s} . This however is one of the open questions which I have on this subject.

3. Proof of the theorem.

We begin by recalling some facts about the functions $F_{X,\xi}$, $f_{X,\xi}$ from [2].

Lemma 3.1. ([2], Lemma 5.9). For $x \in G$, write

$$(3.1) \quad x \in \kappa(x) \cdot AN, \quad \kappa(x) \in K.$$

Then, for every $X \in \mathfrak{s}$, $\xi \in \mathfrak{a}^*$:

$$(3.2) \quad dF_{X,\xi}(1) = df_{X,\xi}(1) \circ L_x, \quad \text{where}$$

$$(3.3) \quad \tilde{X} = \text{Ad } \kappa(\exp X)^{-1}(X)$$

and L_X is the linear isomorphism: $\mathfrak{k} \rightarrow \mathfrak{k}$ given by

$$(3.4) \quad L_X = \frac{\sinh \text{ad } \tilde{X}}{\text{ad } \tilde{X}} \circ \text{Ad } \kappa(\exp X)^{-1}.$$

Lemma 3.2. ([2], Lemma 1.1). Let $X \in \mathfrak{s}$ and let $\xi \in \mathfrak{a}^*$ correspond to $H = H_\xi \in \mathfrak{a}$ via the Killing form. Then

$$(3.5) \quad df_{X,\xi}(1) = 0 \Leftrightarrow [X, H] = 0.$$

If $[X, H] = 0$ then $\exp X \in G_H^0$, a connected reductive subgroup with an Iwasawa decomposition, the components of which are contained in K , resp. A , resp. N . So $\kappa(\exp X) \in G_H^0$ and $[\tilde{X}, H] = 0$ if \tilde{X} is as in (3.3). Using Lemma 3.1 we conclude that $dF_{X,\xi}(1) = 0 \Leftrightarrow df_{X,\xi}(1) = 0$. Using that

$$(3.6) \quad \frac{d}{dt} F_{X,\xi}(k \cdot \exp tY)_{t=0} = dF_{\text{Ad } k^{-1} X, \xi}(1)(Y), \quad k \in K, Y \in \mathfrak{k},$$

and the same formula with F replaced by f , it follows that $F_{X,\xi}$ and $f_{X,\xi}$ have the same set of critical points.

Lemma 3.3. ([2], Cor. 5.2). If $X \in \mathfrak{s}$, $\xi \in \mathfrak{a}^*$, then

$$(3.7) \quad \frac{d}{dt} F_{tX,\xi}(1) = f_{X,\xi}(\kappa(\exp tX)).$$

Now we look at the 1-parameter family of functions

$$(3.8) \quad F_{X,\xi}^{(t)} = \frac{1}{t} F_{tX,\xi}, \quad F_{X,\xi}^{(0)} = f_{X,\xi}, \quad F_{X,\xi}^{(1)} = F_{X,\xi}.$$

We see that the set of critical points of $F_{X,\xi}^{(t)}$ is equal to the set of critical points of $\frac{1}{t} f_{tX,\xi} = f_{X,\xi}$ so to the set of critical points of $f_{X,\xi}$, for all $t \in \mathbb{R}$. Moreover

$$(3.9) \quad F_{X,\xi}^{(t)}(1) = \frac{1}{t} \int_0^t f_{X,\xi}(\kappa(\exp sX)) ds,$$

If $dF_{X,\xi}^{(t)}(1) = 0$ then $\kappa(\exp sX)$ is a critical point for $f_{X,\xi}$ for all $s \in [0, t]$, so $f_{X,\xi}(\kappa(\exp sX)) = f_{X,\xi}(1)$, that is

$$(3.10) \quad F_{X,\xi}^{(t)}(1) = f_{X,\xi}(1) \text{ if } dF_{X,\xi}^{(t)}(1) = 0.$$

Using that $F_{X,\xi}^{(t)}(k) = F_{\text{Ad } k^{-1} X, \xi}^{(t)}(1)$, we get that $F_{X,\xi}^{(t)}$ and $f_{X,\xi}$ have the same values at the critical points. Now we try to find a diffeomorphism $\phi_{X,\xi}^{(t)} : K \rightarrow K$ depending smoothly on t , such that $\phi_{X,\xi}^{(0)} = \text{identity}$ and

