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Semi-classical analysis for Harper’s equation. III : Cantor structure of the spectrum


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SEMiclassal Analysis for Harper's Equation III
Cantor Structure of the Spectrum

by

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RESUME. Dans ce travail nous continuons notre étude de l'opérateur de
Harper, \( \cosh D + \cos x \) dans \( L^2(\mathbb{R}) \), par des méthodes d'analyse microlocale et
de renormalisation. On obtient une description assez complète du spectre dans
le cas où \( h/2\pi \) est irrationnel avec un développement en fraction continue :
\( h/2\pi = 1/(q_0 + 1/(q_1 + ...)) \), si \( q_j \in \mathbb{Z} \), \( |q_j| \geq C_0 \) et \( C_0 > 0 \) est assez grand. En
particulier le spectre est un ensemble de Cantor de mesure 0. Nos résultats
sont aussi valables pour certaines perturbations de l'opérateur de Harper et on
donne une application à l'opérateur de Schrödinger magnétique périodique sur
\( \mathbb{R}^2 \).

ABSTRACT. In this paper we continue our study of Harper's operator
\( \cosh D + \cos x \) in \( L^2(\mathbb{R}) \), by means of microlocal analysis and renormalization.
A rather complete description of the spectrum is obtained in the case when
\( h/2\pi \) is irrational and has a continued function expansion :
\( h/2\pi = 1/(q_0 + 1/(q_1 + ...)) \) with \( q_j \in \mathbb{Z} \), \( |q_j| \geq C_0 \), provided that \( C_0 > 0 \) is
sufficiently large. In particular, the spectrum is a Cantor set of measure 0.
Our results are also valid for certain perturbations of Harper's operator and an
application to the periodic magnetic Schrödinger operator on \( \mathbb{R}^2 \) is given.

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0. Introduction.

This work is a continuation of our study, started in [HS1,2], of the spectrum of Harper's operator, by the use of semi-classical methods. If \( h \in \mathbb{R} \), \( h \neq 0 \), then the problem is to study the union of the spectra, when \( \theta \) varies in \( \mathbb{R} \), of the operators in \( \mathcal{I}(l^2(\mathbb{Z}), l^2(\mathbb{Z})) \), given by,

\[
H_\theta u(n) = \frac{i}{2} (u(n+1) + u(n-1)) + \cos(hn + \theta) u(n).
\]

As a set, this union of spectra coincides with the spectrum of,

\[
P_\theta = \cos(hD_x) + \cos(x)
\]

in \( \mathcal{I}(L^2(\mathbb{R}), L^2(\mathbb{R})) \), where \( D_x = i^{-1} \partial / \partial x \), so that \( \cos(hD_x) = \frac{i}{2}(\tau_h + \tau_{-h}) \), where \( \tau_h u(x) = u(x-h) \). Inspired by ideas of Wilkinson [W1], we obtained a partial Cantor structure result for the spectrum, \( \text{Sp}(P_\theta) \) of \( P_\theta \) under the assumption that \( h/2\pi \) is irrational and,

\[
h/2\pi = 1/(q_1 + 1/(q_2 + \ldots)) \quad q_j \in \mathbb{Z}, 1 \leq j < \infty,
\]

and

\[
|q_j| \geq C_0,
\]
for some sufficiently large constant $C_0$. (See Théorème 1 in [HS1].) Roughly, our result was that if $\varepsilon_0 > 0$ and if $C_0 > 0$ is sufficiently large (as a function of $\varepsilon_0$), then outside $[-\varepsilon_0, \varepsilon_0]$ the spectrum of $P_0$ is contained in a union of intervals, $J_j$ of width $\exp(-\sim |q_1|)$ and such that the separation between neighboring intervals (on the same side of $[-\varepsilon_0, \varepsilon_0]$) is $\sim 1/|q_1|$. If $X_j$ is the increasing affine function that maps $J_j$ onto $[-2, 2]$, then outside $[-\varepsilon_0, \varepsilon_0]$, the set $X_j(J_j \cap \text{Sp}(P_0))$ can again be localized into a finite union of closed intervals, having widths and separations of the same order of magnitude as for the $J_j$, but in terms of $q_2$ instead of $q_1$. This procedure can then be continued indefinitely.

The proof of this result was obtained by applying first microlocal analysis near a "potential well", i.e. a component of $\cos(\xi) + \cos(x) = \mu$, where $\mu \in [-2, 2] \setminus [-\varepsilon_0, \varepsilon_0]$, in order to obtain certain discrete eigenvalues, well defined up to $O(e^{-1/\varepsilon_0})$. It then followed that $\text{Sp}(P_0)$ is localized to certain intervals, $J_j$, exponentially close to these eigenvalues. After that we analyzed the tunnel effect between the potential wells, and this permitted us to describe $\text{Sp}(P_0) \cap J_j$ as the spectrum of a certain infinite "interaction" matrix. Exploiting certain translation invariance properties of the resulting matrix, we could then reduce the study of its spectrum to that of $P(x, h'D_x)$, the Weyl quantization of $P(x, h'\xi)$, (and by definition, the $h'$--Weyl quantization of $P(x, \xi)$, where $P = P_j, h$ is a small perturbation of $P_0 = \cos(\xi) + \cos(x)$. Here $h'/2\pi = 1/(q_2 + 1/(q_3 + ...))$. For $P$ we could then start over again ... .

In this paper, we shall be able to eliminate the intervals, $[-\varepsilon_0, \varepsilon_0]$, and obtain a fairly complete description of $\text{Sp}(P_0)$, under the assumption (0.3), (0.4) with $C_0 > 0$ sufficiently large. When trying to make this improvement at the first level of the iteration scheme, an obvious difficulty is that for $\mu \approx 0$, $P_0^{-1}(\mu)$ is close to the union of the lines $\xi = \pm x + (2k+1)\pi$, $k \in \mathbb{Z}$, and there is no more obvious localization into potential wells.
As before, we can however study microlocal solutions of the homogeneous equation, \((P_0 - \mu)u = 0\), and as a matter of fact, this was done heuristically already by Azbel [Az]. Away from the saddle points, \((k\pi, l\pi)\), \(k+1 \in \mathbb{Z} + 1\), the characteristic set, \(P_0^{-1}(\mu)\) is a smooth analytic curve, and near a point in this part of the set, the microlocal kernel of \((P_0 - \mu)\) is a one-dimensional space, generated by a standard WKB solution. Near a saddle point the space of microlocal solutions is two dimensional, and can be computed more or less explicitly. If we choose for instance the point \((0, \pi)\), then an element of the microlocal kernel near that point, is determined by its behaviour near the open segments \([-\pi, 2\pi), (0, \pi)[\) and \([0, \pi), (\pi, 0)\]. Using a microlocal study of \(P_0 - \mu\), we can then obtain a globally defined, well posed "Grushin" problem,
\[(0.5) \quad (P_0 - \mu)u + R_- u^- = v, \quad R_+ u = v^+,
\]
where, \(u, v \in L^2(\mathbb{R}), u^-, v^+ \in \mathcal{D}'(\mathbb{R}^2; \mathbb{C}^2)\). Roughly (thinking of the case, \(v = 0\)), the condition \(R_+ u = v^+\) fixes the microlocal behaviour of \(u\) near all segments of the form \([-((k-1)\pi, (1+1)\pi), (k\pi, l\pi)\), \(k+1 \in \mathbb{Z} + 1\), and \(R_- u^-\) provides a one-dimensional inhomogeneity near each segment of the form, \([-((k+1)\pi, (1+1)\pi), (k\pi, l\pi)\]. Denoting the solution by,
\[(0.6) \quad u = E_+ v + E_- v^+, \quad u^- = E_- v + E_+ v^+,
\]
where all operators depend on \(\mu\), it is easy to show that \(\mu\) belongs to the spectrum of \(P_0\) if and only if \(0\) belongs to the spectrum of \(E_- +\). Now \(E_- +\) may be viewed as a block matrix, \(E_- + (\alpha, \beta))\), \(\alpha, \beta \in \mathbb{Z}^2\), where each entry is a \(2 \times 2\) matrix. By the same procedure as for the matrix \(W\) above, we then see that \(0 \in \text{Sp}(E_- +\)) iff \(0 \in \text{Sp}(P)\), where \(P\) is a \(2 \times 2\) matrix of \(h\)-pseudodifferential operators. After rescaling, we see that (in the most interesting spectral region) \(P\) falls into a certain class of "strong type 2 operators". We also define strong type 1 operators, as scalar \(h\)-pseudodifferential operators, satisfying certain commutation relations and which are close to \(P_0(x, hD)\). Fortunately, the study of strong type 2 operators is often very close to the study of \(s\)-type 1 operators, and we can again divide the problem into certain potential well cases and a branching case.

An interesting feature is that we loose the linear dependence of the spectral parameter, already after considering the first branching problem, so we shall systematically work with operators \(P = P_\mu\), and define \(\mu - \text{Sp}(P)\) as the set of \(\mu\) such that \(0\) belongs to the spectrum of \(P_\mu\). Theorem 6.2 below shows that the study of the \(\mu\)-spectrum of a strong type 1 \(h\)-pseudodifferential operator sufficiently close to \(P_0\) can, when \(h\) is sufficiently small, be localized into a union of closed disjoint intervals, such that the further study of the \(\mu\)-spectrum in each of these intervals leads to an operator either of \(s\)-type 1 or 2. Theorem 9.2 gives the corresponding result for \(s\)-type 2 operators. Theorem 9.3 is a combination of the Theorems
6.2 and 9.2 and says that if we start with a strong type 1 operator sufficiently close to $P_0$ and if (0.3), (0.4) hold with $C_0$ sufficiently large, then we get a complete description of the $\mu$-Spectrum by means of an infinite sequence of localizations into finite disjoint unions of closed subintervals and rescalings. From the additional quantitative informations stated after the Theorems 6.2 and 9.2 about the lengths and separations of the various intervals appearing in those theorems, combined with Theorem 9.3, we obtain the following result (which expresses only a small part of the very precise information that our methods produce).

**Theorem 0.1.** Let $P=P(x,hD^\alpha)$ be a self-adjoint $h$-pseudodifferential operator such that the corresponding Weyl symbol, $P(x,\xi)$ extends holomorphically to the "band" $|\text{Im}(x,\xi)| < 1/e$, and satisfies:

- (0.7) $P(x,\xi+2\pi\alpha) = P(x,\xi)$, for all $\alpha \in \mathbb{Z}^2$,
- (0.8) $P(\xi,-x) = P(\xi,x)$,
- (0.9) $|P(x,\xi) - (\cos(\xi) + \cos(x))| \leq \varepsilon$, when $|\text{Im}(x,\xi)| < 1/e$.

If (0.3), (0.4) hold with $C_0 > 0$ sufficiently large, and if $0 < \varepsilon < \varepsilon_1$ with $\varepsilon_1 > 0$ sufficiently small, then $\text{Sp}(P)$ is of Lebesgue measure 0, has no isolated points and is nowhere dense. (The last statement means that $\text{Sp}(P)$ is dense in no non-trivial open interval.)

As already mentioned, the method produces a much more precise description of the spectrum, which is unfortunately rather lengthy to formulate in terms of a theorem, but the interested reader will be able to extract that information from the proofs. This refined description will no doubt be useful when studying the Hausdorff dimension of the spectrum as a function of the sequence $(q_j)$. From the point of view of applications, it is important that our results apply also to small perturbations of Harper's operator. In appendix e we show that under suitable assumptions, the spectrum of a periodic magnetic Schrödinger operator is near the bottom a Cantor set of measure 0.

In [HS2], the results of [HS1] were extended to the case when for some $N$: $|q_j| \geq C_N(q_1, \ldots, q_N, \varepsilon_0)$ for $j \geq N+1$, but still with the same incompleteness as in [HS1]. We believe that the techniques of the present paper rather automatically lead to a more complete Cantor structure result also in that case. One could probably generalize the result even to the case when there is a sequence $1 \leq j_1 < j_2 < j_3 < \ldots$ of integers such that $|q_{j_k}| \geq C_{j_k-j_{k-1}}(q_{j_k-1}+1, \ldots, q_{j_k-1})$, for suitable functions, $C_N$.

The plan of the paper is the following:

**Section 1.** Here we introduce and study certain auxiliary operators.
**Section 2.** contains a formal study of the iteration steps that we will encounter, and we show that certain crucial symmetries are conserved.
**Section 3.** Here we treat the potential well problem for $s$-type 1 operators by suitable modifications of the methods in [HS1].
**Section 4** treats the branching problem for strong type 1 operators and we obtain a "renormalized" 2×2 system of $h'$-pseudodifferential operators. **Sections 5 and 6** contain some preliminary results for the renormalized operator in section 4. It is showed that after rescaling and depending on the spectral region, the renormalized operator is either of $s$-type 1 or 2. Theorem 6.2 gives the main iteration step for $s$-type 1 operators. **Section 7** treats the totally degenerate potential well case, which is the only case genuinely non-scalar case for $s$-type 2 operators. Here we use some ideas from [HS2]. **Section 8** treats the non-degenerate potential well case for $s$-type 2 operators. **Section 9** is devoted to the branching case for $s$-type 2 operators. Theorem 9.2 gives the main iteration step for $s$-type 2 operators.

Various results are collected into 5 appendices:

**Appendix a** contains various general results in microlocal analysis. The paragraph a.1 recollects the approach of [SI] to analytic microlocal analysis via FBI-transforms. We refer to that book for a more thorough treatment. In paragraph a.2 we develop a simple functional calculus for analytic pseudodifferential operators. Paragraph a.3 may be of independent interest. It gives a refined correspondence between unitary Fourier integral operators and canonical transformations.

**Appendix b**. Here we give local normal forms for self-adjoint pseudodifferential operators when the symbol has a saddle point or a minimum. We only allow unitary conjugations and taking functions of the operator. We believe that the results of this appendix will be useful in other contexts.

**Appendix c**. Here we show that certain 2×2 systems of pseudodifferential operators can be reduced to the case when the diagonal terms are scalars. This is of use in section 7. See also [HS2].

**Appendix d** contains some justifications of the arguments in section 4.

**Appendix e** gives an application to magnetic Schrödinger operators. This is a modification of the corresponding arguments in [HS1]. Since the symmetry (0.8) was never assumed in [HS1], we have to add an extra symmetry to the magnetic and electric fields and check that this leads to (0.8).

Some of the results of the present paper have been announced in [HS3]

We would finally like to thank A. Grigis for a large number of interesting and stimulating dicussions with the authors during the preparation of this long work.
1. Various operators with commutation relations.

In this section, we introduce various auxiliary operators, that will play an important role later, and we study their commutation relations. Some of this was already done in [HS1.2], but we think it is convenient to have all at the same place. Let $h \in \mathbb{R}$, $h \neq 0$. All operators will act on $L^2(\mathbb{R})$. The first operators we study are natural $h$-quantizations of the translations:

$$(x, \xi) \rightarrow (x, \xi) + 2\pi \alpha, \alpha \in \mathbb{Z}^2 \text{ (and sometimes even in } \mathbb{R}^2).$$

Let $\tau = \tau_{2\pi}$ denote the operator of translation by $\tau_{2\pi}$; $\tau u(x) = u(x-2\pi)$, let $\tau^*$ denote the operator of multiplication by $e^{2\pi i x / h}$, and put

$$(1.1) \quad T_\alpha = \tau_\alpha \tau_\alpha^*,$$

for $\alpha \in \mathbb{Z}^2$.

Sometimes, we shall also use that there is a natural extension of the definition of $T_\alpha$ to the case when $\alpha \in \mathbb{R}^2$, since there is an obvious definition of real powers of $\tau$ and $\tau^*$. In a way, the crucial phenomenon that causes all the interesting phenomena for Harper's operator, is that $\tau$ and $\tau^*$ do not commute in general. In fact, $\tau \cdot \tau^* = \exp(-i(2\pi^2 / h) \tau \cdot \tau)$. Let $h' \in \mathbb{R}$, be a number such that,

$$(1.2) \quad 2\pi / h = k + h'/2\pi,$$

for some integer $k$. Then $\tau \cdot \tau^* = \exp(-ih') \tau^* \tau$, and more generally we get,

$$(1.3) \quad T_\alpha T_\beta = e^{ih'\alpha_2 \beta_1} T_{\alpha + \beta},$$

$$(1.4) \quad T_\alpha T_\beta = e^{ih'\sigma(\alpha, \beta)} T_{\beta} T_\alpha,$$

for $\alpha, \beta \in \mathbb{Z}^2$, where $\sigma$ denotes the standard symplectic form on $\mathbb{R}^2$, given by $\sigma(x, \xi; y, \eta) = \xi y - x \eta$. (1.3) and (1.4) remain valid for $\alpha, \beta \in \mathbb{R}^2$, provided that we replace $h'$ by $(2\pi^2 / h)$. The next operator we introduce is the unitary Fourier transform $\mathcal{F} h = \mathcal{F}$, which can be viewed as an $h$-quantization of the map $\kappa: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by,

$$(1.5) \quad \kappa(x, \xi) = (\xi, -x).$$

Later on we shall also need the maps $\kappa_t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined as rotation by the angle $t$, so that $\kappa_t x = \kappa - \pi/2$. By definition,

$$(1.6) \quad \mathcal{F} u(\xi) = (2\pi h)^{-1/2} \int e^{-i x \xi / h} u(x) \, dx, \quad h > 0,$$

and as already mentioned, $\mathcal{F}: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is unitary, $\mathcal{F}^{-1} = \mathcal{F}^*$, where $\mathcal{F}^*$ denotes the complex adjoint in the $L^2$-sense. It is easy to check,

(starting with the operators $\tau$ and $\tau^*$) that,

$$(1.7) \quad \mathcal{F} \cdot T_\alpha = e^{-ih'\alpha_2 \alpha_1} T_{\kappa(\alpha)^*} \mathcal{F},$$

for $\alpha \in \mathbb{Z}^2$, and the same relation with $h'$ replaced by $(2\pi)^2 / h$, when $\alpha \in \mathbb{R}^2$.

It will be useful in the following, to recall the relation between $\mathcal{F}$ and the unitary group associated to the harmonic oscillator, $R = \frac{1}{2}(h^2 \partial^2 + \alpha^2 - h)$, $D = -i(\partial / \partial x)$. Let $U_t = e^{i t R / h}$. Since $u_0 = e^{-\alpha^2 / 2h}$ is in the kernel of $R$, we have $U_t U_0 = U_0$. On the other hand, we know (Lesur [L]), that $U_{-\pi/2}$ and $\mathcal{F}$ are metaplectic (unitary) operators with the same canonical transformation; $\kappa_{-\pi/2}$, and hence that $U_{-\pi/2} = \omega \mathcal{F}$, for some $\omega$ of modulus 1. Since
$U_{-\pi/2}U_0 = U_0$ and $\mathcal{F}U_0 = U_0$, it follows that $\omega = 1$;
(1.8) $U_{-\pi/2} = \mathcal{F} = \mathcal{F} h$, $h > 0$.

For later reference, it will also be convenient to know $U_{\pi/4}$ explicitly. Using
that the phase $\varphi(x, y) = -x^2/2 + (2)^{1/2}xy - y^2/2$ generates the correct
canonical transformation, namely $X_{\pi/4}$, we first see that
(1.9) $U_{\pi/4}U(x) = Ch^{-1/2} \int e^{i\varphi(x, y)/h}u(y) \, dy$,
for some constant $C$. In order to determine this constant, we again use that
$U_{\pi/4}U_0 = U_0$ and that the corresponding integral in (1.8) can be evaluated
exactly, to obtain that, $C = 2^{-1/4} \pi^{-1/2} e^{i\pi/8}$.

Let $\Gamma$ denote the antilinear operator of complex conjugation; $\Gamma u = \bar{u}$. To
this operator we associate the transformation of phase-space;
$(x, \xi) \rightarrow (x, -\xi)$. We notice that this transformation is anti-canonical, in the
sense that the Jacobian is equal to $-1$. As a general rule we shall associate
anti-canonical transformations to antilinear operators. The present
association is justified by the following fact. Let $A = A(x, hD)$ be the $h$-Weyl
quantization of the symbol $A(x, \xi) \epsilon S^0(\mathbb{R}^2) = \{a \in C^\infty(\mathbb{R}^2); \text{ for all } j, k \epsilon \mathbb{N}, \text{ there exists } C_{j,k} \text{ such that } |\partial_x^j \partial_\xi^k A(x, \xi)| \leq C_{j,k}, \text{ for all } (x, \xi) \epsilon \mathbb{R}^2 \}$, defined by,
(1.10) $Au(x) = (2\pi h)^{-1} \int e^{i(x - y)\theta/h}A((x + y)/2, \theta)u(y) \, dyd\theta$, $h > 0$,
so that $A$ is $O(1)$ as a bounded operator on $L^2(\mathbb{R})$ by standard theorems. (See
for instance [HS1] for a non standard proof.) When we want to distinguish more
clearly between the operator and its symbol, we shall sometimes write $Op_h(A)$
or simply $Op(A)$ for the operator. The justification of the association is then
given by,
(1.11) $\Gamma O p(A) = O p(B) \Gamma$,
where $B(x, \xi) = \widehat{A}(x, -\xi)$. Notice that $\Gamma^2 = id$, so that (1.11) may take many
equivalent forms.

Thus in a way, $\Gamma$ is a natural quantization of reflection in the $x$-axis.
We shall also need quantizations of other reflections, such as in some of the
lines $l_\theta = \{t(\cos \theta, \sin \theta); t \epsilon \mathbb{R}\}$. To define such reflections, it is natural to
rotate $l_\theta$ by $\pi - \theta$ to the $x$-axis ($l_0$), then reflect in the $x$-axis, and finally
rotate back again. More precisely, the quantization of reflection $\gamma_\theta$ in $l_\theta$, is
defined by,
(1.12) $\Gamma_\theta = \Gamma_0 \Gamma U_{-\theta}$,
so that $\Gamma_0 = \Gamma$. This corresponds to $\gamma_\theta = \kappa_\theta \gamma_0 \kappa_{-\theta}$. From the definition of $U_\theta$,
it is easy to verify that,
(1.13) $\Gamma U_\theta = U_{-\theta} \Gamma$ (and classically, $\gamma_0 \kappa_\theta = \kappa_{-\theta} \gamma_0$),
which gives rise to several obvious equivalent forms of (1.12). We get the
general relations,
(1.14) $\Gamma_b U_a = U_{\alpha \beta} \Gamma_\beta$, $\gamma_a \kappa_b = \kappa_a \gamma_b$, if $2b - a = \alpha + 2\beta$. 
Now it is a general fact, that
(1.15) $U_{-\theta} O p(a) U_\theta = O p(a + \kappa_\theta)$,
and combining this with (1.11), we get,

(1.16) \( A \ast \Gamma_\theta = \Gamma_\theta \ast B \), if on the symbol level, \( A \ast \gamma_\theta = \overline{B} \).

Again, we notice that \( \Gamma_\theta^2 = \text{id} \), so (1.16) can take many equivalent forms.

Later, we shall make a particular use of \( V = \text{def.} \Gamma_{\pi/4} \), whose associated transformation is \( \gamma_{\pi/4} = \text{def.} \varsigma: (x, \xi) \rightarrow (\xi, x) \). Using that \( V = U_{\pi/2} \ast \Gamma = \Gamma \ast U_{-\pi/2} \), it is easy to check, using (1.7), (1.6) that,

(1.17) \( VT_\alpha = e^{it\gamma'\alpha_1 \alpha_2 T} S(\alpha) V \).

We finally notice that,

(1.18) \( (\Gamma_\theta u, v) = (u, \Gamma_\theta v), u, v \in L^2 \).
2. Formal study of the iteration process.

In the process that we are going to study, there will appear infinitely often one of the following two types of operators, namely:

**Type 1.** $P = P(x,\hbar D_x)$ is a scalar pseudodifferential operator with the following properties:

(2.1) $P$ commutes with the operators $T_\alpha$, $\alpha \in \mathbb{Z}^2$.

This means that the Weyl symbol; $P(x,\xi)$ of $P$ is $2\pi$-periodic both in $x$ and in $\xi$.

(2.2) $P^* = P^*$, $P^* = P^*P$.

This means for the Weyl symbol, that $P \cdot \xi = \bar{P}$, where $\xi = x - \pi/2$ is the map of rotation of $\mathbb{R}^2$ by the angle $-\pi/2$, i.e. the canonical transformation associated to $\mathcal{F}$. We also assume a symmetry under reflection in $x = \xi$:

(2.3) $PV = VP$.

Here we recall that $V$ quantizes the reflection map $(x,\xi) \rightarrow (\xi, x)$, and is an anti-linear operator. At the very first iteration step, $P$ will be selfadjoint (for real values of the spectral parameter $\lambda$), but that property will be lost later on in the iteration and will be replaced by:

(2.4) $P_1^*P$ and $PP_2^*$ are selfadjoint,

where $P_1$ and $P_2$ are bounded pseudo-differential operators, satisfying (2.1) and:

(2.5) $P_1^*P = P_1^*P = P_1^*P$.

(2.6) $P_1^*P = P_1^*P = P_1^*P$.

In the beginning of the iteration, we take $P_1$ and $P_2$ to be the identity operator, and later, they will be elliptic near the characteristics of $P$.

**Type 2.** $P$ is a 2x2 system of pseudodifferential operators:

$L^2(\mathbb{R},\mathbb{C}^2) \rightarrow L^2(\mathbb{R},\mathbb{C}^2)$, where the subscripts indicate the coordinate indices that we use for the two different copies of $\mathbb{C}^2$. We shall always think of these indices, as defined modulo 4, and in order to simplify the notation we write from now on $\mathbb{C}^2_{\text{odd}} = \mathbb{C}^2_{(1,3)}$, $\mathbb{C}^2_{\text{even}} = \mathbb{C}^2_{(2,4)}$. Let $T$ denote one of the two operators $T_1: \mathbb{C}^2_{\text{odd}} \rightarrow \mathbb{C}^2_{\text{even}}$ or $T_2: \mathbb{C}^2_{\text{even}} \rightarrow \mathbb{C}^2_{\text{odd}}$, where each one is defined by the general formula $(Tx)_j = x_{j-1}$. Then the operator $T^2$, defined either as $T_1T_2$ or $T_2T_1$ is given in the standard basis of $\mathbb{C}^2$ by the matrix:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

We then assume that $P$ satisfies (2.1) and

(2.7) $P^*T = T^*P^*$, $P^*T = T^*P$.

(2.8) $VP = VP T^2$.

Again, we assume that there are operators $P_1$ and $P_2$ satisfying (2.1), such that (2.4) holds and we replace (2.5) and (2.6) by:

(2.9) $P_1^*T = T^*P_1^*$, $P_2^*T = T^*P_2^*$,

(2.10) $VP_j = P_jVT^2$, $j = 1, 2$. 

It will be important later on to observe that these two relations and (2.4) remain valid, if we replace $P_1, P_2$, by $P_1 + tP, P_2 + tP$ for any real number $t$.

In this section, we study formally certain Grushin type problems for the operator $P$ either of type 1 or of type 2, and we shall see that we obtain in each case a new "renormalized" operator which is again of one of the two types. For each type there are two types of Grushin problems; either a "potential well" problem or a "branching" problem. As we shall see, the potential well problem leads always to a renormalized operator of type 1, (regardless of whether we start from an operator of type 1 or of type 2), and a branching problem always leads to a renormalized operator of type two. Hence there are 4 cases to check. The richest one is the branching problem for an operator of type 2 (leading to a new operator of type 2), and the other 3 cases are simple adaptations of this one. For an explanation of the terminology "branching" and "potential well", we refer to later sections dealing with more substantial (microlocal) analysis.

Case 1. The branching problem for a type 2 operator. Let $f_0, j \in L^2(R; C_{2 \times 2}^2)$ be a function, whose form will be specified later, and in some sense (that we do not need to specify yet) concentrated near the segment $\{(0, \pi), (\pi, 0)\}$. We put $f_0, j = (\sigma \tau)^{1/2} f_0, j$, for $j \in \mathbb{Z}_4 = \mathbb{Z}/(4)$, and then $f_{\alpha, j} = T f_0, j$, for $\alpha \in \mathbb{Z}^2$. We put,

$$ R_+ u(\alpha, j) = (u \mid f_{\alpha, j}), u \in L^2, (\alpha, j) \in \mathbb{Z}^2 \times (1, 3), $$

From the choice of the $f_{\alpha, j}$, it will be clear that $R_+$:

$$L^2(R; C_{0}^{2}) \rightarrow L^2(\mathbb{Z}^2 \times (1, 3))$$

is a bounded operator. Similarly, we define a bounded operator

$$R_- : L^2(\mathbb{Z}^2 \times (2, 4)) \rightarrow L^2(R; C_{2 \text{even}}^2),$$

by

$$ R_- u = \sum_{\mathbb{Z}^2 \times (2, 4)} u_{\alpha, j} f_{\alpha, j}. $$

From the definition of the $f_{\alpha, j}$ and from the commutation relations, that we have studied earlier, it follows that,

$$ T_{\gamma} f_{\alpha, j} = \exp(i \gamma_2 \alpha_1 h') f_{\alpha + \gamma, j}. $$

We conclude that,

$$ T_{\gamma} R_- = R_- T_{\gamma},$$

$$ T_{\gamma} R_+ = R_+ T_{\gamma},$$

where $\tau_{\gamma} : L^2(\mathbb{Z}^2 \times (1, 3)((2, 4))) \rightarrow L^2(\mathbb{Z}^2 \times (1, 3)((2, 4)))$ is defined by,

$$ \tau_{\gamma} w(\alpha, j) = \exp(i h' \gamma_2 (\alpha_1 - \gamma_1)) w(\alpha - \gamma, j). $$

Using this together with (2.1), we obtain,

$$ T_{\gamma} \begin{pmatrix} 0 \\ T_{\gamma} \end{pmatrix} = \begin{pmatrix} T_{\gamma} \\ 0 \end{pmatrix} \tau_{\gamma}, $$

where,
Throughout this section, we shall assume that $V$ is bijective $L^2 \times L^2 \to L^2 \times L^2$, and we denote the inverse by,
\[ \mathcal{E} = \begin{pmatrix} E & E^+ \\ E^- & -E^- \end{pmatrix}. \]
Then, (2.16) remains valid, if we replace $\mathcal{P}$ by $\mathcal{E}$. As a general fact, we notice that if $E_+^-$ is bijective with a bounded inverse, then $P: L^2 \to L^2$ has the same property, and the inverse of $P$ is given by $E^- E_+^- (E_+^+)^{-1} E^-$. Conversely, if $P$ is bijective with a bounded inverse, then the same holds for $E_+^-$, and the inverse is given by $-R_+ R^-^{-1}$. In particular, we know that $0$ belongs to the spectrum of $P$ if it belongs to the spectrum of $E_+^-$. The fact that $\mathcal{E}$ satisfies (2.16) implies in particular that
\[ \mathcal{E}^{-1} E_+^- = E_+^- \mathcal{E}^{-1} \mathcal{E} \in \mathbb{Z}^2. \]
Now $E_+^-$ has a matrix $E(\alpha, j; \beta, k)$, $\alpha, \beta \in \mathbb{Z}^2$, $j = 2, 4$, $k = 1, 3$, and if we identify $L^2(\mathbb{Z}^2 \times \{1, 3\}) \oplus L^2(\mathbb{Z}^2 \times \{2, 4\}) = L^2(\mathbb{Z}^2; C_{\text{even}}(\mathbb{Z}^2))$, then we can also think of the matrix of $E_+^-$ as an infinite matrix of $2 \times 2$ blocks; $E_+^-(\alpha, \beta) = (E_j(\alpha, j; \beta, k))_{j = 2, 4, k = 1, 3}$. Analyzing the relations (2.19), we then get the equivalent statement,
\[ E_+^-(\alpha, \beta) = \exp(ih' \beta_2 (\alpha_1 - \beta_1)) f(\alpha - \beta), \]
where $f$ is a matrix valued function on $\mathbb{Z}^2$. We will always be in the situation when $f$ is exponentially decreasing; $\|f(\alpha)\| \leq C \exp(-|\alpha|/C)$, for some $C > 0$, so we assume this property from now on. Noticing as in [HS1], section 6, that $E_+^-$ is a convolution in the variables $\alpha_1$ and that after a suitable conjugation, $E_+^-$ may be viewed as a convolution in the variables $\alpha_2$, we can show that $0 \in \text{Spec}(E_+^-)$ iff $0 \in \text{Spec}(Q)$, where $Q$ is the $h'$-Weyl quantization of the symbol,
\[ Q(x, \xi) = \sum f(\alpha) \exp(-i\alpha_1 x_1 h'/2) e^{-i<\delta(\alpha), (x, \xi)>}, \]
where $\delta(x, \xi) = (\xi, x)$. Here we assume that $0 < h' < 2\pi$. (When $h' = 0$, $E_+^-$ is a convolution operator and the condition that $0$ belongs to the spectrum is equivalent to: $0 \in \text{Q}(\mathbb{R}^2)$.) In view of the exponential decrease of $f$, we see that $Q$ is a well defined analytic matrix-valued function, $2\pi$-periodic both in $x$ and in $\xi$, so $Q$ satisfies (2.1), if the $T_{\alpha_1}$ (as always on the renormalized operator level) are defined with $h$ replaced by $h'$. What we shall verify (in each of the four cases), is that $Q$ inherits all the invariance properties of $P$. Before that, it will be convenient to make some general remarks about the renormalization map $E_+^- \to Q$, namely that this map respects composition, and passage to the adjoint.
Proposition 2.1. Let \( e: \mathbb{Z}^2 \rightarrow \text{Mat}(n \times 1) \), and \( f: \mathbb{Z}^2 \rightarrow \text{Mat}(1 \times m) \) be exponentially decreasing functions, so that we can define the composition \( E \circ F \), of the two matrices,

\[
(2.22) \quad E(\alpha, \gamma) = \exp(ih' \gamma \sigma_2(\alpha_1 - \gamma_1)) \ e(\alpha - \gamma),
\]

\[
(2.23) \quad F(\gamma, \beta) = \exp(ih' \beta \sigma_2(\gamma_1 - \beta_1)) \ f(\gamma - \beta).
\]

Then,

\[
(2.24) \quad (E \circ F)(\alpha, \beta) = \exp(ih' \beta \sigma_2(\alpha_1 - \beta_1))(e \circ f)(\alpha - \beta),
\]

where by definition,

\[
(2.25) \quad e \circ f(\alpha - \gamma) = \sigma_\gamma \ \exp(ih' \gamma \sigma_2(\alpha_1 - \gamma_1)) \ e(\alpha - \gamma)f(\gamma - \beta).
\]

Moreover, if we denote by \( Q_e, Q_f, Q_{e \circ f} \), the renormalized symbols, defined as in (2.21), then for the corresponding \( h' \)-quantizations, we have,

\[
(2.26) \quad Q_e \circ Q_f = Q_{e \circ f}.
\]

Proof. The proof is by straightforward calculation: We have,

\[
E \circ F(\alpha, \beta) = \sigma_\gamma \ \exp(ih' \gamma \sigma_2(\alpha_1 - \gamma_1) + \beta_2(\gamma_1 - \beta_1)) \ e(\alpha - \gamma)f(\gamma - \beta),
\]

and in order to obtain (2.24), (2.25), it is enough to check that,

\[
\gamma_2(\alpha_1 - \gamma_1) + \beta_2(\gamma_1 - \beta_1) - \beta_2(\alpha_1 - \beta_1) = (\gamma_2 - \beta_2)(\alpha_1 - \gamma_1).
\]

If we compose the corresponding \( h' \)-Weyl quantizations, we find that

\[
Q_e \circ Q_f = \Sigma \ e(\alpha) \ f(\beta) \ \exp(-i(\alpha_1 \alpha_2 + \beta_1 \beta_2)h'/2) \ A_{\alpha, \beta},
\]

\[
A_{\alpha, \beta} = e^{-i\alpha_2 x/2} e^{-i\alpha_1 h'D} e^{-i(\alpha_2 + \beta_2)x/2} e^{-i\beta_1 h'D} e^{-i\beta_2 x/2}.
\]

Using the relations:

\[
e^{-i\alpha_1 h'D} e^{-i\beta_2 x/2} = e^{i\beta_2 \alpha_1 h'/2} e^{-i\beta_2 x/2} e^{-i\alpha_1 h'D},
\]

\[
e^{-i\alpha_2 x/2} e^{-i\beta_1 h'D} = e^{-i\alpha_2 \beta_1 h'/2} e^{-i(\alpha_2 + \beta_2)h'/2} \ \sigma_{\gamma_1} \ e^{-i(\alpha_1 + \beta_1)h'/2}.
\]

we see that,

\[
A_{\alpha, \beta} = e^{i(\beta_2 \alpha_1 - \alpha_2 \beta_1)h'/2} e^{i(\alpha_2 + \beta_2)h'/2} \ \sigma_{\gamma_1} \ e^{-i(\alpha_1 + \beta_1)h'/2}.
\]

where \( Q_{h'}(...), Q_{h'}(...) \) denotes the \( h' \)-Weyl quantization of \( ... \). Hence,

\[
Q_\gamma = \Sigma \ g(\gamma) \ e^{-i\gamma_1 h'D} e^{-i(\gamma_2 + \beta_2)x/2} \ \sigma_{\gamma_1} \ e^{-i(\alpha_1 + \beta_1)h'/2}.
\]

Rewriting (2.25) on the form,

\[
e \circ f(\gamma) = \Sigma_{\alpha + \beta = \gamma} e^{-i\alpha_1 h'D} e^{-i(\alpha_2 + \beta_1 + 2 \beta_2)h'/2} e^{-i(\alpha_1 \alpha_2 + \beta_1 \beta_2 - \beta_2 \alpha_1 + \alpha_2 \beta_1)h'/2}.
\]

we see that \( g = e \circ f \), which gives (2.26). \[\square\]

For the passage to the adjoint, we have,

Proposition 2.2. Let \( F(\alpha, \beta) = e^{i(\alpha_2 + \beta_1)h'(\alpha_1 - \beta_1)} \ f(\alpha - \beta) \), where \( f \) is an exponentially decreasing function with values in the space of \( n \times m \)
matrices. If \( F^* \) denotes the complex adjoint of \( F \) in the sense of infinite matrices, then,
\[
F^*(\alpha, \beta) = e^{ih'Z_2(\alpha_1 - \beta_1)}g(\alpha - \beta),
\]
where \( f(\alpha)^* \) denotes the complex adjoint of \( f(\alpha) \) for each fixed \( \alpha \). If \( Q_f \) and \( Q_g \) denote the corresponding renormalized symbols, then we have
\[
(2.28) \quad Q_g(x, \xi) = Q_f(x, \xi)^*,
\]
and hence the corresponding relation for the \( h' \)-quantizations,
\[
(2.29) \quad Q_g = Q_f^*.
\]
In other words, the map \( F \to Q_f \) respects passage to the complex adjoints.

