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On some series of representations related to symmetric spaces


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ON SOME SERIES OF REPRESENTATIONS RELATED TO SYMMETRIC SPACES.

by

H. Schlichtkrull

In this paper, the series of representations constructed by M. Flensted-Jensen in [3] and [4] are considered. The main results of [8], on lowest \( K \)-types and Langlands parameters of the representations of [3] in the equal rank case, are generalized to the other series as well. The representations are identified with subquotients of parabolically induced representations. The parabolic subgroup we use, \( P = MAN \), is cuspidal, and moreover, the symmetric space \( M/M \cap H \) satisfies the equal rank condition. The inducing representation \( \tau \Theta \cup \Theta 1 \) of \( MAN \) is given by a Flensted-Jensen representation \( \tau \) of \( M \), and thus the determination of Langlands parameters is reduced to Flensted-Jensen representations of \( M \). Further, these results imply unitarity of the representations under certain conditions (see Theorem 4).

Since the proofs of some of our results are rather straightforward generalizations of those of [8], we do not give all the details in these cases, but refer to [8] in stead.

Our results generalize some results of G. Ólafsson [5], [6] (in fact, Theorem 1 and 3 below were obtained before we received [5] and [6]).

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1. Notation. Let $G/H$ be a semisimple symmetric space with $G$ and $H$ connected and linear. Let $\tau$ be the corresponding involution, and let $\theta$ be a commuting Cartan involution. Denote by $g = h \oplus q$ and $g = k \oplus p$ the corresponding decompositions of the Lie algebra $g$, and let $K$ be the maximal compact subgroup of $G$ with Lie algebra $k$. Let $G_0$ denote the analytic subgroup of $G$ with Lie algebra $g_0 = k \cap h + p \cap q$.

Choose a $\theta$-invariant maximal abelian subspace $a^0$ of $q$, and put $t = a^0 \cap k$. Let $\Delta = a^0_+$ be the set of roots of $a^0$ in $g$, and choose a positive system $\Delta^+$ which is $\theta$-compatible, i.e. $\alpha \in \Delta^+$ and $\alpha|_{\tau} = 0$ implies $\theta \alpha \in \Delta^+$. Put $\rho = \rho(\Delta^+) = \frac{1}{2} \sum_{\alpha \in \Delta^+} (\dim g^0_\alpha) \alpha \in a^0_+$.

Let $\ell = g^+$ be the centralizer of $t$ in $g$, and let $\ell$ denote the orthocomplement of $t$ in $\ell$ (w.r.t. the Killing form of $g$). Choose $t_2$ maximal abelian in $\ell \cap h \cap q$, then $t = t + t_2$ is maximal abelian in $k \cap q$. Let $\Delta_c = \Delta(\ell \cap k)$, $\Delta_{c,1} = \{\alpha \in \Delta_c | \alpha|_{\ell} = 0\}$ and $\Delta_{c,2} = \{\alpha \in \Delta_c | \alpha|_{\ell} \neq 0\}$. Put $\Delta^{+}_{c,2} = \{\alpha \in \Delta^{+}_c | \exists \beta \in \Delta^{+} : \beta|_{\ell} = \beta|_{t_2}\}$ and choose a positive system $\Delta^{+}_{c,2}$ for the root system $\Delta^{+}_{c,2}$, then $\Delta^{+} = \Delta^{+}_c \cup \Delta^{+}_{c,2}$ is a positive system for $\Delta_c$. Define $\rho_c = \rho(\Delta^{+}_c) = \frac{1}{2} \sum_{\alpha \in \Delta^{+}_c} (\dim k^0_\alpha) \alpha \in i\ell^*$ and $\rho_{c,1} = \rho(\Delta^{+}_{c,1})$ similarly. Notice that $\rho_{c,1}$ does not vanish in general, but at least we have:

**Lemma 1.** $<\rho_{c,1}, \alpha> = 0$ for all $\alpha \in \Delta^{+}_{c,2}$.

**Proof:** Let $\alpha \in \Delta^{+}_{c,2}$, and denote by $s_\alpha$ reflection in $\alpha$. Then $s_\alpha(\Delta^{+}_{c,1}) = \Delta^{+}_{c,1}$ and hence the lemma.

