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NOLAN R. WALLACH

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THE ASYMPTOTIC BEHAVIOR OF HOLOMORPHIC REPRESENTATIONS

By

Nolan R. Wallach*

Abstract. In this article the Jacquet module of a holomorphic representation is computed by a direct and elementary method. The preliminary results involve the study of the notion of opposite parabolic subalgebra.

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Introduction.

Let $\underline{g}_o$ be a semi-simple Lie algebra over R. Let $\theta:\underline{g}_o \to \underline{g}_o$ be a Cartan involution of $\underline{g}_o$ and let $\underline{k}_o = \{X \in \underline{g}_o \mid \theta X = X\}$. Let $\underline{g}$ (resp. $\underline{k}$) denote the complexification of $\underline{g}_o$ (resp. $\underline{k}_o$). Then a $(\underline{g}, \underline{k})$ - module, M, is said to be holomorphic if there is a θ-stable Borel subalgebra, $\underline{b}$, such that M is in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for $\underline{b}$. It is not hard to show that if there exists an infinite dimensional holomorphic $(\underline{g},\underline{k})$-module, then $\underline{g}_o$ contains a θ-stable simple ideal $\underline{g}_o'$ such that $(\underline{g}_o', \underline{k}_o \cap \underline{g}_o')$ is a symmetric pair of Hermitian type.

The purpose of this article is to give a description of the Jacquet module of an irreducible holomorphic representation. No doubt many of the results of this article are known to several specialists in the field (for example, Casselman and Zuckerman have communicated certain less precise results to us). However, there is no place in the literature where one can find a reference. The importance of these results now stems from the fact that the unitarizable holomorphic representations have been classified ([1]).

As it turns out the determination of the Jacquet module of a holomorphic representation is relatively easy once one understands the notion of "opposite parabolic". The first half of this paper is devoted to a rather detailed study of opposite parabolics and the relationship between their categories $\mathcal{O}$. We give here an example for $\underline{s\ell}(2,\mathbb{C}) = \underline{g}$. Let

$$H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then $\underline{b} = \mathbb{C}H + \mathbb{C}X$ and $\underline{b}' = \mathbb{C}h + \mathbb{C}x$ are opposite parabolics. Indeed

$$\underline{b} \cap \underline{b}' = \mathbb{C}\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix}.$$

Using $\mathbb{C}\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix}$ as a Cartan subalgebra of $\underline{g}$ one sees that $\underline{b}'$ is the opposite Borel subalgebra in the usual sense.

HOLOMORPHIC REPRESENTATIONS

As a consequence of our results in the second part of this paper we describe the Langlands parameters of holomorphic representations. Collingwood has informed us that he knows how to do this also (although we have not seen his results or methods).

1. Remarks on the Category $\mathcal{O}$.

Let $\underline{g}$ be a reductive Lie algebra over $\mathbb{C}$. Let $\underline{b} \subset \underline{g}$ be a Borel subalgebra of $\underline{g}$. We define the category $\mathcal{O}'(\underline{b})$ to be the subcategory of the category $\mathcal{M}(\underline{g})$ of $\underline{g}$-modules consisting of those $M \in \mathcal{M}(\underline{g})$ such that

(1) M is finitely generated as a $U(\underline{g})$-module.

(2) If $m \in M$, then $\dim U(\underline{b})\cdot m < \infty$.

<u>Lemma 1.1</u>. Let $\underline{h} \subset \underline{b}$ be a Cartan subalgebra of $\underline{b}$. If $M \in \mathcal{O}'(\underline{b})$ and $\underline{h}$ acts semi-simply on M then if $\underline{h}' \subset \underline{b}$ is a Cartan algebra of $\underline{g}$ then $\underline{h}'$ acts semi-simply on M.

<u>Proof</u>. Let $\underline{n}(\underline{b})$ be the nil radical of $\underline{b}$. Then there is $X \in \underline{n}(\underline{b})$ such that $e^{\mathrm{ad}X}\cdot\underline{h} = \underline{h}'$. By (2) we see that X acts locally nilpotently on M (i.e. if $m \in M$ there is $k = k(m)$ such that $X^{k(m)}m = 0$). Thus if $t \in \mathbb{C}$ we can form

$$T(t)\cdot m = \sum (t^n/n!)X^n m\ , \quad m \in M\ .$$

The sum is actually finite for all $m \in M$. By the obvious formal relations one has

$$T(t+S) = T(t)\ T(S)$$

$$\text{So}\ \ T(-1)\ T(1) = T(1)\ T(-1) = I\ .$$

Set $T = T(1)$. Then T is bijective on M. Also if $Y \in \underline{g}$ then

$$T(t)\, Y\, m = (e^{t\,\mathrm{ad}X}\, Y)\, T(t)\, m\ .$$

Hence

$$Th\cdot m = (e^{\mathrm{ad}X}\, h)Tm\ , \quad h \in \underline{h}\ .$$

This clearly implies the result.

We can thus define the category $\mathcal{O}(\underline{b})$ to be the subcategory of $\mathcal{O}'(\underline{b})$ consisting of these objects M of $\mathcal{O}'(\underline{b})$ that are semi-simple relative to some Cartan subalgebra $\underline{h} \subset \underline{b}$.

Let $\mathcal{B}$ denote the space of all Borel subalgebras of $\underline{g}$. If $\underline{b} \in \mathcal{B}$ we denote by $\bar{\mathcal{B}}(\underline{b})$ the subset of all $\bar{\underline{b}} \in \mathcal{B}$ such that $\bar{\underline{b}}\ \ \underline{b}$ is a Cartan subalgebra of $\underline{g}$.

We describe $\bar{\mathcal{B}}(\underline{b})$. Let $\underline{h} \subset \underline{b}$ be a Cartan subalgebra. Let $\Phi(\underline{g},\underline{h})$ be the root system of $\underline{g}$ with respect to $\underline{h}$. Let Φ^+ be the system of positive roots of $\Phi(\underline{g},\underline{h})$ corresponding to $\underline{b}$. Then $\underline{n}(\underline{b}) = \bigoplus_{\alpha \in \Phi^+} \underline{g}_\alpha$. Set $\bar{\underline{n}}(\underline{b},\underline{h}) = \sum_{\alpha \in \Phi^+} \underline{g}_{-\alpha}$. Then $\bar{\underline{b}}(\underline{b},\underline{h}) = \underline{h} + \bar{\underline{n}}(\underline{b},\underline{h}) \in \mathcal{B}$. Clearly $\bar{\underline{b}} \cap \underline{b} = \underline{h}$.