The proof of this result is just a routine calculation, starting from the fact that \( F^*(\alpha, \beta) = (F(\beta, \alpha))^* \), and we omit the details.

We now continue the study of the case 1, and we shall next look at the Fourier invariance. From the commutation relations between the \( T \) and \( \mathcal{F} \), and the definition of \( f_{\alpha, j} \), we obtain,
\[
(2.30) \quad (\mathcal{F}T)f_{\alpha, j} = e^{-ih'\alpha_1\alpha_2}f_{\kappa(\alpha), j-1},
\]
where we recall that \( \kappa = \kappa - \pi/2 \), denotes rotation by the angle \( \pi/2 \). From this and the definition of \( R_{\pm} \), it is easy to show that
\[
(2.31) \quad R_+\mathcal{F}T = 9R^* \quad \text{(implying \( \mathcal{F}TR_- = R^*G \)),}
\]
\[
(2.32) \quad \mathcal{F}TR^*_+=R_9G \quad \text{(implying \( 9R_+ = R^*\mathcal{F}T \)).}
\]
Here \( G \) denotes (the only possible) one of the two operators:
\[
\{ 2(\mathbb{Z}^2 \times (1, 3)) \to 2(\mathbb{Z}^2 \times (2, 4)), 2(\mathbb{Z}^2 \times (2, 4)) \to 2(\mathbb{Z}^2 \times (1, 3)) \}
\]
given in both cases by,
\[
(2.33) \quad g_{w}(\alpha, j) = e^{ih'\alpha_1\alpha_2}w(\kappa^{-1}(\alpha), j+1).
\]
Combining (2.7) with (2.31),(2.32), we get,
\[
\begin{pmatrix}
\mathcal{F}T & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
f_{\alpha, j}
\end{pmatrix}
= \begin{pmatrix}
f_{\alpha, j}
\end{pmatrix}^*
\begin{pmatrix}
\mathcal{F}T & 0 \\
0 & 0
\end{pmatrix}^*,
\]
which implies the corresponding relations for \( \mathcal{E} \):
\[
(2.34) \quad \begin{pmatrix}
\mathcal{F}T & 0 \\
0 & 0
\end{pmatrix}\mathcal{E} = \mathcal{E}^* \begin{pmatrix}
\mathcal{F}T & 0 \\
0 & 0
\end{pmatrix},
\]
\[
(2.35) \quad \mathcal{E} = \mathcal{E}^*,
\]
In particular, we obtain,
\[
(2.36) \quad g_{E} = \mathcal{E}_{-+} = \mathcal{E}^*_{-+} = \mathcal{E}^*_{-+} = \mathcal{E}_{-+} = \mathcal{E}_{-+}.
\]
More explicitly, the last relation means that,
\[
(2.37) \quad E(\alpha, \beta) = e^{ih'(\alpha_1 \alpha_2 - \beta_1 \beta_2)} T \cdot E(\kappa(\beta), \kappa(\alpha))^* \cdot T^{-1},
\]
or in view of (2.20),
\[
(2.38) \quad f(\kappa(\alpha)) = T \cdot f(\alpha)^* \cdot T^{-1}.
\]
Using the relations, \( t\kappa = -\kappa \), \( t\delta = \delta \), \( \kappa \delta = -\delta \kappa \), we get for the corresponding renormalized operator,
\[ Q \cdot \chi(x, \xi) = \sum f(\alpha) e^{-i\alpha_1 \alpha_2 h' / 2} e^{-i \langle \delta(\alpha), \chi(x, \xi) \rangle} = \sum f(\alpha) e^{-i\alpha_1 \alpha_2 h' / 2} e^{-i \langle \delta(\alpha), \chi(x, \xi) \rangle} = \sum f(\alpha) e^{-i\alpha_1 \alpha_2 h' / 2} e^{-i \langle \delta(\alpha), \chi(x, \xi) \rangle} = \sum f(\alpha) e^{-i\alpha_1 \alpha_2 h' / 2} e^{-i \langle \delta(\alpha), \chi(x, \xi) \rangle} = T(\sum f(\alpha) e^{-i\alpha_1 \alpha_2 h' / 2} e^{-i \langle \delta(\alpha), \chi(x, \xi) \rangle})^{* T^{-1}} = TQ^{* T^{-1}}, \]

(2.37) \quad Q \cdot \chi = TQ^{* T^{-1}}.

For the \( h' \)-quantizations, this means that \( \mathcal{S}^{-1} Q \mathcal{S} = TQ^{* T^{-1}} \), or rather,

(2.38) \quad Q(\mathcal{S} T) = (\mathcal{S} T) Q^{*}.

This is precisely the relation (2.7) for the renormalized operator, (with the only difference, that now \( \mathcal{S} \) denotes the \( h' \)-quantization).

We shall next study the invariance under reflection in the line \( x = \xi \). In view of (2.8), we shall assume that it is possible (subject to verification later), to choose \( f_0, 1 \) such that,

(2.39) \quad \forall T^2 f_0, 1 = f_0, 1.

Using the various commutation relations between the operators \( \mathcal{S}, T_{\alpha}, \) and \( V \), it is easy to check first that \( V f_0, 2 = f_0, 4 , \) \( V T^2 f_0, 3 = f_0, 3 \) and then more generally that,

(2.40) \quad V T^2 f_{\alpha, j} = e^{ih' \alpha_1 \alpha_2} f_{V(\alpha, j)}, \text{ for } j \text{ odd},

\[ V f_{\alpha, j} = e^{ih' \alpha_1 \alpha_2} f_{V(\alpha, j)}, \text{ for } j \text{ even,} \]

where \( V(\alpha, j) = (\delta(\alpha), 2 - j) \). From this it follows that,

(2.41) \quad V R^- = R' V', \quad R_+ V T^2 = V R_+ ', \text{ where the antilinear operator } V' \text{ is defined by,}

(2.42) \quad V' w(\alpha, j) = e^{ih' \alpha_1 \alpha_2} \bar{w}(v(\alpha, j)).

We notice that \( V'^2 = \text{id} \). Combining (2.41), (2.42), (2.8), we get,

\[ (V O) \circ \mathcal{C} = \mathcal{C} (V T^2 O), \]

which implies,

(2.43) \quad \mathcal{C} \begin{pmatrix} V & 0 \\ 0 & V' \end{pmatrix} = \begin{pmatrix} V T^2 & 0 \\ 0 & V \end{pmatrix} \mathcal{C}.

In particular, we have,

(2.44) \quad V' E_{-} = E_{-} V',

or more explicitely,

(2.45) \quad E(v(\alpha, j), (v(\beta, k)) = e^{ih' (\alpha_1 \alpha_2 - \beta_1 \beta_2)} \bar{E}(\alpha, j, \beta, k).

For the corresponding matrix \( f(\alpha) = (f(\alpha, j, k)) \), we then get,

(2.46) \quad f(\delta(\alpha), 2 - j, 2 - k) = e^{ih' \alpha_1 \alpha_2} \bar{f}(\alpha, j, k),

which implies for the corresponding renormalized symbol, that

(2.47) \quad Q(-\xi, -\xi, 2 - j, 2 - k) = \bar{Q}(x, \xi, j, k).

This does not use quite the reflection that we want, but iterating (2.37), we get \( Q \cdot \chi^2 = T^2 Q T^{-2} \), and this relation can be written explicitely as,

\( Q(-x, -\xi, 2^2, k + 2, k + 2) \), and combining this with (2.46), we get
Here the left hand side is the symbol of the operator $V\circ Q \circ V$, where we now use the $h'$-Weyl quantization, so on the renormalized operator-level, \(2.48\) can be written as:

\[
VQ = Q V T^2.
\]

This is the same relation (for the $h'$-quantizations) as \(2.8\).

To finish completely the case 1, we have to find natural operators $Q_1$ and $Q_2$, satisfying the obvious analogues of \(2.4\), \(2.9\), \(2.10\). Under the hypotheses above, we notice that the adjoint of $P$ gives the Grushin problem,

\[
P^* u + \mathcal{R}^* u_+ = v, \quad \mathcal{R}^* u = v_-
\]

which is well posed with inverse given by $\mathcal{S}^*$;

\[
u = E^* v + E^- v_-
\]

\[
u_+ = E^* v + E^- v_-
\]

Let us take $v = 0$ in these relations, and multiply the first equation of \(2.50\), by $P_2$. Using that $P_2 P^* = P P^*$, we get $P P^* u = -P_2 R^* u_+$. We can view this as the first equation of a Grushin problem for $P_2^* u$, and hence,

\[
P_2^* u = -E P_2 R^* u_+ + E_+ R_+ P_2^* u,
\]

\[
0 = -E_+ P_2 R^* u_+ + E^- R_+ P_2^* u.
\]

Substituting \(2.51\) with $v = 0$ into the last equation of \(2.52\), we obtain,

\[
E P_2 R^* E_+ v_+ = E_+ R_+ P_2^* E^- v_-, \text{ for all } v_-, \text{ and this means that}
\]

\[
E_+ (E_+ P_2 R^*_+)^* \text{ is self-adjoint. If we replace } (P, \mathcal{S}, P_2) \text{ in these arguments by } (P^*, \mathcal{S}^*, P_1^*) \text{, we also get that } E^- (E_+ P_1^* R^-)^* \text{ is self-adjoint, or equivalently, that } (R^* P_1 E_+) (R^* P_1 E^-)^* \text{ is self-adjoint. Using the various translation-invariance properties, it is easy to check that } E_+ P_2 R^*_+ \text{ and } R^* P_1 E_+ \text{ commute with } \mathcal{S}^*, \text{ so we can define the corresponding renormalized operators (by the same procedure as } E_+ \to Q) \text{; } Q_2, \text{ and } Q_1.
\]

From the propositions 2.1 and 2.2, it is then clear that $Q_1^* Q$ and $Q_2^* Q$ are self-adjoint and that $Q_1, Q_2$ commute with the operators $T^\gamma$ (where we now use the $h'$-quantizations). Moreover, if we repeat the arguments leading to the other invariance properties for $Q$, we obtain the natural analogues of \(2.9\) and \(2.10\), namely that,

\[
Q_1 T^\gamma T^2 = T^\gamma Q_1^*, Q_2 T^\gamma T^2 = T^\gamma Q_2^*,
\]

\[
(2.54) \quad V Q_j = Q_j V T^2, \quad j = 1, 2.
\]

In fact, with $A_1 = R^* P_1 E_+$, $A_2 = E_+ P_2 R^*_+$, we first find $A_1 \mathcal{S} = \mathcal{S} A_2^*$, $\mathcal{S} A_1 = A_2^* \mathcal{S}$. In general, if $M$ and $M'$ commute with the $\mathcal{S}^*$ and $\mathcal{S} \mathcal{M} = \mathcal{M} \mathcal{S}$, then for the corresponding renormalized operators, we find,
Q_m(\mathcal{T})^{-1}=(\mathcal{T})^{-1}Q_m', and this implies (2.53). To see that we have (2.54), we first notice that (V'u|v)2=(u|V'v)2, and that the analogous relation for \mathcal{V} has been obtained in section 1. The commutation relations for R_\pm and V_\mathcal{V}T^2,V', then give, R_*V=V'R_*, VT^2R_+=R_+V'. Using this and (2.43), we see that A_j commutes with V', j=1,2. From this we get (2.54) by the argument leading to (2.49).
This completes the study of the case 1.

**Case 2. The potential-well problem for a 2\times 2 system.** Assume as in the preceding case, that P is a type 2–operator. There are two subcases, corresponding to whether the potential wells are associated naturally to 2\pi\mathbb{Z}^2 or to 2\pi\mathbb{Z}^2+(\pi,\pi). Let us first reduce completely the second case to the first one. Given P, P_j, we put P'=S^{-1}PS, P_j'=S^{-1}P_jS, where S=T(2,1). Using the commutation relations between T_\mathcal{V}, T_\mathcal{V}, V, it is easy to check that (P',P_j',P_j') satisfies the relations (2.1),(2.4),(2.7)-(2.10).
Since S is a natural quantization of the translation by (\pi,\pi), we get a complete reduction to the subcase n°1. To describe the kind of Grushin–problem that we shall study in this case, let f_{0,1} be an L^2–function suitably concentrated to the component of the characteristic set of P in \mathbb{R}^2, which is "close to' (0,0) (in a sense that will be made precise later). As we shall see later, it will be possible to choose f_{0,1} in such a way that,
(2.55) (\mathcal{T})^2f_{0,1}=\pm f_{0,1},
which we here take as an assumption. We now let the index j vary in \mathbb{Z}_2=\mathbb{Z}/(2), and put f_{0,2}=(\mathcal{T})^{-1}f_{0,1}, where for convenience T' is defined as T when the + sign is valid in (2.55) and defined as iT in the other case. Then in general, f_{0,j}=(\mathcal{T})^{k_j-1}f_{0,k}, and since T^4=Id, we also have (T')^4=Id. Following closely the case 1, we define f_{\alpha,j}=T_\alpha f_{0,j} and the operators R_\pm:L^2(\mathbb{R};\mathbb{C}^2)\to L^2(\mathbb{Z}^2), R_-:L^2(\mathbb{Z}^2)\to L^2(\mathbb{R};\mathbb{C}^2), by:
(2.56) R_+u(\alpha)=(u|f_{\alpha,1}),
(2.57) R_-w=\Sigma w(\alpha)f_{\alpha,2}.
As in the earlier case, we shall assume that the corresponding operator \mathcal{P} is bijective and has bounded inverse \mathcal{E}=
\begin{pmatrix}
E & E_+ \\
E_+ & E_-
\end{pmatrix}.
Repeating the argument of the case 1, we then obtain,
(2.58) T_\mathcal{V}R_-=R_-T_\mathcal{V}, R_+T_\mathcal{V}=T_\mathcal{V}R_+,
(2.59) \mathcal{V}E_+=E_+\mathcal{V},
where \mathcal{V}:L^2(\mathbb{Z}^2)\to L^2(\mathbb{Z}^2) is given by the obvious modification of (2.15), and where we now notice that E_+ is given by an infinite matrix.
(E-+(a, β)), where E-+(a, β) are scalars rather than 2x2 blocks. We still have (2.20) and we can define the renormalized operator as before, having the crucial property that 0 belongs to the spectrum of P iff it belongs to the spectrum of Q. Now Q is a scalar pseudodifferential operator instead of a 2x2 system. Repeating the earlier arguments, we also find,

(2.60) \( T^*R_+ = R_+^* \), \( R_+^*T^* = gR_+ \),

(2.61) \( gE_{-+} = E_{++}g \),

where \( g : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2) \) is defined by the obvious simplification of (2.33). As before this leads to the relation Q*K = Q for the renormalized symbol, and hence for the corresponding h'-quantizations,

(2.62) \( Q^* = S^*Q^* \).

In order to treat the invariance under reflection in the line x=ξ, we add the assumption that

(2.63) \( VT^2f_0,1 = f_0,1 \).

Then using the commutation relations and (2.63), (2.55), we get after a simple computation,

(2.64) \( Vf_{0,2} = f_{0,2} \).

Using again the commutation relations, we obtain from (2.63), (2.64),

(2.65) \( VT^2f_\alpha,1 = e^{ih^'\alpha_1\alpha_2} f_\delta(\alpha),1 \), \( Vf_\alpha,2 = e^{ih^'\alpha_1\alpha_2} f_\delta(\alpha),2 \),

and as in the case 1, this leads to

(2.66) \( VR_+ = R_+^*V' \), \( R_+^*VR_+ = V'R_+ \),

where the antilinear operator V' is defined by,

(2.67) \( V'w(\alpha) = e^{ih^'\alpha_1\alpha_2} \bar{w}(\delta(\alpha)) \).

Continuing as in case 1, we then obtain (2.44), which now means more explicitely that,

(2.68) \( E(\delta(\alpha), \delta(\beta)) = e^{ih^'(\alpha_1\alpha_2 - \beta_1\beta_2)} \bar{E}(\alpha, \beta) \).

For the corresponding matrix f(\alpha), we then get,

(2.69) \( f(\delta(\alpha)) = e^{ih^'\alpha_1\alpha_2} f(\alpha) \),

which implies for the corresponding renormalized symbol, that

(2.70) \( Q(-\xi, -x) = \bar{Q}(x, \xi) \).

This does not use quite the reflection that we want, but iterating the relation Q*K = Q, we get Q*K^2 = Q, and this relation can be written explicitely as,

(2.71) \( Q(-x, -\xi) = Q(x, \xi) \),

and combining this with (2.70), we get,

(2.72) \( \bar{Q}(\xi, x) = Q(x, \xi) \).

Here the left hand side is the symbol of the operator V\*Q\*V, where we now use the h'-Weyl quantization, so on the renormalized operator-level, (2.72) can be written as:

(2.73) \( QV = VQ \).

This is the same relation (for the h'-quantizations) as (2.3).

To finish completely the case 2, we have to find natural operators Q_1 and Q_2, satisfying the natural analogues of (2.4)-(2.6). The first part of
the corresponding argument in case 1 is completely general, and as there, we see that \( E_-(E_2 R^*)^* \) and \( (R^* P_1 E_+)^* E_- \) are self-adjoint. Again \( E_2 R^* \) and \( R^* P_1 E_+ \) commute with \( \mathcal{G}_Y \), so we can define the corresponding renormalized operators \( Q_2 \) and \( Q_1 \). It is then clear that \( Q_1^* Q_2 \) and \( QQ_2^* \) are self-adjoint and that \( Q_1, Q_2 \) commute with the operators \( T_\gamma \) (where we now use the \( h' \)-quantizations). Moreover, if we repeat the arguments leading to the other invariance properties for \( Q \), we obtain the natural analogues of (2.5) and (2.6), namely that,

\[
(2.74) \quad Q_1 \mathcal{G} = \mathcal{G} Q_2^*, \quad Q_2 \mathcal{G} = \mathcal{G} Q_1^*,
\]

\[
(2.75) \quad \forall \mathcal{G}_j = \mathcal{G}_j V, \quad j=1,2.
\]

This completes the study of the case 2.

**Case 3. The branching problem for a type 1-operator.** This case is a complete repetition of the case 1, with the obvious simplification that all the invariance relations before renormalization, do not involve any operators \( T' \). As in case 1, the renormalized operator is of type 2.

**Case 4. The potential well problem for a type 1-operator.** This case is a complete repetition of the case 2, with the obvious simplification that all the invariance relations before renormalization, do not involve any operators \( T' \). As in case 2, the renormalized operator is of type 1.
3. The potential-well problem for a type 1 operator.

Let \( P \) be of type 1, as defined in the section on the formal study of the iteration process, and let \( P_1, P_2 \) be corresponding operators such that \( P^*P, \) \( PP^* \) are self adjoint etc. We shall assume that \( P, P_j \) depend analytically on a real parameter \( \mu \in [-4, 4] \), and more precisely that

\[
(3.1) \quad P = O_P(P(\mu, x, \xi)), \quad P_j = O_P(P_j(\mu, x, \xi)),
\]

where \( P, P_j \) are holomorphic in \( D(\varepsilon) = \{ \mu \mid \mu | < 4 \times ((x, \xi) \in \mathbb{C}^2 ; |\text{Im}(x, \xi)| < 1/\varepsilon) \} \),

for some sufficiently small \( \varepsilon > 0 \). In this domain and with the same \( \varepsilon \), we also assume that,

\[
|P - (\cos \xi + \cos x - \mu)| \leq \varepsilon, \quad |P_j - 1| \leq \varepsilon, \quad \text{for} \quad (\mu, x, \xi) \in D(\varepsilon).
\]

We formalize this by introducing the following definition.

**Definition 3.1.** We say that \((P, P_1, P_2)\) (or simply \( P \)) is of strong type 1, if it is of type 1 in the sense of section 2, and has the properties above. If \((P, P_1, P_2)\) is of strong type 1, we let \( \epsilon(P) \) (or rather \( \epsilon(P, P_1, P_2) \)) be the infimum of all \( \varepsilon > 0 \), such that \((3.2)\) holds.

In this definition, \( P(x, \xi) \) and \( P_j(x, \xi) \) may depend on many more parameters than \( \mu \). Then \( \epsilon(P) \) will depend on these parameters also. The goal of this section (and the following ones) is to obtain results which are uniform with respect to all these additional parameters, valid when \( \epsilon(P) \) is sufficiently small. Among the additional parameters we may also have \( h \), but the most important \( h \)-dependence, and the only one that we explicitly take into account in our notation, is the one resulting from the fact that we work with \( h \)-quantizations.

**Definition 3.2.** If the operator \( P \) is bounded in some Hilbert space, and depend on a parameter \( \mu \in \mathbb{M} \), then we define \( \mu - \text{Sp}(P) \) as the set of \( \mu \) in \( \mathbb{M} \) such that \( 0 \in \text{Sp}(P) \).

In this section we shall assume that \((P, P_1, P_2)\) is of strong type 1, and that \( 0 < h \leq 2\pi \). Notice that if \( \epsilon(P) \) is sufficiently small, the operators \( P_j \) are bijective on \( L^2 \), so \( 0 \) belongs to the spectrum of \( P \) iff it belongs to the spectrum of \( H_1 = P^*P \) (or equivalently, if it belongs to the spectrum of \( H_2 = PP^* \)). Recall that \( H_1 \) and \( H_2 \) are selfadjoint, and intertwined by \( \mathcal{F} \). In particular,

\[
(3.3) \quad H_1 \mathcal{F}^2 = \mathcal{F}^2 H_1,
\]

where we recall that \( \mathcal{F}^2 \) is a quantization of \( x^2 \) which is the reflection in \( (0, 0) \). Also recall that,

\[
(3.4) \quad \mathcal{V} H_1 = H_1 \mathcal{V},
\]

where \( \mathcal{V} \) is a quantization of reflection in the line \( x = \xi \). \( \mathcal{V} \) is antilinear and \( \mathcal{V}^2 = I \). From \((3.2)\) it follows that \( H_j \) also satisfy \((3.2)\) with a new \( \epsilon > 0 \), in a region defined as before with this new \( \epsilon \). The new \( \epsilon \) tends to 0, when \( \epsilon(P) \) tends to zero. The Weyl symbol \( H_1 \) is real valued for real \( \mu \) and \((3.3)\) and \((3.4)\) imply that \( H_1(\mu, x, \xi) \) is invariant under reflections in \( (0, 0) \) and in the line.
\( x = \xi; H_1(-x, -\xi) = H_1(x, \xi), H_1(x, \xi) = H_1(\xi, x) = H_1 \circ \Delta(x, \xi). \) Moreover \( H_1 \) is 2\( \pi \)-periodic in \( x \) and in \( \xi \), and \( H_1(x, \xi) + \mu \) is on the real domain an arbitrarily small perturbation in \( \mathbb{C}^2 \) of the symbol \( \cos \xi + \cos x \). We start by discussing the symbol \( H_1 + \mu \) on the real domain, following closely the argument of section 7 in [HS1]. Modulo \( 2\pi \mathbb{Z}^2 \) there are precisely 4 critical points of this symbol, namely a non-degenerate maximum close to \((0, 0)\), a non-degenerate minimum close to \((\pi, \pi)\) and two non-degenerate saddle points close to \((0, \pi)\) and \((\pi, 0)\) respectively. From the invariance under the map \((x, \xi) \mapsto (-x, -\xi)\), it follows that the point of maximum is equal to \((0, 0)\). Using also the translation invariance, we see that the symbol is invariant under the maps \((x, \xi) \mapsto (2\pi - x, 2\pi - \xi), (x, \xi) \mapsto (-x, 2\pi - \xi), (x, \xi) \mapsto (2\pi - x, -\xi)\), which are respectively the reflections in \((\pi, \pi), (0, \pi)\) and \((\pi, 0)\). From this we conclude that the point of minimum is \((\pi, \pi)\) and that the two saddle points are \((0, \pi)\) and \((\pi, 0)\). The map \( \Delta \) leaves \( H_1 \) invariant and exchanges the two saddle points, hence we have the same critical value at the two saddle points.

Summing up, we have,

**Proposition 3.3.** For real \( \mu \)'s, the symbol \( H_1 \) of \( H_1 = P \tilde{H} P \) is invariant under reflection in any of the points \((k\pi, 1\pi), k \in \mathbb{Z}\), under translation by \( 2\pi \) in \( x \) or in \( \xi \), and under reflection in the line \( x = \xi \) (and more generally in any of the lines \( \xi = \pm x + k2\pi; k \in \mathbb{Z}\)). Modulo \( 2\pi \mathbb{Z}^2 \) there are precisely 4 critical points, all non-degenerate; a maximum at \((0, 0)\) with \( H_1(0, 0) + \mu = 2 + \mathcal{O}(\epsilon) \), a minimum at \((\pi, \pi)\) with \( H_1(\pi, \pi) + \mu = -2 + \mathcal{O}(\epsilon) \), and saddle points at \((0, \pi)\) and \((\pi, 0)\) with \( H_1(0, \pi) + \mu = H_1(\pi, 0) + \mu = \mathcal{O}(\epsilon) \).

Let \( c(\mu) = H_1(\pi, 0) + \mu = \mathcal{O}(\epsilon) \). The discussion below will be uniformly valid, provided that \( \epsilon(\mu) > 0 \) is sufficiently small, that \( \mu \geq c(\mu) + \epsilon_0 \) for some arbitrary but fixed \( \epsilon_0 > 0 \), and finally that \( 0 < h \leq h_0(\epsilon_0) \) for some \( h_0(\epsilon_0) > 0 \). The assumption that \( \mu \geq c(\mu) + \epsilon_0 \) can also be written; \( H_1(\pi, 0) \leq -\epsilon_0 \), so the real characteristic set of \( H_1 \) avoids the saddle points \((k\pi, 1\pi), k \in \mathbb{Z}\).

For \( \lambda > c(\mu) \), let \( U_{\alpha} = U_{\alpha}(\lambda) \) be the component of \((H_1 + \mu - \lambda)^{-1}(0)\) in \( \mathbb{R}^2 \), naturally associated to \( \alpha \in \mathbb{Z}^2 \), and let \( H_1 + \mu \) be a modification of \( H_1 + \mu \) obtained by filling all the potential wells \( U_{\alpha}(c(\mu) + \epsilon_0) \) for \( \alpha \neq (0, 0) \) but leaving \( H_1 \) unchanged near \( U_0 \) and in the compact domain to which \( U_0 \) is the boundary. For the moment, we can do this by standard \( C^\infty \)-theory as in section 2 of [HS1].

We know that the eigenvalues of \( H_1 + \mu \) in the interval \([c(\mu) + \epsilon_0, \infty]\) are simple, bounded from above by \( H_1(0, 0) + \mu + \mathcal{O}(h) \), and mutually separated by a distance \( \geq h/C_0 \) for some constant \( C_0 \). Moreover, if we use (3.2) and ordinary perturbation theory, we see that each such eigenvalue \( \lambda(\mu) \) will depend analytically on \( \mu \) with a holomorphic extension (as well as for the eigenfunction) to the strip \( |\text{Im} \mu| < h/C_0 \). Moreover, in this strip, we have \( d\lambda(\mu)/d\mu = \mathcal{O}(\epsilon) \), so if \( \mu \) is close to \( \lambda(\mu) \) to start with, then there will be a
simple real zero \( \mu = \mu_1 \) of \( \mu - \lambda(\mu) \), and from now on we restrict the attention to an interval \([\mu_1 - h/C_0, \mu_1 + h/C_0]\), where we may choose \( C_0 \) as large as we like. It is then clear that \( \tilde{H}_1 = \tilde{H}_1 + \mu - \mu \) has an inverse of norm \( \Theta(1/|\mu - \mu_1|) \), for \( \mu \) complex with \( 0 < |\mu - \mu_1| < h/C_0 \). If we do the "filling" with some care, using quantizations of the modifying operators based on superpositions of Gaussians, then we also know that the spectrum of \( H_1 + \mu \) in \([\lambda(\mu) - h/C_0, \lambda(\mu) + h/C_0]\) for \( C_0 \) sufficiently large, is concentrated to an exponentially short interval centered at \( \lambda(\mu) \), when \( \mu \) is real. (More details about gaussian quantizations will be given in section 4.) Let \( \pi_\mu \) be the associated spectral projection. Writing \( \pi_\mu \) as a contour integral with the resolvent of \( H_1 + \mu \), we see that the definition of \( \pi_\mu \) extends holomorphically to \( |\text{Im} \mu| < h/C_0 \), and we have \( \|d\pi_\mu /d\mu\| = \Theta(\epsilon/h) \).

We shall now fix \( \mu = \mu_1 \) for a while, and we start by recalling the definition of certain distances as in [HS1]. With \( U_\alpha = U_\alpha(\mu_1) \), we now put \( U_j = \pi_x(U(j,k)) = \) the \( x \)-space projection of \( U(j,k) \). Then for \( x \) real between \( U_0 \) and \( U_1 \), the complex zeros of \( H_1(x,\xi) \) are of the form \( \xi(x) + 2\pi k \pm i \Phi_1'(x) \), \( k \in \mathbb{Z} \), where \( \xi(x) \) is real and \( (x, \xi(x)) \) tends to a point of \( U_0(0,0) \), when \( x \) tends to the right boundary of \( U_0 \), and \( \Phi_1'(x) \geq 0 \). We then extend \( \Phi_1 \) to a \( C^1 \) function on \( \mathbb{R} \), such that \( \Phi_1'(x) \) is \( 2\pi \) periodic and vanishes on \( U_0 \). Finally, we put \( D_1(x,y) = |\Phi_1(x) - \Phi_1(y)| \). Since \( H_1(-x,-\xi) = H_1(x,\xi) \), we obtain that \( \Phi_1 \) is odd, and hence that \( D_1(-x,-y) = D_1(x,y) \). Let us define \( D_2 \) associated to \( H_2 \) in the same way. As we shall see (or rather recall) solutions to the equation \( H_1 u = 0 \) will often have weighted estimates in terms of \( D_1 \). Since \( \varphi \) exchanges \( H_1 \) and \( H_2 \), the Fourier transforms of solutions of \( H_1 u = 0 \) will satisfy estimates in terms of \( D_2 \) and vice versa. Since our weighted estimates will allow for losses \( = \Theta(\epsilon/h) \) for every \( \epsilon > 0 \), we can often replace \( D_1 \) and \( D_2 \) by one single distance function by means of the following observation: We have that \( H_1(x,\xi) = 0 \) implies \( P(x,\xi) \), \( \tilde{P}(x,\xi) \), \( H_2(x,\xi) = \Theta(h) \), simply because on the operator level, \( P = (P)^{-1}H_1 \) and so on, and the composition formula for the full symbols is reduced to multiplication, up to an error \( \Theta(h) \). If we add the assumption that \( d_\xi^2 H_1(x,\xi) \neq 0 \), then there is a unique \( \xi' = \xi + \Theta(h) \) such that \( H_2(x,\xi') = 0 \). Completing this argument with a simple discussion of what happens in a small neighborhood of the \( x \)-projected wells, we see that \( \Phi_2(x) - \Phi_1(x) = o(1)(1+|x|) \) uniformly, when \( h \to 0 \). Thanks to this we will usually be able to replace \( D_1 \) or \( D_2 \) by \( D = \text{def.} \frac{1}{2}(D_1 + D_2) \).

We choose \( \varphi = \varphi_0 \) (where \( 0 = (0, 0) \)) the same way as the function \( g_0 \) in [HS1], section 4, to be a suitable Gaussian of \( L^2 \)-norm \( \Theta(h^{-N_0}) \), such that...
(3.5) \( f_0 = \pi_{\mu_1} \varphi_0 \) has norm 1,
(3.6) \( f_0, \mathcal{F} f_0 = O(e^{-\varepsilon/h}) \),

in the sense that for every \( \varepsilon > 0 \), there is an \( h_{\varepsilon} > 0 \), such that
\[
\| e^{(1-\varepsilon)/h} f_0 \| = O(e^{\varepsilon/h}) \text{ for } 0 < h < h_{\varepsilon},
\]
and similarly for \( \mathcal{F} f_0 \). Here
\[
f(x) = \min_{k \in \mathbb{Z}} D(0, x) + V_0 I_{k1},
\]
where \( 0 < V_0 \leq S_0 = \text{def.} D(0, 2\pi) \). Without changing the growth estimates for \( f_0 \), we shall next modify this function in order to fulfill some symmetry relations, required in our formal iteration scheme. Microlocally, we know that \( f_0 \) is concentrated to a neighborhood of \( U_Q \) and by construction, we also know that \( H^* f_0 \) is exponentially small in \( L^2 \), and in particular, it makes sense to say that \( f_0 \) belongs to the kernel of \( H^* \) microlocally near \( U_Q \). (See appendix a.) Now this kernel is one-dimensional (in a sense that we leave to the reader to define, possibly after reading [HS1]), and invariant under the operator \( \mathcal{F}^2 \). Hence, \( f_0 = \pm \mathcal{F}^2 f_0 \) up to an exponentially small error in \( L^2 \), and without changing any of the essential properties of \( f_0 \), we can then replace this function by \( \frac{1}{2} (f_0 \pm \mathcal{F}^2 f_0) = \pi_{\mu_1} \left( \frac{1}{2} (\varphi_0 \pm \mathcal{F}^2 \varphi_0) \right) \). Since \( \mathcal{F}^4 = 1 \), the new function \( f_0 \) will satisfy:

(3.7) \( \mathcal{F}^2 f_0 = \pm f_0 \).

We can also add symmetry under \( V \). Microlocally near \( U_Q, 0 \), we have \( V f_0 \sim \omega f_0 \) for some complex \( \omega \) of modulus 1, since \( V \) commutes with \( H^* \). Writing this as \( V \omega \frac{1}{2} f_0 \sim \omega \frac{1}{2} f_0 \), (since \( V \) is anti linear,) we get approximate \( V \)-invariance by replacing \( f_0 \) by \( \omega \frac{1}{2} f_0 \), and this does not destroy any of the earlier properties. If \( f_0 \) denotes this new function, we make a last modification by replacing \( f_0 \) by \( \frac{1}{2}(f_0 + V f_0) \). This will not alter (3.7), since \( V \) and \( \mathcal{F}^2 \) commute, and we have gained the property,

(3.8) \( V f_0 = f_0 \).

Recall now that the subscript \( 0 \) stands for \( 0 \) in \( \mathbb{Z}^2 \), and in order to use the notation of section 2, we put \( f_{0,1} = f_0(0,0), 1 = f_0 \), and from this function, we define \( f_{\alpha, j} \) for \( \alpha \in \mathbb{Z}^2, j = 1, 2 \), as in case 4 of that section, (which is an obvious modification of the case 2 in the same section,) as well as the corresponding operators \( R_+ \) and \( R_- \). The corresponding Grushin problem is then,

(3.9) \( Pu + R_- u^- = v, \ R_+ u = v_+ \),

and we shall first study this problem microlocally near the potential well \( U_Q \) for \( \mu = \mu_1 \). We then get the problem,

(3.10) \( Pu + u^- (0) f_{0,2} = v, \)

\( (u|f_{0,1}) = v_+(0) \).

Since \( \mu = \mu_1 \), we know that \( f_{0,1} \) is microlocally in \( \text{Ker} P \), \( \text{Ker} P = \text{Ker} P_0 \), and
$f_{0,2} \in \ker PP^*_2 = (\text{Im } PP^*_2)^\perp = (\text{Im } P)^\perp$. Here we proceed formally and we leave to the reader to give a precise sense to the arguments by introducing the convenient error-estimates and pseudodifferential cutoffs. (See also appendix d, where a more complicated Grushin problem is treated.) In order to solve (3.10), we write $u = u' + x f_{0,1}$, $u' \in (f_{0,1})^\perp$, $v = v' + y f_{0,2}$, $v' \in (f_{0,2})^\perp = \text{Im}(P)$, $x = (u|f_{0,1})$, $y = (v|f_{0,2})$. We notice that $P : (f_{0,1})^\perp \to (f_{0,2})^\perp$ is microlocally a bijection, and that the inverse $E_0$ is of $\mathcal{L}(L^2, L^2)$-norm $\mathcal{O}(h^{-1})$. We then get the microlocal solution,

$$u = E_0 v' + v_+(0) f_{0,1},$$

$$u_-(0) = (v|f_{0,2}),$$

so the inverse of

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix}$$

is microlocally of the form,

$$\mathcal{G}_0 = \begin{pmatrix} E_0 & E_0^* \\ E_0^* & -E_0 & -E_0^* & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{O}(h^{-1}) & \mathcal{O}(1) \\ \mathcal{O}(1) & 0 \end{pmatrix},$$

and by standard microlocal cutoffs and c, (or by microlocal apriori estimates as in appendix d,) we see that $\mathcal{P}$ also globally has an inverse of the form (3.12) up to an error $\mathcal{O}(h^\infty)$, and where the description of $E_0$ is now only microlocal near the wells $U_\alpha$. By ordinary perturbation theory, we see that if we let $\mu$ vary in the complex disc $D(\mu_1, \epsilon_0 h)$ of center $\mu_1$ and radius $\epsilon_0 h$, then $\mathcal{P}$ defined above is still bijective, with an inverse,

$$\mathcal{G} = \begin{pmatrix} E & E_+ \\ E_- & -E_+ \end{pmatrix} = \mathcal{O}(h^{-1}).$$

Representing $\mathcal{G}$ by a perturbation series in the usual way, we see that,

$$E_- = (P - P_0) f_{0,1} f_{0,2} + \mathcal{O}(\epsilon^2) + \mathcal{O}(h^\infty),$$

where we put $\nu = \mu - \mu_1$, $P = P(\mu, x, hD)$, $P_0 = P(\mu_1, x, hD)$. Let us study the first term to the right in (3.14). In view of (3.2), we have :

$$f_{0,1} = P f_{0,1} + \mathcal{O}(\epsilon).$$

Using the selfadjointness of $P f_{0,1}$ and $P_0 P^*_2$ and the fact that $P_0 f_{0,1} = \mathcal{O}(h^\infty)$ (and even exponentially small if we are careful), we see that $P_0 P^*_2 P f_{0,1} = \mathcal{O}(h^\infty)$, so $P f_{0,1} = \omega f_{0,2} + \mathcal{O}(h^\infty)$. From this and (3.15), we conclude that,

$$f_{0,1} = \omega f_{0,2} + \mathcal{O}(\epsilon) + \mathcal{O}(h^\infty).$$

Since $f_{0,2} = (i) \mathcal{S} f_{0,1}$, and $(i) \mathcal{S} f_{0,1} = f_{0,1}$, we can permute $f_{0,1}$ and $f_{0,2}$ in (3.16). It follows that we may assume that $\omega^2 = 1$, so finally,

$$f_{0,1} = \pm f_{0,2} + \mathcal{O}(\epsilon) + \mathcal{O}(h^\infty).$$

(3.2) and Cauchy's inequality give,

$$P - P_0 = -\nu I + \mathcal{O}(\epsilon \nu),$$
and from this and (3.17) we obtain:
(3.19) \((P-P_0)\xi_0,1|\xi_0,2=\bar{\xi}_0v(1+\Theta(\epsilon)+\Theta(h^\infty))\). Substitution in (3.14) gives,
(3.20) \(E_-=(\bar{\xi}_0v+\Theta(\epsilon v))I+\Theta(v^2/h^3)+\Theta(h^\infty)\).