For each $\lambda \in a^0_\ell$ we define $v_\lambda \in \ell_\ell^*$ by the following equations:

$$(1) \quad (v_\lambda + 2\rho_c)|_\ell = (\lambda + \rho)|_\ell \quad \text{and} \quad (v_\lambda + 2\rho_{c,1})|_{t_2} = 0.$$

2. Flensted-Jensen's representations. Let $c > 0$ be the smallest possible constant such that [4] Theorem 1 holds, and define $\Delta_{c,a^0_\ell}$ to be the set of those $\lambda \in a^0_\ell$ satisfying the following conditions (2) and (3):
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(2) \( \text{Re}\langle \lambda, a \rangle > c \) for all \( a \in \Delta^+ \) with \( a \mid \xi = 0 \)

(3) \[
\begin{cases}
  \frac{\langle u_\lambda, a \rangle}{\langle a, a \rangle} \in \mathbb{Z}^+ & \text{for all } a \in \Delta^c \\
  u_\lambda(X) \in \mathbb{Z} & \text{for } X \in \mathfrak{t}, \exp 2\pi i X = e
\end{cases}
\]

For each \( \lambda \in \Lambda \) Flensted-Jensen [4] defines a function \( \psi_\lambda \in C^\infty(G/H) \) by an integral formula (for the dual function on the dual symmetric space \( G^0/H^0 \)), and the following properties hold for these functions:

a) The representation of \( K \) generated by \( \psi_\lambda \) is finite dimensional and irreducible. Denoting by \( \delta_\lambda \) the contragredient of this representation of \( K \), \( \delta_\lambda \) is spherical for \( K/K \cap H \) and has highest weight \( \mu_\lambda \).

(We have not included Condition (9) of [4], since it is redundant by Lemma 1).

b) \( \psi_\lambda \) is a joint eigenfunction for \( U(g)^K \) acting on \( C^\infty(G/H) \) from the left. The eigenvalues are determined as follows: There is a unique homomorphism \( \gamma: U(g)^K \to U(a^0_\mathfrak{k}) \) such that for \( u \in U(g)^K \):

\[
(4) \quad u - \gamma(u) \in (\mathbb{Z} \cap \mathfrak{k}) \cdot U(g) + U(g) \cdot (h_{\mathfrak{c}}^{a_0} + n^0)
\]

where \( n^0 = \sum_{a \in \Delta^+} g_a^0 \). Then \( u\psi_\lambda = \gamma(u)(-\lambda - \rho)\psi_\lambda \).

Remark. In the sequel we use only properties a) and b) of the functions \( \psi_\lambda \). If \( \psi_\lambda \) can be defined (e.g. by analytic continuation in \( \lambda \)), such that a) and b) still hold for some \( \lambda \) which does not satisfy (2), then our results can be extended to these parameters as well.

From a) and b) it follows by [2] Proposition 9.1.10 (iii) that the \( K \)-type \( \psi_\lambda^\vee \) has multiplicity one in the \( g \)-module generated by \( \psi_\lambda \). Consequently, this module has a unique irreducible quotient \( T_\lambda^\vee \) which contains \( \psi_\lambda^\vee \).

If \( \mathfrak{t} \) is maximal abelian in \( \mathfrak{k} \cap \mathfrak{q} \), then \( \psi_\lambda \) is the same as the function defined in [3]. In this case \( c = 0 \), but (2) is not necessary for defining \( \psi_\lambda \). In fact (2) is not serious since one can prove that then \( \psi_{s\lambda} = \psi_\lambda \) for all elements \( s \) from the Weyl group of the root system \( \{ a \in \Delta \mid a \mid \xi = 0 \} \). The series of (\( g,K \)-
modules $T^\lambda$ is in this case called the fundamental series for the symmetric space $G/H$.

If we can choose $a^0$ such that $t = a^0$, we say that $G/H$ satisfies the equal rank condition. If furthermore $\langle \lambda, a \rangle > 0$ for all $a \in \Lambda^+$, then $\psi_\lambda$ is square integrable with respect to invariant measure on $G/H$, and hence $\psi_\lambda$ generates a unitary irreducible representation $\pi^G_\lambda$ of $G$, whose Harish-Chandra module is $T^\lambda$. This was proved under stronger assumptions on $\lambda$ in [3], and subsequently proved in general by T. Oshima (unpublished, cf. however [10] and [13]).

3. Lowest K-types. Let $L = G^t$, then $L$ is connected and has Lie algebra $\mathfrak{t}$. Put $n_1 = \sum_{a \in \Delta_+} g_a$ and $n_2 = \sum_{a \in \Delta_+} g_a^t$, and observe that $\mathfrak{g}_+ + n_1$ is a $t$-stable parabolic subalgebra of $\mathfrak{g}_+$. Choose an Iwasawa decomposition $\mathfrak{t} = \mathfrak{k} \oplus a \oplus \mathfrak{n}_t$ such that $a^0 \cap p = a$ and $n_2 \subset n_t$. Notice that $a$ is $t$-stable, and $a \cap q = a^0 \cap p$ by maximality of $a^0$ in $q$ so that $a = a^0 \cap p + a \cap k$.