Lemma 1.2. Let $\underline{b} \in \mathcal{B}$. Then the following statements are equivalent

(1) $\underline{b}' \in \bar{\mathcal{B}}(\underline{b})$,

(2) there is $\underline{h} \subset \underline{b}$ such that $\underline{b}' = \underline{h} + \bar{\underline{n}}(\underline{b},\underline{h})$,

(3) $\underline{b}' \in \mathcal{B}$ and $\underline{b}' + \underline{b} = \underline{g}$.

Proof. (1) implies (2). Let $\underline{b}' \in \bar{\mathcal{B}}(\underline{b})$. Let $\underline{h} = \underline{b}' \quad \underline{b}$. Then $\underline{n}(\underline{b}) = \bigoplus_{\alpha \in \Phi^+} \underline{g}_\alpha$, $\Phi^+ \subset \Phi(\underline{g},\underline{h})$ a system of positive roots. $\underline{n}(\underline{b}') = \bigoplus_{\alpha \in \Psi} \underline{g}_\alpha$, $\Psi \subset \Phi(\underline{g},\underline{h})$ a system of positive roots. Since $\underline{b}' \cap \underline{b} = \underline{h}$, $\Psi \cap \Phi^+ = \emptyset$. So $\Psi = -\Phi^+$.

That (2) implies (3) is clear.

To prove (3) implies (1) we note that $\dim(\underline{b}'+\underline{b}) = \dim \underline{b}' + \dim \underline{b} - \dim(\underline{b} \cap \underline{b}')$. $\dim \underline{b} = \ell + \frac{\dim \underline{g} - \ell}{2}$ for $\underline{b} \in \mathcal{B}$. Here ℓ is the rank of $\underline{g}$. Thus if $\underline{b}+\underline{b}' = \underline{g}$ $\dim \underline{g} = \dim \underline{g} + \ell - \dim(\underline{b} \cap \underline{b}')$. So $\dim \underline{b} \cap \underline{b}' = \ell$. The Bruhat lemma implies that $\underline{b} \cap \underline{b}'$ contains a Cartan subalgebra of $\underline{g}$. Hence $\underline{b} \cap \underline{b}'$ is a Cartan subalgebra of $\underline{g}$.

If $M \in \mathcal{M}$ we define

$$j_{\underline{b}}(M) = \{\lambda \in M^* \mid \underline{n}(\underline{b})^k \cdot \lambda = 0 \quad \text{for some } k\} .$$

Since $\underline{n}(\underline{b})$ acts nilpotently on $\underline{g}$ under ad it is trivial to see that

(3) $j_{\underline{b}}(M)$ is a $\underline{g}$ submodule of M^*.

Let $\underline{m} \subset \underline{g}$ be a Lie subalgebra such that the action of $\underline{m}$ on $\underline{g}$ relative to ad is completely reducible. Let $\mathcal{C}(\underline{g},\underline{m})$ denote the category of all $(\underline{g},\underline{m})$-modules. That is $M \in \mathcal{C}(\underline{g},\underline{m})$ if as an $\underline{m}$-module M splits into a direct sum of irreducible finite dimensional $\underline{m}$-modules. If $M \in \mathcal{C}(\underline{g},\underline{m})$ then we say that M is admissible if for every finite dimensional $\underline{m}$-module, F,

$$\dim \operatorname{Hom}_{\underline{m}}(F,M) < \infty .$$

Let $\mathcal{A}(\underline{g},\underline{m})$ denote the full category of all admissible objects in $\mathcal{C}(\underline{g},\underline{m})$. If $M \in \mathcal{A}(\underline{g},\underline{m})$ we define $\tilde{M}$ (the admissible dual of M) by

$$\tilde{M} = \{\lambda \in M^* \mid \dim U(\underline{m})\cdot\lambda < \infty\} .$$

Lemma 1.3. The functor $M \rightsquigarrow \tilde{M}$ defines an equivalence of categories between $\mathcal{A}(\underline{g},\underline{m})$ and $\mathcal{A}(\underline{g},\underline{m})^{opp}$ (the opposite category).

Proof. Let $\hat{\underline{m}}$ denote the set of equivalence classes of irreducible finite dimensional representations of $\underline{m}$. If $M \in \mathcal{A}(\underline{g},\underline{m})$, $\gamma \in \hat{\underline{m}}$ let $M(\gamma)$ denote the sum of all $\underline{m}$-submodules of M that are in γ. Then admissibility implies that $\dim M(\gamma) < \infty$. Now it is easy to see that $\tilde{M} = \bigoplus_{\gamma} M(\gamma)^*$ as an $\underline{m}$-module. Thus $\tilde{M} \in \mathcal{A}(\underline{g},\underline{m})$. It is also trivial to check that $(\tilde{M})^{\sim} = M$ for $M \in \mathcal{A}(\underline{g},\underline{m})$ (in the sense that $M \subset (M^*)^*$ naturally), and $M \rightsquigarrow \tilde{M}$ is an exact functor from $\mathcal{A}(\underline{g},\underline{m})$ to $\mathcal{A}(\underline{g},\underline{m})$. The result now follows.

We note that if $\underline{b} \in \mathcal{B}$ and if $\underline{h} \subset \underline{b}$ is a Cartan subalgebra then

$$\mathcal{O}(\underline{b}) = \mathcal{O}'(\underline{b}) \cap A(\underline{g},\underline{h}).$$

Lemma 1.4. Let $\underline{b} \in \mathcal{B}$, $\underline{b}' \in \bar{\mathcal{B}}(\underline{b})$. If $M \in \mathcal{O}(\underline{b})$ then $j_{\underline{b}}(M) = \tilde{M}$ where $\tilde{M}$ is defined relative to $\underline{h} = \underline{b} \cap \underline{b}'$.

Proof. $\underline{b} = \underline{h} + \underline{n}$, $\underline{b}' = \underline{h} + \bar{\underline{n}}(\underline{b},\underline{h}) = \underline{h} + \bar{\underline{n}}$. Since $M \in \mathcal{O}'(\underline{b})$, M is finitely generated as a $U(\bar{\underline{n}})$-module. Now $j_{\underline{b}'}(M) = \bigcup_{k=1}^{\infty} (M/\bar{\underline{n}}^k M)^*$. Thus since $\dim M/\bar{\underline{n}}^k M < \infty$, $j_{\underline{b}'}(M) \subset \tilde{M}$. Let $M = \bigoplus_{\xi \in \underline{h}^*} M(\xi)$ $(\underline{h}^* = \hat{\underline{h}})$. If $\lambda \in \tilde{M}$ then $\lambda = \sum_{\xi \in S_\lambda} \lambda_\xi$, $S_\lambda \subset \underline{h}^*$ a finite set, $\lambda_\xi \in M(\xi)^*$. We note that $(\bar{\underline{n}}^k M) \cap M(\xi) = (0)$, $\xi \in S_\lambda$ for k sufficiently large. Thus $\bar{\underline{n}}^k \lambda = 0$ for k sufficiently large. Hence $\tilde{M} \subset j_{\underline{b}'}(M)$.