This gives interesting information about the behaviour of the diagonal part of \(E_-\) when \(v\ll h^3\), and this region contains the possible values of \(v\) in \([-h/C_0,h/C_0]\) such that 0 is in the spectrum of \(P\). We restrict the attention to this region from now on. After a translation by \(\Theta(h^\infty)\) (and even by an exponentially small quantity), we may assume also that the diagonal part of \(E_-\) vanishes for \(v=0\). This is possible since the diagonal part of \(E_-\) is constant by translation invariance, real by reflection invariance with respect to \(x=\xi\) (see section 2, case 4, which is the analogue of case 2), and given by (3.20). We recall (3.6) and the subsequent definition of \(f_0\) and of \(v_0\). Thus, if \(z(v)I\) is the diagonal part of \(E_-\), we know that \(z(v)\) is real for real \(v\), (since \(z(v)=f_0(0)\) and we have (2.69),) and we may assume that (3.21) \(z(v)=(\bar{\xi}_0v+\Theta(\epsilon v))I+\Theta(v^2/h^3)\).

The next problem is to study the off diagonal part of \(E_-\). Using the techniques of section 7 in [HS1], it is clear that we have the following result:

Let \(F\) be a real \(C^\infty\)-function with all derivatives in \(L^\infty\), such that
(3.22) \(|F'(x)|\leq(|D'(x)v|+\delta)_+,\)
(3.23) \(|F(U_k)-F(U_j)|\leq(1-\delta)v_0|k-j|,\)
for some \(\delta>0\). Then if \(F^*((j,k))=F(U_j)\), \(F\) is bijective from \(L^2(R;e^{2F(x)/h})\times L^2(e^{2F^*/h})\) onto itself for sufficiently small \(h\), and the inverse \(\delta\) is \(\Theta(h^{-1})\) in the associated operator norm. (Here \(L^2(e^{2F^*/h})=(w;\text{ we}eF^*/h)\). We also have an analogous result with weighted spaces on the Fourier transform side, and using both results, we obtain,
(3.24) \(E_-(\alpha,\beta)=\Theta(\epsilon^-(1-\delta)v_0|\alpha-\beta|\omega/h),\)
for every \(\delta>0\). Here \(|\alpha|_\infty=\max(|\alpha_1|,|\alpha_2|)\). From this it follows that 0 can belong to the spectrum only if \(|v_1|\) is exponentially small.

Recall that \((R^*P_1E_+)^*E_-\) and \(E_-(E_-P_2R_+^*)^*\) are self adjoint. Put \(A_1=R^*P_1E_+, A_2=E_-P_2R_+^*\). The matrix of \(A_1\) is given by
\(A_1(\alpha,\beta)=(P_1E_+\delta_\beta |f_\alpha,2)=\Theta(\epsilon^-(1-\delta)v_0|\alpha-\beta|\omega/h)\) for all \(\delta>0\), and by translation invariance, \(A_1(\alpha,\alpha)=A_1(0,0)\). For \(v=0\), we get,
\(A_1(0,0)=(P_1f_0,1|f_0,2)\) ignoring errors which are \(\Theta(h^\infty)\), so \(|A_1(0,0)|\geq\text{const.}>0\). The reflection symmetry in \(x=\xi\), implies that,
\(A_1(\delta(\alpha),\delta(\beta))=e^{ih}(\alpha_1\alpha_2-\beta_1\beta_2)\overline{A_1}(\alpha,\beta),\)
which implies that \(A_1(\alpha,\alpha)\) is real valued. For the corresponding renormalized operators \(Q_j\), it follows that
\(Q_j=A_j(0,0)+\Theta(\epsilon^{-1/C_0h})\) in a complex strip \(|\text{Im}(x,\xi)|\leq1/C_0h\), and replacing \(Q_j\)
by $A_j(0,0)^{-1}Q_j$ this will imply after rescaling in the $\mu$-variable, that the part of (3.2) concerning $Q_j$ is satisfied with $\epsilon$ replaced by $C_0h$.

Following section 4 in [HS1], we shall now improve the estimates on $E_{-+}(\alpha, \beta)$ when $\nu=0$, so that $z(\nu)=E_{-+}(\alpha, \alpha)=0$. Define $S_\alpha \in L^2(\mathbb{Z}^2)$ by $S_\alpha(\beta)=1$ when $\beta=\alpha$ and $=0$ otherwise. Put $g_{\alpha,1}=E_+(S_\alpha)$, $g_{\alpha,2}=E_-(S_\alpha)$, and notice that $g_{\alpha,1}$ and their Fourier transforms satisfy the same growth estimates as $f_{\alpha,1}$. Since $R_+E_+=I$, we get,

$$\langle g_{\alpha,1} \phi, \phi \rangle = \langle E_+ S_\alpha \phi, \phi \rangle = \langle R_+ E_+ S_\alpha \phi, \phi \rangle = \langle S_\alpha, \beta \rangle \text{ (Kronecker's delta):}$$

(3.25) $\langle g_{\alpha,1} \phi, \phi \rangle = \delta_{\alpha, \beta}$.

Similarly, using that $R_+E_+=I$:

(3.26) $\langle g_{\alpha,2} \phi, \phi \rangle = \delta_{\alpha, \beta}$.

Applying, $P\phi+R_+E_+=0$ to $S_\beta$, we get,

(3.27) $P g_{\beta,1} + S_\alpha \neq \beta E_{-+}(\alpha, \beta) f_{\alpha,2} = 0$,

which with (3.26) implies that,

(3.28) $P g_{\beta,1} \phi, g_{\alpha,2} \phi = -E_{-+}(\alpha, \beta)$.

Taking the adjoint of the identity $E_+P+E_+R_+=0$, and applying it to $S_\beta$ we get,

(3.29) $P^* g_{\beta,2} + S_\alpha \neq \beta E_{-+}(\alpha, \beta) f_{\alpha,1} = 0$,

which also implies (3.28).

We are looking for improved estimates for small values of $\alpha$ and $\beta$, so it will be natural to adopt the following terminology: we say that

$A(\alpha, \beta) = \hat{\delta}(e^{\frac{-a(\alpha, \beta)}{h}})$, where $a$ is a real-valued function, if for every $\delta>0$, and for every bounded set $K$ in $\mathbb{Z}^2$, there is a constant $C>0$, such that

$$|A(\alpha, \beta)| \leq C \delta^{a(\alpha, \beta)}/h$$,

we shall also use the analogous terminology for functions of one variable $\alpha$, and for functions of a real variable $x$ we change the earlier convention, into: $u = \hat{\delta}(e^{\frac{-f(x)}{h}})$, if for every $\delta>0$ and every compact set $K$ in $\mathbb{R}$, the $L^2$-norm of $ue^{\frac{-f(x)}{h}}$ on $K$ is bounded by a constant times $e^{\frac{\delta}{h}}$, for sufficiently small $h$. In all that follows, the bounds near infinity in $x$ or in $\alpha, \beta$ will be unchanged, but play an important role in the iteration process. Let us assume by induction that,

(H.1) $E_{-+}(\alpha, \beta) = \hat{\delta}(e^{\frac{-a(\alpha, \beta)}{h}})$, where $a: \mathbb{N} \to \mathbb{R}$ satisfies,

(H.2) $0 \leq a(k+1)-a(k) \leq v_0$,

(H.3) $|a(1)| \leq S_0 = \text{def.} D(0, 2\pi)$.

(H.4) $|a(k)| \geq v_0k$.

The assumption, (H) is satisfied with $a(k)=v_0k$, and we also recall (3.24), which gives additional information about $E_{-+}$ near infinity. Our object is to increase the function $a$. By translation invariance it is enough to assume (H.1) for $\alpha=0$ or for $\beta=0$, and also to prove (H.1) for a new function $a$ for such values of $\alpha, \beta$. Let us first study $P g_{0,1}$ and $P^* g_{0,2}$ with the help of (3.27) and (3.29). The two cases are completely parallel, so we concentrate on $P g_{0,1}$.
We recall that \( g_\alpha, f_\alpha = \hat{\mathcal{O}}(e^{-f(x-2\pi \alpha\xi)/h}) \). It follows from (3.27) and (H.1) that,

\[
P_{g_\alpha, f_\alpha} = \hat{\mathcal{O}}(e^{-(a(1)+f(x))/h}) + \sum_{\beta_1 \neq 0} \hat{\mathcal{O}}(e^{-\left[ a(1)\beta_1 + f(x-2\pi \beta_1) \right]/h}),
\]
so

\[
P_{g_\alpha, f_\alpha} = \hat{\mathcal{O}}(e^{-F(x)/h}),
\]
where \( F \) is the even function given by,

\[
F(x) = \min(a(1)+f(x), \min_{\beta_1 \neq 0}(a(1)\beta_1 + f(x-2\pi \beta_1))).
\]

Thanks to (H.2)-(H.4), we get for \( 2\pi k \leq |x| \leq 2\pi (k+1) \):

\[
F(x) = \min(a(k)+D(2\pi k, |x|), a(k+1)+D(2\pi (k+1), |x|)).
\]

Here we use the convention \( a(0) = a(1) \). We also have the same estimate for \( P^*g_\alpha, f_\alpha \). Now recall (3.28) for \( \beta = 0 \):

\[
(E_{-}\alpha, \alpha) = -(P_{g_\alpha, f_\alpha} g_\alpha, f_\alpha).
\]

Assume \( \alpha_1 \neq 0 \) and for instance that \( \alpha_1 > 0 \). We write,

\[
(E_{-}\alpha_1, \alpha_1) = (XP_{g_\alpha, f_\alpha} g_\alpha, f_\alpha) + (g_\alpha, f_\alpha (1-\chi)P^*g_\alpha, f_\alpha) - (g_\alpha, f_\alpha P^*g_\alpha, f_\alpha) = I + II + III.
\]

Here \( \chi = [1-\infty, \lambda] \) is the characteristic function of \([1-\infty, \lambda] \), where \( \lambda \) is to be chosen conveniently in \([0, 2\pi \alpha_1] \).

**Case 1.** \( \alpha_1 = 1 \). We choose \( \lambda = \pi \), which gives the same estimate for I and II:

\[
I + II = \hat{\mathcal{O}}(1)e^{-\left(a(1)+v_0\right)/h},
\]
where now \( \hat{\mathcal{O}}(1) \) simply means \( \mathcal{O}(e^{\delta}/h) \) for every fixed \( \delta > 0 \). To estimate III, we also follow [HS1], but with the following slight simplification, based on the observation that \([P^*, \chi] = (1-\chi)P^*\chi - \chi P^*(1-\chi) \). This means that we can rely on the boundedness of \( P \) in weighted \( L^2 \)-spaces, developed in [HS1], section 7, and we obtain, \([P^*, \chi]g_\alpha, f_\alpha = \hat{\mathcal{O}}(e^{-D(x, 2\pi)/h}) \) for \( x \leq \pi \), and \( = \hat{\mathcal{O}}(e^{-D(x, 0)/h}) \) for \( x \geq \pi \). From this we see that III = \( \hat{\mathcal{O}}(e^{-So}/h) \), and hence, \( E_{-}\alpha_1, \alpha_1 = \hat{\mathcal{O}}(1)e^{-\min((a(1)+v_0, So)/h)} \).

**Case 2.** \( \alpha_1 \) is even \( \geq 2 \). If we choose \( \lambda = 2\pi = \alpha_1/2 \), then we will get the same bound for I and II, but for III we only get \( \hat{\mathcal{O}}(e^{-v_0\alpha_1}/h) \), which is no improvement compared to the initial function \( a \). Instead, we take \( \lambda = (\alpha_1+1)/2 \), which will produce a worse bound for I and a better bound for II, so it is enough to estimate I and III: We find I (and II) = \( \hat{\mathcal{O}}(\exp(-(a(1)/2)+v_0\alpha_1/2)/h)) \), and III = \( \hat{\mathcal{O}}(\exp(-(v_0(\alpha_1-1)+S_0)/h)) \), so

\[
E_{-}\alpha_1, \alpha_1 = \hat{\mathcal{O}}(1)e^{-\min((a(1)+v_0, \alpha_1-1)/2, v_0\alpha_1/2))/h}.
\]

**Case 3.** \( \alpha_1 \) is odd \( \geq 3 \). We choose \( \lambda = \alpha_1/2 = [\alpha_1/2] + 1/2 \). Then

\[
I, II = \hat{\mathcal{O}}(1)e^{-\left(a(\alpha_1/2)+v_0(\alpha_1-[\alpha_1/2])/h, \right)}
\]

III = \( \hat{\mathcal{O}}(1)e^{-((\alpha_1-1)v_0+S_0)/h} \), so

\[
E_{-}\alpha_1, \alpha_1 = \hat{\mathcal{O}}(1)e^{-\min((\alpha_1-1)v_0+S_0, a(\alpha_1/2)+v_0(\alpha_1-[\alpha_1/2])/h)/h}.
\]

Notice that this estimate also covers the case II.

Working with Fourier transforms, we get the same estimates with \( \alpha_1 \) replaced by \( \alpha_2 \), so finally,
where,
\[ b(1) = \min(a(1) + \nu_0, \gamma_0), \]
\[ b(j) = \min(\gamma_0 + (j-1)\nu_0, a(j/2) + \nu_0( j-(j-1)/2)) / h, \quad j \geq 2. \]

As in [HS1], we check that (H) is verified with \( b \) replaced by \( a \), and that after a finite number of iterations of this procedure, we get a function \( a(j) \) with \( a(1) = \gamma_0, \; a(j) > \gamma_0 \) for \( j > 1 \). (An infinite iteration leads to \( a(j) = \gamma_0 + (j-1)\nu_0 \).) In other words, we have achieved that,
\[ E_{-+}(\alpha, \beta) = \hat{O}(e^{-\gamma_0 / h}) \quad \text{for} \quad |\alpha - \beta|_{\infty} = 1, \quad \text{and} \quad \hat{O}(e^{-(\gamma_0 + \varepsilon_0) / h}) \quad \text{for} \quad |\alpha - \beta|_{\infty} \geq 2, \]
for some \( \varepsilon_0 > 0 \). Moreover, for \( \alpha_1 = 1 \), we have,
\[ E_{-+}(\alpha, 0) = (\gamma_0, 1) [P^*, \chi] g_{\alpha, 2} + \hat{O}(1) e^{-\gamma_0 / h} =
-(i [P, \chi] g_{\alpha, 1}) + \hat{O}(1) e^{-\gamma_0 / h} \]
with \( \chi = 1_{-\infty, \mathbb{t}} \), with \( \mathbb{t} \) close to \( \pi \). In order to study the last scalar product, we shall slightly modify the function \( \chi \) (as in section 7 of [HS1]). Let
\[ \pi_\gamma(x) = C_1 h^{-3/2} e^{-\gamma(x) / h}, \]
where \( \gamma(x) = \gamma_0(x - s), \; \gamma_0(x) = i C_0 x^2 \), with \( C_0 > 0 \) sufficiently large but fixed, and \( C_1 = C_1(C_0) \) the constant such that \( \int_{\mathbb{C}} \pi_\gamma(x) \, dx = 1 \). Putting \( K = \int_{\mathbb{C}} \pi_\gamma(x) \, dx \), it is easy to check, using the growth estimates on
\[ g_{\alpha, 1}, g_{\alpha, 2}, \; P g_{\alpha, 1}, \; P^* g_{\alpha, 2}, \]
that we can replace \( [P, \chi] \) in (3.33) by \( \psi_k\pi_k \psi \), provided that \( C_0 \) is sufficiently large. Here \( \psi \in C_0^{\infty} \) has its support near \( \pi \) and is equal to 1 near that point. (We now take \( \mathbb{t} = \pi \).) As in section 7 of [HS1], we obtain on the other hand,
\[ [P, \pi_k] = h \hat{\partial} \psi_k / \partial s + R_k, \quad \text{for} \quad |x - s| + |y - s| \leq 1, \; x, y \in \mathbb{C}, \]
where,
\[ Q_k = C h^{-3/2} \int \left( (x - y)^\theta + \frac{1}{2} (\psi_k(x) + \psi_k(y)) / h \right) Q_k(x + y, \theta, \frac{1}{2} (x + y) - s, h) d\theta, \]
\[ R_k = h^{-3/2} \int \left( (x - y)^\theta + \frac{1}{2} (\psi_k(x) + \psi_k(y)) / h \right) R_k(x + y, \theta, \frac{1}{2} (x + y) - s, h) d\theta, \]
where \( Q \) is a realization of a classical analytic symbol of order 0, defined for \( x, y \) in the region in (3.34), and for \( |\text{Im} \theta| \leq C(\varepsilon) \), where \( \varepsilon \) is the parameter in (3.2) and \( C(\varepsilon) \) tend to \( \infty \) when \( \varepsilon \) tends to 0. \( R \) is holomorphic and
\[ = \hat{O}(e^{-\gamma_0 / h}), \]
where \( C_0 \) depends on \( C_0 \). We have,
\[ (3.35) \quad \psi_k[P, K] \psi = h \hat{\partial} \psi_k / \partial s + R_k \psi \]
In view of the properties of the symbol of \( R_k \), we see that the contributions from the integral in (3.35) to \( \psi_k[P, K] \psi \) (replacing \( [P, \chi] \) in (3.33)) is negligible, and we obtain,
\[ (3.36) \quad E_{-+}(\alpha, 0) = -h (\psi_k \pi_k \psi g_{\alpha, 1} g_{\alpha, 2}) + \hat{O}(e^{-(\gamma_0 + \varepsilon_0) / h}). \]

Now recall from [HS1] that \( Q_k \) is the solution of a division problem, and that the leading part \( q \) of the symbol \( Q_k \), is obtained by:
\[ (3.37) \quad P(x, \theta + i \sigma) - P(x, \theta - i \sigma) = C_0 \sigma q(x, \theta, \sigma). \]
As in [HS1], section 5, one can show that $g_{0,1} = c_0(h)a(x,h)e^{i\varphi(x)/h}$ on $[x_+ + \delta, \pi + \delta 0]$, $g(1,0,2) = c_1(h)b(x,h)e^{i\psi(x)/h}$ on $[\pi - \delta 0, \pi - x_+ - \delta]$, for every $\delta > 0$ and for some fixed $\delta 0 > 0$, where $a, b$ are realizations of elliptic analytic symbols of order 0, $c_0, c_1, c_0^{-1}, c_1^{-1} = \hat{0}(1)$, $T_x(U(0,0)) = \{x_+, x_\pm\}$, and $\varphi, \psi$ are the solutions of the eiconal equations, $H_j(x, \psi') = 0, H_j(x, \psi) = 0$, satisfying $\text{Im} \varphi' > 0, \text{Im} \psi' < 0, \varphi(x) \to 0, \psi'(x) \to \xi_+$, when $x \to x_+$, $\psi(x) \to 0, \psi'(x) \to \xi_-$, when $x \to 2\pi + x_-$. Here $\xi_\pm$ are the unique values, such that $(x_\pm, \xi_\pm) \in U(0, 0)$. By symmetry considerations, we have

$$(3.38) \quad \text{Im} \varphi(x) + \psi(x) = 0, \text{Re} \varphi(x) = \text{Re} \psi(x) + \text{const.}.$$

Since, $\partial \psi H_j(x, \psi') \neq 0$, we obtain from (3.37) that $q(x, \psi') \neq 0$, and by analytic stationary phase (see [S1]), we obtain

$$(3.39) \quad \psi Q \pi \psi g_{0,1} = c_2(h)d(x,h)e^{i(\varphi(x) + \psi(x))/h}, \quad \text{near } x = \pi,$$

where $c_2$ has the same properties as $c_0$ and $c_1$, and $d$ is a realization of an elliptic analytic symbol of order 0. By the continuity of $Q \pi$ in weighted $L^2$-spaces, we also have

$$(3.40) \quad \psi Q \pi \psi g_{0,1} = \hat{0}(e^{i(\varphi(x) + \psi(x))/h}).$$

Combining this with the WKB-form of $g(1,0,2)$, and analytic stationary phase, we get,

$$(3.41) \quad (\psi Q \pi \psi g_{0,1} | g(1,0,2)) = c_3(h)e^{-S_0/h}, \quad c_3, c_3^{-1} = \hat{0}(1).$$

When $\alpha_2 \neq 0$, we use that $g_{\alpha,2} = e^{-i\alpha_2/2\pi}e^{i\alpha_2 x/2}g(1,0,2)$, and by contour deformation, we obtain,

$$(3.42) \quad (\psi Q \pi \psi g_{0,1} | g_{\alpha,2}) = \hat{0}(e^{-(S_0 + \xi_0)/h}),$$

for some $\xi_0 > 0$.

Summing up the discussion so far, we have for $j = \mu_1$ in addition to (3.24), that

$$(3.43) \quad E_{-1}(\alpha, 0) = \hat{0}(e^{-(S_0 + \xi_0)/h}), \quad \text{for } |\alpha| > 1,$$

$$(3.44) \quad E_{-1}(\alpha, 0) = a(\alpha)e^{-S_0/\hbar}, \quad \text{for } |\alpha| = 1, \quad \text{where } a(\alpha), 1/a(\alpha) = \hat{0}(1).$$

Here $|\alpha| = |\alpha|_1 + |\alpha|_2$, and $\xi_0 > 0$. The corresponding renormalized operator, $Q$ which is invariant under $\Gamma^2$ and $V$, is then of the form

$$(3.45) \quad Q = Q_0 + R,$$

where on the symbol level, $Q_0 = e^{-S_0/\hbar}(a(1,0)e^{-i\varphi} + a(-1,0)e^{i\varphi})$, and $R = \hat{0}(e^{-(S_0 + \xi_0)/h})$ in a complex strip $|\text{Im}(x, \xi)| \leq \xi_0/\hbar$. From the invariance under $\Gamma^2$, we deduce that $a(-\alpha) = a(\alpha)$, and from the $V$-invariance, that $a(1,0) = a(-1,0)$, so with $a_0 = a(1,0)$, we get,

$$(3.46) \quad Q_0 = 2e^{-S_0/\hbar}(a_0 e^{i\varphi} + a_0 e^{-i\varphi}),$$

where $Q_0$ and $QQ_2$ are self-adjoint and on the symbol-level,
Q_j=1 + O(e^{-\varepsilon_0/h}) \text{ for } |\text{Im}(x, \xi)| \leq \varepsilon_0/h. \text{ This implies that } \text{Im}(a_0) = O(e^{-\varepsilon_0/h}), \\
so after multiplication of Q_j by the real non-vanishing factor, 
\exp^{So/h(2\text{Re}a_0)^{-1}}, we are reduced to the case when 
(3.46) \quad Q = \cos \xi + \cos x + O(e^{-\varepsilon_0/h}).

So far \( \lambda \) has been fixed = \( \lambda_1 \), and it only remains to study \( E_- \) for neighboring values. Combining (3.24) with (3.43), (3.44) and the Cauchy inequalities, we get for \( |\lambda - \lambda_1| \leq \varepsilon_0 h \), 
(3.47) \quad E_-(\lambda, o) = O(e^{-v_0}/h) + O(|\lambda - \lambda_1|/h) \delta(e^{-v_0}/h), \text{ for } |\lambda| \geq 2,

Also recall that \( z(v) = E_+(0, 0) \) has a simple zero at \( v = 0 \), i.e. for \( \lambda = \lambda_1 \), and that the behaviour near \( v = 0 \) is given by (3.21). From these three facts, we see that \( 0 \notin \text{Spec}(E_-) \) if \( |\lambda - \lambda_1| \geq \varepsilon_h(z_0 + \delta)/h \) and \( h \leq \varepsilon_h \), where \( \delta > 0 \) may be arbitrarily small. We then restrict the attention to the values of \( \lambda \) such that 
(3.48) \quad E_+(\lambda, o) = a(\lambda)e^{-v_0}/h + O(|\lambda - \lambda_1|/h) \delta(e^{-v_0}/h), \text{ for } |\lambda| \leq 1.

This together with the information already obtained about the \( Q_j \) shows that 
(\( Q, Q_1, Q_2 \)), is of strong type 1 with \( \varepsilon(Q) = h/c_0 \).

Let us sum up the results of this section in the following

**Proposition 3.4.** Let \((P, P_1, P_2)\) be of strong type 1 and \( 0 < h \leq 2\pi \). Then for 
\( 0 \leq \varepsilon(P) \leq \varepsilon_0 \) sufficiently small, the symbol \( H \) of \( P \) has non-degenerate saddle points, all with the same critical value, \( c(\lambda) - \lambda \), where \( c(\lambda) = O(\varepsilon(P)) \) is holomorphic in \( D(0, 4) = \{ \lambda \in \mathbb{C} : |\lambda| < 4 \} \). There is a unique real value \( \lambda_0 = \lambda_0(P) = O(\varepsilon(P)) \), such that \( \lambda_0 - c(\lambda_0) = 0 \). For every \( \varepsilon > 0 \), there exists \( c_1 > 0 \), such the following holds when \( 0 \leq \varepsilon(P) \leq \varepsilon_0 \), \( 0 < h \leq 1/c_1 \): 
\( \mu - \text{Sp}(P) \subset \bigcup_{-N \leq j \leq N} J_j \), where \( J_j \) are closed disjoint intervals labelled in increasing order (so that \( J_{j+1} \) is the neighbor to the right of \( J_j \)), with the following properties:

1° \quad \text{If } a = \inf J_{-N}, b = \sup J_N, \text{ then } a = -2 + O(\varepsilon(P)) + O(h), 
\quad b = 2 + O(\varepsilon(P)) + O(h).

2° \quad The distance between \( J_j \) and \( J_{j+1} \) is of the order of magnitude \( h \).

3° \quad J_0 = [\mu_0 - \varepsilon_1 + O(h), \mu_0 + \varepsilon_1 + O(h)], \text{ and } \partial J_0 \subset \mu - \text{Sp}(P).
For $j \neq 0$, the length of $J_j$ is $e^{-l_j/h}$, where $l_j \sim 1$ (that is of the same order of magnitude as 1), and if $\mu \rightarrow X_j(\mu) = \mu'$ is a suitable increasing affine map, then $X_j(\mu - \text{Sp}(P) \cap J_j) = \mu' - \text{Sp}(Q)$, where $Q = Q_j$ is the $h'$-quantization of a strong type 1 symbol, with $\varepsilon(Q) \rightarrow 0$ as $h \rightarrow 0$. Here $2\pi/h = h'/2\pi \mod \mathbb{Z}$, $0 < h' \leq 2\pi$. (When $h' = 0$, the $h'$-quantization is defined as the multiplication by $Q$ on $L^2(\mathbb{R}^2)$.)
4. The branching problem for a type 1 operator.

Let \((P, P_1, P_2)\) be of strong type 1 with \(\varepsilon(P) \leq \varepsilon > 0\) sufficiently small, \(0 < h \leq 2\pi\). We then recall from Proposition 3.3, that the Weyl symbol \(H_1\) of \(H_1 = P^*P\) has saddle points at \((\pi, 0), (0, \pi)\) with the same critical value \(c(\mu) - \mu\), where \(c(\mu) = \Theta(\varepsilon)\). In section 3, we studied potential well Grushin problems for \(P\) when \(|c(\mu) - \mu| \leq \varepsilon_0\), for any fixed \(\varepsilon_0 > 0\), provided that \(\varepsilon > 0\), and \(h > 0\) are small enough. In this section, we study the case when \(|c(\mu) - \mu| \leq \varepsilon_0\), and \(\varepsilon_0 > 0, \varepsilon > 0, h > 0\) are sufficiently small. The real characteristics of \(H_1\) are then included in a thin neighborhood of the union of all segments \(s(\alpha, j), \alpha \in \mathbb{Z}^2, j \in \mathbb{Z}_4\), where \(s(0, 1)\) is the segment \([(0, \pi), (\pi, 0)]\), \(s(0, j) = \kappa^1 - j(s(0, 1))\), \(s(\alpha, j) = (2\alpha \pi + s(0, j))\), where as before \(\kappa\) denotes rotation around \((0, 0)\) by the angle \(-\pi/2\).

The new difficulty is then to make a microlocal study of the operators \(P, H_1\) and \(H_2\) near the saddle points \((k\pi, l\pi), k, l \in \mathbb{Z}, k + l \text{ odd}\). Because of the invariance properties of our operators, it will be essentially enough to concentrate on what happens near the point \((0, \pi)\). Let us start with \(H_1\). We recall from Proposition 3.3 that the symbol \(H_1\) is invariant under reflection in the point \((0, \pi)\). On the operator level, we have, \(CH_1 = H_1C\), where \(C = T^{0,1}_0 C_0\) is a quantization of reflection in \((0, \pi)\), and \(C_0 = \mathcal{F}^2\) quantizes the reflection in \((0, 0)\). The real symbol \(H_1 + \mu - c(\mu)\) has critical value 0 at \((0, \pi)\) and applying the results of the section b of the appendix, we obtain a real valued analytic symbol \(f(\mu, t, h) = f_0 + f_1 h + \ldots\) of order 0, defined for \(\mu, t\) near \((0, 0)\), with \(f_0(\mu, 0) = 0, \partial f_0(\mu, 0) / \partial t > 0\), and a unitary analytic Fourier integral operator \(U\), whose associated canonical transformation \(\kappa_U\) maps a neighborhood of
(0,0) onto a neighborhood of (0,\pi), such that,
\begin{equation}
U^{-1}f(\mu, H_1 + \mu - c(\mu), h) U = P_0 = \frac{1}{2}(x h D + h D x).
\end{equation}

Moreover, we can arrange so that,
\begin{equation}
CU = UC_0,
\end{equation}

and so that \( K_U \) maps the part of the negative \( \xi \)-axis in the domain of definition, into a small neighborhood of \( s(0, 1) \). Formally, we observe that if \( u = Uv \) and \( u \) is a microlocal solution of \( H_1 u = 0 \) (which is equivalent to \( Pu = 0 \)), then \( P_0 v = U^{-1} f(\mu, \mu - c(\mu), h) U v = f(\mu, \mu - c(\mu), h) v \) microlocally near (0,0).

The map
\begin{equation}
(4.3) \quad \mu \mapsto \mu' = f(\mu, \mu - c(\mu), h)
\end{equation}
is invertible and its inverse is given by,
\begin{equation}
(4.4) \quad \mu = g(\mu', h),
\end{equation}
where \( g \) is a classical analytic symbol of order 0.

The relation (4.1) will also allow us to treat the inhomogeneous equation \( H_1 u = w \) microlocally near \((0, \pi)\). Indeed, let \( t \mapsto k(\mu, t, h) \) be the inverse map (where \( k \) is an analytic symbol) of the map \( s \mapsto f(\mu, s + \mu - c(\mu), h) \). Then from (4.1), we get,
\begin{equation}
(4.5) \quad U^{-1} H_1 U = k(\mu, P_0, h).
\end{equation}

Now \( k(\mu, \mu', h) = 0 \), where \( \mu'(\mu, h) \) is given by (4.3), and we can factorize:
\[ k(\mu, t, h) = l(\mu, t, h)(t - \mu'(\mu, h)), \]
where \( l(\mu, t, h) \) is an elliptic analytic symbol of order 0. Using this in (4.5), we get,
\begin{equation}
(4.6) \quad U^{-1} H_1 U = l(\mu, P_0, h)(P_0 - \mu').
\end{equation}

Here \( l(\mu, P_0, h) \) is an elliptic operator, so we see that the microlocal inversion problem for \( H_1 \) can be reduced to the corresponding one for \( P_0 - \mu' \).

We now write \( U = U_1 \) and we shall see how to obtain an operator \( U_2 \) which reduces \( H_2 \). Let \( V \) be the antilinear operator, introduced in section 1, and put \( A = V \sigma^1, B = T_0, \sigma^2 V \sigma^1 \), which are antilinear realizations of the reflections in the \( \xi \)-axis and in the line \( \xi = \pi \) respectively. Using the invariance properties for type I operators and the appropriate commutation relations of section 1, it is easy to check that
\begin{equation}
(4.7) \quad A^2 = I, B^2 = I, \quad B = CA = AC,
\end{equation}
\begin{equation}
(4.8) \quad AH_1 = H_2 A, \quad BH_1 = H_2 B.
\end{equation}
Put \( A_0 = V = \Gamma \sigma^1 \), and let \( B_0 = U_3 \sigma^1 / 2 \Gamma \) be a quantization of reflection in \( x + \xi = 0 \). Then we have,
\begin{equation}
(4.9) \quad A_0^2 = I, B_0^2 = I, B_0 = C_0 A_0,
\end{equation}
\begin{equation}
(4.10) \quad [A_0, P_0] = [B_0, P_0] = 0.
\end{equation}
Let \( U_2 \) be the unitary operator defined by,
\begin{equation}
(4.11) \quad U_1 A_0 = A U_2.
\end{equation}
It is then easy to verify that we also have,
\begin{equation}
(4.12) \quad U_1 B_0 = B U_2,
\end{equation}
The next step will be to study $P_0 - \mu'$, and we start by exhibiting suitable solutions to the homogeneous equation. The equation 
\[
(\frac{1}{2}(xD+Dx) - \alpha)u = 0
\]
has four solutions $u_+, u_-, v_+, v_-$, given by:
\[
\begin{align*}
(4.15) & \quad u_\pm(x) = H(\pm x)|x|^{-\frac{1}{2}} + iu_1 v_\pm(x) = H(\pm \xi)|\xi|^{-\frac{1}{2}} - i\alpha_1 = \bar{u}_\pm(\xi),
\end{align*}
\]
where $F_1$ denotes the unitary Fourier transform defined in section 1 for $\hbar = 1$.

The last relation can also be written $v_\pm = F_1^{-1} \Gamma u_\pm$. The general solution in $\mu'$ is of the form $u = \alpha_+ u_+ + \alpha_- u_-$, where the coefficients are related by,

\[
(4.16) \quad \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = A_\alpha \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix},
\]

\[
(4.17) \quad A_\alpha = \Gamma(\frac{1}{2} + i\alpha)(2\pi)^{-\frac{1}{2}} \left( \exp(\pi\alpha / 2 - \pi i / 4) \right)_{1,1} + \left( \exp(-\pi\alpha / 2 + \pi i / 4) \right)_{1,2} + \left( \exp(-\pi\alpha / 2 + \pi i / 4) \right)_{2,1} + \left( \exp(\pi\alpha / 2 - \pi i / 4) \right)_{2,2},
\]

where $[a]_{j,k}$ denotes the $2 \times 2$ matrix whose only non-vanishing element is $a$ in the $j$:th row and $k$:th column. Using the reflection identity

\[
\Gamma(\frac{1}{2} + i\alpha)\Gamma(\frac{1}{2} - i\alpha) = \pi / \sin(\pi\alpha),
\]

we see that $A_\alpha$ is unitary (when $\alpha$ is real).

In the case of solutions to $(P_0 - \mu')u = 0$, we can apply the above with $\alpha = \mu'/\hbar$, but it will be convenient to make also two renormalizations. The first one is due to the fact, that we prefer to work with $F = F_\hbar$ instead of $F_1$, and the second renormalization is due the fact that we wish $u_+$ to enjoy additional approximate symmetry under reflections in the line $x = \xi$, when $\mu' > 0$. Assuming $\mu' > 0$ (which is no essential restriction, as we shall see below), this leads to the choice,

\[
(4.18) u_{\mu}'(x, \mu') = e^{-i\mu'(1 - \log \mu')/2\hbar + i\pi / 2}(\pm x)|x|^{-\frac{1}{2}} + i\mu'/\hbar,
\]

where

\[
\begin{align*}
\psi_{\mu}' &= F^{-1} \Gamma u_{\mu}' = \nu_{\mu}',
\end{align*}
\]

Then using the method of stationary phase, (or (4.20) combined with the complex version of Stirling's formula, that we shall recall later in this section,) we check that,

\[
(4.19) \quad \nu_{\mu}' = (1 + O(h))\psi_{\mu}' + O(h)\psi_{\mu}',
\]

uniformly for $\mu'$ in any compact interval in $]0, \infty[$. From (4.16),(4.17), we obtain that the general (temperate) solution $u$ of $(P_0 - \mu')u = 0$ can be written

$u = \alpha_+ u_+ + \alpha_- u_-$,

where

\[
(4.20) \quad \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \frac{h}{\mu'} \frac{1}{i\mu'/\hbar + \pi / 2} A_{\mu'/\hbar} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}.
\]

As a second preparation concerning $P_0 - \mu'$, we study the
Inhomogeneous equation. We look for an inverse of this operator which propagates singularities in the forward Hamilton field direction. Following a well known procedure, we put,
\[ E = (i/h) \int_0^\infty e^{-it(P_0 - \mu')} \langle \eta \rangle \, dt = (i/h) \int_0^\infty U_t \, dt, \]
where \( U_t \) can be constructed by the standard WKB-procedure for the strictly hyperbolic Cauchy problem. We get,
\[ U_t U(x) = (2\pi h)^{-1} e^{\int_0^x \eta(y) \, dy} e^{t(\mu'/h - \frac{1}{2})} \langle x \rangle \, d\eta = e^{t(\mu'/h - \frac{1}{2})} u(e^{-t \eta}). \]
When \( \mu' \) is real, there is no hope that \( E \) should be a globally \( L^2 \)-bounded operator, but if we put \( H_s(\mathbb{R}) = \{ u \in \mathcal{A}'; <\eta> \in L^2, <\xi> = (1 + \xi^2)^{1/2} \}, \)
\[ L_s^2(\mathbb{R}) = \{ u \in \mathcal{A}'; <\eta> \in L^2 \}, \]
then we shall see that \( E \) is bounded:
\[ H_s^2 + s \rightarrow L_s^2(\mathbb{R}^n), \text{ for every } s > 0. \]

The \( L_s(\mathbb{R}^n) \)-norm of \( U_t \) is equal to the \( L_s(L^2) \)-norm of the integral operator with kernel,
\[ k_t(x, \eta) = (2\pi h)^{-1} e^{t(\mu'/h - \frac{1}{2})} \langle x \rangle \, d\eta = e^{t(\mu'/h - \frac{1}{2})} u(e^{-t \eta}). \]

Using this in (4.21), we obtain:
\[ \|U_t\|_{L_s(L^2)} \leq \|C\|_{L^2} \langle \xi \rangle^{-3/2}, \text{ if } Im\mu'/h + \frac{1}{2} > s > 0. \]

In appendix d we have defined wavefront sets for microlocally defined operators, and computed \( WF'(E) \).