Define $\rho_\mathfrak{t} \in \mathfrak{a}^*$ by $\rho_\mathfrak{t} = \frac{1}{2} \text{Tr} \text{ad}$, then it follows easily that $\rho_t|_{\mathfrak{a} \cap q} = \rho|_{\mathfrak{a} \cap p}$. Define for each $\lambda \in \mathfrak{a}_+^0$ an element $v_\lambda^L \in \mathfrak{a}^*$ by

$$(5) \quad v_\lambda^L|_{\mathfrak{a} \cap q} = -\lambda|_{\mathfrak{a} \cap p} \quad \text{and} \quad v_\lambda^L|_{\mathfrak{a} \cap h} = \rho_\mathfrak{t}|_{\mathfrak{a} \cap h}.$$ 

Theorem 1. Assume $\lambda \in \Lambda$ and

$$(6) \quad \langle (\lambda + \varphi)|_t, a|_t \rangle > 0 \text{ for all } a \in \Delta^+.$$ 

Then $\nu^\mathfrak{t}_\lambda$ is a lowest K-type of $T^\lambda$, and $T^\lambda$ has no other lowest K-types.

Proof: Let $\overline{V}_\lambda$ denote the spherical representation of $\overline{L}$ (the analytic subgroup with Lie algebra $\overline{\mathfrak{t}}$) with parameter $v_\lambda^L \in \mathfrak{a}_+^*$, and denote by $V_\lambda$ the representation of $L$ which extends $\overline{V}_\lambda$ with the character $e^{\mu_\lambda - 2\rho(n \cap p)}$ on $\exp i \overline{\mathfrak{t}}$ (then $V_\lambda$ is well-defined, cf. [8] Lemma 5.5 and the succeeding remark).

Let $X(\mathfrak{t}, n_1, V_\lambda, \nu_\lambda)$ be the $(g, K)$-module induced from $V_\lambda$ in the sense of [11], then one can conclude by comparing actions of $U(\mathfrak{g})^K$ on $\nu_\lambda$ that the module $T^{\lambda\nu}$, contragradient to $T^\lambda$, is equivalent to $X(\mathfrak{t}, n_1, V_\lambda, \nu_\lambda)$, (cf. [8] Lemma 5.6 where $T^{\lambda\nu}$ has been interchanged with $T^\lambda$).
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When \( t = a^0 \) Theorem 1 is exactly [8] Theorem 5.4, and the general case follows in the same way as there, the only complication being the analogue of [8] (5.10), but at that point one can apply Lemma 1 above.

4. Definition. The symmetric space \( G/H \) is said to satisfy Condition D, if the subgroup \( \widetilde{L} = G^\tau \) is compact or, equivalently, if

\[
\text{rank } G/H = \text{rank } G/G_0 = \text{rank } K/K \cap H.
\]

Notice that if \( G/H \) satisfies Condition D, then \( \text{rank } G = \text{rank } K \), so that the discrete series of \( G \) is nonempty. In fact, by [8] Theorem 6.1, \( \pi^G_\lambda \) belongs in this case to the discrete series of \( G \) whenever \( \langle \lambda, \sigma \rangle > k \) for all \( \sigma \in \Delta^+ \), where \( k \) is a certain nonnegative constant explicitly determined. However, for "smaller" \( \lambda \) it happens that \( \pi^G_\lambda \) no longer belongs to the discrete series of \( G \) (cf. [8] Example 7.5), and we do not know in general the Langlands parameter \( \nu \) of \( \pi^G_\lambda \) in this case.

Examples. 1° \( G*G/d(G) \) satisfies Condition D if and only if rank \( G = \text{rank } K \).