It is well known that if $M \in \mathcal{O}(\underline{b})$ then M has finite length. Thus if $\bar{\underline{h}} \subset \underline{b}$ is a Cartan subalgebra, $\tilde{M}$ is also finitely generated. Combining this observation with the previous results we have

Proposition 1.5. Let $\underline{b} \in \mathcal{B}$. Let $\underline{b}' \in \bar{\mathcal{B}}(\underline{b})$. Then

(1) If $M \in \mathcal{O}(\underline{b})$, $j_{\underline{b}'}(M) \in \mathcal{O}(\underline{b}')$.

(2) The functor $M \mapsto j_{\underline{b}'}(M)$ defines an equivalence of categories between $\mathcal{O}(\underline{b})$ and $\mathcal{O}(\underline{b}')^{\mathrm{opp}}$.

Let now $\underline{q} \subset \underline{g}$ be a parabolic subalgebra. Let $\mathcal{B}(\underline{q}) = \{\underline{b} \in \mathcal{B} \mid \underline{b} \subset \underline{q}\}$. We set $\mathcal{O}(q) = \bigcap_{\underline{b} \in \mathcal{B}(\underline{q})} \mathcal{O}(\underline{b})$.

Lemma 1.6. Let $\underline{q} = \underline{m} + \underline{n}(\underline{q})$ ($\underline{n}(\underline{q})$ the nilradical of $\underline{q}$ and $\underline{m}$ a Levi factor of $\underline{q}$). Let $\underline{h} \subset \underline{m}$ be a Cartan subalgebra of $\underline{m}$. Let $\underline{b} \in \mathcal{B}(\underline{q})$ be such that $\underline{h} \subset \underline{b}$. Then

$$\mathcal{O}(\underline{q}) = \mathcal{O}(\underline{b}) \cap C(\underline{g},\underline{m}).$$

Proof. Let $M \in \mathcal{O}(\underline{q})$. $\underline{b} = \underline{h} \oplus \bigoplus_{\alpha \in \Phi^+} \underline{g}_\alpha$, $\Phi^+ \subset \Phi(\underline{g},\underline{h})$ a system of positive roots. $\underline{m} = \underline{h} \oplus \bigoplus_{\alpha \in \Phi(\underline{m},\underline{h})} \underline{g}_\alpha$. Let $\sum = \Phi^+ - \Phi(\underline{m},\underline{h})$. Then $\underline{b}' = \underline{h} \oplus \bigoplus_{\alpha \in \sum} \underline{g}_{\alpha} \oplus \bigoplus_{\alpha \in \Phi^+ - \sum} \underline{g}_\alpha \in \mathcal{B}(\underline{q})$. Set $\underline{n}' = \bigoplus_{\alpha \notin \sum} \underline{g}_\alpha$, $\bar{\underline{n}}' = \bigoplus_{\alpha \notin \sum} \underline{g}_{-\alpha}$. Then

$$\underline{m} = \bar{\underline{n}}' + \underline{h} + \underline{n}'.$$

If $m \in M$ then $\dim U(\underline{b}'')m < \infty$, $\underline{b}'' \in \mathcal{B}(\underline{q})$. Thus $\dim U(\underline{m})m < \infty$ since $\bar{\underline{n}}' \subset \underline{b}'$, $\underline{h} + \underline{n}' \subset \underline{b}$. Since $\underline{h}$ acts semi-simply on M we see that

$$M \in \mathcal{O}(\underline{b}) \cap C(\underline{g},\underline{m}).$$

Let $M \in \mathcal{O}(\underline{b}) \cap \mathcal{C}(\underline{g}.\underline{m})$. Let $\underline{b}' \in \mathcal{B}(\underline{g})$. Since $\underline{m}+\underline{b} = \underline{g}$ we see that if $m \in M$, $\dim U(\underline{g})\cdot m < \infty$. Thus $\dim U(\underline{b}')m < \infty$, $m \in M$. Since $\underline{b} \cap \underline{b}' \supset \underline{h}'$, a Cartan subalgebra of $\underline{g}$, Lemma 1.1 implies $M \in \mathcal{O}(\underline{b}')$.

Let now $\mathcal{P}$ be the set of all parabolic subalgebras of $\underline{g}$. If $\underline{q} \in \mathcal{P}$ define $\bar{\mathcal{P}}(\underline{q})$ to be the set of all $\underline{q}' \in \mathcal{P}$ such that $\underline{q} \cap \underline{q}'$ is a Levi factor of both $\underline{q}$ and $\underline{q}'$.

We describe the elements of $\bar{\mathcal{P}}(\underline{q})$. Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$, $\underline{h} \subset \underline{q}$. Let $\underline{n}(\underline{q})$ be the nilradical of $\underline{q}$. Let $\Phi(\underline{q},\underline{h}) = \{\alpha \in \Phi \mid \underline{g}_\alpha \subset \underline{q}\}$. Set

$$\Psi(\underline{q},\underline{h}) = \{\alpha \in \Phi(\underline{q},\underline{h}) \mid \underline{g}_\alpha \subset \underline{n}(\underline{q})\} .$$

Let $\underline{m} = \underline{m}(\underline{q},\underline{h}) = \underline{h} \oplus \bigoplus_{\alpha \in \Phi(\underline{q},\underline{h})-\Psi(\underline{q},\underline{h})} \underline{g}_\alpha$. Then $\underline{m}$ is a Levi factor of $\underline{q}$.

Set $\bar{\underline{n}}(\underline{q},\underline{h}) = \bigoplus_{\alpha \in \Psi(\underline{q},\underline{h})} \underline{g}_{-\alpha}$. Then $\bar{\underline{q}} = \underline{m} \oplus \bar{\underline{n}}(\underline{q},\underline{h}) \in \bar{\mathcal{P}}(\underline{q})$.

Lemma 1.6. Let $q \in \mathcal{P}$. Then $\underline{q}' \in \bar{\mathcal{P}}(\underline{q})$ if and only if $\underline{q}' \in \mathcal{P}$ and there exists $\underline{h} \subset \underline{q}$ a Cartan subalgebra of $\underline{g}$ such that $\underline{q}' = \underline{m}(\underline{q},\underline{h}) \oplus \bar{\underline{n}}(\underline{q},\underline{h})$.