Let \( U_0, 1 \) be a WKB solution of \( Pu = 0 \), defined microlocally near the interior, \( Int(s(0, 1)) \) of the segment \( s(0, 1) \) defined above. Since \( P \) is of real principal type in this region, and since we work microlocally, and hence neglecting exponentially small errors, we know that \( U_0, 1 \) (in this region) is unique up to a constant factor, and can therefore be expressed as a multiple of \( U_1 w^0 \). Let \( \chi_{0, 1} \in C^\infty_0(\mathbb{R}^2) \) be equal to 1 near \( (0, \pi) \) the end point of the segment \( s(0, 1) \), equipped with the orientation of the Hamilton field of \( \cos \xi + \cos x \), and with support in the disc of radius \( (3/4)(2)^{3/2} \pi \), centered at \( (0, \pi) \). Following a general terminology, \( (\pi, 0) \) will be called the starting point of the segment \( s(0, 1) \).

Define \( \chi_{0, 1} \) and more generally \( \chi_{\alpha, 1} \), by the relations,
\[ \chi_{0, 1} = \chi_{\alpha, 1} = \chi_{0, 1}(x, \xi) = \chi_{0, 1}(x, \xi - 2\pi \alpha). \]

We also choose \( \chi_{0, 1} \) invariant under reflection in the point \( (0, \pi) \), which implies that \( \chi_{\alpha, j} = \chi_{\alpha, k} \) if the segments \( s(\alpha, j) \) and \( s(\beta, k) \) have the same last end point. Taking a suitable quantization of \( \chi_{\alpha, j} \) using superpositions of gaussians, (see below for more details,) we can also arrange so that on the operator level,
\[ \chi_{\alpha, j} \star 1 = \chi_{\alpha, j} \star 1 = \chi_{0, 1}, \quad \chi_{\alpha, j} \star \alpha = \chi_{\alpha, j} \star \alpha = \chi_{\alpha, j} \star \alpha. \]

and again \( \chi_{\alpha, j} = \chi_{\alpha, k} \) if the segments \( s(\alpha, j) \) and \( s(\beta, k) \) have the same end.
point. The scalar product \( (\|H_1x_0,1\|u_0,1\|u_0,1) \) is well defined modulo an exponentially small term, since we see that the essential contribution to this expression must come from the intersection between the cut-off region, \( x_0,1(x,\xi) \notin (0,1) \) and the real characteristics of \( H_1 \), close to \( s(0,1) \). It is clear that this scalar product can be evaluated by stationary phase, and we get \( (\|H_1x_0,1\|u_0,1\|u_0,1)>0 \). After multiplication of \( u_0,1 \) by an elliptic analytic symbol, depending on \( \hbar \) and \( \mu' \) only, we may assume that,

\[
(4.28) \quad (\|H_1x_0,1\|u_0,1\|u_0,1)=1.
\]

We shall then say that \( u_0,1 \) is a normalized microlocal solution of \( H_1u=0 \), defined near \( \text{int}(s(0,1)) \). We notice that (4.28) is essentially independent of the choice of \( x_0,1 \), because if \( x \in C_b^\infty \) vanishes near both endpoints of \( s(0,1) \), and if \( \chi \) is a corresponding gaussian quantization of \( x \),

\[
(i\|H_1\|u_0,1\|u_0,1\|u_0,1)=i(\chi u_0,1\|H_1u_0,1\|u_0,1\|u_0,1)=0,
\]

since \( H_1u_0,1=0 \) microlocally near \( \text{int}(s(0,1)) \).

We now put \( f_0,1=\|H_1x_0,1\|u_0,1, \) which makes sense (modulo an exponentially small uncertainty) as a globally defined function on \( \mathbb{R} \), exponentially decreasing outside the interval \( [\delta,\pi-\delta] \), for some \( \delta>0 \), and generate \( f_{\alpha,j} \) and \( u_{\alpha,j} \) as in section 2,

\[
(4.29) \quad f_{0,j}=\xi^{1-j}f_{0,1}, \quad f_{\alpha,j}=T\alpha f_{0,j},
\]

\[
(4.30) \quad u_{0,j}=\xi^{1-j}u_{0,1}, \quad u_{\alpha,j}=T\alpha u_{0,j}.
\]

Here \( u_{\alpha,j} \) is defined microlocally near \( \text{int}(s(\alpha,j)) \). Using the appropriate invariance and commutation relations of sections 1, 2, it is easy to see that \( u_{\alpha,j} \) is a normalized microlocal solution of \( H_ju=0 \) if \( j \) is odd and of \( H^ju=0 \), if \( j \) is even. (More precisely, for \( j \) even we have \( (\|H_2, x_{\alpha,j}\|u_{\alpha,j}\|u_{\alpha,j})=1 \), and for \( j \) odd we have the same relation with \( H^j \) replaced by \( H_1 \).) Moreover, we have,

\[
(4.31) f_{\alpha,j}=\|H_1, x_{\alpha,j}\|u_{\alpha,j}, \quad f_{\alpha,j}=\|H_2, x_{\alpha,j}\|u_{\alpha,j},
\]

for \( j \) odd and even, respectively.

With these functions \( f_{\alpha,j} \), we define \( R_+, R_- \) as in the case 3 of section 2. Microlocally, \( f_{\alpha,j} \) is non-orthogonal to \( \text{Ker} \ P=\text{Ker} \ H_1 \), when \( j \) is odd, and non-orthogonal to \( \text{Ker} \ P^*=\text{Ker} \ H_2 \), and hence not in \( \text{Im} \ P \) when \( j \) is even. This makes it plausible that the corresponding operator \( P \) is bijective. In order to be completely in the case 3 of section 2, we have also to arrange so that (cf. (2.39)),

\[
(4.32) \quad f_{0,1}=\|H_1, x_0,1\|u_0,1.
\]

To have this we choose \( x_0,1 \) real-valued and such that \( x_0,1*\delta=1-x_0,1 \) near \( s(0,1) \). We also need to be more explicit about the choice of Gaussian quantization. Let \( I=\int I^\pi d\alpha \) be a resolution of the identity, where \( I^\pi_I=\pi^\alpha, h \) has the kernel,

\[
(4.33) \quad \text{Ch}^{-3/2}z^e[(x-y)^{\alpha}_{\xi}+i(x-\alpha)^2/2+i(y-\alpha)^2/2]/h=
\]

\[
\text{Ch}^{-3/2}v_{\alpha}(x)\bar{v}_{\alpha}(y).
\]
Here \( C > 0 \). By definition, the Gauss quantization of \( \chi_{0,1} \) is then,

\[
\chi_{0,1} = \int \chi_{0,1} (\alpha) \pi \alpha d\alpha.
\]

Using that \( \psi_0 \) is invariant under \( \mathcal{F}, \mathcal{V}, \Gamma \), and that \( \psi_{\alpha} = T_{\alpha/2\pi} \psi_0 \), we check that \( T_{\alpha/2\pi} \pi \alpha T_{\alpha/2\pi}^{-1} = \pi \alpha + \chi, \mathcal{F} \pi \alpha \mathcal{F}^{-1} = \pi \chi (\alpha), \pi \alpha \mathcal{V} = \pi \delta (\alpha) \). This shows that the Gauss quantization behaves as the Weyl quantization under conjugation by \( T_{\alpha}, \mathcal{F}, \mathcal{V} \), and in particular, by the choice of \( \chi_{0,1} \), we obtain,

\[
V \chi_{0,1} V = I - \chi_{0,1}, \text{near } s(0,1).
\]

Using also that \( V \) commutes with \( \mathcal{H}^0 \), we obtain,

\[
V \psi_{0,1} = -iV[\mathcal{H}_1, \chi_{0,1}] \psi_{0,1} = i[\mathcal{H}_1, \chi_{0,1}] \psi_{0,1}.
\]

After multiplication of \( \psi_{0,1} \) by a suitable scalar, we may assume that \( V \psi_{0,1} = \psi_{0,1} \), and then we get (4.32).

So far, (4.32) is only a microlocal relation, but we can make it global and exact by replacing \( f_{0,1} \) by \( \frac{i}{2}(f_{0,1} + V f_{0,1}) \), and then we have modified \( f_{0,1} \) only by an exponentially small quantity.

We shall next see how certain WKB-considerations near the branching points, imply the wellposedness of our Grushin problem, and give the possibility to compute the leading contributions to \( E_- \) appearing in the inverse \( \mathcal{C} \) of \( \mathcal{P} \). First we notice that microlocally, \( P_2^* : \text{Ker}(\mathcal{H}_2) \to \text{Ker}(\mathcal{H}_1), P_1 : \text{Ker}(\mathcal{H}_1) \to \text{Ker}(\mathcal{H}_2) \), and that these maps are bijective. Microlocally, near \( \text{Int}(s(0,1)) \), we have the function \( \psi_{0,1} \), defined as a multiple of \( \psi \mathcal{W}_0 \), and since the function \( \psi \mathcal{W} \) is defined globally, we can extend the definition of \( \psi_{0,1} \) in \( \text{Ker}(\mathcal{P}) \) to a full neighborhood of the branching point \((0,\pi)\), and then to neighborhoods of \( \text{Int}(s(0,2)) \) and \( \text{Int}(s((0,1),4)) \), by standard WKB constructions. We then have a microlocal solution \( \psi_{0,1} \), defined in a neighborhood of \( \text{Int}(s(0,1)) \cup (0,\pi) \cup \text{Int}(s(0,2)) \cup \text{Int}(s((0,1),4)) \), which has its wavefront set (defined in appendix d) concentrated to a much smaller neighborhood of this set. Outside the point \((0,\pi)\), \( \psi_{0,1} \) is of simple WKB-form, and there are constants \( \alpha, \beta, \gamma, \delta \), such that,

\[
\psi_{0,1} = \beta P_2^* \psi_{0,2} \text{ near } \text{Int}(s(0,2)),
\]

\[
\psi_{0,1} = \alpha P_2^* \psi_{(0,1),4} \text{ near } \text{Int}(s((0,1),4)),
\]

\[
\psi_{(0,1),3} = \gamma P_2^* \psi_{0,2} \text{ near } \text{Int}(s(0,2)),
\]

\[
\psi_{(0,1),3} = \delta P_2^* \psi_{(0,1),4} \text{ near } \text{Int}(s((0,1),4)).
\]

Here, we have extended \( \psi_{(0,1),3} \) to a neighborhood of \((0,\pi)\), in the same way. When \( \mu \) is real, it is clear from the reduction to \( P_0 - \mu' \), that \( \alpha, \beta, \gamma, \delta \) exist and are \( O(1) \), and we shall compute these coefficients later.

Now consider the Grushin problem,

\[
 Pu + R_u^{-} = 0, R_+ u = \delta_{0,1},
\]

where \( \delta_{\alpha,j} (\beta, k) = 1 \) if \( (\beta, k) = (\alpha, j) \), and \( = 0 \) otherwise. Microlocally, near \( \text{Int}(s(0,1)) \), an approximate solution is given by \( u = u_{0,1} \), still with \( u^{-} = 0 \). In
order to get a global approximate solution, we extend $u_{0,1}$ across the branching points $(0, \pi)$ and $(\pi, 0)$ and then we truncate $u_{0,1}$ suitably on the segments $s(0, 2), s((0,1),4), s(0,4), s((1,0),2)$. Let us discuss the details of this only near $(0, \pi)$. Let $\tilde{X}_{0,2}$ be a cutoff operator, $=I$ microlocally near $(-\pi, 0)$, and $=0$ near $(0, \pi)$, such that $\tilde{X}_{0,2}P^2_2 = P^2_2 X_{0,2}$. Near $s(0,2)$, we put $u = (1 - \tilde{X}_{0,2})u_{0,1}$. Then near the interior of that segment,

$$
P u = \beta P(1 - \tilde{X}_{0,2})P^2_2 u_{0,2} = -\beta P X_{0,2} P^2_2 u_{0,2} = -\beta P^2_2 X_{0,2} u_{0,2} = -\beta P^2_2 X_{0,2} u_{0,2} = -\beta [H_2, X_{0,2}] u_{0,2} = \beta f_{0,2}.
$$

Similarly, let $\tilde{X}_{0,1,4}$ have its support near $(\pi, 2\pi)$, such that $\tilde{X}_{0,1,4}P^2_2 = P^2_2 X_{0,1,4}$. Then near $s((0,1),4)$, we put, $u = (1 - \tilde{X}_{0,1,4})u_{0,1}$, and essentially the same calculation gives $Pu = i\alpha f_{0,1,4}$ near that segment.

Similarly, we can extend $u$ to a neighborhood of $(-\pi', 0)$, such that $Pu = \text{const.} f_{0,1,4}$, and $Pu = \text{const.} f_{(1,0),2}$ respectively near the corresponding segments. Summing up the discussion so far, we have solved the problem (4.37) with exponentially small errors, and the corresponding $u^-$ has only 4 non-vanishing components, out of which, we have computed 2. In appendix d we show how to obtain from this discussion via some apriori estimates the following result.

**Proposition 4.1.** For real $\mu$, $P$ is bijective with an inverse $\delta = [E]_{1,1} + [E^+]_{1,2} + [E^-]_{2,1} + [E^-]_{2,2} = O(\epsilon^{-3/2})$. Moreover, $E_{-+} = O(1)$, and there is an $\epsilon_0 > 0$, such that,

\begin{align}
E_{-+}(\alpha, j; \beta, k) &= O(e^{-(1 + |\alpha - \beta|)\epsilon_0/h}) \text{ uniformly for all } \\
& \text{(\alpha, j), (\beta, k) such that } s(\alpha, j) \cap s(\beta, k) = \emptyset.
\end{align}

\begin{align}
E_{-+}((0,0), 2; (0,0), 1) &= -i\beta, \quad E_{-+}((0,1), 4; (0,0), 1) = -i\alpha, \\
E_{-+}((0,0), 2; (0,1), 3) &= -i\gamma, \quad E_{-+}((0,1), 4; (0,1), 3) = -i\delta.
\end{align}

Here the last two relations of (4.39) are proved as above by considering the problem (4.37) with $\delta_{0,1}$ replaced by $\delta_{(0,1),3}$.

In block matrix form we can write,

\begin{align}
E_{-+}(\alpha, \beta) = e^{ih\beta} \delta_{2,1} f(\alpha, -\beta),
\end{align}

where $f(\alpha)$ is exponentially small for $|\alpha| > 1$. Using the invariance properties (2.36), (2.46) (valid also in the case 3), we get,

\begin{align}
f((1,0)) = [a]_{2,1} + [b]_{2,3} + [\bar{b}]_{4,1} + [b]_{4,3}, \text{ and modulo } O(e^{-\epsilon_0/h}): \\
f((1,0)) = [a]_{2,1}, f((-1,0)) = [a]_{4,3}, f(0,1) = [\bar{a}]_{4,1}, f(0, -1) = [a]_{2,3}.
\end{align}

Here, we see that,

\begin{align}
a &= E((0,1), 4; (0,0), 1) = E((0, -1), 2; (0,0), 3), \\
\bar{a} &= E((1,0), 2; (0,0), 1) = E((-1,0), 4; (0,0), 3), \\
b &= E((0,0), 2; (0,0), 1) = E((0,0), 4; (0,0), 3), \\
b &= E((0,0), 2; (0,0), 3) = E((0,0), 4; (0,0), 1).
\end{align}

Comparing this with (4.39), we get, $-i\alpha = a$, $-i\beta = b$. Using also (4.40), we get, $-i\gamma = f((0, -1); 2, 3) = a$, $-i\delta = f((0, 0); 4, 3) = b$, and hence,
Before the actual computation, we shall see to what extent the matrix,
\[
\begin{pmatrix}
  \beta & \gamma \\
  \alpha & \delta
\end{pmatrix}
= \begin{pmatrix}
  \beta & \alpha \\
  \gamma & \delta
\end{pmatrix}
= i \begin{pmatrix}
  b & a \\
  a & b
\end{pmatrix},
\]
is unitary. Let \( \nu_{0,1} \) be a normalized WKB solution of \( H_2 \nu = 0 \) in the sense of (4.28), defined near \( \text{Int}(s(0,1)) \), and generate \( \nu_{\alpha,j} \), the same way as we generated the \( u_{\alpha,j} \). Then, in view of (2.2), (2.5), \( \nu_{\alpha,j} \) is a normalized solution of \( H_2 \nu = 0 \) when \( j \) is odd, and of \( H_1 \nu = 0 \), when \( j \) is even. Near \( \text{Int}(s(0,1)) \), we can write \( P_1 u_{0,1} = \lambda \nu_{0,1} \), where \( \lambda, 1/\lambda = \mathcal{O}(1) \). Using (2.5), we get
\[
P_1 u_{\alpha,j} = \lambda \nu_{\alpha,j}, \quad \text{when } j \text{ is odd, and } P_2 u_{\alpha,j} = \lambda \nu_{\alpha,j}, \quad \text{when } j \text{ is even, so from (4.36), we get,}
\]
(4.43) \[
\begin{align*}
  u_{0,1} &= \lambda \beta \nu_{0,2} \text{ near } \text{Int}(s(0,2)), \\
  u_{0,1} &= \lambda \alpha \nu_{0,1,4} \text{ near } \text{Int}(s((0,1),4)), \\
  u(0,1,3) &= \lambda \delta \nu_{0,2} \text{ near } \text{Int}(s(0,2)), \\
  u(0,1,3) &= \lambda \delta \nu_{0,1,4} \text{ near } \text{Int}(s((0,1),4)).
\end{align*}
\]
Consider the general solution \( u = x_1 u_{0,1} + x_3 u(0,1,3) = y_2 u_{0,2} + y_4 u(0,1,4), \) of \( H_1 u = 0 \), defined near \( (0,\pi) \). Here we extend \( u_{0,1} \) near \( (\pi,0) \) and \( \nu_{0,1} \) near \( (0,\pi) \) and \( (\pi,0) \) the same way as we did with \( u_{0,1} \) near \( (0,\pi) \), by using microlocal models. Then the coefficients \( x \) and \( y \) are related by
\[
y = \lambda \begin{pmatrix}
  \beta & \gamma \\
  \alpha & \delta
\end{pmatrix} x.
\]
Let \( \chi \) be a pseudodifferential cutoff, \( = 1 \) near \( (0,\pi) \) and with support close to that point. From the trivial identity \( 0 = (i[H_1, \chi] u, u) \), we get after summing the non-exponentially small contributions from each of the four segments, meeting at \( (0,\pi) \): \( 0 = |x_1|^2 + |x_3|^3 - |y_2|^2 - |y_4|^4 + \mathcal{O}(e^{-c_0/h}) \).
This means that,
(4.44) \[
\begin{pmatrix}
  \beta & \gamma \\
  \alpha & \delta
\end{pmatrix}
= i \begin{pmatrix}
  b & a \\
  a & b
\end{pmatrix}
\]
is unitary up to an exponentially small error, in the sense that the adjoint of this matrix differs from the inverse by an exponentially small term. It is then an easy exercise to see that there are \( a', b' \) with \( a - a' \) and \( b - b' \) exponentially small, such that \( \lambda \begin{pmatrix}
  b' & a' \\
  a' & b'
\end{pmatrix} \) is unitary.

We now attack the WKB problem. We first notice that the problem (4.36) is independent of the normalization of the function \( u_{0,1} \) (and of condition (4.32)), so we may assume from now on that,
(4.45) \[
u_{0,1} = u_1 w_0 \text{ near } \text{Int}(s(0,1)).\]
We choose,
(4.46) \[
u_{0,1} = u_2 w_0.
\]
This may affect \( \lambda \) in (4.43) by a factor of modulus 1, since \( u_2 u_1^{-1} \) maps
normalized solutions of $H_1 u = 0$ onto normalized solutions of $H_2 v = 0$. Our main problem will then be to determine $\lambda \alpha$, $\lambda \beta$ in (4.43). We recall the definition of the operators $A, B, C, A_0, B_0, C_0$, earlier in this section and the symmetry relations between these operators and $H_1, H_2, P_0, U_1, U_2$. Put,

(4.47) \[ \tilde{u}_{0,2} = \lambda u_{0,1} \in \ker(H_2), \]
(4.48) \[ \tilde{v}_{0,2} = \lambda v_{0,1} \in \ker(H_1). \]

The problem of determining $\lambda \beta$, can then be decomposed into two problems:

(4.49) Find $\lambda_1$ such that $u_{0,1} = \lambda_1 \tilde{u}_{0,1} = \lambda_1 \tilde{u}_{0,2} \text{ near } \text{Int}(s(0,2))$,

(4.50) find $\lambda_2$ such that $v_{0,1} = \lambda_2 \tilde{v}_{0,1} = \lambda_2 \tilde{v}_{0,2} \text{ near } \text{Int}(s(0,2))$.

Then we will have $\lambda \beta = \lambda_1 \lambda_2$. Notice that $|\lambda_2| = 1$ since $\tilde{u}_{0,1}$ and $\tilde{v}_{0,2}$ have the same normalization. In view of all our symmetry relations, the first problem reduces to a similar one on the model, namely to find $\lambda_1$ such that,

(4.51) \[ w_{0,-} = \lambda_1 u_{0,-} \text{ near } \Gamma_2, \]

where $\Gamma_2$ is the open negative $\xi$-axis, and $\Gamma_j = x_j^{-1}(\Gamma_1)$. Now the solution of (4.51) is $\lambda_1 = \alpha_-$ given by (4.20), with $\alpha_- = 1$, $\alpha_+ = 0$, and we get,

(4.52) \[ \lambda_1 = (\mu'/h) \mu'/h(2\pi)^{-1} \Gamma(\frac{1}{2} - i \mu'/h) e^{-i \mu'/h + \pi \mu'/2h}. \]

For $x > 0$, we can write,

(4.53) \[ u_0^0(x,\mu') = a(x,\mu',h) e^{i \phi_0(x,\mu')/h}, \]

where $a$ is an analytic elliptic symbol, defined for $x > 0$, $|\mu'| \leq \text{const}.$, of the form,

\[ a = e^{i \pi/4} |x|^{-\frac{1}{2}} c(\mu, h), \]

where $c(\mu, h) > 0$ is a normalization factor, and $\phi_0 = \frac{1}{2} \mu' (1 - \log(\mu')) + \mu' \log |x|$. We have analogous representations for $u_0^-, \psi w_0^+$ for all $\xi$. In particular,

\[ \psi w_0^0(\xi) = \tilde{a}(\xi, \mu', h) e^{-i \phi_0(\xi, \mu')/h}, \]

Thus $\psi w_0^0 = \tilde{a} e^{-i \phi_0/2}$ when $\xi > 0$, and hence

\[ \psi w_0^{-1} = \tilde{a} e^{-i \phi_0/2}, \]

so $v_0 = u_2 \psi (\tilde{a} e^{-i \phi_0/2})$ near $s(0,1)$. By stationary phase, we then obtain the representation,

(4.54) \[ v_{0,1} = a(x, \mu, h) e^{i \phi(x, \mu)/h}, \]

microloocally near $\text{Int}(s(0,1)) \setminus \{ \text{a neighborhood of the starting point of this interval} \}$. Here the new analytic symbol, $a$ is defined for $|\mu'| \leq \text{const}$., and we have the following geometric description of $\psi$: Recall (4.14), which implies,

\[ U_2 g(\mu') U_2 = P_0 - \mu', \]

where, $g(t) = g(\mu', t, h) = f(\mu + t + \mu - c(\mu), h) - f(\mu + c(\mu), h)$, so that $g'(0) > 0$, $g(0) = 0$. Let $X_{U_2}$ be an associated canonical transformation in the precise sense of section a in the appendix. We may choose $X_{U_2}$ satisfying the natural intertwining relation resulting from (4.13), and so that the corresponding phase in $U_2$ (generating $X_{U_2}$) is a classical analytic symbol. It is only after these choices that the symbol and the phase in (4.54) are completely defined by the stationary phase method, and if $\Delta^\nu = (x, \nu(x))$, then
$A \psi = x_{U_2} \cdot K(A \psi)$. Since the symbol of $g(H_2)$ is $g \cdot H_2 + \Theta(h^2)$, and $x_{U_2}$ is a well adapted canonical transformation, we get $g \cdot H_2 \cdot x_{U_2} = x \xi - \mu' + \Theta(h^2)$, $g \cdot H_2 \cdot x_{U_2} \cdot x = -(x \xi + \mu') + \Theta(h^2)$. It follows that $\psi$ satisfies an eiconal equation: $H_2(x, \psi \lambda) = 0$, where $H_2 = H_2 + \Theta(h^2)$ is defined by $g \cdot H_2 \cdot x_{U_2} = x \xi - \mu'$. Since $\mu' > 0$, the real characteristics of $H_2$ near $(0, \pi)$ is the union of two disjoint curves, one which is close to $s(0, 1) \cup s(0, 2)$ and the other which is close to $s((0, 1), 3) \cup s((0, 1), 4)$. It is then clear that we can extend the definition of $\psi$ to negative $x$ as a solution of the eiconal equation. Let $\Phi(x, y) = \Phi(x, y, \mu', h)$ be a generating function for the transformation $x_{K} \cdot x$. Since $x_{K}(0, 0) = (0, \pi)$, we have $\partial_{x} \Phi(0, 0) = \pi$, $\partial_{y} \Phi(0, 0) = 0$, and we may assume that $\Phi(0, 0) = 0$. Then $\psi(x) = v_{c} \Phi(0, y) - \psi_{0}(y)$), and for $x = 0$, the critical point $y(\mu')$ is the projection of $(y_{0}, \mu')$; the intersection in the $4$th quadrant of the curves $y_{1} + \mu' = 0$ and $(x_{K} \cdot x)^{-1}(x_{1} - \text{axis})$. The last curve is of the form, $y_{1} = z(y, \mu')$, with $z$ analytic, $z(0, 0) = 0$, $\partial_{y} z(0, 0) = 0$ close to $1$. Using the reflection symmetries with respect to the points $(0, \pi)$ and $(0, 0)$, we also see that $z$ is an odd function of $y$. It follows that $y(\mu') = \mu' \frac{1}{2} g(\mu')$, where $g$ is analytic with $g(0) > 1$, and a simple calculation gives, (4.55) $\psi(0) = O(\mu')$, and this is again an analytic function of $\mu'$. 

The transport equation for $a_{0}$ will conserve the argument of $a_{0}$, thanks to the fact that the eiconal equation for the phase makes use of the full Weyl symbol $H_2 + \Theta(h^2)$. (See appendix a.) Examining the geometry, we also see that $\partial_{y} \Phi > 0$, and with the representation of $\Phi^{-1} \cdot w_{0}$ above, this leads to: $\arg(a_{0}) = \pi / 8$, if $a_{0}$ denotes the leading part of $a$ in (4.54). Here we assume that the leading amplitude in $U_{2} \cdot \Phi^{-1}$ at $(0, 0)$ is $>0$. (Subtracting $\psi(0) = O(\mu')$ from the generating function of this operator, we may even assume that the right hand side in (4.55) is $0$.)

Let $(x_{\mu}, y_{\mu})$ be the unique point of this form on the characteristics of $H_2$, close to $(\pi / 2, \pi / 2)$. Let us study (4.50) at the point $-x_{\mu}$. Since $A u(x) = \tilde{u}(-x)$, we get, (4.56) $A_{0}v_{01}(-x_{\mu}) = s(x_{\mu}, \mu, h)e^{-i\psi(x_{\mu}, \mu) / h}$. On the other hand, by the method of stationary phase, we find, (4.57) $v_{02}(-x_{\mu}) = s^{-1}v_{01}(-x_{\mu}) = J(\mu, h)(a(x_{\mu}, \mu, h) + \Theta(h))e^{i\psi(x_{\mu}, \mu) / h - \pi i / 4 - ix_{\mu}^{2} / h}$, where $J > 0$ is an elliptic symbol of order $0$, defined for $|\mu'| < \text{const}$. We already know that $|\lambda_{2}| = 1$, and if we compare (4.56), (4.57), (4.50), we get, (4.58) $\arg(\lambda_{2}) = \frac{1}{h} - 2\psi(x_{\mu}, \mu) / h + O(1)$, where "$O(1)$" hides a real valued analytic symbol of order $0$, defined for $|\mu'| < \text{const} > 0$. (With the additional normalizations indicated after (4.55), we may reduce the $\Theta(1)$ term to $\Theta(h)$.)
Here we can make the following geometric interpretation: Let $A(\mu)$ be the area of the domain limited by the $\xi$-axis, $H_2^{-1}(0)$ and the line $\xi=x$. Let $B(\mu)$ be the area of the additional region, which is required to fill up the triangle with corners, $(0,0),(0,\pi),(\pi/2,\pi/2)$. (We think of the appropriate component of $H_2^{-1}(0)$ as lying below $s(0,1)$, otherwise just modify what we just said by introducing domains with negative area). Then $B(\mu)+A(\mu)=\pi^2/4$. Since $\mathcal{V}(0,\mu)=\mathcal{O}(\mu')$, we have $\mathcal{V}(\mathcal{C}_\mu)=\int_0^{\pi} \mu'\mu' dx + \mathcal{O}(\mu')=A(\mu)+\mu'/2+\mathcal{O}(\mu')$. Hence, 
(4.59) \[ \arg(\lambda_2) = -(2A(\mu)+\mathcal{O}(\mu'))/h+\mathcal{O}(1). \]
Combining this with (4.52), we get, 
(4.60) \[ \lambda_2 = \lambda_1^2 = \frac{(\mu'/h)\mu'/h(2\pi) - \frac{i}{2} \Gamma(\frac{1}{2} - i\mu'/h)e(-i\mu'/2 - i2A(\mu) + i\mathcal{O}(\mu'))/h + \mathcal{O}(1)}{e(i(\mu'\log 1/h + 2B(\mu))/h + \pi\mu'/2h - ih*/B + i\mathcal{O}(\mu')/h + i\mathcal{O}(1))}, \]
(With additional normalizations indicated above, we may replace $\mathcal{O}(1)$ by $\mathcal{O}(h)$, and suppress completely the $\mathcal{O}(\mu')$ term.) Here, we write, 
$-2A(\mu)/h = -\pi^2/2 + 2B(\mu)/h$, $\pi^2/2 + h* = h*/B + k\pi/2 \mod (2\pi\mathcal{Z})$, $k \in \mathbb{Z}_4$.

(4.61) \[ \lambda_2 = (2\pi)^{-\frac{1}{2}} \Gamma(\frac{1}{2} - i\mu'/h) \times \frac{e(i(\mu'\log 1/h + 2B(\mu))/h + \pi\mu'/2h - ih*/B + i\mathcal{O}(\mu')/h + i\mathcal{O}(1))}{e(\mu'\log 1/h + 2B(\mu))/h + \pi\mu'/2h + i\mathcal{O}(\mu')/h + i\mathcal{O}(1)}}, \]
where $\mathcal{O}(1)$ indicates a real valued analytic symbol of order 0, defined for $\mu'$ in a neighborhood of 0, and $\mathcal{O}(\mu')$ indicates a real valued analytic function defined near 0. In order to evaluate the singularity of $B(\mu)$ at $\mu'=0$, we let $\gamma = \chi_\mu$ (negative $\xi$-axis), and let $B'(\mu)$ be the area obtained the same way as $B(\mu)$, but with $s(0,1)$ replaced by $\gamma$. Then, 
(4.62) \[ B'(\mu) - B(\mu) = \mathcal{O}(e), \]
is an analytic function of $\mu$, and $\mathcal{E}$ is the parameter in the condition (3.2). By canonical transformation, we see that $B'(\mu)$ is equal to the sum of an analytic function $=\mathcal{O}(\mu')$, and the area of $0 \leq y \leq 1$, $0 \leq \eta \leq \min(z(y,\mu),\mu'/y)$, which gives, 
(4.63) \[ B'(\mu) = -\frac{1}{2} \mu'\log \mu' + \mu'f(\mu'), \]
where $f$ is analytic. We conclude that, 
(4.64) \[ 2B(\mu) + \mu'\log \mu' = 2f(\mu')\mu'/\mathcal{O}(e), \]
so (4.61) gives, 
(4.65) \[ \lambda_2 = (2\pi)^{-\frac{1}{2}} \Gamma(\frac{1}{2} - i\mu'/h) \times \frac{e(i(\mu'\log 1/h + 2B(\mu))/h + \pi\mu'/2h + i\mathcal{O}(\mu') + \mathcal{O}(1))/h - ih*/B + i\mathcal{O}(1))}{e(i\mu'\log 1/h + 2B(\mu))/h + \pi\mu'/2h + i\mathcal{O}(\mu')/h + i\mathcal{O}(1)}}, \]
where $\mathcal{O}(\mu')$, $\mathcal{O}(e)$ are analytic functions of $\mu'$, and $\mathcal{O}(1)$ is a real valued analytic symbol of order 0.

We now recall Stirling's formula in the complex (see [0]): 
(4.66) \[ (2\pi)^{-\frac{1}{2}} \Gamma(z) = e^{-\frac{1}{2} z^2} \sum_{n=0}^{\infty} \frac{1}{z^{n+1/2}} \theta_{z^2}^{n+1} (1+1/12 z+1/208 z^2+...), \]
valid uniformly asymptotically when $|z| \to \infty$, $\arg(z) \leq \pi - \delta$, for any $\delta > 0$.

Writing the asymptotic sum in the parenthesis as $e^{k(z)/z}$, where $k$ is holomorphic and bounded in any domain $|arg z| \leq \pi - \delta$, $|z| \geq const.,$ and real-valued on the real axis, we get,
\[ (2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2} - i\mu'/h\right) = e^{-\frac{1}{2}i\mu'/h} - \mu'/h + i\pi/2 + i\mu'/h - \mu'/2i(h/\mu')^2(\mu'/\mu') \]

Here, \( \log(\frac{1}{2} - i\mu'/h) = \log(\mu'/h) - i\pi/2 + i\mu'/h - \mu'/2i(h/\mu')^2(\mu'/\mu') \), where \( \log \) is holomorphic near 0, and real-valued on the real. We then get,

\[ (4.67) \quad (2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2} - i\mu'/h\right) = e^{i(h/\mu')\log(\mu'/h) - \mu'/2h + i(h/\mu')F(\mu'/h)}, \]

where \( F \) is a bounded holomorphic function, defined in a domain, \( |\arg z| < \text{const.} \), \( |z| > \text{const.} \). Substitution into (4.65), gives for \( \mu'/h > \text{const.} > 0 \):

\[ (4.68) \quad \lambda_\beta = e^{-i\mu'/h}\log(\mu') + i\Theta(\mu'/h + i\Theta(\mu')/h + i\Theta(1) - i\mu'/\Theta + i(h/\mu')F(\mu'/h)). \]

Here we recall that the \( \Theta \)-terms indicate real-valued analytic symbols of order 0, defined for \( \mu' \) in a small neighborhood of 0.

The reflection identity, \( \Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \), gives

\[ \Gamma\left(\frac{1}{2} - i\mu'/h\right)\Gamma\left(\frac{1}{2} + i\mu'/h\right) = \pi/\ch(\mu'/h). \]

Combining this with (4.67) and its complex conjugate, we get for \( \mu'/h > \text{const.} > 0 \):

\[ e^{T\mu'/h}/2\ch(\mu'/h) = e^{i(h/\mu')(F(\mu'/h) - \bar{F}(\mu'/h))}, \]

from which we conclude that,

\[ (4.69) \quad \text{Im}(F(\mu'/h)) = \Theta(\mu'/h)e^{-2\pi\mu'/h} \]

\[ \mu'/h > \text{const.} > 0. \]

Reinsertion in (4.68), then shows that,

\[ (4.70) \quad |\lambda_\beta| = 1 + O(e^{-2\pi\mu'/h}). \]

Naturally, all our calculations, so far are valid only up to some exponentially small error, \( O(e^{-1/\ch}) \). After modification of \( \lambda_\beta, \lambda_\alpha \) by such exponentially small terms, we know that the matrix \( \begin{pmatrix} \lambda_\beta & \lambda_\alpha \\ \lambda_\alpha & \lambda_\beta \end{pmatrix} \) is unitary.

This means that

\[ (4.71) \quad |\lambda_\beta|^2 + |\lambda_\alpha|^2 = 1, \quad \arg(\lambda_\beta) - \arg(\lambda_\alpha) = \pm \pi/2. \]

Combining (4.65) and the relation above, we know on the other hand, that,

\[ (4.72) \quad |\lambda_\beta|^2 = \frac{1}{2}(\ch(\mu'/h))^{-1}e^{\pi\mu'/h} \]

so from the first part of (4.71), we conclude that,

\[ (4.73) \quad |\lambda_\alpha|^2 = \frac{1}{2}(\ch(\mu'/h))^{-1}e^{-\pi\mu'/h}. \]

Combining this with (4.65) and the second part of (4.71), we get,

\[ (4.74) \quad \lambda_\alpha = (2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2} - i\mu'/h\right) \]

\[ e^{i\mu'/h}\log(1/h)\theta(h/\mu')\theta(h/\mu') - 2h + i(\Theta(\mu') + \Theta(1))/h - i\theta(1/\Theta) + i\Theta(1)/\pi/2, \]

where the \( \Theta \)-terms are the same as in (4.65).

We shall next extend our asymptotic results to the case \( \mu'/0 \), by symmetry arguments. For \( \alpha \in \mathbb{R}^2 \), let \( \mathcal{F}_\alpha = T_\alpha T_\alpha^{-1} \), so that \( \mathcal{F}_\alpha \) is a quantization of rotation around \( 2\pi \alpha \) by the angle \( -\pi/2 \). Put \( \tilde{\mathcal{F}} = \mathcal{F}_{\alpha = 0} \mathcal{F}_0^{-1} \) and define \( \tilde{T}_j \), \( j = 1, 2 \), similarly. Define \( \tilde{H}_j \) as before. Then \( \tilde{H}_j \) can also be obtained from \( H_j \) by conjugation by \( \mathcal{F}_0 \mathcal{F}_0^{-1} \). It is easy to check that, \( \tilde{T}_j \mathcal{F}_\alpha = e^{ih\Theta(\gamma, \alpha)} T_\gamma \mathcal{F}_{\alpha = 1}(\gamma) \), where the function \( \Theta \) does not need to be explicited. From this relation we immediately obtain,
In order to investigate the Fourier invariance, we first find that
\( P, P_j \) commute with \( T_y \) for all \( y \).