2° From the list of [1] exactly the following spaces with \( G \) classical satisfy Condition D:

\[
\begin{align*}
\text{SU}(2r,q)/\text{SU}(r,k)^+\text{SU}(r,q-k)^+, & \quad \text{SU}(p,q)/\text{SO}(p,q), \\
\text{SU}(2r,2s)/\text{Sp}(r,s), & \quad \text{SU}(n,n)/\text{SL}(n,n)^+\mathbb{R}, \\
\text{SO}^*(4n)/\text{SU}^*(2n)^+\mathbb{R}, & \quad \text{SO}(2r,q)/\text{SO}(r,k)^+\text{SO}(r,q-k)^+, \\
\text{SO}(2r,2s)/U(r,s), & \quad (r \text{ and } s \text{ not both odd}), \\
\text{Sp}(n,\mathbb{R})/\text{SL}(n,\mathbb{R})^+\mathbb{R}, & \quad \text{Sp}(2r,q)/\text{Sp}(r,k)^+\text{Sp}(r,q-k)^+, \\
\text{Sp}(p,q)/U(p,q). &
\end{align*}
\]

5. \( \tau^\lambda \) as induced representation. Let \( a \) be as defined in Section 3, let \( \lambda = \exp a \) and let \( P = MAN \) be a cuspidal parabolic subgroup of \( G \) with \( A \) as its split component.

Observe that \( M \) is invariant under \( \tau \), and that \( \tau \) is a maximal abelian subspace of \( m \cap q \), where \( m \) denotes the Lie algebra of \( M \). Moreover, \( M/(M\widehat{H})_e \) (where subscript \( e \) means "identity component") satisfies Condition D (which is generalized to non-connected reductive groups in the obvious fashion).
Let \( \Delta_m \subset i\mathfrak{t}^* \) (resp. \( \Delta_{mc} \subset i\mathfrak{t}^* \)) consist of the roots of \( t \) in \( \mathfrak{m}_C \) (resp. in \( \mathfrak{m}_C \cap \mathfrak{k}_C \)), let \( \Delta_m^+ = \Delta_m \cap \{ \alpha |_t | \alpha \in \Delta^+ \} \) and \( \Delta_{mc} = \Delta_m^+ \cap \Delta_{mc}^+ \), and put \( \rho_m = \sum_{\alpha \in \Delta_m^+} (\dim \mathfrak{m}_C^\alpha) \alpha \) and \( \rho_{mc} = \sum_{\alpha \in \Delta_{mc}^+} (\dim \mathfrak{m}_C^\alpha) \alpha \).

For \( \lambda \in \mathfrak{t}_C^* \), \( \nu^m = \mathfrak{t}_C^* \) is defined by \( \nu^m = \nu + \rho_m - 2\rho_{mc} \). By the following lemma we get for \( \lambda \in \Delta_0^* \) that \( \nu^m |_t = \nu^m |_t \).

**Lemma 2.** \( \rho |_t - 2\rho_c |_t = \rho_m - 2\rho_{mc} \).

**Proof:** Suppose \( \beta \) is a weight of \( \mathfrak{t}^* \) a in \( \mathfrak{g}_C \), and assume \( \beta |_t \in \{ \alpha |_t | \alpha \in \Delta^+ \} \). The claim is that if \( \beta |_t = 0 \) then \( \beta |_t \) contributes nothing to \( (\rho - 2\rho_c) |_t \). This follows from the fact that then \( \beta \mathfrak{t} \) is also a weight and \( \beta |_t \in \{ \alpha |_t | \alpha \in \Delta_c^+ \} \).

Let \( \lambda \in \Lambda \). Since the highest weight \( \nu^\lambda \) of \( \mathfrak{t} \) has multiplicity one in \( \delta^\lambda \), it follows from Lemma 1 that the multiplicity of the weight \( \nu |_t \) of \( \mathfrak{t} \) in \( \delta^\lambda \) is also one. Therefore, \( \delta^\lambda \) contains a unique irreducible subrepresentation \( \delta^\lambda_M \) of \( M \cap K \) of highest weight \( \nu |_t \). Assuming

\[
\langle \lambda |_t, \alpha \rangle > 0 \quad \text{for all } \alpha \in \Delta^*_m
\]

it follows from the last paragraph of Section 2 above that \( \lambda |_t \) determines a Flensted-Jensen representation \( \pi^M_\lambda \) of \( M \) in the discrete series of \( M/(M \cap H) \) (here one should also take into account the possibility that \( M \) is not semisimple or not connected. In the latter case \( \pi^M_\lambda \) is determined by \( \delta^M_\lambda \) rather than by \( \lambda |_t \). See [6] Section 4.8).

**Theorem 2.** Let \( \lambda \in \Lambda \) and assume (8). Define \( \nu^v_\lambda \in \Delta^*_C \) by (5).

(i) \( \nu^v_\lambda \) is a lowest \( K \)-type of \( \text{Ind}_F^G(\pi^M_\lambda \otimes \nu^L_\lambda \otimes 1) \) where it occurs with multiplicity one.