Proof. Let $\underline{q}' \in \bar{\mathcal{P}}(\underline{q})$. Let $\underline{m} = \underline{q}' \cap \underline{q}$. Let $\underline{h} \subset \underline{m}$ be a Cartan subalgebra of $\underline{g}$. Then $\underline{m} = \underline{m}(\underline{q},\underline{h})$, $\underline{n}(\underline{q}') = \bigoplus_{\alpha \in \Psi(\underline{q}',\underline{h})} \underline{g}_\alpha$, $\underline{n}(\underline{q}) = \bigoplus_{\alpha \in \Psi(\underline{q},\underline{h})} \underline{g}_\alpha$. Let $\underline{b}_1 \supset \underline{h}$ be a Borel subalgebra of $\underline{m}$. Let $\bar{\underline{b}}_1(\underline{b},\underline{h})$ be as above. Then $\underline{b}_1 \oplus \underline{n}(\underline{q}) = \underline{b}$ and $\bar{\underline{b}}_1 \oplus \underline{n}(\underline{q}') = \underline{b}'$ are Borel subalgebra of $\underline{g}$. Now $\underline{b}' \cap \underline{b} \subset \underline{q}' \cap \underline{q} = \underline{m}$. Hence $\underline{b}' \cap \underline{b} = \bar{\underline{b}}_1 \cap \underline{b}_1 = \underline{h}$. Thus $\underline{b}' \in \bar{\mathcal{B}}(\underline{b})$. Hence $\Phi(\bar{\underline{b}}_1,\underline{h}) \cup \Psi(\underline{q}',\underline{h}) = -\Phi(\underline{b},\underline{h}) \cup -\Psi(\underline{q},\underline{n})$. Thus $\Psi(\underline{q}',\underline{h}) = -\Psi(\underline{q},\underline{h})$. Hence $\underline{n}(\underline{q}') = \bar{\underline{n}}(\underline{q},\underline{h})$. Q.E.D.

If $\underline{q} \in \mathcal{P}$, $M \in \mathcal{M}$ define

$$j_{\underline{q}}(M) = \{\lambda \in M^* \mid \underline{n}(\underline{q})^k \cdot \lambda = 0 \text{ for some } k\} .$$

Then $j_{\underline{q}} : \mathcal{M} \rightsquigarrow \mathcal{M}^{opp}$ is a functor.

Lemma 1.7. Let $\underline{q} \in \mathcal{P}$, $\underline{q}' \in \bar{\mathcal{P}}(\underline{q})$ and $\underline{q}'' \in \mathcal{P}$ such that $\underline{q}' \supset \underline{q}''$. Then

$$j_{\underline{q}'}(M) = j_{\underline{q}''}(M) \text{ for all } M \in \mathcal{O}(\underline{q}) .$$

Proof. Since $\underline{n}(\underline{q}'') \supset \underline{n}(\underline{q}')$ it is clear that $j_{\underline{q}''}(M) \subset j_{\underline{q}'}(M)$.

Let $\underline{h} \subset \underline{q}$ be a Cartan subalgebra such that $\underline{q}' = \underline{m}(\underline{q},\underline{h}) + \bar{\underline{n}}(\underline{q},\underline{h})$.

Since $\underline{g} = \bar{\underline{n}}(\underline{q},\underline{h}) + \underline{m}(\underline{q},\underline{h}) + \underline{n}(\underline{q})$, Lemma 1.6 implies that M is finitely generated as a $U(\underline{n}(\underline{q}'))$-module. Thus

$$\dim M/\underline{n}(\underline{q}')^k M < \infty , \quad k = 1,2,\ldots,$$

This implies that $(\underline{m} = \underline{m}(\underline{q},\underline{h}))$,

(1) If $\lambda \in j_{\underline{q}'}(M)$ then $\dim U(\underline{m})\cdot\lambda < \infty$.

Now $\underline{n}(\underline{q}'') = \underline{n}(\underline{q}'') \cap \underline{m} + \underline{n}(\underline{q}')$. Using (1) we see that if $\lambda \in j_{\underline{q}'}(M)$ then

$$(\underline{n}(\underline{g}")\cap \underline{m})^{k}\cdot\lambda = 0 \quad \text{for some } k \quad .$$

Thus $\underline{n}(\underline{g}")^{k'}\cdot\lambda = 0$ for some k'. Thus $j_{\underline{g}'}(M) \subset j_{\underline{g}"}(M)$.

Proposition 1.8. Let $\underline{g} \in P$, $\underline{g}' \in \bar{P}(\underline{g})$. Then the functor $M \rightsquigarrow j_{\underline{g}'}(M)$ is an equivalence of categories between $\mathcal{O}(\underline{g})$ and $\mathcal{O}(\underline{g}')^{opp}$.

Proof. Let $\underline{b}' \in \mathcal{B}(\underline{g}')$. Then there exists $\underline{b} \in \mathcal{B}(\underline{g})$ such that $\underline{b}' \in \bar{\mathcal{B}}(\underline{b})$. By Lemma 1.7, $j_{\underline{g}'}(M) = j_{\underline{b}'}(M) \in \mathcal{O}(\underline{b}')$. Hence $j_{\underline{g}'}(M) \in \mathcal{O}(\underline{g}')$. The rest follows from Lemma 1.7 and Proposition 1.5.

The last results we need have to do with parametrizing the irreducibile objects $\mathcal{O}(\underline{b})$ for $\underline{b} \in \mathcal{B}$. Let $L \in \mathcal{O}(\underline{b})$ be irreducible. Then

$$\dim L^{\underline{n}(\underline{b})} = \{v \in L \mid \underline{n}(\underline{b})\ v = 0\} = 1 \quad .$$

Hence $\underline{b}/\underline{n}(\underline{b})$ acts on $L^{\underline{n}(\underline{b})}$ by $\Lambda \in (\underline{b}/\underline{n}(\underline{b}))^*$. As is well known Λ determines L up to equivalence. If $\Lambda \in (\underline{b}/\underline{n}(\underline{b}))^*$ then Λ defines a one dimensional representation C_Λ of $\underline{b}$. Let $M^\Lambda = U(\underline{g}) \underset{U(\underline{b})}{\otimes} C_\Lambda$ (the usual Verma module). Let L^Λ be the irreducible non-zero quotient of M^Λ. Then $(L^\Lambda)^{\underline{n}(\underline{b})} = C_\Lambda$ as a $\underline{b}$-module.

We set $L^\Lambda = L^\Lambda_{\underline{b}}$ for $\Lambda \in (\underline{b}/\underline{n}(\underline{b}))^*$.

If $\underline{h} \subset \underline{b}$ is a Cartan subalgebra then $\underline{b}/\underline{n}(\underline{b}) \cong \underline{h}$. Thus we may look upon the roots of $\underline{h}$ on $\underline{b}$ as elements of $(\underline{b}/\underline{n}(\underline{b}))^*$ denoted $\Phi(\underline{b})$. We can also pull back the Killing form to get a non-degenerate form on $\underline{b}/\underline{n}(\underline{b})$ which is clearly independent of the choice of $\underline{h}$. We therefore get a form (,) on $(\underline{b}/\underline{n}(\underline{b}))^*$.