In order to check the reflection invariance, we first see that,
\[ V(0, \frac{1}{2}) = T_\frac{1}{2}, 0^\sigma \Gamma T_\frac{1}{2}, 0^V, \]
which implies that, \( V(0, \frac{1}{2})^{-1} = T_\frac{1}{2}, 0^\sigma \Gamma T_\frac{1}{2}, 0^V \). Then we get,

\[ V\tilde{P} = T_\frac{1}{2}, 0^\sigma \Gamma T_\frac{1}{2}, 0^P, \]
where \( K = T_\frac{1}{2}, 0^\sigma \Gamma T_\frac{1}{2}, 0^\sigma T_\frac{1}{2}, 0^V \). Using the translation invariance of \( P \), we get \( KPK^{-1} = \tilde{P} \), and hence, \( V\tilde{P} = \tilde{V} \). This is the analogue of (2.3) and the analogue of (2.6) is obtained the same way.

In conclusion, we have checked that \( \tilde{P} \) together with \( P_j \) is of type 1.

Since the symbols of these operators are obtained by rotation by \(-\pi/2\) around \((0, \pi)\), we also see that \(-\tilde{P}, P_j\) satisfy the more precise assumption (3.2) with \( \mu \) replaced by \(-\mu\). To make this even more precise, we also introduce \( \tilde{U}_1 = \sigma_0, \frac{1}{2} U_1 \sigma^{-1}, \) which is a unitary Fourier integral operator such that \( C \tilde{U}_1 = U_1 C \), and whose associated canonical transformation is close to that of \( U_1 \). Since \( P_0 \) and \( \sigma \) anticommute, we get,

\[ \tilde{U}_1^{-1} \tilde{P} \tilde{U}_1 = -1(\mu, -P_0, h)(P_0 + \mu'), \]
where \( l \) is the same function as in (4.6). This shows that all the results, obtained above for \( P \) when \( \mu' > 0 \), are valid also for \( \tilde{P} \) when \( \mu' < 0 \).

Recall that the general microlocal solution of \( H_1 u = 0 \) near \((0, \pi)\), is of the form
\[ u = x_1 u_0, 1 + x_2 u_0, 1, 3 = y_2 v_0, 2 + y_4 v(0, 1), 4, \]
where
\[ t(y_2, y_4) = (\lambda \beta, 1, 1 + [\lambda, \alpha], 1, 2 + [\lambda, \alpha], 2, 1 + [\lambda, \beta], 2, 2)^t(x_1, x_3). \]
Using that the matrix here is unitary, we get \( x_1 = \tilde{\alpha} \tilde{\beta}, x_3 = \tilde{\alpha} \tilde{\beta}, \) if \( y_2 = 0, y_4 = 1 \). In other words,

\[ v(0, 1), 4 = \tilde{\alpha} \tilde{\beta} u_0, 1 \text{ near } \text{Int}(s(0, 1)), \]
and \[ v(0, 1), 4 = \tilde{\alpha} \tilde{\beta} u(0, 1), 3 \text{ near } \text{Int}(s((0, 1), 3)). \]
The functions \( \tilde{u}_0, 1 = \sigma_0, \frac{1}{2} v(0, 1), 4 \), \( \tilde{v}_0, 1 = \sigma_0, \frac{1}{2} u(0, 1), 4 \) are microlocally in the kernel of \( \tilde{H}_2 \) and \( \tilde{H}_1 \) respectively. (We get here a permutation of the indices 1, 2, that will not cause any essential difficulty in the following.) From these functions, we generate \( \tilde{v}_{\alpha, j}, \tilde{v}_{\alpha, j} \) as before. The analogue of problem (4.43) is then to determine \( \tilde{v}_{\beta, \alpha} \) such that,

\[ \tilde{u}_0, 1 = \tilde{\beta} \tilde{v}_0, 2 \text{ near } \text{Int}(s(0, 2)), \]
\[ \tilde{u}_0, 1 = \tilde{\alpha} \tilde{v}(0, 1), 4 \text{ near } \text{Int}(s((0, 1), 4)), \]
\[ \tilde{u}(0, 1), 3 = \tilde{\alpha} \tilde{v}_0, 2 \text{ near } \text{Int}(s(0, 2)), \]
Since \( \mu' \) has been exchanged with \(-\mu'\), we can apply the earlier results in the case when \( \mu'<0 \), and obtain,
\[
\begin{aligned}
\tilde{\beta} &= (2\pi)^{-1/2} \Gamma(\frac{1}{2} + i\mu'/h) \times \\
e^{-i\mu' \log(1/h)/h - \pi \mu'/2h - i\Theta(\mu')/h + i\Theta(\varepsilon)/h + 1\Theta(1)} 
\end{aligned}
\]
where as before the \( \Theta \)-terms are real valued analytic symbols, of order 0, defined for \( \mu' \) in a neighborhood of 0.

Let us now relate \( \tilde{\beta} \) and \( \tilde{\alpha} \). Applying \( \mathcal{F} \) to the relation of (4.77), we see that we need to compare \( \tilde{\nu}_{0,2} \) and \( \mathcal{F}_{0,2} u_{0,1} \). By definition, 
\[
\tilde{\nu}_{0,2} = (\mathcal{F}_{0,2} T_0, \mathcal{F}_{-1/2,1/2} u_{0,1} = T_{-1/2,1/2} \mathcal{F}_{-1/2,1/2} u_{0,1} \),
\]
while \( \mathcal{F}_{0,2} = e^{-ih^*/4} T_{1/2,-1/2} \mathcal{F} \), so we get,
\[
(4.80) \quad \mathcal{F}_{0,2} u_{0,1} = e^{-ih^*/4} \tilde{\nu}_{0,2}.
\]
Comparing this with (4.77) and (4.78), we get,
\[
(4.81) \quad \tilde{\alpha} = \frac{i}{2} e^{-ih^*/4},
\]
which together with (4.79) gives for \( \mu'<0 \),
\[
(4.82) \quad \lambda \alpha = (2\pi)^{-1/2} \Gamma(\frac{1}{2} - i\mu'/h) \times \\
e^{-i\mu' \log(1/h)/h - \pi \mu'/2h - i\Theta(\mu')/h + i\Theta(\varepsilon)/h + 1\Theta(1)} \pm \pi/2.
\]
This is the same type of expression as (4.74), but it is not immediately clear that the analytic symbols hiding in the \( \Theta \)-terms are the same. Using again the unitarity, we obtain from (4.81) that a formula of type (4.65) is valid also for \( \mu'<0 \), although we can not be absolutely certain that the \( \Theta \)-terms contain the same analytic symbols.

After deformation to the case when \( P = H_1 = H_2 = \cos(hD) + \cos(x) - \mu \), \( P_j = 1 \), and some more detailed computations following the same ideas, it is possible to show in the model case and hence in general, that (4.83) the \( + \) sign is valid in (4.71), hence the \(-\) sign is valid in (4.74).

In this model case, we can take \( u_{0,1} = \nu_{0,1}, \lambda = 1 \). For \( \mu = \mu' = 0 \), we have
\[
\tilde{\beta} = -P, \text{ so } \tilde{\alpha} = \alpha, \tilde{\beta} = \beta, \text{ and (4.81) gives, } |\alpha| = |\beta|. 
\]
\[
\text{arg}\alpha + \text{arg}\beta + h^*/4 = 2\pi k, k \in \mathbb{Z}. \text{ Since arg}\beta = \text{arg}\alpha + \pi/2, \text{ we get,} \\
\text{arg}\alpha = \pi k - \pi/4 - h^*/8, \text{ arg}\beta = \pi k + \pi/4 - h^*/8 \text{ in this special case.}
\]

We shall next extend our results to the case when \( |\text{Im}\mu'| \leq \varepsilon_1 h \) for some \( \varepsilon_1 < \frac{1}{2} \). Recall that \( P, P_j \) depend holomorphically on \( \mu \). For complex \( \mu \), we put,
\[
H_1(\mu) = P_1^*(\mu)P(\mu), \quad H_2(\mu) = P(\mu)P_2(\mu) \text{ so that } H_j \text{ depend holomorphically on } \mu.
\]
Restricting the attention to a square \( |\text{Im}\mu| < \varepsilon_0, |\text{Re}\mu| < \varepsilon_0 \), (where \( \varepsilon_0 > 0 \) is small but fixed and independent of "\( \varepsilon \)" in (3.2)), we first notice that we can choose the WKB-solution \( u_{0,1} \), so that the phase depends analytically on \( \mu \), and so that the symbol is a classical analytic symbol with \( \mu \) and \( x \) as base variables. Here \( x \) is restricted to some neighborhood of \( \pi/2 \). (Earlier in this section, the normalization of \( u_{0,1} \) was chosen in such a way that the phase had a logarithmic singularity at \( \mu' = 0 \), but this was for computational reasons...
only and did not affect the final results concerning the problem (4.43).) We may also arrange so that \( u_{0, \lambda} \) is normalized. We now restrict the attention to a "strip" \(|\text{Im}\mu'| \leq C_1, |\text{Re}\mu'| \leq \varepsilon_0. We have,

\[ \lambda = (u_{0, \lambda}(\mu), \chi_0, \lambda u_{0, \lambda}(\mu)). \]

Define \( f_{\alpha, \lambda} \) and \( u_{\alpha, \lambda} \) as for real values of \( \mu \). Again \( f_{\alpha, \lambda} \) is no more a microlocal function, but we may arrange so that it depends holomorphically on \( \mu \), and so that we have the same growth estimates for \( f_{\alpha, \lambda} \) and \( \mathcal{S} f_{\alpha, \lambda} \) as in the case of real \( \mu \). Then (4.31) remains valid, and \( u_{\alpha, \lambda} \) is a microlocal solution of \( H_j u = 0 \) when \( j \) is odd and of \( H_2 u = 0 \), when \( j \) is even. We extend the definition of \( R_- \) in the obvious way, and we extend \( R_+ \) holomorphically, by putting

\[ (R_+ u)(\alpha, \lambda) = (u | u_{\alpha, \lambda}(\mu)). \]

Restricting \( \mu \) further, by imposing \(|\mu'| \leq C_1 h \), with \( 0 < C_1 < \frac{1}{2} \), we see that the existence of an inverse for the corresponding Grushin problem can be proved the same way as in the case of real \( \mu \), and the inverse \( \mathcal{S} \) is \( \mathcal{O}(h^{-3/2}) \) uniformly. Moreover, we see as before that the matrix elements \( E_{+,\lambda}(\alpha, \beta, \mu) \) satisfy (4.38). The coefficients \( a, b \) defined in (4.41) are now holomorphic functions of \( \mu \), and if we choose \( v_0, v_1 \) by (4.45), (4.46), then \( \lambda = e \text{ Re}(\mu, h) / h \), where \( f \) is an analytic symbol of order 0, such that \( \text{Im} f \) is of order \( \leq -1 \). This implies that \( \lambda \beta \) and \( \lambda \alpha \) are holomorphic functions of \( \mu \) of at most temperate growth when \( h \to 0 \), for \(|\mu'| \leq C_1 h \), \(|\text{Re}\mu'| \leq \varepsilon_0 \). Put \( u = \lambda \beta \).

According to (4.65) there is a holomorphic function \( u_+ \), defined on the same rectangle and of temperate growth there, such that \( u - u_+ \) is uniformly of exponential decrease on the intersection \( \Gamma_+ \) of our rectangle and the positive real axis. Similarly, as indicated after (4.82), there is a function \( u_- \) with the same properties, such that \( u - u_- \) is exponentially small on \( \Gamma_- \), which is the intersection of the negative real axis with our rectangle. From the maximum principle, it follows that if we decrease slightly our rectangle by decreasing \( \varepsilon_0 \) and \( \varepsilon_1 \), then \( u - u_{\pm} \) is exponentially decreasing in

\[ U_{\pm} = \{ \mu' \in \mathbb{C} \text{ dist}(\mu', \Gamma_{\pm}) \leq \varepsilon_1 h \}. \]

For instance to get such an estimate for \( u - u_+ \) in a disc \( D(0, \varepsilon_2 h) \), with \( \varepsilon_2 \) slightly smaller than the original \( \varepsilon_1 \), we introduce the subharmonic function \( g = \log |u - u_+| \), and after some changes of scales, we get a subharmonic function \( \tilde{g} \) with only finitely many log-singularities, defined in a neighborhood of the closure of the domain \( \Omega = \{ z \in \mathbb{C}; |z| < 1, 0 < \text{arg}(z) < 2\pi \} \), satisfying \( \tilde{g}(z) \leq -1 \) for \( 0 \leq x < 1, \tilde{g}(z) \leq 0 \) for \( |z| = 1 \). By the maximum principle, (comparing with the harmonic solution of the corresponding Dirichlet problem,) we then obtain, \( \tilde{g} \leq \alpha < 0 \) for \( |z| \leq \varepsilon_2 / \varepsilon_1 \), and after rescaling back, we get the required exponential decrease.

We also know that \( u_+ / u_- = e \alpha(\mu', h)/h \), where \( a \) is a real valued analytic symbol of order 1. From the exponential decrease of \( u - u_{\pm} \) on the interval \( [-\varepsilon_1 h, \varepsilon_1 h] \), it follows that \( a \) is exponentially small on the same
interval (after adding an integer multiple of 2π). We conclude that a is a realization of the 0 symbol, hence a is exponentially small on the whole rectangle. The conclusion is then that we have extended the validity of (4.65) to the whole rectangle. The same then holds for λα, and we have proved, Proposition 4.2. There is λ = e^(if(μ, h)/h), where f is an analytic symbol of order ≤ 0 such that Im f is of order ≤ -1, such that the following holds for α = a/i, β = b/i: Let ε₀ > 0 be sufficiently small, and let 0 < ε₁ < ε₀. Then (4.65) and (4.82) hold uniformly in the rectangle |Im μ'| < ε₁ h, |Re μ'| < ε₀, where as before, the Ω-terms indicate realizations (possibly with an exponentially small imaginary part) of real valued classical analytic symbols of order 0, which are the same in both expressions. Moreover up to exponentially small errors, we have (4.71), where the + sign is valid.

We shall next study the new self-adjoint operators. Let A₁ = R⁺P₁E⁺, A₂ = E⁻P₂E⁺, so that the operators Q₁, Q₂, defined in section 2, are the renormalizations of A₁, A₂. Repeating the arguments of that section, we get the following results (forming intermediate steps in the verification of the properties (2.7), (2.8) for Q₁, Q₂."

\[(4.85)\quad 9A₁ = A₁ 9, \quad A₁ 9 = 9A₁,\]
\[(4.86)\quad V ' A_j = A_j V'.\]

The fact that \(A_j\) commute with the \(\Sigma_j\), implies on the block-matrix level, that,
\[(4.87)\quad A_j(α, β) = e^{ih'β_j(α - β)} A_j(α - β),\]
and exploiting the properties (4.85), (4.86), we get,
\[(4.88)\quad a_1(κ(α)) = T * a_2(α) * T⁻¹, \quad a_1(κ⁻¹(α)) = T⁻¹ * a_2(α) * T,\]
\[(4.89)\quad a_υ(δ(α); 2⁻, j⁻, k⁻) = e^{ih'να} a_υ(α; j, k).\]

As with \(E_+\), we see that \(a_υ(α)\) is exponentially small, except for \(|α| ≤ 1\), and for \(|α| = 1\), the only non-exponentially small entries are \(a_υ((1, 0); 2, 1), a_υ((0, 1); 4, 1), a_υ((-1, 0); 4, 3), a_υ((0, -1); 2, 3)\). Using (4.88), (4.89), it is easy to check that we have,
\[(4.90)\quad a_1(0) = a_2(0) = [x]_{2, 1} + [\bar{x}]_{2, 3} + [\bar{x}]_{4, 1} + [x]_{4, 3}.\]

Similarly, we get for \(|α| = 1\),
\[(4.91)\quad a_υ((1, 0)) = [z]_{2, 1}, \quad a_υ((0, -1)) = [z]_{2, 3}, \quad a_υ((-1, 0)) = [\bar{z}]_{2, 3}, \quad a_υ((0, 1)) = [\bar{z}]_{4, 1}, \quad \text{modulo } Ω(e⁻�α/h) \text{ for some } ε₀ > 0.\]

(These relations could also have been obtained directly, by using (2.7), (2.8) for \(Q_j\) and studying the trigonometric polynomial of degree 1 in the Fourier series expansion of the symbols \(Q_j\). Looking at higher order contributions, we also see that there is no reason to suspect that \(Q_1 = Q_2\).)

The next problem is then to study \(x = A_υ((0, 2; 0, 1)), z = A_υ((0, 1); 4, 0, 1)\). Applying the definitions of the various operators, we get,
\[x = (P₁E⁺(δ_{0, 1}) | f₀, 2) = λ β(P₁(1 - F_{0, 2})ν₀, 2 | f₀, 2),\]
Recall that the corresponding coefficients $b, a$ for $E_{-+}$ are given by $b = i^{-1} \beta$, $a = i^{-1} \alpha$, so we get the same coefficients, multiplied by a common factor,

$$\theta = i \lambda (P_1(1 - \tilde{X}_{0,1}^2)v_{0,2}f_{0,2}) = i \lambda (P_1(1 - \tilde{X}_{0,1}^2)v_{0,1}f_{0,1}).$$

Recalling that $\lambda v_{0,2} = P_{-}^2 u_{0,2}$, we can also write

$$\theta = i (P_1(1 - \tilde{X}_{0,1}^2)P_{-}^2 u_{0,2} f_{0,2}) = i (P_{-}^2(1 - \tilde{X}_{0,1})P_1 u_{0,1} f_{0,1}) =$$

$$i (P_{-}^2 P_1 (1 - \tilde{X}_{0,1}) u_{0,1} f_{0,1}),$$

where $\tilde{X}_{0,1} P_1 = P_1 X_{0,1}$. Here $P_{-}^2 P_1$ maps $\text{Ker}(H_1)$ to itself, so $P_{-}^2 P_1 u_{0,1} = \mu u_{0,1}$ for some complex number $\mu$. Since $V$ commutes with $P_j$ and $P_{-}$, and since $V u_{0,1} = u_{0,1}$, we find $P_{-}^2 P_1 u_{0,1} = \mu u_{0,1}$, and hence $\mu$ is real. Using (1.18) and the fact that $V^2 = 1$,

$$(P_{-}^2 P_1 (1 - \tilde{X}_{0,1}) u_{0,1} f_{0,1}) = (V P_{-}^2 (1 - \tilde{X}_{0,1}) u_{0,1} f_{0,1}) =$$

$$= (V P_{-}^2 (1 - \tilde{X}_{0,1}) u_{0,1} f_{0,1}) = (P_{-}^2 P_1 (1 - \tilde{X}_{0,1}) u_{0,1} f_{0,1}).$$

Here we also used that $V X_{0,1} V^{-1} = 1 - X_{0,1}$ and that $V u_{0,1} = u_{0,1}$, $V f_{0,1} = f_{0,1}$. Putting $\xi = (P_{-}^2 P_1 (1 - \tilde{X}_{0,1}) u_{0,1} f_{0,1})$, it follows that

$$\xi + \tilde{\xi} = (P_{-}^2 P_1 u_{0,1} f_{0,1}) = \mu (u_{0,1} f_{0,1}) = \mu,$$

(proving again that $\mu$ is real). Since $\mu \neq 0$ by the ellipticity of $P_j$, we deduce that, $\text{Re} \xi \neq 0$, and hence that $\text{Im} \theta \neq 0$. More precisely, we have showed that in the same rectangle as in Proposition 4.2, we have, $x = b \theta + \Theta (e^{-1/\text{Coh}})$,

$$z = a \theta + \Theta (e^{-1/\text{Coh}}),$$

where $\Theta$ is an analytic symbol of order 0, such that $\text{Im} \Theta$ is elliptic. Here $C_0 > 0$. Recall that the renormalization $Q$ of $E_{-+}$ is the $h'$-quantization of the symbol $Q$, given by (2.21), and that the renormalizations $Q_j$ are obtained from $A_j$ by the formula,

$$(4.92) \quad Q_j(x, \xi) = \sum_{\alpha} a_j(\alpha) e^{-i\alpha_1 \alpha_2 h'/2} e^{-i < \xi, (x, \xi)>}.$$ 

We then get,

**Proposition 4.3.** There is a constant $C_0 > 0$, such that for $|\text{Im} (x, \xi)| \leq 1/\text{Coh}$, we have,

$$(4.93) \quad Q(x, \xi) = [b + a e^{-i \xi}]_{2,1} + [\bar{b} + a e^{i \xi}]_{2,3} + [\bar{b} + a e^{-i \xi}]_{4,1} + [b + a e^{i \xi}]_{4,3} + \Theta (e^{-1/\text{Coh}}),$$

$$(4.94) \quad Q_j(x, \xi) = [\Theta b + \Theta a e^{-i \xi}]_{2,1} + [\Theta b + \Theta a e^{i \xi}]_{2,3} + [\Theta b + \Theta a e^{-i \xi}]_{4,1} +$$

$$+ [\Theta b + \Theta a e^{i \xi}]_{4,3} + \Theta (e^{-1/\text{Coh}}).$$

Here it is understood that $\bar{a}$, $\bar{b}$, $\bar{\Theta}$ denote the holomorphic extensions of the complex conjugates of $a$, $b$, $\Theta$ on the real domain.

We have already observed in section 2, that $Q_j$ may be replaced by any real linear combination of $Q$ and $Q_j$ with coefficients that do not depend on $j$.

Since $\text{Im} \Theta$ is elliptic, this means that we may assume (4.94), where $\Theta$ is any (convenient) complex number. Let us denote by $Q^0$ and $Q^0_j$ the explicit matrices
appearing to the right in (4.93),(4.94). After modification of $a, b$, by exponentially small quantities, which do not affect the validity of the last proposition, we may assume that,
\begin{equation}
\text{arg}(b) = \text{arg}(a) + \pi/2,
\end{equation}
so that $b \bar{a}/i > 0$, and $b^2 - a^2 = (|b|^2 + |a|^2)e^{i2\text{arg}(b)}$. Then we get,
\begin{equation}
\text{det}Q(x, \xi) = 2i(|b|^2 + |a|^2) \sin(2\text{arg}(b)) + 2b \bar{a} (\cos(\xi) + \cos(x)) =
= 2b \bar{a} ((b \bar{a}/i)^{-1}(|b|^2 + |a|^2) \sin(2\text{arg}(b)) + \cos(\xi) + \cos(x)),
\end{equation}
\begin{equation}
\text{det}Q(x, \xi) =
= 2|a|^2 b \bar{a} ((b \bar{a}/i)^{-1}(|b|^2 + |a|^2) \sin(2\text{arg}(b)) + \cos(\xi) + \cos(x)).
\end{equation}
Recalling that $b = \beta/i = (\lambda \beta)/(\lambda i)$, and similarly for $a$, we obtain from our earlier results,
\begin{equation}
b = (2\pi)^{-i/2} \Gamma(\lambda - i\mu'/h) e^{i\mu' \log(1/h)}/h + \mu'/2h + \mathcal{O}(\mu'/h)/h,
\end{equation}
\begin{equation}
a = (2\pi)^{-i/2} \Gamma(\lambda - i\mu'/h) e^{i\mu' \log(1/h)}/h - \mu'/2h + \mathcal{O}(\mu'/h)/h - i\pi/2,
\end{equation}
where $\mathcal{O}$ is a classical analytic symbol of order $\leq 0$, such that $\text{Re}(\mathcal{O})$ is of order $\leq 1$. From this it follows that,
\begin{equation}
|a|^2 + |b|^2 = e^{2\text{Re}(\mathcal{O})}/h, \quad b \bar{a}/i = e^{2\text{Re}(\mathcal{O})}/h/2\text{ch}(\pi \mu'/h),
\end{equation}
when $\mu$ is real. Using this in (4.96), we get:
\begin{equation}
\text{det}(Q^0) = 2b \bar{a} (2\text{ch}(\pi \mu'/h) \sin(2\text{arg}(b)) + \cos(\xi) + \cos(x))
\end{equation}
The formulas (4.98),(4.99) are a little less precise than earlier corresponding ones, but as we shall see later, they carry enough information, in order to continue the renormalization procedure.

Let us finally formulate the main result of this section,

**Proposition 4.4.** Let $(P, P_1, P_2)$ be a strong type 1 operator with $0 < h \leq 2\pi$ and with $0 \leq \varepsilon(P) \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small. Define $\mu_0(P) = \Theta(\varepsilon(P))$ as in Proposition 3.4. There exists $h_1 > 0$, such that for $0 < h \leq h_1$, we have an analytic diffeomorphism $\mu \rightarrow \mu'$ from a neighborhood of $\mu_0$ to a neighborhood of $0$ such that $\mu'(-\mu_0, h) = \Theta(h)$. $\mu'$ and its inverse are given by classical analytic symbols of order 0. Moreover, for $h_1 > 0$ and $\varepsilon_1 > 0$ sufficiently small, the $\mu'$-spectra of $P$ and $Q$ coincide in the interval $[-\varepsilon_1, \varepsilon_1]$, if $Q$ denotes the $h'$-quantization of the matrix symbol $Q_1$, given by (4.93). Here $2\pi/h = h'/2\pi \mod(\mathbb{Z})$, $0 < h' \leq 2\pi$, and $a, b$ are given by (4.98), (4.99) (satisfying also (4.100)). For every $\varepsilon \in \mathbb{C}$, there exist symbols $Q_j$ satisfying (4.94), such that $(Q, Q_1, Q_2)$ is a type 2 operator in the sense of section 2. (Here $Q_j$ also denote the corresponding $h'$-quantizations.)
5. Preliminaries for the renormalized operator of section 4.

In section 4, we proved that the renormalization of $E_-$ is given by:

\[(5.1) \quad Q(x,\xi) = [b + \alpha e^{-i\xi}]_{2,1} + \{b + a e^{i\xi}\}_{2,3} + [b + a e^{-i\xi}]_{4,1} + [b + \alpha e^{i\xi}]_{4,3} + \Theta(e^{-1/C_0h}), \]

when $|\Re \mu'| < \varepsilon_0$, $|\Im \mu'| < \varepsilon_1 h$, for some sufficiently small $\varepsilon_0$ and $h>0$, when $\varepsilon_1$ is fixed in $[0,\frac{1}{2}]$. Here,

\[(5.2) \quad b = (2\pi)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} - i\mu'/h\right) e^{i(\mu'/h) \log(1/h) + \pi \mu'/2h + i\Theta(\mu',h)/h}, \]

\[(5.3) \quad a = (2\pi)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} - i\mu'/h\right) e^{i(\mu'/h) \log(1/h) - \pi \mu'/2h + i\Theta(\mu',h)/h - i\pi/2}, \]

where $f$ is a classical analytic symbol of order $\leq 0$, such that $\Im f$ is of order $\leq -1$. After multiplication of $Q$ by $e^{\Theta(h)}$ times the identity matrix, we may assume from now on that $f$ is real valued. Then for real $\mu'$:

\[(5.4) \quad |a|^2 + |b|^2 = 1, \quad \arg(b) - \arg(a) = \pi/2, \]

\[(5.5) \quad |b| = e^{\pi \mu'/2h + e^{-\pi \mu'/h} - 1/2}, \]

\[(5.6) \quad |a| = e^{-\pi \mu'/2h + e^{-\pi \mu'/h} - 1/2}, \]

\[(5.7) \quad |a| |b| = 1/2 ch(\pi \mu'/h). \]

Using the complex version of Stirling's formula, we also had,

\[(5.8) \quad b = e^{i\mu'/h - i(\mu'/h) \log(\mu') + i(h/\mu' + 1/2) + i(\mu'/h) - 1/2} + \Theta(1), \]

for $\mu'/h \geq \text{const.} > 0$, where $F(z)$ is a bounded holomorphic function in a sector $|\Im z| < \Re(z)/C$, $\Re(z) > C$, and,

\[(5.9) \quad \Im F(\mu'/h) = \Theta(1)(\mu'/h) e^{-2\pi \mu'/h}, \]

when $\mu'/h > C$. This relation can also be differentiated with respect to $\mu'$.

Consider the function $B = i^{-1} \log(b)$. Then for $|\Re \mu'/h| \leq \text{const.}$,

\[(5.10) \quad \partial_{\mu'} B = h^{-1} \log(1/h) + \Theta(h^{-1}) \Rightarrow 0 \rightarrow 0, \]

for $|\Re \mu'/h| \leq \text{const.} \leq 1$, we can apply (5.8):

\[(5.11) \quad \partial_{\mu'} B = h^{-1} \log(1/\mu') + \Theta(h^{-1}) \Rightarrow 0. \]

Restricting this to the real axis, we get for $\mu'/h \geq \text{const.} > 0$:

\[(5.12) \partial_{\mu'} B = h^{-1} \log(1/\mu') + \Theta(h^{-1}) \Rightarrow 0. \]

For complex $\mu'$ with $|\Re \mu'/h| \geq \text{const.} > 0$, we get,

\[(5.13) \partial_{\mu'} B = h^{-1} \log(1/\mu') + \Theta(h^{-1}), \]

which we can write as,

\[(5.14) \partial_{\mu'} B = h^{-1} \log(1/\mu') + \Theta(h^{-1}). \]

Combining (5.10) and (5.12), we get,

\[(5.15) \partial_{\mu'} B = h^{-1} \log(h + |\mu'|^{-1}) + \Theta(h^{-1}), \]

and this relation is also valid for $|\Re \mu'/h| \leq \text{const.} < 0$, and hence everywhere in the rectangle $|\Re \mu'| < \varepsilon_0$, $|\Im \mu'| < \varepsilon_1 h$. In view of (5.11), we also know that

\[(5.16) \partial_{\mu'} \Im B = \Theta(h^{-1} e^{-\mu'/C_0h}) \text{ for } \mu'/h \geq \text{const.} > 0. \]

Put,

\[(5.17) \quad Q_0(x,\xi) = [b + \alpha e^{-i\xi}]_{2,1} + \{b + a e^{i\xi}\}_{2,3} + [b + a e^{-i\xi}]_{4,1} + [b + \alpha e^{i\xi}]_{4,3}. \]

Then for real $\mu'$, we get from (4.101) and (5.7), (5.4),
(5.15) \[ \det Q_0(x, \xi) = (1/ch(\pi \mu' / h))[2ch(\pi \mu'/ h) \sin(2\arg(b)) + \cos \xi + \cos x]. \]

The real characteristic set of \( Q_0 \) is non-empty precisely when,

(5.16) \[ \text{ch}(\pi \mu' / h) \sin(2\arg(b)) \in [-1,1]. \]

Cases of special interest appear when,

(5.17) \( \text{ch}(\pi \mu' / h) \sin(2\arg(b)) = -1 \), then \( (\det Q_0)^{-1}(0) = 2\pi \mathbb{Z}^2 \),

(5.18) \( \text{ch}(\pi \mu' / h) \sin(2\arg(b)) = 1 \), then \( (\det Q_0)^{-1}(0) = 2\pi \mathbb{Z}^2 + (\pi, \pi) \),

(5.19) \( \text{ch}(\pi \mu' / h) \sin(2\arg(b)) = 0 \), then \( (\det Q_0)^{-1}(0) \) is the union of the lines \( \xi = (2k+1)\pi, \ k \in \mathbb{Z} \).

In order to understand the structure of the set of real \( \mu' \) such that (5.16) holds, we split the discussion into two cases: 1°, \( |\mu'/h| \leq C_0 \), 2°, \( |\mu'/h| > C_0 \). Here \( C_0 \) will be chosen sufficiently large in the discussion of case 2°. In the discussion of the case 1° it may be arbitrarily large, provided that \( h > 0 \) is sufficiently small.

1° \( |\mu'/h| \leq C_0 \). Put \( \zeta(\mu', h) = \text{ch}(\pi \mu'/ h) e^{2i\arg(b)} \). Since \( \partial_{\mu'} \arg(b) = (1 + o(1))h^{-1} \log(1/h) \), and by Cauchy's inequalities,

\( (\partial_{\mu'} \arg(b)) = O(h^{-1} \log(1/h)) \), it follows that,

\[ \partial_{\mu'}^2 \zeta(\mu', h)/\zeta(\mu', h) = 2i(1 + o(1))h^{-1} \log(1/h), \]

\[ \partial_{\mu'}^2 \zeta(\mu', h)/\zeta(\mu', h) = -(1 + o(1))(1 + o(1))h^{-2} \log(1/h))^2. \]

The curvature of \( \mu' \rightarrow \zeta(\mu', h) \in \mathbb{C} \approx \mathbb{R}^2 \), is given by,

\[ |\partial_{\mu'}^2 \zeta(\mu', h)|^2 Re(\partial_{\mu'}^2 \zeta(\mu', h)) = 8(1 + o(1))h^{-1} \log(1/h)^2, \]

so for \( h > 0 \) small enough the curvature is strictly positive. Moreover \( |\zeta(\mu', h)| = \text{ch}(\pi \mu'/ h) \geq 1 \) with equality precisely for \( \mu' = 0 \). Now (5.16) holds precisely when \( \text{Im} \zeta(\mu', h) \in [-1,1] \), so it follows that the set of \( \mu' \) satisfying (5.16) is a union of closed intervals of length \( \sim h/\log(1/h) \). All these intervals are disjoint except possibly two. This exceptional situation can appear only when \( \sin(\arg(b)) = \pm 1 \) for \( \mu' = 0 \), i.e. when \( \arg(b(0, h)) = \pi/4 + n\pi/2, n \in \mathbb{Z} \). Then the two intervals have \( \mu' = 0 \) as a common boundary point. In a region, where \( |\mu'/h| \geq \text{const.} > 0 \), the distance between two consecutive intervals is of the same order of magnitude as the length of these intervals. In the region where \( |\mu'/h| \) is small, if \( \mu' \) is between two intervals, then the distance between the two intervals is of the order of magnitude, \( (2h/\log(1/h)) |\text{sh}(\pi \mu'/ h)| \).

Also, notice that \( \mu' = 0 \) always belongs to one of the intervals.

2° \( |\mu'/h| \geq C_0 > 0 \). Then \( \text{ch}(\pi \mu'/ h) \geq \text{const.} > 1 \), and we rewrite (5.15), (5.16), as,

(5.15)' \[ \det Q_0 = i[2\sin(2\arg(b)) + (\text{ch}(\pi \mu'/ h))^{-1}(\cos \xi + \cos x)], \]

(5.16)' \[ \sin(2\arg(b)) \in [-1/\text{ch}(\pi \mu'/ h), 1/\text{ch}(\pi \mu'/ h)]. \]

In the region \( |\mu'/h| \geq \text{const.} > 0, |\sin(2\arg(b))| \leq \text{const.}<1 \), we have,

\( |\partial_{\mu'} \sin(2\arg(b))| \sim h^{-1} \log(1/|\mu'|) > \partial_{\mu'} (1/\text{ch}(\pi \mu'/ h)), \) so the set of \( \mu' \) satisfying (5.16)' is a union of closed disjoint intervals, such that if \( \mu' \) is in the separation of two such intervals, then the separation is of the order of magnitude \( h/\log(1/|\mu'|) \), and if \( \mu' \) is in such an interval, then the length of
that interval is \((2+o(1))he^{-\pi|\mu'|/h(\log(1/|\mu'|))^{-1}}\) when \(h \to 0\), \(|\mu'| \to 0\), \(h/|\mu'| \to 0\).

Still in case 2°, we notice that either \(|b|/|a| \geq \text{const.} \cdot 1\), or \(|a|/|b| \geq \text{const.} \cdot 1\), and \(Q_0(x,\xi)\) can never vanish completely for real \(x,\xi\). In other words, the worse that can happen is that \(Q_0(x,\xi)\) (or \(Q(x,\xi)\)) has a 1-dimensional kernel for some real \(x,\xi\).

In the case 1°, let us look for the special situations when \(Q_0\) may vanish completely for some real \(\mu', x, \xi\). Assuming we are at such a point, since both \(a\) and \(b\) are \(\neq 0\), we get from \(b + e^{-i\xi} = b + e^{i\xi} = 0\), that \(e^{-i\xi} = e^{i\xi}\), so \(\xi = k\pi, k \in \mathbb{Z}\). Similarly, we must have \(x = l\pi, l \in \mathbb{Z}\). We then get \(b + (-1)^k a = 0, b + (-1)^l a = 0\), and comparing the last equation with the complex conjugate of the first one, we see that \((-1)^k = (-1)^l\), so \(k\) and \(l\) must have the same parity. Since \(Q_0\) is \(2\pi\)-periodic both in \(x\) and in \(\xi\), it suffices to study the cases of complete degeneration at \((0,0)\) and at \((\pi, \pi)\). The complete degeneration at \((0,0)\) appears iff \(b + a = 0\), and it is easy to see that this happens precisely when \(\mu' = 0\), and \(\arg(b(0,h)) = 3\pi/4 + n\pi, n \in \mathbb{Z}\). Complete degeneration at \((\pi, \pi)\) appears iff \(b - a = 0\), which happens precisely when \(\mu' = 0\) and \(\arg(b(0,h)) = \pi/4 + n\pi\). The complete degeneration is a rather exceptional case, but if \(|\mu'/h|\) is small, then \(|a| \approx |b|\), and \(Q_0(0,0)\) and \(Q_0(\pi, \pi)\) become small respectively, when \(\arg(b) \approx 3\pi/4 + n\pi\), and \(\arg(b) \approx \pi/4 + n\pi\). These cases appear near the end-points of the intervals given by \((5.16)'\).

We end this section by giving some qualitative statements, which are more easy to carry on in a general iteration scheme. Let \(P_0(a,b;x,\xi)\) denote the matrix given by \((5.14)\).

**Definition 5.1.** The triple of \(h\)-pseudodifferential operators \((P, P_1, P_2)\) is of strong type 2, if it is of type 2 and if for the symbols, we have \(|P(x,\xi) - P_0(a,b;x,\xi)| \leq \varepsilon, |P_j(x,\xi) - P_0(ia,ib;x,\xi)| \leq \varepsilon\) for \(|\text{Im}(x,\xi)| < 1/\varepsilon, |\mu| < 4\). Here it is further assumed that \(P, P_j, a, b\) depend holomorphically on \(\mu\), that \(a = a(\mu)\) and \(b = b(\mu)\) satisfy,

\[
|a|^2 + |b|^2 = 1, \quad |\arg(b) - \arg(a)| = \pi/2, \quad \text{for } \mu \text{ real, and}
\]

\[
|\mu| < 4.
\]

And

\[
b(\mu) = b(0)(1 + \mathcal{O}(\varepsilon))e^{i\mu(1 + \mathcal{O}(\varepsilon))},
\]

where each \(\mathcal{O}(\varepsilon)\) indicates a holomorphic term of modulus \(\leq \varepsilon\), which is real when \(\mu\) is real. We define \(\varepsilon(P)\) to be the infimum of all \(\varepsilon\) satisfying the above inequalities, and we define \(C(P)\) as \(\max(1/|a(0)|, 1/|b(0)|)\).

In this definition the number 4 could easily be replaced by any fixed strictly positive number.