(ii) \( T \) is equivalent to the irreducible subquotient of \( \text{Ind}_F^G(\pi^M_\lambda \otimes \nu^L_\lambda \otimes 1) \) containing \( \nu^v_\lambda \).

We prove (i) in the next section and (ii) in Section 7.

6. **Langlands parameters.** For \( \lambda \in \Lambda \) let \( P^G_\lambda = M^G \times \mathbb{A} \rightleftharpoons \mathbb{A}^G_\lambda \rightleftharpoons \mathbb{A}^G_\lambda \) and \( P^M_\lambda = M^M \times \mathbb{A} \rightleftharpoons \mathbb{A}^M_\lambda \rightleftharpoons \mathbb{A}^M_\lambda \) be cuspidal parabolic subgroups of \( G \) and \( M \),

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respectively, associated to the K-type $\delta^\vee_\lambda$, respectively the $M\cap K$-type $\delta^M_\lambda$ by \cite{[12]}. Proposition 5.3.3, and let $\sigma^G_\lambda$ and $\sigma^M_\lambda$ be the associated discrete series representations of $M^G_\lambda$ and $M^M_\lambda$, (cf. \cite{[12]} Lemma 6.6.12). Notice that only the associate classes of $P^G_\lambda$ and $P^M_\lambda$ are uniquely determined.

**Lemma 3.** We can choose $P^G_\lambda$ and $P^M_\lambda$ such that $P^G_\lambda \subset P$ and $P^M_\lambda = P^G_\lambda \cap M$. Then $M^G_\lambda = M^M_\lambda$ and moreover $\sigma^M_\lambda = \sigma^G_\lambda$.

The proof is similar to the proof of \cite{[8]} Lemma 6.5, and we omit it.

In particular $a^G_\lambda = a^M_\lambda \otimes a$.

Assume (8) and let $\nu^G_\lambda \in (a^G_\lambda)^*$ and $\nu^M_\lambda \in (a^M_\lambda)^*$ be the Langlands parameters of $T^\lambda$ and $\pi^M_\lambda$, respectively.

**Proof of Theorem 2 (i):** Since by definition $\pi^M_\lambda$ is a subquotient of $\text{Ind}_{P^M_\lambda}^{M^M_\lambda} (\sigma^M_\lambda \otimes \nu^M_\lambda \otimes 1)$, the composition factors of $\text{Ind}_{P^G_\lambda}^{G^M_\lambda} (\sigma^G_\lambda \otimes \nu^G_\lambda \otimes 1)$ are also composition factors of $\text{Ind}_{P^G_\lambda}^{G^M_\lambda} (\sigma^G_\lambda \otimes (\nu^M_\lambda + \nu^L_\lambda) \otimes 1)$ using induction by stages. Theorem 2 (i) then follows from Lemma 3.

Though Theorem 2(ii) is still to be proved, we observe the following corollary to this and the preceding proof of Theorem 2 (i):

**Corollary:** $\nu^G_\lambda = \nu^M_\lambda + \nu^L_\lambda$.

Thus the determination of Langlands parameters of Flensted-Jensen's representations is reduced to the case of symmetric spaces satisfying Condition D.

For "large" values of $\lambda$, $\pi^M_\lambda$ is itself in the discrete series of $M$ (cf. Section 4), so $\sigma^M_\lambda = \pi^M_\lambda$ and thus Theorem 2 (ii) implies:

**Theorem 3.** There is a constant $c_1 > 0$ such that if $\lambda \in \Lambda$ and

$$(9) \quad <\lambda|_\xi, \alpha|_\xi> > c_1 \text{ for all } \alpha \in \Delta^+ \text{ with } \alpha|_\xi \neq 0$$

then $P$, $\pi^M_\lambda$, $\nu^L_\lambda$ and $\nu_\lambda$ constitute a set of Langlands parameters for $T^\lambda$ (i.e. $T^\lambda \sim J^G_\xi (P, \pi^M_\lambda, \nu^L_\lambda, \nu_\lambda)$ in the notation of \cite{[8]} Section 3).
Since we need Theorem 3 in our proof of Theorem 2 (ii), we indicate how to prove the former without reference to the latter.

**Proof:** The proof follows that of [8] Lemma 6.7 with only minor modifications (see also [11], proof of Proposition 4.13). In short, since $T^{\lambda v} \simeq X(\ell^{\lambda v}_c + \eta, V_{\lambda}, u_{\lambda})$, (cf. the proof of Theorem 1), the $a$-parameters of $T^{\lambda v}$ and $V_{\lambda}$ in the Langlands classification coincide when $u_{\lambda}$ is sufficiently "large", which is ensured by (9). $V_{\lambda}$ however, has the same $a$-parameter as $\overline{V}_{\lambda}$, and since $\overline{V}_{\lambda}$ is spherical this is $-\nu_{\lambda}$.