Let now $\underline{g} \in P$. Then $\mathcal{O}(\underline{g}) = \bigcap_{\underline{b} \in \mathcal{B}(\underline{g})} \mathcal{O}(\underline{b})$. If $\underline{b} \in \mathcal{B}(\underline{g})$ then $\underline{b}/\underline{n}(\underline{g}) \in \mathcal{B}(\underline{g}/\underline{n}(\underline{g}))$. Clearly $\Phi(\underline{b}/\underline{n}(\underline{g})) \subset \Phi(\underline{b})$. One has $L^\Lambda_{\underline{b}} \in \mathcal{O}(\underline{g})$ if and only if

$$2(\Lambda,\alpha)/(\alpha,\alpha) \in \mathbb{N} = \{0,1,2,\ldots\}$$

for $\alpha \in \Phi(\underline{b}/\underline{n}(\underline{g}))$.

It is also convenient to choose $\underline{h} \subset \underline{b}$, $\underline{h}$ a Cartan subalgebra of $\underline{b}$. Then if $\Lambda \in \underline{h}^*$ we may extend Λ to $\underline{b}$ by $\Lambda(\underline{n}(\underline{b})) = 0$. Thus we have $L^\Lambda_{\underline{b}} = L^\Lambda_{\underline{b},\underline{h}}$, $\Lambda \in \underline{h}^*$. Let $\underline{b}' \in \bar{\mathcal{B}}(\underline{b})$. And let C be an inner automorphism of $\underline{g}$ such that $C\,\bar{\underline{b}}(\underline{b},\underline{h}) = \underline{b}'$. Then $\underline{b}' = C\underline{h} \oplus C\,\bar{\underline{n}}(\underline{b},\underline{n})$.

Lemma 1.9. $j_{\underline{b}'}(L^\Lambda_{\underline{b},\underline{h}}) \cong L^{-\Lambda\cdot C^{-1}}_{\underline{b}',C\underline{h}}$.

Proof. Proposition 1.5 implies that $L = j_{\underline{b}'}(L^\Lambda_{\underline{b}.\underline{h}})$ is irreducible and in $\mathcal{O}(\underline{b}')$. It is a simple matter to see that $L \cong L^{-\Lambda}_{\underline{b}'}$. Thus we need only see that $L^{-\Lambda}_{\underline{b}'} = L^{-\Lambda\cdot C^{-1}}_{\underline{b}',C\underline{h}}$. But $C\underline{h}$ acts on $(L^{-\Lambda}_{\underline{b}'})^{\underline{n}(\underline{b}')}$ by $-\Lambda$ pulled back from $\underline{b}'/\underline{n}(\underline{h}')$ to $C\underline{h}$. The lemma now follows.

2. Applications to Holomorphic Representations.

Let $\underline{g}_o$ be a real reductive Lie algebra. Let $\theta: \underline{g}_o \to \underline{g}_o$ be a Cartan involution. Let $\underline{k}_o = \{X \in \underline{g}_o \mid \theta X = X\}$, $\underline{r}_o = \{X \in \underline{g}_o \mid \theta X = -X\}$. We apologize to the reader for using $\underline{r}$ in place of the more customary $\underline{p}$ in order to save the notation $\underline{p}$ for parabolic subalgebras. If $\underline{c}_o$ is a Lie algebra or vector space over R we denote by $\underline{c}$ the complexification of $\underline{c}_o$. We say that $(\underline{g}_o, \underline{k}_o)$ is an irreducible symmetric pair if the action of $\underline{k}_o$ on $\underline{r}_o$ under ad is irreducible. We say that an irreducible symmetric pair is Hermitian if the action of $\underline{k}$ on $\underline{r}$ reduces. As is well known this is equivalent to saying that there exists $H \in \underline{k}$ which is central in $\underline{k}$ and

(1) $\text{ad } H\big|_{\underline{r}}$ has eigenvalues 1 and -1.

Set $\underline{r}^+ = \{X \in \underline{r} \quad \text{ad}H \cdot X = X\}$. Then $\underline{q} = \underline{k} + \underline{r}^+$ is a parabolic subalgebra of $\underline{g}$. Fix $\underline{b}_k \subset \underline{k}$ a Borel subalgebra. Then $\underline{b} = \underline{b}_k + \underline{r}^+$ is a Borel subalgebra of $\underline{g}$ which is θ-stable. Fix $\underline{h} \subset \underline{b}_k$ a Cartan subalgebra of $\underline{g}$. Let $\Phi^+ = \Phi(\underline{b}, \underline{h})$, $\Phi_k^+ = \Phi(\underline{b}_k, \underline{h})$, $\Phi_n = \{\alpha \in \Phi(\underline{g}, \underline{h}) \mid \alpha(H) \neq 0\}$, $\Phi_n^+ = \Phi_n \quad \Phi^+ = \Psi(\underline{g}, \underline{h})$. Fix a linear order on $\Phi = \Phi(\underline{g}, \underline{h})$ such that $\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\}$. We define $\gamma_1 > \dots > \gamma_r$ by the usual recipe. γ_1 is the largest element of Φ_n^+. $\Psi_1 = \{\alpha \in \Phi_n^+ \mid \alpha \pm \gamma_1 \notin \Phi \cup \{0\}\}$. If $\Psi_1 \neq \emptyset$, γ_2 is the largest element of Ψ_1 and $\Psi_2 = \{\alpha \in \Psi_1 \mid \alpha \pm \gamma_2 \notin \Phi \cup \{0\}\}$, if γ_{j-1} and Ψ_{j-1} have been defined as above then γ_j is the largest element of Ψ_{j-1} and $\Psi_j = \{\alpha \in \Psi_{j-1} \mid \alpha \pm \gamma_j \notin \Psi \cup \{0\}\}$. This gives $\Psi_1 \supset \Psi_2 \supset \dots \supset \Psi_{r-1} \supset \Psi_r = \emptyset$.

Set $\underline{g}_{\gamma_j} + \underline{g}_{-\gamma_j} + [\underline{g}_{\gamma_j}, \underline{g}_{-\gamma_j}] = \underline{\ell}_j$. Then $\underline{\ell}_j$ is isomorphic with $\underline{s\ell}(2,\mathbb{C})$.