From the discussion above and Proposition 4.4, it is easy to obtain the following result,

**Proposition 5.2.** Let \((P, P_1, P_2)\) be a strong type 1 operator with \(0 < h \leq h_0\) and \(\varepsilon(P) \leq \varepsilon_0\), with \(h_0, \varepsilon_0 > 0\) sufficiently small, so that we can define the new
\( \mu' \)-variable as in Proposition 4.4. Fix \( 0 \leq C_0 < C_1 \). For \( C_0 h \leq |\mu_0'| \leq C_1 h \), we put
\[ \mu'' = (\mu' - \mu_0') h^{-1} \log(h^{-1}) \].
Then for \( 0 < h \leq h_1 > 0 \) sufficiently small, depending on \( C_1 \) and for \( |\mu''| < 4 \), we can express \( \mu - \text{Sp}(P) \) in the \( \mu'' \)-variables as
\( \mu'' - \text{Sp}(Q) \), where \( (Q, Q_j) \) is a strong type 2 \( h' \)-pseudodifferential operator (with \( \mu \) replaced by \( \mu'' \) in the definition above) with \( c(Q) \to 0 \) as \( h \to 0 \) and with \( c_-(C_0) \leq c(Q) \leq c_+(C_1) \), where \( c_\pm \) only depend on \( C_0 \) and \( C_1 \) respectively, and where \( c_-(C_0) \to +\infty \) when \( C_0 \to +\infty \). We also recall that \( 2\pi/h \equiv h'/2\pi \mod(\mathbb{Z}) \), \( 0 < h' \leq 2\pi \).
6. Reduction to type 1 operators.

Let Q be as in section 5. We shall here study the case when |μ'/h| is bounded from below by some sufficiently large constant. In the case when μ' > 0, we have |b| >> |a| and Q is close to a constant matrix. In the case μ' < 0, |a| >> |b|, and the variable part of Q dominates. We shall first exhibit some symmetries that show that the second case is actually equivalent to the first one.

As before, we denote by P₀(a,b;x,ξ) the expression (5.14) and we shall use the same letters to denote the corresponding h'-Weyl quantizations. We recall that the matrix P₀ maps C²_{odd} into C²_{even}, and define L:

\[ L = \begin{bmatrix} e^{i(x-ξ)/2} & 4 + e^{i(x+ξ)/2} \\ e^{-i(x-ξ)/2} & 1 + e^{-i(x+ξ)/2} \end{bmatrix}, \quad M = \begin{bmatrix} e^{i(x+ξ)/2} & e^{i(x-ξ)/2} \\ e^{-i(x+ξ)/2} & e^{-i(x-ξ)/2} \end{bmatrix}. \]

Recalling the definition of T: C²_{odd} → C²_{even}, C²_{even} → C²_{odd} in section 2, we check that,

(6.1) \[ TL = LT. \]

A long but straightforward computation shows that for the h'-Weyl quantizations,

(6.2) \[ L P₀(a,b) = P₀(e^{-ih'/4b}, e^{-ih'/4a}) M. \]

This gives in particular a unitary equivalence between the cases |b| >> |a| and |a| << |b|. In order to complete the symmetry discussion, we let P, P₁, P₂ be of type 2 as defined in section 2, and we define P', P₁', P₂' by:

(6.3) \[ LP = P'M, \quad LP_1 = P_1'M. \]

It is then straightforward to check that P', P₁', P₂' commute with the T, that P₁'P₂' and P₂'P₁' are self-adjoint, and with the help of (6.1) and (2.7), (2.9), we also obtain (2.7), (2.9) for P', P₁'. Finally, we check that,

(6.4) \[ VL = LV, \quad ML = M'L, \]

and using (2.8), (2.10), we obtain the same relations for P', P₁'. Hence P', P₁' is of type 2 and if \[ P = P₀(a,b)+O(ε), \quad P₁ = P₀(θa, θb)+O(ε), \]

then \[ P' = P₀(e^{-ih'/4b}, e^{-ih'/4a})+O(ε), \quad P₁' = P₀(e^{-ih'/4b}, e^{-ih'/4a})+O(ε). \]

The conclusion of this discussion is that it suffices from now on to consider the case when μ'/h > 0.

Let μ be as in section 5. We shall now study Q for μ in the region \( C ≤ μ ≤ 1/C \), where C is some sufficiently large constant. Let μ₀ be in this interval with \( \sin(2\arg(b(μ₀))) = 0 \), so that

(6.5) \[ \arg(b(μ₀)) = nπ/2, \]

for some \( n ∈ ℤ \). We rescale by introducing a new variable μ'₀,

(6.6) \[ μ' = μ'₀ = (h/\log(μ₀^{-1}))μ'', \]

and restrict the attention to a region where |μ''| is bounded by a constant.
For $1/C$ and $h$ small enough, we then have,
\begin{align}
(6.7) \quad b(\mu', h) &= (1 + o(h)) b(\mu'_0, h) e^i(1 + o(h))
\mu'', \\
(6.8) \quad a(\mu', h) &= (1 + o(h)) a(\mu'_0, h) e^i(1 + o(h))
\mu'',
\end{align}
where $\epsilon > 0$ can be chosen arbitrarily provided that $C$ and $1/h$ are sufficiently large. Put $\delta = 1 a(\mu'_0, h)$, then $|b(\mu'_0, h)| = (1 - \delta)^{1/2}$, and choosing $C$ (and $1/h$) large enough, we may assume that,
\begin{align}
(6.9) \quad Q(x, \xi) &= Q_0(x, \xi) + o(\delta^2), \quad |\text{Im}(x, \xi)| \leq 1/\epsilon.
\end{align}
Recall (4.101), (5.7):
\begin{align}
(6.10) \quad \det(Q_0) &= 2 b(0) |a(\mu'_0, h)| e^i(1 + o(h))
\mu'',
\end{align}
This quantity is non vanishing for real $(x, \xi)$ except when $\mu'' = o(\delta)$, and the same is true for $\det(Q)$, in view of (6.9).

Let $b_0 = e^i m/n$, $P_{0, 0} = P_0(0, b_0)$. Then $P_{0, 0}$ is independent of $(x, \xi)$ and $\text{Ker}(P_{0, 0}) = (e_0)$, where $e_0 = (2^{-1}, (-1)^{n+1} 2^{-1/2})$, and we also notice that $P_{0, 0}$ is equal to $i^n$ times a self adjoint matrix. We now define
\begin{align}
R_+: L^2(\mathbb{R}; \mathbb{C}^2) &\to L^2(\mathbb{R}), \quad R_- : L^2(\mathbb{R}) &\to L^2(\mathbb{R}; \mathbb{C}^2),
\end{align}
by $R_+ u(x) = (u(x)|e_0) e_0$, $R_- u^{-1}(x) = u^{-1}(x) e_0$. Let $\mathcal{P} = [P_{0, 0}]_{1, 1} + [R_-]_{1, 2} + [R_+]_{2, 1} : L^2(\mathbb{R}; \mathbb{C}^2) \times L^2(\mathbb{R}) \to L^2(\mathbb{R}; \mathbb{C}^2) \times L^2(\mathbb{R})$, and define $\mathcal{P}$ similarly with $P_{0, 0}$ replaced by $Q$. In view of (6.7)–(6.9),
\begin{align}
(6.11) \quad \mathcal{P} - \mathcal{P}_0 &= o(\delta + |\mu''|),
\end{align}
in the sense of $L^2$ bounded operators, and also in the sense of Weyl symbols defined on $|\text{Im}(x, \xi)| \leq 1/\epsilon$. (Here we work with the $h'$-quantization.) Now $\mathcal{P}_0$ has a bounded inverse, $\mathcal{E}_0 = [E_0]_{1, 1} + [E_0]_{1, 2} + [E_0, -]_{2, 1}$, so it follows from (6.11) that the same is true for $\mathcal{P}$. We write
\begin{align}
\mathcal{P}^{-1} = \mathcal{E} = [E]_{1, 1} + [E_+]_{1, 2} + [E_-]_{2, 1} + [E_+]_{2, 2}. \quad \text{As earlier, } 0 \text{ belongs to the spectrum of } \mathcal{P} \iff \text{it belongs to the spectrum of } E_+. \quad \text{We have the Neumann series},
\end{align}
\begin{align}
(6.12) \quad \mathcal{E} &= \mathcal{E}_0 (I - (\mathcal{P} - \mathcal{P}_0) \mathcal{E}_0 + ((\mathcal{P} - \mathcal{P}_0) \mathcal{E}_0)^2 - \ldots).
\end{align}
In view of (6.11) we get:
\begin{align}
(6.13) \quad \mathcal{E} - \mathcal{E}_0 &= - \mathcal{E}_0 (\mathcal{P} - \mathcal{P}_0) \mathcal{E}_0 + o(\delta^2 + |\mu''|),
\end{align}
for the symbols in the strip $|\text{Im}(x, \xi)| \leq 1/\epsilon$. In particular,
\begin{align}
(6.14) \quad E_+ - E_0 &= - (Q - P_0, 0) E_0, + o(\delta^2 + |\mu''|),
\end{align}
Here $E_0, + u \to u(x) e_0$, $E_0, - v \to (v(x)|e_0)$, so the first term of the last member of (6.14) is equal to
\begin{align}
(6.15) \quad -\left( (P_0(x, \xi) - P_0, 0) e_0 | e_0 \right) &= - (P_0(x, \xi) | e_0) e_0 = \\
&= - [b + (-1)^n 1^2 + \bar{a} \cos(\xi) + (-1)^n 1^2 \cos(x)].
\end{align}
Using that $b(\mu'_0) = (1 - \delta^2)^{1/2} n_1$, $a(\mu'_0) = \delta^{1/2} n_1$, we get from (6.7), (6.8):
\begin{align}
(6.16) \quad b(\mu') &= (1 + o(\epsilon \mu')) (1 - \delta^2)^{1/2} n_1 e^i (1 + o(\epsilon)) \mu'', \\
(6.17) \quad a(\mu') &= (1 + o(\epsilon \mu')) \delta^{1/2} n_1 e^i (1 + o(\epsilon)) \mu''.
\end{align}
Here all the $\Theta$-terms are real for $\mu''$ real, and the ones in the exponents are the same. Combining this with (6.15) and (6.14), we get,

$$E_+(x,\xi) = i^{n-1}[2\sin((1+\Theta(\epsilon))\mu'') + (-1)^n\xi(\cos(\xi) + \cos(x))] + \Theta(\xi^2 + |\mu''|^2),$$

and possibly after increasing $\epsilon$, this relation remains valid in the complex strip, $|\text{Im}(x,\xi)| \leq 1/\epsilon$.

For $|\mu''| \gg \delta$, the first term in (6.18) dominates, so $E_+$ is bijective in this region. When $|\mu''| \leq (\text{Const.})\delta$, we have,

$$E_- = i^{n-1}[2\sin((1+\Theta(\epsilon))\mu'') + (-1)^n(\cos(\xi) + \cos(x))] + \Theta(\delta),$$

for $|\text{Im}(x,\xi)| \leq 1/\epsilon$. Introducing $\pm(2/\delta)\sin((1+\Theta(\epsilon))\mu'')$ as a new spectral parameter, we see that $E_-/(i^{n-1}\delta)$ satisfies the first part of the condition (3.2) for type 1 operators with a new parameter $\epsilon$, that can be chosen arbitrarily small. As in section 2 we see that $(R^*P_1E_+)\star E_-\star$ and $E_-\star E_\star R^*P_1\star$ are self-adjoint. Choosing $\Theta = i$ for $Q$ it is easy to check by the same argument as above, that the self-adjointing operators $R^*P_1E_\star$ and $i^{n-1}E_-\star Q\star R^*$ after multiplication by a common real factor satisfy the second part of (3.2). Moreover, it is easy to see that all the invariance properties for a type 1 system are satisfied, so $E_+$ can again be studied by applying the results of section 3 and 4. More precisely, we have, Proposition 6.1. There exist $h, \epsilon_0, C_0 > 0$ such that the following holds: Let $(P, P_1, P_2)$ be a strong type 1 operator with $c(P) \leq \epsilon_0$, and $0 < h_0 \leq h$. We can then introduce the new $\mu'$-variable as in Proposition 4.4, and for $C > C_0$, we have $\mu' - \text{Sp}(P) \cap (\mu'; C \leq \mu' \leq 1/C) \subset \bigcup_{1 \leq j \leq n} J_j$, where $J_j$ are closed disjoint intervals, labelled in increasing order such that the width of $J_j$ is of the order of magnitude $h\pi \mu'/h(\log(1/\mu'))^{-1}$ (where $\mu' \in J_j$), and the separation between two consecutive intervals is of the order of magnitude $h \log(1/\mu')^{-1}$ (where $\mu'$ is in the separation). Moreover, if $x_j: \mu' \rightarrow \mu''$ is a suitable increasing affine map, then $x_j(\mu' - \text{Sp}(P)) = \mu'' - \text{Sp}(Q)$, where $Q = Q_j$ is a strong type 1 $h'$-pseudodifferential operator with $c(Q) \rightarrow 0$ as $h, 1/C \rightarrow 0$, uniformly with respect to $j$ and with respect to the choice of $P$ as above, with $\epsilon(P) \leq \epsilon_0$. The analogous result holds in the region $-1/C \leq \mu' \leq -C$. Here $2\pi/h = h'/2\pi \mod(\mathbb{Z})$, $0 < h' \leq 2\pi$.

It also follows from the discussion in this section, that if $Q$ is a strong type 2 $h'$-pseudodifferential operator, $0 < h' \leq 2\pi$, (and writing $\mu'$ instead of $\mu$), then if $C(Q) \geq C_0$, where $C_0$ is sufficiently large, and if $\epsilon(Q) \leq \alpha(C(Q))$, for some strictly positive decreasing function, $\alpha$ on $[C_0, \infty]$, then $\mu' - \text{Sp}(Q) = \bigcup J_j$, where $J_j$ are closed disjoint intervals such that for each $j$, there is an affine map $x_j: \mu' \rightarrow \mu''$, such that $x_j(J_j \cap \mu' - \text{Sp}(Q)) = \mu'' - \text{Sp}(Q)$, where $Q = Q_j$ is a strong type 1 operator with $c(Q) \leq \beta(C(Q))$. Here $\beta(C) \rightarrow 0$ as $C \rightarrow +\infty$.

Combining this with the propositions 3.4, 4.1 and 5.2, we get,
Theorem 6.2. There exist $\varepsilon_0 > 0$ and functions, $F:[0,1] \to [1,\infty[$, $h_0:[0,1] \to [0,1]$, $\alpha: [0,1]^2 \to [0,1]$ with $\alpha(\varepsilon,h) \to 0$ when $h \to 0$ for every fixed $\varepsilon$, such that if $\varepsilon \in [0,1]$, and $P$ is of strong type 1 with $\varepsilon(P) \leq \varepsilon_0$, $0 < h \leq h_0(\varepsilon)$, then

$$\mu - \text{Sp}(P) \subset \bigcup_{N \leq j \leq N_+} J_j,$$

where $J_j$ are closed disjoint intervals, labelled in increasing order, such that for each $j$, there is an affine increasing map $K_j: \mu \to \mu''$ such that:

a) For $j \neq 0$, $K_j(J_j \cap \mu - \text{Sp}(P)) = \mu'' - \text{Sp}(Q)$, where $Q = Q_j$ is a strong type 1 $h'$-pseudodifferential operator with $\varepsilon(Q) \leq \beta(C)$.

b) For $j = 0$, $K_0(J_0 \cap \mu - \text{Sp}(P)) = \mu'' - \text{Sp}(Q)$, where for every $\mu'' \in K_0(J_0)$, $Q$ is a strong type 2 $h'$-pseudodifferential operator with $\varepsilon(Q) \leq \alpha(\varepsilon,h)$, $C(Q) \leq F(\varepsilon)$, as a function of $\mu'' = \mu'' - \mu'' \in \mathbb{R}^1$.  

This is the main result, concerning type 1 operators. We also notice that in terms of the variables $\mu'$ introduced in Proposition 4.4, the length of the interval $J_0$ is of the order of magnitude $h$ (when $C$ is fixed) and that the lengths and separations of the other $J_j$'s in the domain of the $\mu'$-variables are given by Proposition 5.2. Finally outside the domain of the $\mu'$-variables, the lengths are $e^{-C_j/h}$ with $C_j$ of the order of magnitude 1 and the separations are of the order of magnitude $h$. 


7. Type 2 operators close to the totally degenerate case.

Here and in the next two sections we shall consider a general h-pseudodifferential triple, \((P,P_1,P_2)\) of strong type 2, and the corresponding (original) spectral parameter will be denoted by \(\mu\). Let \(P_0(a,b;x,\xi)\) be as in Definition 5.1. Using that \(|a|^2+|b|^2=1, |\arg(b)-\arg(a)|=\pi/2\) we get for real values of \(\mu\),

\[
(7.1) \quad \det(P_0(a,b;x,\xi)) = 2i\alpha(\alpha^{-1}\sin(2\arg(b))+\cos(\xi)+\cos(x)),
\]

where \(\alpha=\frac{a\overline{b}}{i}\) is real and of absolute value \(\leq 1/2\).

In this section, we are interested in the case when \(P\) may come close to the zero matrix for some real \((x,\xi)\). As we saw in section 5, \(P_0(a,b;x,\xi)\) may vanish completely, only if \(|a|=|b|\). In the case, \(\arg(b)-\arg(a)=\pi/2\), we saw in section 5 that \(P_0(a,b;x,\xi)\) vanishes completely at \((x,\xi)=(0,0)\), when \(\arg(b)=3\pi/4+n\pi\), and at \((\pi,\pi)\), when \(\arg(b)=\pi/4+n\pi\). These are the only points of complete degeneration in that case. When \(|a|=|b|=2^{-1/2}\), \(\arg(b)-\arg(a)=-\pi/2\), the same discussion shows that \(P_0(a,b;x,\xi)\) vanishes completely at \((x,\xi)=(0,0)\), when \(\arg(b)=\pi/4+n\pi\), and at \((\pi,\pi)\), when \(\arg(b)=3\pi/4+n\pi\), and that these are the only points where \(P_0\) vanishes.

The eight cases can be treated the same way and in order to fix the ideas, let us assume that

\[
(7.2) \quad \arg(b(\mu))-\arg(a(\mu))=\pi/2,
\]

and we wish to study \(P\) in a region where \(|b|\approx 2^{-1/2}, \arg(b)\approx 3\pi/4\). More precisely, we shall assume that,

\[
(7.3) \quad \arg(b(0))=3\pi/4, |b(0)|-2^{-1/2}\leq \varepsilon,
\]

and the discussion below will be uniformly valid for \(\mu\) in some small fixed neighborhood of 0, provided that \(\varepsilon(P), \varepsilon, \varepsilon>0\) are sufficiently small. Without any loss of generality, we may assume that,

\[
(7.4) \quad \varepsilon(P)\leq \varepsilon.
\]

Combining (7.3), (7.1) and Definition 5.1, we get,

\[
(7.5) \quad \det P_0(x,\xi)=i(1+\Theta(\varepsilon))[\cos(\xi)+\cos x-2+4\mu^2+\Theta(\varepsilon)+\Theta(\mu^4)],
\]

for \(|(x,\xi)|, |\mu|,l \leq 1/C_0\), where \(C_0\) is large but independent of \(\varepsilon\). When \(\mu\) is real, we also know that the expression inside the bracket is real. When \(\mu^2/\varepsilon\) is larger than some constant, we see that \((x,\xi)\rightarrow \det P_0(x,\xi)\) vanishes on a Jordan curve around \((0,0)\) which is close to a circle of radius \(\delta |\mu|\).

It is easy to check that \(\det P_j\neq 0\), for \(|\mu|,l \leq 1/C_0, |(x,\xi)| \leq 1/C_0\). Consider then the self-adjoint operator, \(H_1=P_1P\). On the symbol level,

\[
(7.6) \quad H_1(x,\xi)=P_1(x,\xi)P(x,\xi)+\Theta(h).
\]

\(H_1(x,\xi)\) is a selfadjoint matrix for \(\mu\) and \((x,\xi)\) real. Combining (7.5), (7.6) and the fact that \(\det P-\det P_0=\Theta(\varepsilon)\), (which follows from (7.4),) we get,

\[
(7.7) \quad \det H_1(x,\xi)=f(x,\xi,\mu)[\cos(\xi)+\cos x-2+4\mu^2+\Theta(\varepsilon)+\Theta(\mu^4)+\Theta(h)],
\]

where \(f\) is a non-vanishing real-valued analytic function. Assuming from now on that \(0<\varepsilon\leq 1/C_0\), we see that when \(\mu\) is real and \(\mu^2/(\varepsilon+h)\) is large, then
det$H_l$ vanishes on a Jordan curve around $(0,0)$, close to a circle of radius $\frac{1}{C_0}|\mu|$. According to our results on normal forms for systems in appendix c, there exist classical analytic symbols $Z(x,\xi,\mu,h), C_1(\mu,h), C_2(\mu,h)$ of order 0, defined for $|\mu|\leq 1/C_0$, $|l(x,\xi)|\leq 1/C_0$ (and varying in a bounded set of analytic symbols, if $P,P_j$ depend on additional parameters, but remaining within the bounds that we have specified for $P$, and $P_j$), such that in the sense of classical analytic pseudodifferential operators,

\begin{equation}
    U^*H_lU = [C_1]_{1,1} + [C_2]_{2,2} + [Z]_{1,2} + [Z^*]_{2,1}.
\end{equation}

Here $U$ is an elliptic classical analytic Fourier integral operator of order 0 as in Proposition c.3, depending analytically $\mu$ in the sense that it can be realized with a phase $\varphi(x,y,\theta,\mu)$ which is analytic in all variables, and with an amplitude, $a(x,y,\theta,\mu,h)$, which is a classical analytic symbol. The principal part $\xi(x,\xi,\mu)$, (of $Z$) satisfies $|\xi|\sim l(x,\xi)$ and $i^{-1}(\xi,\xi)>0$, and $C_j$ are real-valued. Again, if $\mu^2/(e+h)$, is large, the determinant of the symbol of (7.8) vanishes roughly on a circle of radius $|\mu|$ around the origin, and we conclude that in the same region,

\begin{equation}
    C_1(\mu,h)C_2(\mu,h)\approx \mu^2.
\end{equation}

Using that $P(0,0)=O(e)$, when $\mu=0$, we also get,

\begin{equation}
    C_j(0,0)=O(e+h).
\end{equation}

Let $c_j(\mu)$ denote the principal part of $C_j$. Considering the Taylor expansions $c_j(\mu)=c_j(0)+c_j'(0)\mu+O(\mu^2)$, we get from (7.9), that,

\begin{equation}
    c_j'(0)c_2'(0)\geq \text{const}>0.
\end{equation}

We may assume that $c_j'(0)\geq \text{const}>0$, and we are then in the situation when $C_j$ are both strictly increasing functions of $\mu$ changing sign somewhere in the interval $[-C(e+h),C(e+h)]$.

In order to study the operator (7.8), we first consider the operator $Z^*Z$. According to our results on normal forms for scalar pseudodifferential operators in appendix b, we can find a real valued analytic symbol $F=F(\mu,t,h)$ of order 0, and a unitary Fourier integral operator $W$, such that,

\begin{equation}
    W^{-1}F(\mu,Z^*Z,h)W = P_0 = \frac{1}{2}(hD)^2 + x^2 - h).
\end{equation}

Using this, we can define the $k$:th eigenvalue of $Z^*Z$ (microlocally) by,

\begin{equation}
    F(\mu,\lambda_k,h) = kh,
\end{equation}

provided that $kh\leq 1/C_0$, (where $C_0>0$ is some new fixed constant). Since $Z$ has a one-dimensional kernel (microlocally), it is clear that $\lambda_0=0$, and hence,

\begin{equation}
    F(\mu,0,h)=0.
\end{equation}

Recalling that $\partial_tF(\mu,0,h)>0$, we let $G=G(\mu,s,h)$ be the inverse of the map $t\to F$, so that $G$ is also a classical analytic symbol of order 0. Then (7.13), can also be written,

\begin{equation}
    \lambda_k = G(\mu, kh,h).
\end{equation}

A normalized eigenfunction $u_k$ of $Z^*Z$ is obtained (microlocally) as a multiple of
\( \text{Wu}^k \), where \( u^k \) is the \( k \)-th normalized eigenfunction of the harmonic oscillator, \( P_0 \):

\[
Z^* u_k = \lambda_k u_k.
\]

Then,

\[
Z Z^* (Z u_k) = \lambda_k (Z u_k),
\]

and for \( k \geq 1 \), we have \( \| Z u_k \| \leq (h/\text{Const.})^{\frac{1}{2}} \), so we conclude that \( \lambda_1, \lambda_2, \ldots \) are also among the eigenvalues of \( Z Z^* \). Conversely, let \( Z Z^* v_k = \mu_k v_k \), microlocally, near \((0,0)\). Then \((Z^* Z) Z^* v_k = \mu_k Z^* v_k\), and since \( \| Z^* v \| \leq (h/\text{Const.})^{\frac{1}{2}} \| v \| \), we conclude that \( \mu_k \) is an eigenvalue of \( Z^* Z \). The conclusion of this discussion is that the low eigenvalues of \( Z Z^* \) (defined modulo \( \Theta(e^{-1}h) \) by reduction of \( Z Z^* \) to \( P_0 \)) are the values \( \lambda_1, \lambda_2, \ldots, \lambda_k, \ldots \), \( \text{kh} \leq 1/\text{Const.} \). From here to (7.38), our arguments are slightly heuristic. They could easily be made rigorous (to the price of a few more pages ...), by introducing microlocal parametrix operators, starting by introducing \( \pi_k u = (u | u_k) u_k \) and \( E_k \) (self adjoint), such that \( I = \pi_k + E_k(Z^* Z - \lambda_k) \). Let \( J(Z Z^*, \lambda_k) \), \( k \geq 1 \), and \( J(Z^* Z, \lambda_k) \), \( k \geq 0 \), denote the corresponding 1-dimensional eigenspaces. If \( v_k \) denotes a normalized element of \( J(Z Z^*, \lambda_k) \), then,

\[
Z u_k = \alpha_k v_k, \quad Z^* v_k = \beta_k u_k, \quad \alpha_k \beta_k = \lambda_k,
\]

for \( k \geq 1 \). Since \( \alpha_k = (Z u_k | v_k) = (u_k | Z^* v_k) = \beta_k \), we may assume after changing \( v_k \), that,

\[
\alpha_k = \beta_k = \lambda_k^{1/2}.
\]

Suppose now that for some value of \( \mu \), the kernel of the operator (7.8) is microlocally nonempty. In other words, there exist a normalized vector \( (f,g) \), such that microlocally,

\[
C_1 f + Z g = 0, \quad Z^* f + C_2 g = 0.
\]

Here \( C_1 \) and \( C_2 \) are scalars, so if we apply \( Z^* \) to the first equation, and use the second one, we get, \( Z^* Z g = C_1 C_2 g \), and similarly, \( ZZ^* f = C_1 C_2 f \). This gives the necessary condition,

\[
C_1 C_2 = \lambda_k
\]

for some \( k \geq 0 \), and when this condition is verified with \( k \geq 1 \), we must have \( f = x_k v_k, g = y_k u_k \) for some \( x_k, y_k \in \mathcal{C} \). The problem (7.20) is then equivalent to the system,

\[
C_1 x_k + \alpha_k y_k = 0, \quad \beta_k x_k + C_2 y_k = 0,
\]

which has a 1-dimensional space of solutions, since \( \alpha_k \beta_k = C_1 C_2 \) in this case. When \( k = 0 \), we have \( C_1 C_2 = 0 \), and we notice in this case that we must have \( C_2 = 0 \). In fact, otherwise \( C_2 \neq 0 \), \( C_1 = 0 \), and (7.20) becomes,

\[
Z g = 0, \quad Z^* f + C_2 g = 0.
\]

Applying \( Z \) to the second equation and using the first, we get \( ZZ^* f = 0 \), and
since \(Z Z^*\) is bijective, we get \(f = 0\). Then since \(C_2 \neq 0\), we get \(g = 0\) from the second part of (7.23).

On the other hand, if \(C_2 = 0\), (7.20) becomes

\[
(7.24) \quad C_1 f + Z g = 0, \quad Z^* f = 0,
\]

and the one-dimensional solution space is generated by \(f = 0, g = u_0\).

It follows from the earlier discussion, and in particular from (7.11), that the function \(\mu \to C_1 C_2\) has a strictly positive second derivative, and that the minimum is \(\leq 0\). Using also the fact that \(\partial_{\mu}^{j} \lambda_k = O(\lambda_k)\) uniformly for every fixed \(j\), we see that the values of \(\mu\) such that \(C_1 C_2 = \lambda_k\) for some \(k \geq 1\), or \(C_2 = 0\), are isolated and the gap between two consecutive values is of an order of magnitude varying between \(\hbar\) and \(\hbar^2\), with the exception of one gap (neighboring the value where \(C_2 = 0\)) whose order of magnitude is at least of the order of \(\hbar^2\). At these values the microlocal kernel of \(H_1\) is 1-dimensional.

From now on we shall work near one of these values, \(\mu_0\), but our arguments will be uniform, with respect to all such possible choices. For \(\mu\) close to \(\mu_0\), \(H_1\) has a simple isolated (microlocally defined) eigenvalue \(E = E(\mu)\). We want to study \(\partial_{\mu} E(\mu_0)\). The arguments in the section on reduction to normal forms of systems still work, if we add one more parameter \(E \in \mathbb{R}\), and give,

\[
(7.25) \quad U^*(H_1 - E)U = [C_1]_{1,1} + [C_2]_{2,2} + [Z]_{1,2} + [Z^*]_{2,1},
\]

where now \(Z, C_j\) are analytic symbols in \(x, \xi, \mu, E\) and \(U\) depends analytically on \(\mu, E\) in the same way as explained after (7.8). Differentiating (7.25) with respect to \(E\), and putting \((x, \xi) = (0, 0)\), we get,

\[
(7.26) \quad \partial_{E} C_j < 0,
\]

and more precisely, that \(\partial_{E} C_j\) are negative elliptic symbols of order 0.

When \(k = 0\), \(E(\mu) = E_0(\mu)\) is determined by \(C_2(\mu, E_0(\mu), h) = 0\), and in view of (7.26) and the fact that \(\partial_{\mu} C_2 > 0\), we see that,

\[
(7.27) \quad \partial_{\mu} E_0(\mu, h) > 0,
\]

and more precisely, that \(\partial_{\mu} E_0\) is a positive elliptic analytic symbol of order 0.

When \(k \geq 1\), \(E = E_k(\mu, h)\) is determined by,

\[
(7.28) \quad C_1(\mu, E, h)C_2(\mu, E, h) = \lambda_k(\mu, E, h).
\]

(Actually, there are two such solutions.) It is easy to see that \(\partial_{\mu} \lambda_k, \partial_{E} \lambda_k = O(\lambda_k)\), and if we differentiate (7.28), we get,

\[
C_1 \partial_{\mu} C_2 + C_2 \partial_{\mu} C_1 + (C_1 \partial_{E} C_2 + C_2 \partial_{E} C_1) \partial_{\mu} E_k = O(\lambda_k).
\]

Here \(C_1\) and \(C_2\) have the same sign, and we may assume for instance that they are both positive. Then \(C_1 \partial_{\mu} C_2 + C_2 \partial_{\mu} C_1\) and \((-C_1 \partial_{E} C_2 + C_2 \partial_{E} C_1)\) are both positive and of the same order of magnitude as \(C_1 + C_2 \geq \lambda_k^{1/2}/\text{Const.}\). It then follows that,

\[
(7.29) \quad \partial_{\mu} E_k = -(C_1 \partial_{\mu} C_2 + C_2 \partial_{\mu} C_1 + O(\lambda_k))/(C_1 \partial_{E} C_2 + C_2 \partial_{E} C_1) \sim 1.
\]
In order to control also the domain of definition of $E_k$ as a holomorphic function of $\mu$, we add some arguments. Let $f(\mu, E) = \lambda_1(\mu, E) - \lambda_2(\mu, E)$, which is a bounded holomorphic function, defined in a 2-disc of fixed radius centered at $(\mu_0, 0)$, where $\mu_0 = \mu_0(0)$ is one of the two real solutions of $\lambda_1(\mu, 0) - \lambda_2(\mu, 0) = 0$. The computation above is then valid for $\mu = \mu_0$, $E = 0$, and shows that $a = \frac{\partial}{\partial E} f(\mu_0, 0) \sim \frac{\partial}{\partial E} f(\mu_0, 0)$ is at least of the same order of magnitude as $\lambda_k^{1/2}$. Moreover $|\partial E| \leq \text{const}$. Then for $|\mu - \mu_0| \leq \delta_1 h^{1/2}$, $|E| \leq \delta_2 h^{1/2}$, where $\delta_1/\delta_2$, and $\delta_2$ are sufficiently small, we have $|\partial E f(\mu, E) - \partial E f(\mu_0, 0)| < \alpha$, $|f(\mu, 0)| \leq (\text{Const.}) \delta_1 h^{1/2}$, and it follows that there is a unique solution $E = E_k(\mu)$ with $|E_k| \leq \delta_2 h^{1/2}$ of the equation $f(\mu, E) = 0$.

Thus in addition to (7.29) we know that $E_k$ is a well defined holomorphic function of $\mu$ in $|\mu - \mu_0| \leq h^{1/2}/\text{Const.}$ such that $|E_k| \leq (\text{Const.}) h^{1/2}$.

If we let $\psi_0$ be a normalized element (defined microlocally near $(0, 0)$) of the kernel of $H_1$, then),(7.30) \[ \partial_{\mu} E(\mu, 0) = (\partial_{\mu} H_1(\mu) \psi_0) \psi_0. \]

Next, we recall some invariance properties. From (2.7), (2.9), we get

(7.31) \[ H_1 \psi \psi = \psi \psi H_2, \quad H_2 \psi \psi = \psi \psi H_1. \]

Using (2.8), (2.10), we also get,

(7.32) \[ H_1^2 \psi = \psi H_1^2, \quad H_2^{\psi \psi} = \psi H_2^{\psi \psi}. \]

If $\psi_0 = \psi_0^* \psi_0$, then,

(7.33) \[ H_2 \psi_0^* = 0, \quad \text{microlocally near } (0, 0), \]

and $\psi_0^*$ is also normalized. Here, we recall that $T' = T$, when the $+$ sign is valid in the general identity $\psi_0 = \pm (T')^2 \psi_0$, and that $T' = iT$ in the $-$ sign case. Since $P_1 \psi_0 = 0$, $P_1^* \psi_0 = 0$, we have $P_1^* P_1^* \psi_0 = P_1^* P_1 \psi_0 = 0$, so

(7.34) \[ P_1 \psi_0 = \alpha \psi_0^*, \]

for some non-vanishing $\alpha \in \mathbb{C}$. Similarly,

(7.35) \[ P_2^* \psi_0 = \beta \psi_0^*, \]

for some non-vanishing $\beta \in \mathbb{C}$. Combining this with (7.30), we get for $\mu = \mu_0$ (with $\partial = \partial_{\mu}$): $\partial E = (\partial_{\psi} P_1 \psi_0 | \psi_0) = (\partial_{\psi} P_1 \psi_0 | \psi_0)$ = $0 + (\partial_{\psi} P_1 \psi_0 | P_1 \psi_0) = \overline{\partial}(\partial_{\psi} P_1 \psi_0 | \psi_0^*)$. Then using also (7.29), we conclude that,

(7.36) \[ |\partial_{\mu} (P_0 \psi_0 | \psi_0^*)| \sim 1, \quad \text{for } \mu = \mu_0. \]

We now have to pass from microlocal results near $(0, 0)$ to global results. Still with $\mu = \mu_0$, let us define $U_\beta$, $\beta \in \mathbb{C}$ as the real component close to $2\pi \beta$ of $\text{det} H_1(x, \xi) = 0$. In order to define a suitable distance, let us recall that the "leading parts" of $P, P_j$ are $P_0(a, b; x, \xi)$, and $P_{0, j} = P_0(ia, ib; x, \xi)$ respectively, and that with $\overline{b} = i\alpha$, $\alpha > 0$, we have,

$\text{det} P_0 = 2i(\sin(2\text{arg}(b)) + \alpha \cos \xi + \alpha \cos x)$.
Since we are far from the "branching case" when $\sin(2\arg(b)) = 0$, we see that $\det P_j \neq 0$, near the complex characteristics of $P$. As in section 3, we then define $D(x, y) = |\Phi(x) - \Phi(y)|$, where $\Phi'(x) = 0$ on the projections of the potential wells, and $\Phi'(x) = \Im \zeta(x)$, between these projections, where $\zeta(x)$ is the complex root with positive imaginary part of the equation $\det(H^{(x, \zeta)}) = 0$, depending continuously on $x$, and such that $(x, \zeta(x))$ tends to a point in $U_j$, when $x$ decreases towards the projection of $U_j$. Here $U_j$ is the closest projected well to the left of $x$. In this definition, we have privileged $H_1$ over $H_2$, but as in section 3, we use the fact that the roots of $\det(H_1)$ are close to those of $\det(H_2)$, and that $H_1$ and $H_2$ are intertwined by $\mathcal{F} T$, so defining $D$ from $H_2$ would give no essential difference. Notice that $D(x, 0)$ is an even function of $x$, since the characteristics of $H_1$ are invariant under reflection in $(0, 0)$.

Because of the presence of possible real characteristics of $H_1$ far away from those of $P$, we shall avoid the use of spectral projections of $H_1$. The treatment will therefore be slightly different from that in section 3. Let $\mathcal{P}_0$, $\mathcal{P}_0^\ast$ denote realizations of $\mathcal{P}_0$, $\mathcal{P}_0^\ast$, obtained by superpositions of Gaussians. (To be more precise, let $S: L^2(\mathbb{R}) \to H(\mathbb{C}; e^{-2\Phi_0/h})$ be a globally defined unitary FBI–transform with a quadratic phase (see appendix a and [S1]). Then $S\mathcal{P}_0$ and $S\mathcal{P}_0^\ast$ are well defined (modulo exponentially small contributions) as elements of $H_{10c}(e^{-2\Phi_0/h})$ in a neighborhood of $\pi_x^* \times_S (U_0, 0)$, and exponentially small outside that set. If $\chi \in C_0^\infty(\mathbb{C})$ has its support close to $\pi_x^* \times_S (U_0, 0)$, and is equal to 1 near that set, then we can take as realizations, $S^* (\chi S \mathcal{P}_0)$ and $S^* (\chi S \mathcal{P}_0^\ast)$.) Then we may arrange so that,

$$f(x) \geq v_0 D(0, x) / D(0, 2\pi) \quad \text{and} \quad |f'(x)| \leq |\partial_x D(x, 0)|.$$ (To get this near $\pi_x^* (U_0, 0)$ we repeat the arguments of the proof of Proposition 5.1 in [HS1].) Here $0 < v_0 D(0, 2\pi)$. Recall that in section 3 we had a similar function $f$ with the crucial property that $f(x) = D(0, x)$, in a neighborhood of $[-\pi, \pi]$.