**Remark.** In particular, Theorems 1 and 3 generalize the results of [8] to the fundamental series for $G/H$. For these representations, the results have been obtained independently by G. Ólafsson [6], where they are also generalized to arbitrary real reductive linear groups (in the sense of [12] p. 1).

7. Proof of Theorem 2 (ii). From Theorem 3 the statement of Theorem 2 (ii) immediately follows for sufficiently large values of $\lambda$. We will now prove Theorem 2 (ii) in general by explicit construction of a $C^\infty$-vector for the induced representation $\text{Ind}_{P_1}^{G}(T^{\lambda v} \otimes V_{\lambda} \otimes 1)$, generating a subrepresentation which contains $T^{\lambda v}$ as a quotient.

Consider the K-type $\delta_{\lambda}$ of highest weight $u_{\lambda}$. Let $U_{\lambda}$ be a representation space for $\delta_{\lambda}$, and assume that $\delta_{\lambda}$ is unitary on $U_{\lambda}$. Let $u_0$ and $u_{\lambda}$ in $U_{\lambda}$ be a $K \cap H$-fixed vector and a vector of weight $u_{\lambda}$ respectively, normalized to $(u_{\lambda}, u_0) = 1$.

Define $\zeta_p \in \mathfrak{a}^*$ by $\zeta_p = \frac{1}{2} \text{Tr} \ ad_n$. Guided by [3] Eq. (3.18) we attempt a definition of a function $\varphi_{\lambda}$ on $G$ for $\lambda \in \Lambda$:

$$\varphi_{\lambda}(kxh\alpha) = \int_{(M \cap K_0H)_{\xi}} \langle \zeta_p, H(x^{-1}) \rangle e^{\frac{1}{4} \langle \zeta_p, \log a \rangle} d_e$$

for $k \in K$, $x \in (M \cap G_0)_{\xi}$, $h \in (M \cap H)$, $a \in A$ and $n \in N$. The term $H(x^{-1})$ appearing in (10) is defined using the Iwasawa projection corresponding to $\Delta^*$ of the dual group $G^0$ - see [3] or [4].

**Proposition 1.** Eq. (10) defines a nonzero $C^\infty$-function $\varphi_{\lambda}$ on $G$ which is $K$-finite of the irreducible type $\delta_{\lambda}$. When (8) holds the function $m \mapsto \varphi_{\lambda}(gm)$ on $M$ belongs to $L^2(M/(M \cap H)_{\xi})$ for each $g \in G$, and is in the representation space of $\tau_{\lambda}^M$. 284
Proof: For connected semisimple $M$ it follows from [9] Example 3.5 that the formula
\[
\psi_\lambda(kxh) = \int_{(M\cap N)H} \delta_\lambda(kl)u_\lambda e \quad dl
\]
for $k \in M \cap K$, $x \in (M\cap N)G_0$ and $h \in (M\cap N)H$, gives a well defined $U_\lambda$-valued $C^\infty$-function on $M$ satisfying $\psi_\lambda(km) = \delta_\lambda(k)\psi_\lambda(m)$ for $k \in M \cap K$, $m \in M$. Moreover, when (8) holds the function $m \mapsto (\psi_\lambda(m), u_0)$ is in $L^2(M/(M\cap H)H)$ and generates $\tau_\lambda$.

The preceding remarks are easily generalized to the general nonconnected reductive $M$.

From (11) we have that (10) is equivalent to
\[
\psi_\lambda(kmn) = (\delta_\lambda(k)\psi_\lambda(m), u_0)e
\]
for $k \in K$, $m \in M$, $a \in A$ and $n \in N$. From this Proposition 1 follows.

From Proposition 1 we see that we may regard $\psi_\lambda$ as a $C^\infty$-vector for $\text{Ind}_p^G(\pi_\lambda \otimes \psi_\lambda \otimes 1)$. Since $\psi_\lambda$ is $K$-finite of type $\nu_\lambda$ which has multiplicity one, $\psi_\lambda$ is a joint eigenvector for $\mu_\lambda$.

Proposition 2. The eigenvalues for $U(g)^K$ of $\psi_\lambda$ and $\psi_\lambda$ are equal.

Proof: Let $u \in U(g)^K$. We will first prove the existence of an element $u_1 \in U(a^0)$ such that $u\psi_\lambda = u_1(\lambda)\psi_\lambda$ for all $\lambda \in \Lambda$.