Clearly, $\theta \underline{\ell}_j = \underline{\ell}_j$. If $(\underline{\ell}_j)_o = \underline{\ell}_j \cap \underline{g}_o$. Then it is easy to see that there exists an isomorphism $\eta_j : (\underline{\ell}_j)_o \to \underline{s\ell}(2, \mathbb{R})$ such that $\eta_j(\theta X) = -{}^t\eta_j(X)$. We can thus choose $H_j \in [\underline{g}_{\gamma_j}, \underline{g}_{-\gamma_j}]$, $X_j \in \underline{g}_{\gamma_j}$, $Y_j \in \underline{g}_{-\gamma_j}$ such that

$$\eta_j(H_j) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\eta_j(X_j) = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

$$\eta_j(Y_j) = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

Then

$$\eta_j(X_j+Y_j) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Set $\underline{a}_o = \sum_{i=1}^{r} \mathbb{R}(X_i+Y_i)$. Then $\underline{a}_o \subset \underline{r}_o$. It is part of the standard theory of symmetric pairs that $\underline{a}_o$ is a maximal abelian subalgebra of $\underline{g}$ subject to the condition that it is contained in $\underline{r}_o$.

Let C (the Cayley transform) be defined by

$$C = \prod_{i=1}^{r} \exp(\mathrm{ad}X_i)\ \exp(\tfrac{1}{2}(\log 2)\ \mathrm{ad}H_i)\ \exp(-\mathrm{ad}Y_i)$$

Then $CH_i = -(X_i+Y_i)$, $i = 1,\ldots,r$.

It is also standard that $C\bar{\underline{n}}(\underline{b},\underline{h}) \cap \underline{g}_o = \underline{n}_o$ is a maximal nilpotent subalgebra of $[\underline{g}_o,\underline{g}_o]$. Set $\bar{\underline{b}} = \bar{\underline{b}}(\underline{b},\underline{h}) = \underline{h} + \bar{\underline{n}}(\underline{b},\underline{h})$. Then the formula for C immediately implies

(2) $C\bar{\underline{b}} \in \bar{\mathcal{B}}(\underline{b})$.

Now applying Lemma 1.9 we have

(3) $j_{C\underline{b}}(L^{\Lambda}_{\underline{b},\underline{h}}) \cong L^{-\Lambda\circ C^{-1}}_{C\underline{b},C\underline{h}}$.

Define $\underline{m}_o = C_{\underline{g}_o}(\underline{a}_o) = \{X \in \underline{g}_o \mid [X,\underline{a}_o] = 0\}$. Then $\underline{m}_o \oplus \underline{n}_o = \underline{p}_o$ is a minimal parabolic subalgebra of $\underline{g}_o$. One checks that

(4) $\underline{p} \supset C\bar{\underline{b}}$.

Set $\bar{\underline{q}} = \underline{k} + \underline{r}^-$, $\underline{p}' = C\bar{\underline{q}}$. Then

(5) $\underline{p}' \in \mathcal{P}(\underline{q})$ by the formula for C.

Applying Lemma 1.7 we have

(6) If $M \in \mathcal{O}(\underline{q})$ then
$j_{\underline{p}'}(M) = j_{C\bar{\underline{b}}}(M)$.

<u>Lemma 2.1</u>. $\underline{p}' \supset \underline{p}$.

<u>Proof</u>. Let $\tilde{\underline{m}}_1 = \{X \in \underline{g} \mid [X,H_i] = 0,\ i = 1,\ldots,r\}$. Let $\Psi = \Phi(\tilde{\underline{m}}_1,\underline{h})$. If $\alpha,\beta \in \Phi_n^+$ then $(\alpha+\beta)(H) = 2$. Thus $\alpha+\beta \notin \Phi$. This implies that if $\alpha,\beta \in \Phi_n$ and $(\alpha,\beta) = 0$ then $\alpha\pm\beta \notin \Phi\cup\{0\}$. Thus $\Psi \cap \Phi_n^+ \supset \cap \Psi_i = \emptyset$. So $\Psi \subset \Phi_k$. Hence $\tilde{\underline{m}}_1 \subset \underline{k}$. Now $\underline{m}_1 = C\tilde{\underline{m}}_1$. Thus $\underline{p} \subset C(\tilde{\underline{m}}_1+\bar{\underline{b}}) \subset C\bar{\underline{q}} = \underline{p}'$.

Now Lemma 1.8 implies

Lemma 2.2. If $M \in \mathcal{O}(\underline{g})$ then $j_{\underline{p}}(M) = j_{\underline{p}'}(M)$.

Lemma 1.8 and (5) now imply

Proposition 2.3. $j_{\underline{p}}: \mathcal{O}(\underline{g}) \hookrightarrow \mathcal{O}(\underline{p}')$ ($\mathcal{O}(\underline{p})$) defines an equivalence of categories between $\mathcal{O}(\underline{g})$ and $\mathcal{O}(\underline{p}')^{opp}$.

Let now $(\underline{p}")_o \subset \underline{g}_o$ be a subalgebra containing $\underline{p}_o$. Then $\underline{p}"$ is a parabolic subalgebra of $\underline{g}$. Let $\underline{p} = \underline{m} \oplus \underline{n}$ as above, then $\underline{p}" = \underline{m}" \oplus \underline{n}(\underline{p}")$ with $\underline{m}"$ a θ-stable Levi factor of $\underline{p}"$ and $\underline{m}" \supset \underline{m}$, $\underline{n}(\underline{p}") \subset \underline{n}$. Set ${}^*\underline{p}" = \underline{p} \cap \underline{m}"$.

Lemma 2.4. If $L \in \mathcal{O}(\underline{p})$ is irreducible then

(1) $L/\underline{n}(\underline{p}")L$ is an irreducible $(\underline{m}", \underline{k} \cap \underline{m}")$-module.

(2) $j_{*\underline{p}"}(L/\underline{n}(\underline{p}")L)$ is an irreducible $\underline{m}"$ module.

Proof. We first show that (2) implies (1). Indeed, $j_{*\underline{p}"}(L/\underline{n}(\underline{p}")L)$ is the real Jacquet module in the sense of [2]. Thus $M \mapsto j_{*\underline{p}"}(M)$ is an exact functor on the category $\mathcal{H}(\underline{m}",\underline{k} \cap \underline{m}")$ of finitely generated admissible $(\underline{m}",\underline{k} \cap \underline{m}")$ modules. Also Casselman's theorem implies that if $M \in \mathcal{H}(\underline{m}", \underline{k} \cap \underline{m}")$, $M \neq 0$ then $j_{*\underline{p}"}(M) \neq 0$. This clearly implies that (2) implies (1).

We now prove (2). $j_{*\underline{p}"}(L/\underline{n}(\underline{p}")L) = j_{\underline{p}}(L)^{\underline{n}(\underline{p}")}$. Since $j_{\underline{p}}(L)$ is irreducible and in $\mathcal{O}(\underline{p})$, $j_{\underline{p}}(L)^{\underline{n}(\underline{p}")}$ is irreducible as an $\underline{m}"$-module. Q.E.D.

To get more refined results we must introduce some more notation and (well known) structural results.