Since $\mathcal{P}_0 = \pm (\mathcal{F} T)^2 \mathcal{P}_0$ microlocally, we may also arrange that the realization $\mathcal{P}_0$ has the same property in the exact sense. This implies that $\mathcal{P}_0^\ast = \pm (\mathcal{F} T)^2 \mathcal{P}_0^\ast$. We can also arrange so that $\mathcal{F} T^2 \mathcal{P}_0 = \mathcal{P}_0$, which implies that $\mathcal{F} T \mathcal{P}_0^\ast = \mathcal{P}_0^\ast$. Put $\mathcal{P}_\alpha = T_\alpha \mathcal{P}_0$, $\mathcal{P}_\alpha^\ast = T_\alpha \mathcal{P}_0^\ast$, $R_\pm u(x) = (u | \varphi_\alpha)$, $R_\pm u = \Sigma u \varphi_\alpha$. Then as in section 3, we see that the corresponding operator $\mathcal{P}$:

$L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bijective for $|\mu - \mu_0| \leq h/C_0$, with an inverse $\mathcal{E} = \mathcal{O}(h^{-1})$.

Let as usual $E_-$ denote the lower diagonal element of the square matrix of $\mathcal{E}$,
and let $E_{-\pm}(\alpha, \beta)$ denote the corresponding matrix. From reflection and translation invariance, we then see that $E_{-\pm}(\alpha, \alpha) = z(\mu)$ is independent of $\alpha$, and real valued when $\mu$ is real. Using (7.36), we obtain as in section 3, that,

$$|\partial_{\mu} z(\mu)| \sim 1, \text{ for } \mu = \mu_0,$$

and it also follows from the construction, that $z(\mu_0)$ is exponentially small.

After an exponentially small real correction of $\mu_0$, we may then assume that $z(\mu_0) = 0$. Cauchy's inequalities imply that $z''(\mu) = O(h^{-3})$, so for $|\mu - \mu_0| \ll h^3$, we have $|\partial_{\mu} z(\mu) - \partial_{\mu} z(\mu_0)| \ll 1$, and it follows that $\mu_0$ is the only zero of $z(\mu)$ in this region, which is the only interesting one since it follows from the results above that $0$ may belong to the spectrum of $P_0$ only if $\mu$ is exponentially close to (one of the values) $\mu_0$. Repeating the argument of section 3, we see that $\mathcal{E} = O(h^{-1})$ as an operator in $L^2(\mathbb{R}; e^{2F(x)/h}) \times L^2(e^{2F^*(x)/h})$, where

$$\begin{align*}
(7.39) & \quad |F(x)| \leq (|\Phi(x)| - \delta)_+, \\
(7.40) & \quad |F(x) - F(2\pi k)| \leq (1 - \delta)(x - 2\pi k), k \in \mathbb{Z},
\end{align*}$$

where $\delta > 0$ may be chosen arbitrarily small, and where $f$ was introduced in (7.37). Moreover $F^*(\alpha) = F(\pi(x) U_\alpha))$. Using the Grushin problem based on $\mathcal{P}_0, \mathcal{P}_0^*$, we shall now improve the functions $\mathcal{P}_0, \mathcal{P}_0^*$. Let $\mu = \mu_0$. As a new function $\mathcal{P}_0$, we take $\mathcal{P}_0^b = aE_{-\pm}(\beta, 0)\mathcal{P}_0$, where $a$ is a normalization constant, exponentially close to 1. Then $\mathcal{P}_0 = \mathcal{P}_0^b$ microlocally near $U_0, 0$, and using the boundedness of $\mathcal{E}$ in weighted spaces plus the fact that,

$$\begin{align*}
(7.41) & \quad \mathcal{P} \mathcal{P}_0^b = -aE_{-\pm}(\beta, 0)\mathcal{P}_0,
\end{align*}$$

we get:

$$\begin{align*}
(7.42) & \quad \mathcal{P} \mathcal{P}_0^b = \mathcal{P}(e^{-f_1/h}),
\end{align*}$$

where $f_1$ is even, and given by

$$\begin{align*}
(7.43) & \quad f_1(x) = \min(D(0, x), v_0 + f(x), \min_{k \in \mathbb{Z}}(0) |k| v_0 + f(x - 2\pi k)).
\end{align*}$$

Here, we also use the fact that $E_{-\pm}(\alpha, \beta) = \mathcal{E}(e^{-v_0 |\alpha - \beta| \infty/h})$, which is a consequence of the boundedness in weighted spaces.

From $\mathcal{P}_0^b$ we generate $\mathcal{P}_0^1, \mathcal{P}_0^1, \ldots$, as before, and we then consider the new Grushin problem with $R_+, R_-$ defined, using these new functions. Let $\mathcal{E}, E_{-\pm}$ correspond to this new problem. After an exponentially small correction of $\mu_0$, it is then clear that $\mathcal{E}, E_{-\pm}$ will have the same properties as before, but with $f$ replaced by $f_1$. Iterating this procedure, we obtain a sequence of functions $\mathcal{P}_0^b, \mathcal{P}_0^{b1}, \ldots$, with,

$$\begin{align*}
(7.44) & \quad \mathcal{P}_0^k, \mathcal{P} \mathcal{P}_0^k = \mathcal{P}(e^{-f_k/h}),
\end{align*}$$

where,

$$\begin{align*}
(7.45) & \quad f_0(x) = v_0 D(0, x)/D(0, 2\pi), \text{ and if we put } f_{\infty}(x) = \min_{k \in \mathbb{Z}}(11 v_0 + D(2\pi, x))
\end{align*}$$
then it is easy to check that $f^0 \leq f^k \leq f^\infty$. We can rewrite (7.45) as

(7.46) \quad f^{k+1}(x) = \min(f^{\infty}(x), \Lambda(f^k)),$

where $\Lambda(f) = \min(y_0 + f(x), \min_{y_0 \neq 0} |y_0| + f(x - 2\pi i))$. Notice that,

$\Lambda(\min(\alpha, \beta)) = \min(\Lambda(\alpha), \Lambda(\beta))$. If $g^0 = f$, $g^{k+1} = \Lambda(g^k)$, then it is easy to see

that, $g^k(x) = k\nu_0 + \min_{-k} |y_0| \leq k\nu_0 - 2\pi i)$. Now we claim that,

(7.47) \quad f^k = \min(f^{\infty}, g^k).$

In fact, this is true for $k = 0$, and if it is true for some $k$, then $f^{k+1} = \min(\infty, f^{\infty}(x), \Lambda(f^k)) = \min(f^{\infty}, \Lambda(f^{\infty}, \Lambda(g^k)) = \min(f^{\infty}, g^{k+1})$, since

$\Lambda(f^{\infty}) = f^{\infty}$. From the behavior of $g^k$, we see that if $0 < \nu_1 < \nu_0$, and $k$ is

sufficiently large, then $f^k \geq \min_{\nu_1} (\nu_1 + D(2\pi i, x))$. Then, after replacing $\nu_0$, by

some smaller number, and $\psi_0$ by $\psi_k$, for some sufficiently large $k$, we may assume that (7.37) holds with

(7.48) \quad f(x) = \min_{\nu_1 \geq \nu_0} (\nu_1 + D(2\pi i, x)).$

Putting $f_0(x) = \psi_0$, $f_0(x) = \psi_0^k$, we are now completely at the same point as in

section 3 at (3.24), and we can now repeat the arguments of that section

without any essential changes until (3.37). $g_{\alpha, \beta}$ have the same growth

estimates as before, and WKB-representations, valid in the same intervals,

(7.49) \quad g_{0,1} = c_0(h)e^{iy(x)/h}a(x), g_{0,2} = c_1(h)e^{iy(x)/h}b(x)$,

where $\det P(x, y(x)) = \det P(x, y(x)) = 0$, $c_1, c_2^{-1} = o(e^{-\delta/h})$, for every $\delta > 0$.

Notice that since on the symbol level, $P(x, \psi(x)) = P(x, \psi(x)), \psi(x)$, we have $\psi'(x) = \psi'(x)$

for real $x$, and hence we may assume that

(7.50) \quad \text{Im}(\psi) = S_0 = D(0, 2\pi), \text{Re}\psi = \text{Re}\psi.$

As in section 3, $a$ and $b$ are classical analytic symbols of order 0, but now

with values in $C^2$. The leading parts $a_0, b_0$ are non-vanishing, and satisfy

(7.51) \quad P(x, \psi'(x))(a_0) = 0, P(x, \psi'(x))(b_0) = P(x, \psi'(x)) = 0.$

We still have (3.39), where $d$ is an analytic symbol of order 0. If $d_0(x)$ is the

leading part, then $d_0(\pi) = \psi'(\pi) = a_0(\pi) = \psi'(\pi) = a_0(\pi)$, where

$C_0$ is a non-vanishing constant. Instead of (3.41), we have,

(7.52) \quad (\psi_0, g_0, g_0, \psi_0) = c_0(h)e^{-S_0/h},$

where $k$ is an analytic symbol of order 0, with leading part,

(7.53) \quad \partial_\xi P(x, \psi'(\pi))(a_0) = 0.$

Here $b_0(\pi) = c_0(\text{Im}(P(\pi, \psi'(\pi)))$ is the expression (7.53) is non

vanishing. In order to prove this, we have to verify that

$\partial_\xi P(\pi, \psi'(\pi))(a_0) \notin \text{Im}(P(\pi, \psi'(\pi)))$. Working at the point $(\pi, \psi'(\pi))$, we

start from $c_0 \partial P = \det(P)I$, which we differentiate, and get,

$(\psi_0, P + c_0 \partial P) = \psi_0 \partial P \notin \text{Im}(P(\pi, \psi'(\pi)))$. If $\partial P(a_0) = 0$, there is a

vector $c_0$, such that $(\partial P)(a_0) = P(c_0)$. Then, $c_0 \partial P(c_0) = (\partial P(a_0) = 0$, or equivalently, $\det(P)c_0 = \partial P(a_0) = 0$. Here the left hand side vanishes, since
det(P)=0, while the right hand side does not, so we have a contradiction. This proves that $k$ is an elliptic symbol of order 0. Then we obtain (3.43), (3.44), exactly as in section 3, and we can finish the discussion as in that section.

Summing up, we have,

**Proposition 7.1.** There exist $\epsilon_0, h_0, \mu_0 > 0$, such that if $P$ is a strong type 2 operator with $\epsilon(P) \leq \epsilon_0$, $0 < h \leq h_0$, $|\arg(b(0)) - 2\pi| \leq \epsilon_0$, $\arg(b(0)) = \pi/4 + n\pi/2$, for some $n \in \mathbb{Z}$, we have $\mu - \text{Sp}(P) \cap [-\mu_0, \mu_0] \subset \mathbb{N} \setminus \mathbb{N}_+ \cup \mathbb{N}_+ \cdot J_j$, where $J_j$ are closed disjoint intervals of width $e^{-1/C_j h}$, where $C_j \sim 1$, and the separation between two consecutive intervals is at least $h/C$ for some fixed $C > 0$. More precisely, $J_j$ is exponentially close to a point $\tilde{J}_j$, where the points $\tilde{J}_j$ are given by $C_1(\tilde{J}_j)C_2(\tilde{J}_j) = G(\tilde{J}_j, |J_j|, h)$ when $j \neq 0$, and by $C_2(\tilde{J}_0) = 0$ when $j = 0$. Here $C_1$, $C_2$ are analytic functions defined in a fixed neighborhood of 0, (varying in a bounded set of such functions,) satisfying $C'_j \geq \text{Const.} > 0$, and $G(\mu, \lambda, h)$ is a real valued analytic symbol of order 0, satisfying $G(\mu, 0, h) = 0$, $\partial_{\lambda} G > 0$. For each $j$, there is an affine increasing map $\kappa_j : \mu \to \mu'$, such that $\kappa_j(\mu - \text{Sp}(P) \cap J_j) = \mu' - \text{Sp}(Q)$, where $Q = Q_j$ is an $h'$-pseudodifferential operator of strong type 1 with $\epsilon(Q) \to 0$ as $h \to 0$, uniformly with respect to the other parameters.
8. Simply degenerate type 2 operators in the potential well case.

Let \((P, P_1, P_2)\) be a strong type two operator with \(C(P) \leq C_0\) where \(C_0 > 0\) is fixed. Let \([\alpha, \beta] \subset [-3, 3]\) be an interval such that,

(8.1) \[|b(\mu)| - 2^{-\frac{1}{2}} + |\text{arg}(b(\mu)) - (\pi/4 + n\pi/2)| \geq 1/C_0,\]

(8.2) \[|\text{arg}(b(\mu)) - n\pi/2| \geq 1/C_0,\]

for every \(\mu \in [\alpha, \beta]\) and every \(n \in \mathbb{Z}\). Assuming \(\varepsilon(P) \leq \varepsilon_0, 0 < h \leq h_0\), where \(\varepsilon_0 > 0, h_0 > 0\), we shall then study \(\mu - \text{Sp}(P) \cap [\alpha, \beta]\). The assumption (8.1) implies that we are not in the totally degenerate case, studied in the preceding section, and the assumption (8.2) implies that we are not in the "branching case", that will be studied in the next section. In order to fix the ideas, we may assume that \(\sin(2\text{arg}(b(\mu))) < 0\) on \([\alpha, \beta]\). Again \(P_j, j=1,2\) are elliptic near the complex characteristics of \(P\). Moreover, the selfadjoint operator \(H_1\) has the property that if \(\det(H_1(x, \xi)) = 0\) for some real \((x, \xi)\) close to the characteristics of \(P\), then \(\text{Ker}(H_1(x, \xi))\) is one dimensional. The same holds for \(H_2\). The part of the real characteristic set of \(H_1\) (for \(\mu\) real) which is close to the characteristics of \(P\), is then either empty, or of the form \(U_{\alpha} \in \mathbb{Z}^2 U_\alpha\), where \(U_\alpha\) is either equal to \(2\pi\alpha\), or equal to a simple Jordan curve around \(2\pi\alpha\). We here assume that all points in \([\alpha, \beta]\) are close to an interval where \(\det(P_0(a, b; x, \xi))\) may vanish for some real \((x, \xi)\). Indeed, if \(\mu\) is far from such a value then \(P\) is elliptic and \(\mu\) is also far from \(\mu - \text{Sp}(P)\). In the case when \(\det(H_1)\) has no real zeros, we see that \(\det(H_1(0, 0))\) is small and we define \(U_\alpha\) to be \((2\pi\alpha)\).

Microlocally, near \(U_0, 0\), we can apply the results on block-decomposition of systems, developed in [HS2] and in section a of the appendix, to see that there is unitary \(2 \times 2\) system of classical analytic pseudodifferential operators, \(U = U(\mu, x, hD, h)\), such that,

(8.3) \[U^*H_1U = [q]_{1, 1} + [h_1]_{2, 2},\]

where \(q, h_1\) are scalar classical analytic selfadjoint pseudodifferential operators of order 0. Moreover \(q\) is elliptic (near \(U_0, 0\)).

We now restrict \(\mu\) to a neighborhood of a real value \(\mu_0\), with the property that \(U_{0, 0} \neq \emptyset\). In the case when \(U_{0, 0} = (0, 0)\) for \(\mu = \mu_0\), then for that value of \(\mu\), we have \(h_1(0, 0) = \Theta(h), h_1''(0, 0) = \Theta(h), h_1''(0, 0) = -1,\) and \(\partial_\mu h_1 \neq 0\). We may assume for simplicity that \(\partial_\mu h_1 > 0\). If \(U_{0, 0}\) is diffeomorphic to a circle around \((0, 0)\) (for \(\mu = \mu_0\)), then \(h_1\) vanishes on a circle of distance \(\Theta(h)\) from \(U_{0, 0}\), that we shall from now on identify with \(U_{0, 0}\). On this circle \(d_x, \xi h_1 \neq 0\), and \(h_1\) is positive inside the circle. Moreover \(\partial_\mu h_1 \neq 0\) on \(U_{0, 0}\) and in order to fix the ideas, we may assume that \(\partial_\mu h_1 > 0\). (All this follows from the fact that \(h_1\) is close to a non-vanishing factor times \(\det(P_0)\).)
Microlocally, near $U_0,0$, the eigenfunctions of $h_1$, associated to small eigen-values can be obtained by WKB-constructions, and they are given by,

$$F(\mu,E,h) = kh, \ k \in \mathbb{N},$$

where $F$ is an analytic symbol of order 0, with leading part $f(\mu,E)$, given by,

$$f(\mu,E) = \text{Vol}((x,\xi); h_1 \geq E).$$

(A more complete statement would require the introduction of microlocal spectral projections and partial parametrices, as we also mentioned in section 7 after (7.17).)

At this point, we recall, that if $T(E) = T(E,\mu) > 0$ is the primitive period of the $h_1$-flow in $h_1 = E$, then $T$ is analytic in $(E,\mu)$ and,

$$\partial_E \text{Vol}(h_1 \geq E) = -T(E).$$

Hence, in the case when $U_0,0$ is close to $(0,0)$, we see how (8.4) follows from the results in the end of Appendix b. In the case when $U_0,0$ is a circle, the formula (8.4) was obtained in the analytic case in [HS1]. In the $C^\infty$ category, it was earlier proved by Helffer-Robert [HR1,2]. When $U_0,0 = U_0,0(\mu_0)$ is a circle it is clear that $f(\mu,E)$ is analytic near $(\mu_0,0)$.

When $U_0,0(\mu_0)$ is a point, we see that $f(\mu,E)$ is analytic near $(\mu_0,E_0)$ when $-E_0$ is small and positive. Integrating (8.6), and using the analyticity of $T(E,\mu)$, we conclude that $f(\mu,E)$ is analytic near $(\mu_0,0)$ in this case too. In both cases we have,

$$\partial_\mu f > 0, \ \partial_E f < 0, \ \text{for} \ \mu = \mu_0, \ E = 0.$$  

Let us now assume that $\mu_0$ is one of the interesting values, namely that,

$$F(\mu_0,0,h) = kh, \ \text{for some} \ k \in \mathbb{N}.$$  

For $|\mu - \mu_0|/h \leq 1/C$, with $C > 0$ large enough, let $E = E(\mu,h)$ be the solution of,

$$F(\mu,E,h) = kh,$$

where $k$ is the same number as in (8.8). In view of (8.7) this definition makes sense, and $E(\mu,h) = \Theta(h)$ is holomorphic for $|\mu - \mu_0| < h/C$. Moreover,

$$\partial_\mu E > 0.$$  

Since $U$ is unitary, we get the same (microlocal) eigenvalues for $H_1$. Let $\varphi_0,0$ be a normalized function defined microlocally near $U_0,0$, such that $H_1\varphi_0,0 = 0$ for $\mu = \mu_0$. We then have (7.30), and from this point on the discussion of section 7 applies without any essential changes. We then obtain,

**Proposition 8.1.** Let $C_0 > 0$. Then there exist $\varepsilon_0 > 0, \ h_0 > 0$ such that if $P$ is a strong type 2 h-pseudodifferential operator with $C(P) \leq C_0, \ \varepsilon(P) \leq \varepsilon_0, \ 0 < h \leq h_0$ and $[\alpha,\beta] \subset [-3,3]$ is an interval on which (8.1),(8.2) hold, (with the same constant, $C_0$,) then $\mu - \text{Sp}(P) \cap [\alpha,\beta] \subset \bigcup_{1 \leq j \leq N} \mathcal{J}_j$, where $\mathcal{J}_j$ are closed disjoint intervals, labelled in increasing order. The separation between $\mathcal{J}_j$ and $\mathcal{J}_{j+1}$ is of the order of magnitude, $h$ , and the width of $\mathcal{J}_j$ is $e^{-1/C_jh}$, where $C_j \sim 1$.

Moreover, for each $j$, there is an increasing affine map $K_j: \mu \rightarrow \mu'$, such that
\chi_{j}(\mu - \text{Sp}(P) \cap J_j) = \mu' - \text{Sp}(Q), \text{ where } Q \text{ is a strong type 1 } h' - \text{pseudodifferential operator (depending on } j) , \text{ such that } \varepsilon(Q) \rightarrow 0, \text{ when } h \rightarrow 0, \text{ uniformly with respect to the other parameters.}
9. The branching problem for type 2 operators.

Let \((P, P_1, P_2)\) be a strong type 2 operator such that,
\[(9.1) \quad \sin(2\arg(b(0))) = 0 ; \quad \arg(b(0)) = n\pi/2, \ n \in \mathbb{Z}.
\]
If \(C(P) \leq C_1\), for some arbitrary but fixed constant \(C_1\), then our study will be uniformly valid for \(|\varepsilon| \leq \varepsilon_0, \ e(P) \leq \varepsilon_0, \ 0 < h \leq h_0\), where \(\varepsilon_0, \varepsilon_0, h_0\) are strictly positive and depend on \(C_1\).

We start by some general remarks on the reduction of certain \(2 \times 2\) systems to scalar ones. Let \(H = H(x, hD)\) be a selfadjoint \(2 \times 2\) system of analytic pseudodifferential operators defined near \((x_0, \xi_0) \in T^*\mathbb{R}\). We assume that \(H(x_0, \xi_0)\) is of rank 1, and to start with, we shall work microlocally near \((x_0, \xi_0)\). Let us repeat some arguments developed for the block decomposition of systems of pseudodifferential operators. See [HS2]. Let \(\gamma\) be a small circle around 0 with positive orientation and put,
\[(9.2) \quad \Pi = (2\pi i)^{-1} \int_{\gamma} (z - H)^{-1} dz.
\]
Then \(\Pi\) is an analytic pseudodifferential operator whose Weyl symbol satisfies,
\[(9.3) \quad \Pi(x_0, \xi_0) + \Theta(h^2) = \text{the orthogonal projection onto the kernel of } H(x_0, \xi_0).
\]
Moreover,
\[(9.4) \quad \Pi^* = \Pi, \quad \Pi^2 = \Pi, \quad H\Pi = \Pi H.
\]
Let \(R\) be a \(2 \times 1\) system of pseudodifferential operators of order 0, such that \(\Pi R\) is elliptic. Then \(R^* \Pi R\) is a positive elliptic scalar pseudodifferential operator of order 0, and we put,
\[(9.5) \quad S = R^* \Pi R (R^* \Pi R)^{-\frac{1}{2}}.
\]
Then,
\[(9.6) \quad S^* S = I, \quad \Pi S = S.
\]
Similarly, we can construct a \(2 \times 1\) system, \(S'\), such that
\[(9.7) \quad S'^* S' = I, \quad (1 - \Pi) S' = S'.
\]
Since \(S'^* S = 0, \ S^* S' = 0\), we see that the \(2 \times 2\) system \(\mathcal{A} = (S, S')\) is isometric, and hence elliptic and unitary. Hence \(\mathcal{A}^* = I\),
\[(9.8) \quad S S^* + S' S'^* = I.
\]
Applying \(\Pi\) to this relation, we get, \(\Pi S S^* = \Pi\). Since \((1 - \Pi) S S^* = 0\), we conclude that,
\[(9.9) \quad SS^* = \Pi.
\]
Similarly,
\[(9.10) \quad S' S'^* = I - \Pi.
\]
Put \(\mathcal{H} = S^* HS\). Then,
\[(9.11) \quad S \mathcal{H} = HS, \quad \mathcal{H} S^* = S^* H.
\]
Here the two relations are equivalent and the first one follows from a straightforward computation: \(S \mathcal{H} = S S^* HS = \Pi HS = \Pi \Pi S = S\).

If \(u\) is a scalar function satisfying \(\mathcal{H}u = 0\) (near \((x_0, \xi_0)\)), then with \(v = Su\), we get \(Hv = S \mathcal{H}u = 0\). Conversely, if \(v\) is a \(C^2\) valued function such that
Hv=0, then Πν=ν, SS*ν=ν, and u=S*v satisfies ˜Hu=0. Hence the problems Hv=0 and ˜Hu=0 are equivalent (microlocally).

We now return to our type 2 operator P,P_1,P_2. We know that P is elliptic outside a small neighborhood of Us(α,j), and after replacing P_j by (Reθ)P+(Imθ)P_j for a suitable θ, we can assume that P_j are elliptic in that region. Define H_1, H_2 as usual. We shall make a reduction of H_j to scalar operators in a neighborhood of Us(α,j). In this region H_j(x,ξ) is of rank 1 whenever det(H_j(x,ξ)) vanishes. The invariance properties of P,P_j imply that,

(9.12) H_j\mathcal{F}T=\mathcal{F}TH_2, H_2\mathcal{F}T=\mathcal{F}TH_1,
(9.13) H_1V\mathcal{T}^2=V\mathcal{T}2H_1, H_2V=VH_2,
(9.14) T_\gamma H_j=H_j T_\gamma, j=1,2, \gamma \in \mathbb{Z}^2.

Notice that (9.12) implies,

(9.15) [H_j,(\mathcal{F}T)^2]=0.

We now concentrate on H_1 for a while. We shall construct S_1 in a neighborhood of Us(α,j) adapted to H_1 as above, such that,

(9.16) (\mathcal{F}T)^2S_1=\iota S_1\mathcal{F}2,
(9.17) V\mathcal{T}^2S_1=\iota S_1V,
(9.18) T_\gamma S_1=S_1 T_\gamma, \gamma \in \mathbb{Z}^2,

where \iota=±1, will be determined below.

Suppose, that we have found a 2×1 system R of order 0, such that Π_1R is elliptic near Us(α,j), where Π_1 is the projection associated to H_1 as in (9.2), and such that,

(9.19) (\mathcal{F}T)^2R=\iota R\mathcal{F}2,
(9.20) V\mathcal{T}^2R=\iota RV,
(9.21) T_\gamma R=RT_\gamma.

Taking the adjoints of these relations, and using the fact that (9.13)-(9.15) carry over to Π_1, we see that R*Π_1R and consequently (R*Π_1R)^{-\frac{1}{2}} commute with V,\mathcal{F}2,T_\gamma, and hence,

(9.22) S_1=Π_1R(R*Π_1R)^{-\frac{1}{2}},

has all the desired properties.

The problem is then to construct R satisfying (9.19)-(9.21). Identifying the Weyl symbol R(x,ξ) with R(x,ξ)(i)∈\mathbb{C}^2, we see that the problem is to construct an analytic vector R(x,ξ) such that Π_1(x,ξ)(R(x,ξ))≠0, and such that,

(9.23) R=\iota T^2R*\tau c_0,
(9.24) R=\iota T^2\Gamma(R*\delta),
(9.25) R*\tau T^2\pi \gamma =R,

where c_0(x,ξ)=(-x,-ξ), \delta(x,ξ)=(ξ,x), \tau_2π\gamma(x,ξ)=(x,ξ)-2π\gamma, and \Gamma denotes complex conjugation in \mathbb{C}^2. Recall that \delta describes the reflection in
where $c_0(x,\xi) = (-x, -\xi)$, $\delta(x,\xi) = (\xi, x)$, $\tau_{2\pi\gamma/\gamma}(x,\xi) = (x,\xi) - 2\pi\gamma$, and $\Gamma$ denotes complex conjugation in $\mathbb{C}^2$. Recall that $\delta$ describes the reflection in the line $x = \xi$, and that $\tilde{\delta} = \delta \circ c_0 = c_0 \circ \delta$ is given by $\tilde{\delta}(x,\xi) = (-\xi, -x)$ and describes reflection in the line $x = -\xi$. $(9.23), (9.24)$ imply, $(9.26) \quad R = \Gamma(R \circ \tilde{\delta})$,
and more generally, any two of $(9.23), (9.24), (9.26)$ imply the third.

The first problem will be to construct a continuous vector, $R(x,\xi)$ satisfying $(9.23)-(9.26)$. Let $\lambda(x,\xi)$ be the small eigenvalue of $H_1(x,\xi)$. The ellipticity requirement will then be fulfilled if we can find $R(x,\xi)$ non-vanishing in the kernel of $H_1(x,\xi) - \lambda(x,\xi)I$, satisfying $(9.23)-(9.26)$.

On the level of Weyl symbols $(9.13)-(9.15)$ give,
$(9.27) \quad H_1 = T^2 \Gamma(H_1 \circ \delta) \Gamma T^2$,
$(9.28) \quad H_1 = T^2 (H_1 \circ c_0) T^2$,
$(9.29) \quad H_1 \circ \tau_{2\pi\gamma} = H_1$.
We conclude that $\Gamma T^2$ maps $\text{Ker}(H_1 - \lambda_1)(x,\xi)$ onto $\text{Ker}(H_1 - \lambda)(\tilde{\delta}(x,\xi))$, and that $T^2$ maps $\text{Ker}(H_1 - \lambda)(x,\xi)$ onto $\text{Ker}(H_1 - \lambda)(c_0(x,\xi))$. From $(9.27), (9.28)$, we also get,
$(9.30) \quad H_1 = \Gamma(H_1 \circ \tilde{\delta}) \Gamma$,
from which we conclude that $\Gamma$ maps $\text{Ker}(H_1 - \lambda)(x,\xi)$ onto $\text{Ker}(H_1 - \lambda)(\tilde{\delta}(x,\xi))$. Let us shorten the notation by writing $\text{Ker}(x,\xi)$ instead of $\text{Ker}(H_1 - \lambda)(x,\xi)$. By periodicity, $\text{Ker}(0,\pi) = \text{Ker}(0, -\pi)$, so $T^2$ maps $\text{Ker}(0,\pi)$ onto itself. The eigenvalues of $T^2$ are 1 and $-1$, so the restriction of $T^2$ to $\text{Ker}(0,\pi)$ is ±1 and we define $\tau$ to be that number. We can compute $\tau$ by putting $\mu' = 0$ and noticing that $\text{Ker}(0,\pi)$ is then close to $\text{Ker}(P_0(0,\pi))$. Hence $\tau$ is also given by the restriction of $T^2$ to this space. When $\mu = 0$, we have, $b = |b| \iota^n$, $a = |a| \iota^n \overline{\iota}$. An easy computation shows that $\text{Ker}(P_0(0,\pi))$ is generated by $\iota(1 - 1)$ when $n$ is even, and by $\iota(1 1)$, when $n$ is odd. Hence $\tau = 1$ when $n$ is odd and equal to $-1$, when $n$ is even.

From the commutative diagram,

\[
\begin{array}{ccc}
\text{Ker}(0,\pi) & \xrightarrow{\Gamma} & \text{Ker}(-\pi,0) \\
\Gamma \downarrow & & \Gamma T^2 \downarrow \\
\text{Ker}(\pi,0) & \xrightarrow{\Gamma} & \text{Ker}(0,-\pi),
\end{array}
\]

where all maps commute with $T^2$, we conclude that $T^2 = \tau$ also on the kernel of $H_1 - \lambda$ at the points $(\pi,0), (0,-\pi), (-\pi,0)$.
We next notice that $\Gamma T^2$ maps $\ker(\pi/2, \pi/2)$ onto itself. This map is antilinear and conserves the $C^2$ norm, so there is a non-vanishing element $u_0$ of $\ker(\pi/2, \pi/2)$ such that $\Gamma T^2 u_0 = u_0$. Put $R(\pi/2, \pi/2) = u_0$. We extend the definition of $R(x, \xi)$ by continuity to the closed segment $s(0,1)$, in such way that (9.24) holds. Then there is a unique way of extending the definition to $s(0,3)$ so that (9.23) holds, and it is also easy to check that (9.24) extends to $s(0,3)$. We then also have (9.26) on $s(0,1)\cup s(0,3)$. Now let $v_0 \in \ker(-\pi/2, \pi/2)$ be a non-vanishing element such that $\Gamma v_0 = v_0$. We put $R(-\pi/2, \pi/2) = v_0$ and extend the definition by continuity to $s(0,2)$ in such a way that (9.26) holds on $s(0,2)$ and so that $R(x, \xi)$ takes the already prescribed values at the endpoints of this segment. After that we extend to $s(0,4)$ by using (9.23). Then (9.23) holds on $Us(0,j)$, and we get (9.26) on $s(0,4)$ and hence on $Us(0,j)$. We have then obtained (9.23), (9.24), (9.26) on $Us(0,j)$, and by construction, we have $R(0,\pi) = R(0,-\pi)$, $R(\pi,0) = R(-\pi,0)$. We can then extend the definition of $R$ to $Us(\alpha, j)$ in such a way that (9.25) holds. If $(x, \xi) = (y, \eta) + 2\pi \alpha$, $(y, \eta) \in Us(0,j)$, then $c_0(x, \xi) = c_0(y, \eta)$ and $S(x, \xi) = S(y, \eta)$ modulo $2\pi \mathbb{Z}^2$, so we have (9.23)–(9.26) for the extension.

Let $R$ be an analytic function defined in a neighborhood of $Us(\alpha, j)$ such that $\widetilde{R} - R$ is small on $Us(\alpha, j)$ and such that (9.25) holds. Then $\widetilde{R}$ is close to $\widetilde{R} = \frac{1}{2}((R + \Gamma T^2(\widetilde{R} \cdot c_0)))$, which satisfies (9.23) and (9.25). If $R' = \frac{1}{2}((R + \Gamma T^2(\Gamma \cdot d)))$, then (9.23), (9.25) remain valid, and we also obtain (9.24). $R'$ can be chosen arbitrarily close to $R$ (on $Us(\alpha, j)$) so $\Pi_1 R'$ is elliptic near $Us(\alpha, j)$. Writing $R$ instead of $R'$ we have then the required properties, and as we have seen, this gives an operator $S_1$ satisfying (9.16)–(9.18) and

(9.31) $S_1 S_1^* = \Pi_1$, $S_1^* S_1 = I$.

Put $\widetilde{H}_1 = S_1^* H_1 S_1$, so that,

(9.32) $S_1 \widetilde{H}_1 = H_1 S_1$, $\widetilde{H}_1 S_1^* = S_1^* \widetilde{H}_1$.

Combining this with (9.16)–(9.18) and (9.13)–(9.15), we get,

(9.33) $[\mathcal{F}^2, \widetilde{H}_1] = 0$,

(9.34) $[V, \widetilde{H}_1] = 0$,

(9.35) $[T, \widetilde{H}_1] = 0$.

In view of (9.12), we next define the isometry $S_2$ by,

(9.36) $\mathcal{F} T S_1 = S_2 \mathcal{F}$.

Then $\mathcal{F} T S_2 = (\mathcal{F} T)^2 S_1^* \mathcal{F}^{-1} = I S_1^* \mathcal{F}$:

(9.37) $\mathcal{F} T S_2 = I S_1^* \mathcal{F}$.

If $\Pi_2$ is the projection associated to $H_2$ as in (9.2), we still have (9.12) after replacing $H_1, H_2$ by $\Pi_1, \Pi_2$. Then $S_2 S_2^* = (\mathcal{F} T S_1 \mathcal{F}^{-1})(\mathcal{F} T S_1 \mathcal{F}^{-1})^* = \mathcal{F} T S_1 \mathcal{F}^{-1} \mathcal{F} S_1^* (\mathcal{F} T)^{-1} = \mathcal{F} T \Pi_1 (\mathcal{F} T)^{-1} = \Pi_2$,

(9.38) $S_2 S_2^* = \Pi_2$, $S_2^* S_2 = I$. 


Using (9.16)-(9.18), we get,
(9.39) \((yT)^2S_2 = tS_2 y^2\),
(9.40) \(VS_2 = S_2 V\),
(9.41) \(T_T S_2 = S_2 T_T\).

Put \(\tilde{H}_2 = S_2^* H_2 S_2\), so that,
(9.42) \(S_2 \tilde{H}_2 = H_2 S_2\), \(S_2^* \tilde{H}_2 = \tilde{H}_2 S_2^*\).

In conclusion, \(\tilde{H}_1, \tilde{H}_2\) have the same invariance properties as the operators \(H_1, H_2\) of section 4.

We next recall that,
(9.43) \(\det(P) = 2|a||b|((\sin(2\arg(b)))/(|a||b|) \pm (\cos(\xi) + \cos(x)))\),
and that \(\det(P) = \det(P_0) + O(\epsilon)\). It is clear that \(\tilde{H}_j\) are modulo \(O(h)\) equal to elliptic factors times \(\det(P)\), so if we consider \(\tilde{\xi}(\sin(2\arg(b)))/(|a||b|)\) as the new spectral parameter, \(\mu\), then,
(9.44) \(\tilde{H}_j = a_j(\xi, \xi)(\cos(\xi) + \cos(x) + O(\epsilon) + O(h) - \mu)\),
where \(a_j\) are elliptic and real valued of order 0. It is no restriction to assume that \(a_j > 0\). Repeating the argument of section 4, we see that the symbol \(\tilde{H}_1\) is invariant under reflection in the point \((0, \pi)\), and that \((0, \pi), (\pi, 0), (0, -\pi), (-\pi, 0)\) are non-degenerate saddle points for \(\tilde{H}_1\) with the same critical value \(d(\tilde{H}_1)\). We have,
(9.45) \(\partial_\mu d(\mu) < 0\),
and \(d(\mu)\) has a simple zero for some \(\mu = O(\epsilon) + O(h)\). After another change of the spectral parameter, we may assume that \(d(\mu) = -\mu\). For \(\mu = 0\), we also know that the real characteristics of \(\tilde{H}_1\) close to the \((0, \pi)\) and \((\pi, 0)\) are contained in an arbitrarily small neighborhood of \(Us(\alpha_i, j)\), when \(\epsilon\) and \(h\) are sufficiently small.

Let \(X_{\alpha_i, j}\) be as in section 4, satisfying (4.26), (4.27), (4.34). Let \(u_{0,1}\) be WKB solution of \(H_1 u = 0\), defined microlocally near \(Int(s(0,1))\), and extended to neighborhoods of \((0, \pi)\) and \((\pi, 0)\), as in section 4. The choice of extension is obvious, if we recall that,
(9.46) \(u_{0,1} \equiv S_1 \tilde{u}_{0,1}\),
where,
(9.47) \(\tilde{H}_1 \tilde{u}_{0,1} = 0\).

We shall assume that \(\tilde{u}_{0,1}\) is a normalized solution of (9.47). An easy computation then shows that,
\((i[\tilde{H}_1, X_{0,1}]u_{0,1}, u_{0,1}) = (i[\tilde{H}_1, S_1^* X_{0,1} S_1] \tilde{u}_{0,1}, \tilde{u}_{0,1}) = 1\),
since \(\tilde{X}_{0,1} = S_1^* X_{0,1} S_1\) is equal to 1 near \((0, \pi)\) and equal to 0 near \((\pi, 0)\).
Hence \(u_{0,1}\) is a normalized solution of \(H_1 u = 0\). Notice that this notion has the same invariance properties as in the scalar case.
Imitating section 4, we define \( f_{\alpha,j}, u_{\alpha,j} \) by, 
\[
(9.48) \quad f_{0,j} = (\mathbf{c} T)^{l-1} f_{0,1}, \quad u_{0,j} = (\mathbf{c} T)^{l-1} u_{0,1},
\]
(9.49) \( f_{\alpha,j} = \mathbf{T} \alpha f_{0,j}, \quad u_{\alpha,j} = \mathbf{T} \alpha u_{0,j} \).