By symmetrization we identify the symmetric algebra $S(k+m)$ with a subspace of $U(g)$. Since $g = n \otimes a \otimes (m+k)$ we can determine elements $v_1, \ldots, v_p$ in $U(a)$ and $w_1, \ldots, w_p$ in $S(k+m)$ such that $u - \sum_{i=1}^p v_i w_i \in nU(g)$ (cf. [2] 2.4.14), and since $a$ and $m \cap k$ commute we may assume that $w_i$ is centralized by $m \cap k$ $(i=1, \ldots, p)$.

Put $\varphi_\lambda^\gamma(g) = \varphi_\lambda^\gamma(yg)$ for $\gamma, g \in G$, then since $u \in U(g)^K$ we have that $(u\varphi_\lambda^\gamma)(y) = (u\varphi_\lambda^\gamma)(g)$ for $\gamma \in K$. Using the decomposition $G = KM_eAN$ we may take $g = man$, $m \in M_e$, $a \in A$, $n \in N$. Since $\varphi_\lambda^\gamma$ is invariant under $N$ and homogeneous under $A$ from the right we get

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To prove our claim that \( \omega \phi = \psi \), for some \( \psi \in U(a^0) \) it is then clearly enough to prove that for each \( w \in S(m+k) \) there exists \( w_0 \in U(t) \) such that

\[
\phi (w) = w_0 \psi (m) \quad (14)
\]

for all \( \lambda \in \Lambda \) and \( m \in M_e \), \( y \in K \).

Let \( w \in S(m+k) \) and write \( w = \Sigma_{j=1}^q a_j \otimes b_j \) where \( a_j \in S(mn) \) and \( b_j \in S(f) \), according to the identification \( S(m+k) = S(mn) \otimes S(f) \). Denote by \( v \rightarrow v' \) the principal antiautomorphism of \( U(g) \). From (12) we then get for \( m \in M_e \) that:

\[
\phi (w) = \prod_{j=1}^q \delta_{\lambda} (y) \delta_{\lambda} (b_j \psi) (a_j \phi) (m) , \quad u_0 .
\]

Let \( M^0 \) denote the group dual to \( M \) by Flensted-Jensen's duality. Let \( f(x) = e^{-\lambda \cdot H(x)} \) for \( x \in M^0 \), and write \( m = k_x h \) where \( k \in (MN)_n \), \( x \in (MN)_n \) and \( h \in (MN)_n \), then (11) gives that

\[
\psi (m) = \int_{(MN)_n} \delta_{\lambda} (kl) u_\lambda f(x^{-1}) dl
\]

and therefore it follows that

\[
\psi (m) = \int_{(MN)_n} \delta_{\lambda} (kl) u_\lambda ([Ad(kl)^{-1}a_j]_L f) (x^{-1}) dl
\]

where \( Ad(kl)^{-1}a_j \) denotes \( Ad(kl)^{-1}a_j \) acting as a left invariant differential operator on \( C^\infty (M^0) \) (cf. [9] Eq.'s (2.3) and (4.6)).

Now we get

\[
\prod_{j=1}^q \delta_{\lambda} (b_j \psi) (a_j \phi) (m) = \int_{(MN)_n} \delta_{\lambda} (kl) \prod_{j=1}^q \delta_{\lambda} (Ad(kl)^{-1}b_j \psi) (a_j \phi) (m) \]

since \( w = \prod a_j \otimes b_j \) commutes with \( kl \).

Using the decompositions

\[
\tau = \tau (m_0 \cap n_0) \quad \xi = \xi (m_0 \cap n_0)
\]

we get

\[
\phi (w) = \int_{(MN)_n} \delta_{\lambda} (kl) \prod_{j=1}^q \delta_{\lambda} (b_j \psi) (a_j \phi) (m) \]

and

\[
k = \nu (\xi n_0) \quad \nu (\tau (n_0, 1))
\]

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where \( n_{c,1} = \sum_{a \in \Sigma_C}^c \mathbf{k}^a_c \), we can define a map \( \eta : S(m+k)^{\mathbf{c}} \rightarrow U(\mathbf{c}) \) uniquely by

\[
\omega - \eta(w) \in (n_{c,1} + \mathbf{k}^c)S(m+k) + S(m+k)(\mathbf{m}^c \cap \mathbf{p}^c + \mathbf{m}^c \cap \mathbf{n}^0 \cap \mathbf{p}^c) .
\]

Using Lemma 1 one can see that \( \delta_\lambda(x)u_\lambda = 0 \) for \( x \in n_{c,1} + \mathbf{k}^c \). Since also \( X_Lf = 0 \) for \( x \in \mathbf{m}^c + \mathbf{m}^c \cap \mathbf{n}^0 \), it follows then that

\[
(\omega^{\lambda})^2(m) = \eta(w)(u_\lambda | x)\omega^{\lambda}(m)
\]
as claimed in (14).