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$$(3.11) \quad F_{X,\xi}^{(t)}(\phi_{X,\xi}^{(t)}(k)) = f_{X,\xi}(k) \text{ for all } t \in [0,1].$$

Differentiating (3.11) with respect to t gives

$$(3.12) \quad \frac{\partial}{\partial t} F_{X,\xi}^{(t)}(\phi_{X,\xi}^{(t)}(k)) + dF_{X,\xi}^{(t)}(\phi_{X,\xi}^{(t)}(k)) \circ \frac{\partial \phi_{X,\xi}^{(t)}}{\partial t}(k) = 0, \text{ which}$$

in fact is equivalent to (3.11) in view of the initial condition $\phi_{X,\xi}^{(0)} = \text{identity}$. The idea is now to find a vector field $v_{X,\xi}^{(t)}$ on K depending analytically on t, X, ξ such that

$$(3.13) \quad \frac{\partial}{\partial t} F_{X,\xi}^{(t)}(k) + dF_{X,\xi}^{(t)}(k) \circ v_{X,\xi}^{(t)}(k) = 0$$

and then obtain $\phi_{X,\xi}^{(t)}$ by solving the ordinary differential equation

$$(3.14) \quad \frac{\partial}{\partial t} \phi_{X,\xi}^{(t)}(k) = v_{X,\xi}^{(t)}(\phi_{X,\xi}^{(t)}(k)), \quad \phi_{X,\xi}^{(0)}(k) = k.$$

I learned this idea from Moser [6] and Mather [5], but it might have a much older history.

In any case, for (3.13) it is a necessary condition that $\frac{\partial}{\partial t} F_{X,\xi}^{(t)}(k) = 0$ if $dF_{X,\xi}^{(t)}(k) = 0$, but this follows from $dF_{X,\xi}^{(t)}(k) = 0 \Leftrightarrow df_{X,\xi}(k) = 0$, in which case $F_{X,\xi}^{(t)}(k) = f_{X,\xi}(k)$, constant in t , as observed above. In Lemma 3.1, 3.2 we have seen that $dF_{X,\xi}^{(t)}(k)$ is proportional to $[\text{Ad } k^{-1}(X), H_\xi]$ by a linear isomorphism depending analytically on t, X, H_ξ . In view of these observations the existence of $v_{X,\xi}^{(t)}$ with the desired properties is ensured by the following

Lemma 3.4. Let $\psi : \mathfrak{s} \times \mathfrak{a} \rightarrow \mathbb{R}$ be analytic such that $\psi(X, H) = 0$ whenever $[X, H] = 0$. Then there exists an analytic map $\chi : \mathfrak{s} \times \mathfrak{a} \rightarrow \mathbb{F}$ such that

$$(3.15) \quad \psi(X, H) = \langle [X, H], \chi(X, H) \rangle \text{ for all } X \in \mathfrak{s}, H \in \mathfrak{a}.$$

If ψ is linear in H for each X then χ can be chosen not depending on H and if ψ depends smoothly on additional parameters then ψ can be chosen to do the same.

Actually χ is obtained by an explicit formula from ψ , from which these properties can be read off. The construction is based on the observation that in $\mathfrak{s} \times \mathfrak{a}$ the relation $[X, H] = 0$ has a reasonably simple description. For $X \in \mathfrak{s}$ write

$$(3.16) \quad X = X_0 + \sum_{\alpha \in \Delta_+} X_\alpha, \quad X_0 \in \mathfrak{a}, \quad X_\alpha \in \mathfrak{s} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}).$$

Then

$$(3.17) \quad [X, H] = - \sum_{\alpha \in \Delta^+} \alpha(H) \cdot JX_{\alpha}$$

where J is the linear isomorphism: $\mathfrak{s} \oplus \mathfrak{a} \rightarrow \mathfrak{f} \oplus \mathfrak{m}$ which sends $Y - \theta Y$ to $Y + \theta Y$ (for $Y \in \mathfrak{n}$). It follows that $[X, H] = 0$ if and only if for each $\alpha \in \Delta^+$ either $X_{\alpha} = 0$ or $\alpha(H) = 0$.