Let $\varepsilon_i \in \underline{a}_o^*$ be defined by $\varepsilon_i(X_j+Y_j) = \delta_{ij}$. Then it is standard that the root system of $\underline{a}_o$ on $\underline{n}_o$ denoted $\Phi(\underline{p}_o,\underline{a}_o)$ consists of the linear functionals

$$\varepsilon_i \pm \varepsilon_j \,, \quad 1 \leq i < j \leq r$$
$$2\varepsilon_i \,, \quad i = 1,\ldots,r$$

and possibly ε_i, $i = 1,\ldots,r$.

If the ε_i do not occur then we see that $\Phi(\underline{p}_o,\underline{a}_o)$ is a positive root system of type C_r. If the ε_i occur then $\Phi(\underline{p}_o,\underline{a}_o)$ is a non-reduced positive root system of type BC_r.

In the C_r case the simple roots are $\varepsilon_1-\varepsilon_2, \ldots, \varepsilon_{r-1}-\varepsilon_r, 2\varepsilon_r$ and in the BC_r case the simple roots are $\varepsilon_1-\varepsilon_2, \ldots, \varepsilon_{r-1}-\varepsilon_r, \varepsilon_r$.

Set $(\underline{a}_i)_o = \sum_{j \leq r} \mathbb{R}(X_j+Y_j)$. Set $(\underline{m}_i)_o = C_{\underline{g}_o}((\underline{a}_i)_o)$, $(\underline{n}_i)_o = \bigoplus_{\substack{\lambda \in \Phi(\underline{p}_o,\underline{a}_o) \\ \lambda|_{(\underline{a}_i)_o} \neq 0}} (\underline{n}_o)_\lambda$.

Set $(\underline{p}_i)_o = (\underline{m}_i)_o + (\underline{n}_i)_o$. Then $(\underline{p}_1)_o \supset (\underline{p}_2)_o \supset \cdots \supset (\underline{p}_r)_o = \underline{p}_o$.

We now give a slightly different description of the $\underline{m}_i$. We preface it with the following well known result.

Lemma 2.5. If $\alpha \in \Phi$ and $(\alpha,\gamma_i) = 0$ then $\alpha \pm \gamma_i \notin \Phi \cup \{0\}$.

Proof. It is well known that $(\gamma_i,\gamma_i) = (\gamma_1,\gamma_1)$ $i = 1,\ldots,r$. Hence there is $S_i \in W(\underline{g},\underline{h})$ (the Weyl group of $\underline{g}$ relative to $\underline{h}$) such that $S_i\gamma_1 = \gamma_i$. Since γ_1 is the largest root in Φ^+ we see that γ_i is the largest root in $S_i\Phi^+$. Hence the result.

Let for $1 \leq j \leq r$,

$$C_j = \prod_{i \leq j} \exp(\operatorname{ad}X_i)\exp(\tfrac{1}{2}(\log 2)\operatorname{ad}H_i)\exp(-\operatorname{ad}Y_i) .$$

C_j is usually called a partial Cayley transform. We note that $C_jH_i = -(X_i+Y_i)$ for $i \leq j$ and if $\underline{h}_j = \{H \in \underline{h} \mid \gamma_i(H) = 0,\ i \leq j\}$, $C_j|_{\underline{h}_j} = I$.

Set $\tilde{\underline{m}}_j = \{X \in \underline{g} \mid [\bar{X},H_i] = 0,\ i = 1,\ldots,j\}$. Then clearly $C_j\,\tilde{\underline{m}}_j = \underline{m}_j$. Now

$$\tilde{\underline{m}}_j = \underline{h} + \bigoplus_{\substack{\alpha \in \Phi \\ (\alpha,\gamma_i)=0,\, i\leq j}} \underline{g}_\alpha .$$

Thus $C_j\,\tilde{\underline{m}}_j = \underline{a}_j \oplus \underline{h}_j \oplus \bigoplus_{\substack{\alpha \in \Phi \\ (\alpha,\gamma_i)=0}} \underline{g}_\alpha$ by Lemma 2.5.

We set $\underline{b}_j = \underline{b} \cap \tilde{\underline{m}}_j$ and $\underline{b}_j' = C_j\,\underline{b}_j = \underline{a}_j \oplus \underline{h}_j \oplus \bigoplus_{\substack{\alpha \in \Phi^+ \\ (\alpha,\gamma_i)=0}} \underline{g}_\alpha$.

Lemma 2.6. Set $\underline{m}_j' = \underline{h}_j \oplus \bigoplus_{\substack{\alpha \in \Phi \\ (\alpha,\gamma_i)=0,\, i\leq j}} \underline{g}_\alpha$.

Then $(\underline{m}_j' \cap \underline{g}_o,\ \underline{m}_j' \cap \underline{k}_o)$ is an irreducible symmetric pair of Hermetian type.

Proof. Set $(\underline{r}_j)_o = \underline{m}_j' \cap \underline{r}_o$, $\underline{r}_j^+ = \bigoplus_{\substack{\alpha \in \Phi_n^+ \\ (\alpha,\gamma_i)=0,\, i\leq j}} \underline{g}_\alpha$.

$\underline{r}_j^- = \bigoplus_{\substack{\alpha \in \Phi_n^+ \\ (\alpha,\gamma_i)=0,\, i\leq j}} \underline{g}_{-\alpha}$. Then $\underline{r}_j = \underline{r}_j^+ \oplus \underline{r}_j^-$.

Thus the result follows if we show that $(\underline{m}_j' \cap \underline{g}_o,\ \underline{m}'_j \cap \underline{k}_o)$ is irreducible.

For this set $*\underline{a}_j = \sum_{i>j} \mathbb{C}(X_i+Y_i)$, $*\underline{n}_j = \underline{n}(\underline{p}_j) \cap \underline{m}_j$. $*\underline{m}_j = \underline{m} \cap \underline{m}_j$. Then $*\underline{m}_j \oplus *\underline{a}_j \oplus *\underline{n}_j$ is a minimal parabolic over $\mathbb{R}$ of $(\underline{m}_j')_o = \underline{m}_j' \cap \underline{g}_o$. The root system of $*\underline{a}_j$ on $*\underline{n}_j$ is easily seen to be either of the form C_{r-j}, BC_{r-j}. In either case it is irreducible. The result now follows.