To finish the proof of Proposition 2 we prove that \( u_1(\lambda) = \gamma(u)(-\lambda - \rho) \) for all \( \lambda \in a_0^* \). Since \( \omega^{\lambda} \) generates the K-type \( \nu^{\lambda} \) in \( \text{Ind}_P^G(\pi \mathbf{M} \otimes \nu^{\lambda} \otimes 1) \) this follows immediately from Theorem 3 when (9) holds. Since \( u_1 \) and \( \gamma(u) \) are polynomials in \( \lambda \) the assertion holds for all \( \lambda \).

Theorem 2 (ii) follows immediately from Proposition 2.

Remark. It would be interesting if one could construct a G-homomorphism from the space

\[
\{ f \in C^\infty(G) \mid f(\text{gman}) = f(\text{g})e \quad \forall m \in (M \cap \mathbf{H})_e, \ a \in A, \ n \in N, \ g \in G \}
\]
to \( C^\infty(G/H) \), taking \( \omega^{\lambda} \) to \( \psi^{\lambda} \). In the special case of \( \sigma = \Theta \), \( \psi^{\lambda} \) is the spherical function, \( \mathbf{P} \) is a minimal parabolic and \( \omega^{\lambda} \) is the function \( g \rightarrow e^{\lambda, H(\mathbf{g})} \), and thus the homomorphism searched for is the Poisson transformation. In general the work of Oshima (cf. [7]) can probably be used to construct such a homomorphism.

8. Unitarity. Let \( \lambda \in \Lambda \) and consider the following condition on \( \lambda \)

\[
\langle \lambda | x, a^*_1 \rangle > 0 \quad \text{for all } a \in \Delta^+ \text{ with } a^*_1 \mathbf{n}^0 = 0 .
\]

Theorem 4. Assume (15), and moreover that \( \lambda \) is purely imaginary on \( a_0^* \cap \mathbf{p} \). Then \( T^{\lambda} \) is unitarizable.
Proof: Choose a parabolic subgroup $\tilde{P} = MAN$ with Langlands decomposition as indicated, such that $M = G^0 np$ and $P \subseteq \tilde{P}$. Then $\tilde{a}$ is $\tau$-invariant, and $\tilde{a} \cap q = a^0 np$ since $a^0$ centralizes $a^0$ and $a^0$ is maximal in $q$. $\tilde{M}$ is invariant under $\tau$ and $\tau$ is a maximal abelian subspace of $\tilde{m} \cap q$, and thus $\tilde{M}/(\tilde{M}H)_{e}$ satisfies equal rank. By (15) $\lambda_{\ell}$ determines a representation $\pi_{\lambda}$ in the discrete series of $\tilde{M}/(\tilde{M}H)_{e}$.

Observe that $a = (a\tilde{m}) \otimes \tilde{a}$. Put $\mathcal{Z} = \ell \cap \tilde{m}$, $\tilde{\mathcal{Z}} = \ell \cap \tilde{Z}$ and $\tilde{\mathcal{Z}} = \frac{1}{2} \text{Tr} \, \text{ad} \, \mathcal{L} \in (a\tilde{m})^*$. It is then easily seen that $\tilde{\mathcal{Z}} = \mathcal{Z} \cap a\tilde{m}$. Therefore $\pi_{\lambda}$ is a subquotient of $\text{Ind}_{\tilde{M}H}^{\tilde{M}}(\pi_{\lambda} \otimes \mathcal{L} | a\tilde{m} \otimes 1)$ by Theorem 2, and using induction by stages and Theorem 2 once more we get that $T_{\lambda}$ is a subquotient of $\text{Ind}_{\mathcal{P}}^{\tilde{M}}(\pi_{\lambda} \otimes \mathcal{L} | \tilde{a} \otimes 1)$.

Now $\tilde{a} = a \cap H \otimes a^0 np$ and $\mathcal{Z} | a\tilde{m} = 0$, therefore $\mathcal{L} | \tilde{a}$ is purely imaginary by (5), and the theorem follows.

Remark. Theorem 4 was proved for the fundamental series for large values of $\lambda$ by Ólafsson ([5]).

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