For $I \subset \Delta^+$, write now

$$(3.18) \quad \Pi_I(X) = X_0 + \sum_{\alpha \in \Delta^+ \setminus I} X_{\alpha}.$$

Then, based on Newton's binomial formula, we can write

$$(3.19) \quad \psi(X, H) = \sum_{I \subset \Delta^+} \sum_{J \subset \Delta^+ \setminus I} (-1)^{|J|} \psi(\Pi_{I \cup J}(X), H).$$

Observing that $\psi(X_0, H) = 0$ by assumption, we concentrate our attention on the term

$$(3.20) \quad \psi_I(X, H) = \sum_{J \subset \Delta^+ \setminus I} (-1)^{|J|} \psi(\Pi_{I \cup J}(X), H).$$

Every term in the right hand side is equal to zero if $\alpha(H) = 0$ for all $\alpha \in \Delta^+ \setminus I$.

Let $\alpha_1, \dots, \alpha_p \in \Delta^+ \setminus I$ be a basis of $\sum_{\alpha \in \Delta^+ \setminus I} \mathbb{R} \cdot \alpha$. Write

$$(3.21) \quad a_j = \{H \in \mathfrak{a}; \alpha_i(H) = 0 \text{ for } i \leq j\}, \quad a_0 = \mathfrak{a},$$

and let π_i be a linear projection from a_{j-1} to a_j . Write

$$(3.22) \quad \pi_j = \tilde{\pi}_j \circ \dots \circ \tilde{\pi}_1 : a \rightarrow a_j,$$

$$(3.23) \quad \psi_I(X, H) = \sum_{j=1}^p \psi_I(X, \pi_{j-1}(H)) - \psi_I(X, \pi_j(H)),$$

and finally

$$(3.24) \quad \begin{aligned} & \psi_I(X, \pi_{j-1}(H)) - \psi_I(X, \pi_j(H)) \\ &= \sum_{J \subset \Delta^+ \setminus I} (-1)^{|J|} [\psi(\pi_{I \cup J}(X), \pi_{j-1}(H)) - \psi(\pi_{I \cup J}(X), \pi_j(H))] \\ &= \sum_{J \subset \Delta^+ \setminus I \cup \{\alpha_j\}} (-1)^{|J|} [\psi(\pi_{I \cup J}(X), \pi_{j-1}(H)) - \psi(\pi_{I \cup J \cup \{\alpha_j\}}(X), \pi_{j-1}(H)) \\ & \quad - \psi(\pi_{I \cup J}(X), \pi_j(H)) + \psi(\pi_{I \cup J \cup \{\alpha_j\}}(X), \pi_j(H))]. \end{aligned}$$

The last expression between square brackets is equal to zero if $X_{\alpha_j} = 0$ or $\alpha_j(H) = 0$. So this expression is of the form

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$$\alpha_j(H) \cdot \langle JX_{\alpha_j}, X_{I,j}(X,H) \rangle$$

for some analytic mapping $\chi_{I,j} : \mathfrak{s} \times \mathfrak{a} \rightarrow \mathbb{K} = \mathbb{K} \cap (\mathfrak{g}_{\alpha_j} + \mathfrak{g}_{-\alpha_j})$. Summing all the terms gives the desired mapping χ .

From χ we get an analytic vector field $v_{X,\xi}^{(t)}$ on K , depending analytically on t, X, ξ satisfying (3.13). Now, observing that

$$(3.25) \quad F_{\text{Ad } l^{-1}(X), \xi}^{(t)}(k) = F_{X, \xi}^{(t)}(lk), \quad k \in K,$$

it follows that $\lambda_1^* v_{\text{Ad } l(X), \xi}^{(t)}$ satisfies (3.13) as well, here $\lambda_1 : k \rightarrow l.k$ denotes left multiplication by l . Because the equation (3.13) is linear in v , also

$$(3.26) \quad \bar{v}_{X, \xi}^{(t)} = \int_K \lambda_1^* v_{\text{Ad } l(X), \xi}^{(t)} dl$$

will satisfy (3.13). This vectorfield has the additional symmetry

$$(3.27) \quad \bar{v}_{\text{Ad } k^{-1}(X), \xi}^{(t)} = \lambda_k^* \bar{v}_{X, \xi}^{(t)},$$

which for the solution $\phi_{X, \xi}^{(t)}$ of (3.14), with v replaced by \bar{v} , will lead to

$$(3.28) \quad \phi_X(lk) = l \cdot \phi_{\text{Ad } l^{-1}(X)}(k), \quad k, l \in K.$$

This proves Theorem 1.1, with $\Psi(X) = \phi_X(1)$, $X \in \mathfrak{s}$.

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