If $\Lambda \in \underline{h}^*$ define $\Lambda^j = - \sum_{i \leq j} \frac{2(\Lambda,\gamma_i)}{(\gamma_i,\gamma_i)} \varepsilon_i$. We look upon Λ^j as an element of $\underline{a}_j^*$. We note

(7) If $L^\Lambda_{\underline{b},\underline{h}} \in \mathcal{O}(\underline{g})$ then

$$-\mathrm{Re}(\Lambda,\gamma_2)/(\gamma_2,\gamma_2)) \leq \cdots \leq -\mathrm{Re}((\Lambda,\gamma_j)/(\gamma_j,\gamma_j)) \ .$$

Indeed, if $\alpha \in \Phi_n^+$ then $\alpha = \gamma_1 - Q$ with Q sum of elements of Φ_k^+. Now if $L^\Lambda_{\underline{b},\underline{h}} \in \mathcal{O}(\underline{g})$ then $2(\Lambda,\beta)/(\beta,\beta) \in \mathbb{N}$ for $\beta \in \Phi_k^+$. Thus if $\alpha \in \Phi_n^+$ then

$$\mathrm{Re}(\Lambda,\alpha) \leq \mathrm{Re}(\Lambda,\gamma_1) \ .$$

Since $(\gamma_i,\gamma_i) = (\gamma_1,\gamma_1)$, $i = 1,\ldots,r$. This implies (7).

If $\mu \in \underline{a}_j^*$, $M \in \mathcal{M}(\underline{m}_j')$ we denote by $C_\mu \otimes M$ the $\underline{m}_j$-module, M with $\underline{m}_j'$ acting on M as given and $\underline{a}_j$ acting by μI.

Lemma 2.7. Let $\Lambda \in \underline{h}^*$ be such that $L^\Lambda_{\underline{b},\underline{h}} \in \mathcal{O}(\underline{g})$. Then

$$L^\Lambda_{\underline{b},\underline{h}} \ / \ \underline{n}_j(\underline{b},\underline{h}) \ L^\Lambda_{\underline{b},\underline{h}} \ \cong \ C_{\Lambda^j} \otimes L^{\Lambda|\underline{h}_j}_{\underline{b}_j,\underline{h}_j}$$

Furthermore, $L^{\Lambda|\underline{h}_j}_{\underline{b}_j,\underline{h}_j} \in \mathcal{O}(\underline{g}_j')$, $\underline{g}_j' = \underline{g} \cap \underline{m}_j'$

Proof. We already know that the left hand side, L, of the asserted equivalence is irreducible. Thus $L \cong C_\mu \otimes L'$, $\mu \in \underline{a}_j^*$, $L' \in \mathcal{M}(\underline{m}_j')$ and L' is irreducible. Applying (3) above we find that $\mu = \Lambda \cdot C^{-1}|_{\underline{a}_j}$. Now

$$(\Lambda \cdot C^{-1})(X_i+Y_i) = -\Lambda(H_i) =$$
$$= -2(\Lambda,\gamma_i)/(\gamma_i,\gamma_i), \quad i \leq j \ .$$

Hence $\mu = \Lambda^j$ as asserted. On the other hand since $C_j\bar{\underline{b}} \in \mathcal{B}(\underline{b})$ we see that $(L^\Lambda_{\underline{b},\underline{h}})^{\underline{n}(\underline{b})} \ / \ \underline{n}(\underline{p}_j) \ L^\Lambda_{\underline{b},\underline{h}} = (0)$ $(\underline{n}(\underline{p}_j) \subset \underline{n}(C_j\bar{\underline{b}}))$. Thus $(L')^{\underline{n}(\underline{b}_j)}$ must have the weight $\Lambda|\underline{h}_j$ relative to $\underline{h}_j$. Q.E.D.

Using Lemma 2.7 it is now a simple matter to describe the Langland's parameters of the $L^\Lambda_{\underline{b},\underline{h}}$ that are in $\mathcal{O}(\underline{g})$ relative to $\theta(\underline{p})$.

Set $\Psi^+ = \Phi_k^+ \cup -\Phi_n^+$. If $L^\Lambda_{\underline{b},\underline{n}}$ is in $\mathcal{O}(\underline{g})$ then we say that $L^\Lambda_{\underline{b},\underline{n}}$ is in the "closure of the discrete series" if

$$\mathrm{Re}(\Lambda+\rho,\alpha) \geq 0, \qquad \alpha \in \Psi^+$$

Here $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ (as usual).

(8) $L^{\Lambda}_{\underline{b},\underline{n}}$ is in the closure of the discrete series if and only if

(i) $2(\Lambda+\rho,\alpha)/(\alpha,\alpha) \in \mathbb{N}-\{0\}, \quad \alpha\in\Phi_k^+$

(ii) $\mathrm{Re}\ 2(\Lambda+\rho,\gamma_1)/(\gamma_1,\gamma_1) \leq 0$.

Indeed, $L^{\Lambda}_{\underline{b},\underline{n}}$ is in $\mathcal{O}(\underline{g})$ if and only if (i) is satisfied (see the end of Section 1, (ii) is clearly necessary. (*) in the proof of Lemma 2.6 implies that it is sufficient (assuming (i)).

<u>Proposition 2.8</u>. Let $\Lambda \in \underline{h}^*$ be such that $L^{\Lambda} = L^{\Lambda}_{\underline{b},\underline{h}} \in \mathcal{O}(\underline{g})$. If L^{Λ} is not in the closure of the discrete series then there exists a unique $1 \leq j \leq r$ such that (here $\rho_{\underline{p}_j}(H) = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}H|_{\underline{n}(\underline{p}_j)}$, $H \in \underline{a}_j$)

(1) $\mathrm{Re}(\Lambda_j-\rho_{\underline{p}_j},\lambda) < 0, \quad \lambda \in \Phi(\underline{p}_j,\underline{a}_j)$

(2) $L^{\Lambda|_{\underline{h}_j}}_{\underline{b}_j,\underline{h}_j}$ is in the closure of the discrete series for $\underline{m}_j'$.

<u>Proof</u>. $\rho_{\underline{p}_j}\big|_{\underline{a}_j} = -\rho\circ C^{-1}\big|_{\underline{a}_j}$. Thus (1) is the same as saying that

$$(*) \qquad \mathrm{Re}(-2(\Lambda+\rho,\gamma_1)/(\gamma_1,\gamma_1)) < \mathrm{Re}(-2(\Lambda+\rho,\gamma_2)/(\gamma_2,\gamma_2)) <$$

$$\cdots < \mathrm{Re}(-2(\Lambda+\rho,\gamma_j)/(\gamma_j,\gamma_j)) < 0$$

and (2) says

$$(**) \qquad \mathrm{Re}(-2(\Lambda+\rho,\gamma_{j+1})/(\gamma_{j+1},\gamma_{j+1})) \geq 0 .$$

Now if L^{Λ} is not in the closure of discrete series then (*) is either true for all $1 \leq j \leq r$ and the result follows since $\underline{m}_r'$ $\underline{k}$. Otherwise (*) is true for a maximal $j \leq r$. Hence (**) is true for $j+1$. Q.E.D.

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N.R. WALLACH
Rutgers College
Dept of Maths
University Heights Campus
NEW BRUNSWICK
NJ 08903
USA