Again, \( u_{\alpha,j} \) is a microlocal solution of \( H u = 0 \) if \( j \) is odd and of \( N u = 0 \), if \( j \) is even. Repeating the arguments in section 4, (using the fact that we can always reduce ourselves to the scalar case near the characteristics of \( P \)) we see that the corresponding Grushin problem for \( P \) is well posed. In order to be completely in the general framework of section 2, we need (2.39) rather than (4.32). We obtain this by using that \( \chi, \delta = 1 - \chi_{0,1} \) near \( s(0,1) \) as in section 4: Since \( VT^2 \) commutes with \( H_1 \), we obtain:
\[
VT^2 f_{0,1} = -i VT^2 [H_1, \chi_{0,1}] u_{0,1} = i [H_1, \chi_{0,1}] VT^2 u_{0,1}.
\]
Here \( VT^2 u_{0,1} \) is also a normalized microlocal solution of \( H u = 0 \). The space of these solutions being one dimensional, we may assume after multiplying \( u_{0,1} \) by a complex scalar, that \( VT^2 u_{0,1} = u_{0,1} \). Then \( VT^2 f_{0,1} = f_{0,1} \) microlocally, and as in section 4, we can turn this into an exact relation.

Introducing again the WKB problem, (4.36), we obtain Proposition 4.1, as well as (4.43),(4.44). Here \( \lambda \) has the same properties as in section 4, so again the problem is to find \( \lambda \alpha, \lambda \beta \) satisfying (4.43). We recall here that \( \chi = \alpha, \delta = \beta \). Again \( v_{0,1} \) is a normalized microlocal solution of \( P^* v = 0 \), and \( v_{\alpha,j} \) are generated from \( v_{0,1} \) the same way as \( u_{\alpha,j} \) were generated from \( u_{0,1} \). Put \( v_{0,1} = S_2 \tilde{v}_{0,1} \), where \( \tilde{v}_{0,1} \) is a normalized solution of \( \tilde{H}_2 v = 0 \). Then it is easy to check that,
\[
(9.50) \quad u_{\alpha,1} = S_1 \tilde{u}_{\alpha,1}, \quad u_{\alpha,2} = S_1 \tilde{u}_{\alpha,2}, \quad u_{\alpha,3} = S_1 \tilde{u}_{\alpha,3}, \quad u_{\alpha,4} = S_2 \tilde{u}_{\alpha,4},
\]
(9.51) \[ v_{\alpha,1} = S_2 \tilde{v}_{0,1}, \quad v_{\alpha,2} = S_1 \tilde{v}_{\alpha,2}, \quad v_{\alpha,3} = S_2 \tilde{v}_{\alpha,3}, \quad v_{\alpha,4} = S_1 \tilde{v}_{\alpha,4}. \]

Substituting this into the problem (4.43), we get the equivalent problem,
\[
(9.52) \quad \tilde{u}_{0,1} = \lambda \beta \tilde{v}_{0,2} \text{ near } \text{Int}(s(0,2)),
\]
\[
\tilde{v}_{0,1} = S_1 \lambda \alpha \tilde{v}(0,1), \quad 4 \text{ near } \text{Int}(s((0,1),4)),
\]
\[
\tilde{u}_{(0,1),3} = S_1 \lambda \alpha \tilde{v}_{0,2} \text{ near } \text{Int}(s(0,2)),
\]
\[
\tilde{v}_{(0,1),3} = S_1 \lambda \beta \tilde{v}(0,1), \quad 4 \text{ near } \text{Int}(s((0,1),4)).
\]

It is then clear that the computation of \( \lambda \beta, \lambda \alpha \) in section 4 applies without any change to \( \lambda \beta, \lambda \alpha \), and we see that Proposition 4.2 remains valid, with only one modification: We can no longer state that the + sign is valid in (4.71). The study of the new selfadjointing operators goes through almost without any change, we just have to replace \( V \) by \( VT^2 \) at certain places, and we see that Proposition 4.3 remains valid. So does the discussion after that proposition, with the obvious changes in (4.95),(4.99) due to the fact that now \( \arg(b) - \arg(a) = \pm \pi/2 \).

Summing up, we have,

**Proposition 9.1.** There exist functions, \( \tilde{\varepsilon}_0 : [1, \infty[ \to [0,1], \quad \tilde{\mu}_0 : [1, \infty[ \to [0,1], \quad \tilde{F} : [0,1] \to [1, \infty[, \quad \tilde{\mu}_0 : [0,1] \times [1, \infty[ \to [0,1], \quad \tilde{\alpha} : [0,1] \times [1, \infty[ \times [0,1] \to [0,1] \), with
$\tilde{\alpha}(\varepsilon, C, h) \to 0$, when $h \to 0$ for every fixed $(\varepsilon, C)$, such that if $0 < \varepsilon \leq 1$, $C \geq 1$, and $P$ is a strong type 2 $h$-pseudodifferential operator with $C(P) \leq C$, $\varepsilon(P) \leq \tilde{\varepsilon}_0(C)$, then $0 < h \leq \tilde{\varepsilon}_0(\varepsilon, C)$, $\arg(b(0)) = \pi/2$ for some $n \in \mathbb{Z}$, then $\mu - \text{Sp}(P) \cap [-\tilde{\mu}_0(C), \tilde{\mu}_0(C)] \subseteq \bigcup_{-N \leq j \leq N^+} J_j$, where $J_j$ are closed disjoint intervals, labelled in increasing order, such that for each $j$, there is an affine increasing map $K_j: \mu \to \mu''$ such that one of the following holds:

a) $K_j(J_j \cap \mu - \text{Sp}(P)) = \mu'' - \text{Sp}(Q)$, where $Q = Q_j$ is a strong type 1 $h'$-pseudodifferential operator with $\varepsilon(Q) \leq \varepsilon$.

b) $K_j(J_j \cap \mu - \text{Sp}(P)) = \mu'' - \text{Sp}(Q)$, where for every $\mu_0'' \in K_j(J_j)$, $Q$ is of strong type 2 with $\varepsilon(Q) \leq \tilde{\alpha}(\varepsilon, C, h)$, $C(Q) \leq \tilde{C}_0(\varepsilon(C))$, as a function of $\mu''' = \mu'' - \mu_0'' \in [-4, 4]$.

Combining this result with the propositions 7.1 and 8.1, we get the following result, which is the analogue of Theorem 6.2 for type 2 operators, Theorem 9.2. There exist functions, $\tilde{\varepsilon}_0: [1, \infty[ \to ]0, 1[$, $F: [0, 1[ \to ]1, \infty[$, $\tilde{\varepsilon}_0: [0, 1[ \times ]0, 1[ \to ]0, 1[$, $\tilde{\alpha}: [0, 1[ \times ]0, 1[ \to ]0, 1[$, with $\tilde{\alpha}(\varepsilon, C, h) \to 0$, when $h \to 0$ for every fixed $(\varepsilon, C)$, such that if $0 < \varepsilon \leq 1$, $C \geq 1$, and $P$ is a strong type 2 $h$-pseudodifferential operator with $C(P) \leq C$, $\varepsilon(P) \leq \tilde{\varepsilon}_0(C)$, $0 < h \leq \tilde{\varepsilon}_0(\varepsilon, C)$, then $\mu - \text{Sp}(P) \cap [-3, 3] \subseteq \bigcup_{-N \leq j \leq N^+} J_j$, where $J_j$ are closed disjoint intervals, labelled in increasing order, such that for each $j$, there is an affine increasing map $K_j: \mu \to \mu''$ such that one of the following holds:

a) $K_j(J_j \cap \mu - \text{Sp}(P)) = \mu'' - \text{Sp}(Q)$, where $Q = Q_j$ is a strong type 1 $h'$-pseudodifferential operator with $\varepsilon(Q) \leq \varepsilon$.

b) $K_j(J_j \cap \mu - \text{Sp}(P)) = \mu'' - \text{Sp}(Q)$, where for every $\mu_0'' \in K_j(J_j)$, $Q$ is of strong type 2 with $\varepsilon(Q) \leq \tilde{\alpha}(\varepsilon, C, h)$, $C(Q) \leq \tilde{C}_0(\varepsilon(C))$, as a function of $\mu''' = \mu'' - \mu_0'' \in [-4, 4]$.

The discussion in section 7-9 also gave information about the sizes of, and separations between the intervals: Let $P$ be as in the theorem with $\varepsilon$ and $C$ fixed. Then the intervals $J_j$ may be subdivided into groups, $\bigcup_{k} J_k \subseteq \bigcup_{I_k}$, where $I_k$ are closed disjoint intervals of width $\sim 1$ and at most $\pi + \mathcal{O}(\varepsilon(P)) + \mathcal{O}(h)$. The separation between two neighboring intervals is at least $\frac{\pi}{2}/(\text{const.})$ and at most $\pi/(1 + \text{const.})$, where "const." indicate two strictly positive constants, which only depend on $C$. In each $I_k$ there is at most one $J_j = J_j(k)$, such that $b$ of the theorem applies, and $J_j(k)$ is of width $\sim h$ and situated at a distance $\mathcal{O}(\varepsilon(P)) + \mathcal{O}(h)$ from the middle point of $I_k$. This interval exists if the middle of $I_k$ is in $[-3, 3]$ and at a distance $\gg \varepsilon(P) + h$ from $\pm 3$. As for the other intervals in $I_k$, their sizes and separations are as described after Theorem 6.2, with only one modification: If $I_k = [a', a]$, and $I_{k+1} = [b', b]$ are two consecutive intervals such that $b - a$ is very small (but larger than $h e/\text{const.}$), and $a, b$
belong to some intervals $J_j$ and $J_{j+1}$ (which we always can assume after shrinking $I_k$ and $I_{k+1}$), then if $\mu > b$ is close to $b$ and in the separation of two consecutive $J$-intervals, the size of that separation is of the order of magnitude, $h/((b-a) + (\mu - b))$. The analogous statement holds for the separation between two consecutive intervals slightly to the left of $a$; just replace $\mu - b$ by $a - \mu$.

The theorems 6.2 and 9.2 may be combined into an infinite iteration scheme. Without loss of generality, we may assume that the function $F$ is the same in both theorems. Let $\varepsilon_0 > 0$ be as in Theorem 6.2 and let $0 < h_1 \leq \min(h_0(\varepsilon_0), h_0(\varepsilon_0, F(\varepsilon_0)))$ be sufficiently small, so that if $0 < h \leq h_1$, then $\max(\alpha(\varepsilon_0, h), \tilde{\alpha}(\varepsilon_0, F(\varepsilon_0), h)) \leq \tilde{\varepsilon}_0(F(\varepsilon_0))$. Then we obtain:

**Theorem 9.3.** Let $0 < h \leq h_1$ and define $h' \in [0, 2\pi]$ by $2\pi/h \equiv h'/2\pi \mod(\mathbb{Z})$.

Then,

(A) Let $P$ be a strong type 1 $h$-operator with $\varepsilon(P) \leq \varepsilon_0$. Then after restriction to suitable subintervals and after suitable affine maps, the study of $\mu - \text{Sp}(P)$ can be reduced to the study of $\mu'' - \text{Sp}(Q) \cap [-3, 3]$, where $Q$ is either a strong type 1 $h'$-operator with $\varepsilon(Q) \leq \varepsilon_0$, or a strong type 2 $h'$-operator with $\varepsilon(Q) \leq \tilde{\varepsilon}_0(F(\varepsilon_0), C(Q) \leq F(\varepsilon_0))$.

(B) Let $P$ be a strong type 2 $h$-operator with $\varepsilon(P) \leq \tilde{\varepsilon}_0(F(\varepsilon_0))$, $C(P) \leq F(\varepsilon_0)$. Then for $\mu - \text{Sp}(P) \cap [-3, 3]$, we have the same conclusion as in (A).

This theorem, together with the more precise quantitative information, that we added after the theorems 6.2 and 9.2, give Theorem 0.1 in the introduction, in the case when all the $q_j$ in (0.3) are positive. The following discussion shows how to extend this result to the case when $q_j$ have arbitrary signs.

Let us first extend the definition of the auxiliary operators to the case case when $h < 0$. The definition of $T_\alpha = T_\alpha, h$ is as before, and we check that:

$\text{det} T_\alpha, -h = T_\chi'(\alpha), h$,

where we recall that $\chi'(\alpha) = (\alpha_1, -\alpha_2)$. The extention of $\mathcal{F}_h$ to negative $h$ is given by:

$\mathcal{F}_h u(\xi) = (2\pi |h|)^{-\frac{1}{2}} \int e^{-i\xi x/h} u(x) \, dx$.

Then,

$\mathcal{F}_h, -h = \mathcal{F}_h^{-1}$,

so if we put $R = h^2 b_x^2 + x^2 - |h|$, then also for negative $h$, we have,

$\mathcal{F}_h = e^{-i\pi R/2h} = U_{-\pi/2, h}$.

Here, we write $U_{t,h} e^{itR/h}$. As before, we put:

$V_h = \Gamma \mathcal{F}_h \mathcal{F}_h^{-1} \Gamma = U_{\pi/4, h} U_{-\pi/4, h}$,

and we notice that,
(9.58) \[ v_h = v_h \mathcal{F}_h^2 = \mathcal{F}_h^2 v_h, \]
where \( \mathcal{F}_h^2 \) is independent of \( h \). If \( P(x,\xi) \) is a symbol, we still define the \( h \)-Weyl quantization of \( P \) as the standard Weyl quantization of \( P(x,\hbar \xi) \). By a change of variables:
\[
(9.59) \quad P(x,\hbar D) u(x) = (2\pi |\hbar|)^{-1} \int J_{\xi}(x-y) \theta(\frac{1}{2}(x+y),\theta) u(y) dy d\theta.
\]
The commutation relations of section 1 then extend to the case, \( h<0 \). An \( h \)-pseudodifferential operator can also be viewed as an \( (-\hbar) \)-pseudodifferential operator, and the relation between the corresponding symbols is simply: \( \tilde{P} = P \cdot e^\xi \). The rules for computing the symbols of of \( \Gamma_{\hbar \xi} \), \( \mathcal{F}_h^{-1} \mathcal{P} \mathcal{F} \) are the same for negative \( h \) (provided that the conjugating operators are quantized with the same \( h \)).

The definition of (strong) type 1 and 2 operators now extends naturally to the case when \( h<0 \), and for type 1 operators, we have:

**Proposition 9.4.** If \( (P,P_1,P_2) \) is of (strong) type 1 in the sense of \( h \)-pseudodifferential operators, then the same is true in the sense of \( (-\hbar) \)-pseudodifferential operators.

**Proof.** Let us first check that the invariance properties are satisfied also in the \( (-\hbar) \)-quantization: Using (9.53), it is obvious that \( P \) and \( P_j \) commute with \( T_{\alpha}, -h \). Let \( \mathcal{F} \), \( V \) denote the \( h \)-quantizations and \( \mathcal{F}_-, V_- \) the \( (-\hbar) \)-quantizations. Then combining (2.2),(2.5) with (9.55), we see that (2.2),(2.5) hold with \( \mathcal{F} \) replaced by \( \mathcal{F}_- \). Since \( P, P_j \) commute with \( V \) (by (2.3),(2.6)) and with \( \mathcal{F}^2 \), they also commute with \( V \mathcal{F}^2 = V_- \). Hence, if \( (P,P_1,P_2) \) is of type 1 as \( h \)-operators, the same is true in the sense of \( (-\hbar) \)-operators. Since \( \cos(\xi) + \cos(x) \) is even in \( \xi \), it is also clear that the notion of strong type 1 operators is independent of the sign of \( h \).

For type 2 operators, the situation is a little more complicated. Recall the definition of the operator \( T \) in section 2, and let \( S \) denote one of the two operators \( S_1: C_0^{\text{odd}} \rightarrow C_0^{\text{even}}, S_2: C_0^{\text{even}} \rightarrow C_0^{\text{odd}} \), both given by the identity matrix, for the natural bases. Then \( S \) is real and,
\[
(9.60) \quad TST = S, \quad S^2 = \mathbb{I}, \quad S^* = S^{-1} = S.
\]

**Proposition 9.5.** If \( (P,P_1,P_2) \) is a type 2 \( h \)-operator, then \( (\tilde{P}, \tilde{P}_1, \tilde{P}_2) = (SP^*S, SP_2^*S, SP_1^*S) \) is a type 2 \( (-\hbar) \)-operator.

**Proof.** \( \tilde{P}_1^* \tilde{P} = SP_2^* S P_2^* S P_1^* S = SP_2^* P_1^* S P_1^* S = SP_2^* \mathcal{P} P_1^* S \), which is self adjoint. Similarly, we see that \( \tilde{P}_1 \tilde{P}_2^* \) is self adjoint. It is obvious that \( \tilde{P}, \tilde{P}_1, \tilde{P}_2 \) commute with \( T_{\alpha} \).

Let's check the Fourier invariance: \( \tilde{P}_1 \mathcal{F}_- T = SP_2^* \mathcal{F} \mathcal{F}_-^{-1} T = SP_2^* T^{-1} \mathcal{F} \mathcal{F}_-^{-1} T^{-1} S = SP_2^* (T \mathcal{F}_- T)^{-1} S \mathcal{F}_-^{-1} T P_1 S = \mathcal{F}_-^{-1} T \mathcal{F}_- T^2 P_1 S = \mathcal{F}_- T^2 T^2 S P_2^* S = \mathcal{F}_- T^2 \mathcal{F}_- T^2 S \). The remaining two Fourier invariance relations are proved the same way. The three reflection invariance relations are also proved the same way, so we only treat one case: \( \tilde{P}_1 V_- T^2 = SP_2^* S \mathcal{F}_- T^2 T^2 = \mathcal{F}_- T^2 S P_2^* S = \mathcal{F}_- T^2 V T^2 S P_2^* S = V_- \tilde{P}_1 \).

\[ \square \]
Proposition 9.6. If $(P, P_1, P_2)$ is of strong type 2 with respect to $\mu, h$, then $(\tilde{P}, -\tilde{P}_1, -\tilde{P}_2)$ (defined in the preceding proposition) is of strong type 2 with respect to $-\mu, -h$, and we have $\varepsilon(\tilde{P}) = \varepsilon(P), C(\tilde{P}) = C(P)$.

Proof. If $P(x, \xi)$ denotes the $h$-symbol of $P$, then the $(-h)$-symbol of $SP*S$ is $SP(x, -\xi)*S$. Here we recall that $S$ is given by the identity matrix for the standard bases, so from: $P(x, \xi) = P_0(a, b; x, \xi) + O(\varepsilon)$, we get

$SP(x, -\xi)*S = P_0(a, b; x, -\xi)* + O(\varepsilon) = P_0(\tilde{a}, \tilde{b}; x, \xi) + O(\varepsilon)$. The same computations work for $-\tilde{P}_1, -\tilde{P}_2$, and the proposition follows. □

Propositions 9.4-9.6 permit to extend Theorems 6.2 and 9.2 to the case when $h$ and $h'$ are in $[-2\pi, 0[$. (In the statements, we simply have to replace all upper bounds on $h$ by upper bounds on $|h|$.) Also, if we start with $h > 0$, and $2\pi/h = h'/2\pi \mod(\mathbb{Z})$ with $|h'| < \pi$, $h' \neq 0$, then the arguments of sections 2-6 work without any changes, so Theorems 6.2 and 9.2 are valid also if we suppress the assumption that $h'$ should be positive. This also holds for Theorem 9.3, and we then get Theorem 0.1.
In this section, we treat various generalities that will be needed in the main text. We shall first recall the general approach to (analytic) microlocal analysis, based on FBI (or generalized Bargmann) transforms in [S1]. Then we develop a simple functional calculus for analytic pseudodifferential operators, and finally, we make some remarks on the Weyl quantization of symbols.

### a.1. FBI transforms and microlocal analysis

We shall essentially recall some of the theory of [S1], with some modifications, in order to treat global questions. As a general rule, the large parameter $\lambda$ in [S1] will here be equal to $1/h$, with $0 < h \leq 2\pi$.

Let $\Omega \subseteq \mathbb{C}^n$ and let $\Phi$ be a continuous real valued function, defined on $\Omega$. A function $u(x,h)$ on $\Omega \times [0,2\pi]$ (or possibly with $2\pi$ replaced by some smaller constant, $> 0$) belongs by definition to the space $H_\Phi^{loc}(\Omega)$, if,

(a.1.1) $u$ is holomorphic in $x$ for every $h$,
(a.1.2) For every compact $K \subseteq \Omega$ and every $\varepsilon > 0$, there is a constant $C > 0$, such that $|u(x,h)| \leq Ce^{(\Phi(x) + \varepsilon)/h}$ on $K \times [0,2\pi]$.

We also define $H_\Phi^p(\Omega)$ to be the space of functions $u$ on $\Omega \times [0,2\pi]$ satisfying (a.1.1), such that $u(.,h)$ belongs to $L^2(\Omega, e^{-2\Phi/hL(dx)})$ for every $h$. Here $L(dx)$ denotes the Lebesgue measure on $\Omega$. Notice that the properties of belonging to $H_\Phi^{loc}$ is an asymptotic property, while the property of belonging to $H_\Phi^p(\Omega)$ is of interest for every $h$, (and sometimes we shall use the property $u \in H_\Phi^p(\Omega)$ for individual $h$'s or for a much smaller set of $h$'s (by abuse of terminology)).

We call $\Phi(x,h) = \sum_{k \in \mathbb{N}} a_k(x) h^k$ a (formal) classical analytic symbol (c.a.s.) of order $0$, if the $a_k$ are holomorphic in the same open set, $\Omega$ and if for every compact $K \subseteq \Omega$, there is a constant $C > 0$, such that $|a_k(x)| \leq C^k + 1_k^k$ for every $x \in K$, $k \in \mathbb{N}$. (Sometimes we may also allow $k$ to vary in the half integers.)

A classical analytic symbol of order $m \in \mathbb{R}$ is an expression of the form $h^{-m} a(x,h)$, where $a$ is a c.a.s. of order $0$. If $a$ is a c.a.s. of order $m$, then for every open $\Omega \subset \subset \Omega$, we can define a realization, $\tilde{a} \in H_\Phi^{loc}(\tilde{\Omega})$, well defined modulo $O(e^{-1/Ch})$ for some $C > 0$. (This is done by summing the first (const.)/$h$ terms in the formal sum giving $a$, and choosing the constant suitably. See [S1], page 3, for more details.) In the main text we will also have lots of parameters. A family of c.a.s. of order $0$, $a_\alpha(x,h), \alpha \in A$ on $\Omega$, is by definition bounded if for every compact $K$ in $\Omega$, we can choose a constant $C$ as above, which is independent of $\alpha$. A c.a.s. $a$, of order $0$ is **elliptic**, if the leading term, $a_0$ in its formal series expansion is non-vanishing.

We next define local FBI transforms (as in [S1], chapter 7). Let $\Phi(x,y)$ be a holomorphic function, defined in a neighborhood of $(x_0,y_0) \in \mathbb{C}^n \times \mathbb{R}^n$, such that,

(a.1.3) $\Phi_y'(x_0,y_0) = -\eta_0 \in \mathbb{R}^n$,
(a.1.4) $\text{Im} \Phi_y''(x_0,y_0) > 0$, (in the sense of symmetric matrices),
(a.1.5) $\det(\Phi_x''(x_0,y_0)) \neq 0$. 

**Microlocal analysis.**

In this section, we treat various generalities that will be needed in the main text. We shall first recall the general approach to (analytic) microlocal analysis, based on FBI (or generalized Bargmann) transforms in [S1]. Then we develop a simple functional calculus for analytic pseudodifferential operators, and finally, we make some remarks on the Weyl quantization of symbols.
Let \( t(x,y,h) \) be a realization of an elliptic c.a.s. Let \( Y \subset \mathbb{R}^n \), \( X \subset \mathbb{C}^n \) be small neighborhoods of \( y_0 \) and \( x_0 \) respectively, and let \( \chi \in C_0^\infty(Y) \) be equal to 1 in a neighborhood of \( y_0 \). Assuming these quantities chosen suitably, we can then define for \( u \in \mathcal{D}'(Y) \), \( x \in X \),
\[
(a.1.6) \quad T_u(x,h) = \int e^{i\varphi(x,y)/h} t(x,y,h) \chi(y) u(y) dy,
\]
and if \( u \) is independent of \( h \), we get \( T_u \in \mathcal{E}^0_\text{loc}(X) \), where,
\[
(a.1.7) \quad \Phi(x) = \sup_{y \in Y} (-\text{Im}(\varphi(x,y))).
\]
The same conclusion is valid, if \( u = u_h \) depends on \( h \), but some Sobolev norm of \( u_h \) is of at most temperate growth when \( h \to 0 \) and bounded on every compact \( h \)-interval.

Viewing \( T \) as Fourier integral operator, we can associate to it the canonical transformation \( K_T : (y, -\varphi'(x,y)) \to (x, \varphi'_x(x,y)) \) from a complex neighborhood of \( (y_0, \eta_0) \) onto a complex neighborhood of \( (x_0, \xi_0) \). Here \( \xi_0 = -2i\partial \Phi(x_0)/\partial x \), and more generally, if we introduce
\[
\Delta \Phi = (x, -2i\partial \Phi(x)/\partial x) ; x \in X \), then (locally near \( (x_0, \xi_0) \)), \( \Delta \Phi = K_T(\mathbb{R}^{2n}) \).

Using FBI-transforms, we get a convenient setting for the microlocal theory of Fourier integral operators. We restrict the discussion to germs, but using classical theorems on the resolution of the \( \bar{\partial} \)-operator, it is easy to extend the discussion (by partitions of unity) to the case of pseudo-convex domains equipped with plurisubharmonic weight functions.

Let \( \varphi(x,y,\theta) \) be a non-degenerate phase function defined near \( (x_0, y_0, \theta_0) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \), in the sense that,

1. \( \varphi \) is holomorphic,
2. \( \partial \varphi/\partial \theta_1, \ldots, \partial \varphi/\partial \theta_n \) are independent on the set \( C_\theta \) defined by \( d_\theta \varphi = 0 \).

We assume that \( (x_0, y_0, \theta_0) \in C_\varphi \), and put \( \eta_0 = -\varphi'_y(x_0, y_0, \theta_0) \), \( \xi_0 = \varphi'_x(x_0, y_0, \theta_0) \). Then \( \Delta \varphi = (x, \varphi'_x(x,y,\theta) ; y, \varphi'_y(x,y,\theta) \); \( (x,y,\theta) \in C_\varphi \) \in \( \mathbb{C}^n(x,\xi) \times \mathbb{C}^n(y,\eta) \) is a canonical relation, that is: a Lagrangian manifold for \( \sigma(x,\xi) - \sigma(y,\eta) \), where \( \sigma(x,\xi) = \Sigma d\xi_j \wedge dx_j \) and similarly for \( \sigma(y,\eta) \). Let us assume that \( \Delta \varphi \) is the graph of a canonical transformation, and let \( \Phi(y) \) be a plurisubharmonic \( C^2 \) function defined near \( y_0 \) satisfying, \( \eta_0 = -2i\partial \Phi(y_0)/\partial y \).

Then \( \chi(\Delta \varphi) \) is of real dimension \( 2n \), and if the projection of this manifold to \( \mathbb{C}^n(x) \) is a local diffeomorphism, then it has the form \( \Delta \Psi \), for some \( C^2 \) function \( \Psi \). If we assume that the hessian \( \Psi''(x_0) \) is plurisubharmonic, then \( \Psi \) is plurisubharmonic everywhere, and after changing \( \Psi \) by a constant, we have \( \Psi(x) = v.c.(y,\theta) (-\text{Im} \varphi(x,y,\theta) + \Phi(y)) \), where \( v.c.(y,\theta) \). means "critical value with respect to the variables \( (y,\theta) \) of the function \( \cdot \).", and where it turns out that the critical value is taken at a non-degenerate critical point of signature \( 0 \), that is at a saddle point. Conversely, if we know that \( (y,\theta) \to -\text{Im} \varphi(x_0,y,\theta) + \Phi(y) \) has a saddle point at \( (y_0,\theta_0) \), then the critical value \( \Psi \), defined as above will be plurisubharmonic, and \( \Delta \Psi = \chi(\Delta \varphi) \).
If \( a(x,y,\theta,h) \) is the realization of an c.a.s. in a neighborhood of \( (x_0,y_0,\theta_0) \), then for suitable neighborhoods \( Y,X \) of \( y_0 \) and \( x_0 \) respectively, we can define the Fourier integral operator,

\[
A_u(x,h) = \int e^{i\Phi(x,y,\theta)/h} a(x,y,\theta,h) u(y,h) \, dyd\theta,
\]

as an operator \( A: H^1_{\text{loc}}(Y) \to H^1_{\text{loc}}(X) \), by choosing a suitable integration contour, and introducing suitable cut-off functions. A different choice of contour and cut-off will change \( A_u \) only by a term which is exponentially small in a neighborhood of \( x_0 \). There is a natural composition result for these Fourier integral operators, and it is also possible to show that the class of operators we get by varying essentially only depends on \( \Phi \) and on the normalizing value \( \Phi(y_0,y_0,\theta_0) \). As a particular case, we get the classical analytic pseudodifferential operators, when \( x_0 = y_0, N = n \) and \( \Phi(x,y,\theta) = (x-y)\theta \). The standard quantization of a symbol \( a \) is then,

\[
A_u(x,h) = (2\pi h)^{-n} \int e^{i(x-y)\theta/h} a(x,\theta,h) u(y) \, dyd\theta,
\]

while the Weyl quantization of a symbol \( b \) is

\[
B_u(x,h) = (2\pi h)^{-n} \int e^{i(x-y)\theta/h} b(\frac{1}{2}(x+y),\theta,h) u(y) \, dyd\theta.
\]

The standard symbol, \( a \) and the Weyl symbol \( b \) are uniquely determined from the operators \( A,B \), and we get the same operator, \( A = B \) if and only if

\[
b = e^{-iD_x D_\theta} / 2a, \quad a = e^{iD_x D_\theta} / 2b,
\]

where we also know (by analytic stationary phase) that \( e \pm iD_x D_\theta / h \) are order preserving maps on the space of formal analytic symbols in an open set. We say that the pseudodifferential operator \( A \) is of order \( m \) if the corresponding symbol (either standard or Weyl) is of order \( m \). In the natural composition result for analytic p.s.d.o. we also know that the order of the composition of two operators is equal to the sum of the orders of each factor. The (standard or Weyl) symbol of the composition, \( C = A \circ B \) is also given by the standard composition formulas,

\[
c = \sum_{k=0}^\infty (k!)^{-1} (\partial_x D_y)^k (a(x,\xi)b(y,\eta)) |(y,\eta) = (x,\xi),
\]

\[
c = \sum_{k=0}^\infty (k!)^{-1} (\frac{1}{2}i\sigma(D_x D_\xi; D_y D_\eta))^k (a(x,\xi)b(y,\eta)) |(y,\eta) = (x,\xi),
\]

for respectively the standard symbols and the Weyl symbols.

If \( A \) is a Fourier integral operator, with an elliptic symbol, associated to the canonical transformation \( \kappa \), with \( \kappa(y_0,\eta_0) = (x_0,\xi_0) \), then we can find a Fourier integral operator \( B \) associated to \( \kappa^{-1} \) such that \( B \circ A = I \), \( A \circ B = I \), in the sense that the pseudodifferential operators \( B \circ A \) and \( A \circ B \), defined respectively near \( (y_0,\eta_0) \) and \( (x_0,\xi_0) \) have symbol 1. (This implies that the corresponding realizations on suitable \( H^1_{\text{loc}} \) -spaces, are simply the identity operator modulo exponentially small errors). Applying this to the FBI-transform, \( T \) in (a.1.6), we can find an "inverse" \( S = T^{-1} \), of the form,

\[
S v(y,h) = \int e^{-i\Phi(x,y)/h} s(x,y,h) v(x) \, dx,
\]

and if we denote by \( m(t) \) and \( m(s) \) the orders of \( t \) and \( s \) respectively, then \( m(t) + m(s) = n \). If \( \tilde{T} \) is a second FBI transform with \( \kappa \tilde{T}(y_0,\eta_0) = (x_0,\xi_0) \), and if \( \Phi \) and \( \tilde{\Phi} \) denote the weightfunctions associated to \( T \) and \( \tilde{T} \) respectively, by
(a.1.7) and it's analogue for \( \bar{T} \), then if \( u = u_h \in \mathcal{S}'(Y) \) is a family of distributions of temperate growth, as specified after (a.1.7), and if \( U \) denotes the Fourier integral operator composition \( \bar{T} \circ S \), we have near \( \tilde{x}_0 \), \( \bar{T} u_h(x, h) = U T u_h(x, h) \) modulo an exponentially small error in \( H_\Phi^{loc} \). Naturally, since the behaviour of \( T u_h(x, h) \) near \( x_0 \) only tells us anything about the phase space behaviour of \( u_h \) near \( (y_0, \eta_0) \), there is no hope to be able to recover \( u_h \) from only that information. This information is however sufficient, to predict the behaviour of \( \bar{T} u_h \) near \( \tilde{x}_0 \), and it justifies the terminology: "Let \( u = u_h \) be a distribution defined microlocally near \( (y_0, \eta_0) \); this we mean simply, that each time we choose an FBI transformation \( T \) as above, then up to an exponentially small uncertainty, we have an element \( T u \) of class \( H_\Phi^{loc} \) defined in a neighborhood of \( x_0 \), the various elements being related by \( \bar{T} u = (\bar{T} \circ T^{-1})(T u) \).

If \( a(y, \eta, h) \) is a c.a.s. defined near \( (y_0, \eta_0) \), then we can associate a formal pseudodifferential operator \( A \), using either the standard or the Weyl quantization. By "Egorov's theorem", which is valid in our setting, \( B = T A T^{-1} \) is then again a pseudodifferential operator with a c.a.s. (either for the standard or for the Weyl quantization) defined near \( (x_0, \xi_0) \), so we can define the action of \( B : H_\Phi^{loc}(\Omega_1) \rightarrow H_\Phi^{loc}(\Omega_2) \), where \( \Omega_1 \) and \( \Omega_2 \) are suitably chosen neighborhoods of \( x_0 \). In other words, we have then defined the action of \( A \) on functions that are defined microlocally near \( (y_0, \eta_0) \). Similarly, if \( \xi \) is a real canonical transformation from a neighborhood of \( (y_0, \eta_0) \) onto a neighborhood of \( (z_0, \xi_0) \), given by the non-degenerate phase function, \( \varphi(z, y, \theta) \), defined near \( (z_0, y_0, \theta_0) \), and if \( a \) is a c.a.s. defined near the same point, then we can define the action of the corresponding Fourier integral operator \( A \), mapping functions that are defined microlocally near \( (y_0, \eta_0) \) to functions defined near \( (z_0, \xi_0) \). Again, we have the natural composition results, including Egorov's theorem.

Somewhere in the proofs of these results in [S1], it is made use of a certain resolution of the identity, which permits to represent a distribution locally and not only microlocally as a superposition of Gaussians. Instead it would have been possible to use a global FBI transform, (which is essentially a Bargmann transform). Such a transform is given by,

\[
(a.1.15) \quad T u(x, h) = \int e^{i\varphi(x, y)/h} t(h) u(y) dy, \quad x \in \mathbb{C}^n, \; u \in L^2(\mathbb{R}^n),
\]

where \( \varphi \) is a second order polynomial, satisfying (a.1.3)-(a.1.5) (which now automatically become global conditions), and where \( t(h) \) is of the form \( Ch^{-m} \), \( C \neq 0 \). The function \( \Phi \), defined as before, now becomes a strictly plurisubharmonic second order polynomial on \( \mathbb{C}^n \). Choosing \( m = -3n/4 \) and \( C \) suitably, we can arrange so that \( T \) becomes isometric: \( L^2(\mathbb{R}^n) \rightarrow H_\Phi(\mathbb{C}^n) \) (by verifying that \( T^* T \) is the identity operator). Using also that the orthogonal projection \( \Pi : L^2(\mathbb{C}^n; e^{-2\Phi/hL}(dx)) \rightarrow H_\Phi(\mathbb{C}^n) \) has a simple explicit integral.
kernel, (that can be obtained by choosing a suitable contour when writing the
identity as a pseudodifferential operator, see [S1]), it is also easy to check
that $T^*T = I$ on $H^s(\mathbb{C}^n)$, so $T$ is unitary. We also notice both in the global
and in the local case, that if $T$ is an FBI transform, then $T^*T$ is also an FBI
transform. Here $\mathcal{F}$ denotes the unitary Fourier transform, defined in section 1.
In the global case, it is possible to choose $\mathcal{F}$ in such a way that $\Phi$ becomes
rotation invariant, and $T^*\Phi = \Phi^T$, where $\Phi v(x) = v(ix)$.

a.2. Functional calculus for analytic pseudodifferential operators.

Let $P$ be an $m_0 \times m_0$ matrix of formal analytic pseudodifferential
operators of order 0, with a c.a.s. defined in a neighborhood of $(x_0, \xi_0) \in \mathbb{R}^{2n}$. Let $p = p_0(x, \xi)$ be the principal symbol, that is the leading term in the
asymptotic expansion in powers of $h$, of the symbol of $P$. (Notice that $p$ does
not depend on whether we take the standard or the Weyl quantization.) If
$z \in \mathbb{C} \setminus \text{Sp}(p(x_0, \xi_0))$, then $(z-P)^{-1}$ is a formal analytic pseudodifferential
operator of order 0 whose symbol is defined in a neighborhood of $(x_0, \xi_0)$. If
$F(z, h)$ is a c.a.s. of order $m$, defined in a neighborhood of $\text{Sp}(p(x_0, \xi_0))$, then
we define $F(P, h)$ as the formal analytic pseudodifferential operator of order $m$, given by,

$$F(P, h) = (2\pi i)^{-1} \int_{\partial \Omega} F(z, h)(z-P)^{-1} dz,$$

where $\partial \Omega$ is the oriented $C^\infty$ boundary of a small neighborhood, $\Omega$, of
$\text{Sp}(p(x_0, \xi_0))$. When $F$ is a polynomial in $z$, it is easy to check that the
definition coincides with what one expects in that case. In the general case,
let $f(z)h^{-m}$ be the leading contribution to $F$ (i.e.; the principal symbol of $F$).
Then the principal symbol of $F(P, h)$ is $f(p(x, \xi))h^{-m}$. If $G$ is a second symbol
with the same properties as $F$, we check, by choosing a smaller $\Omega$ in one of
the representations of $F(P)$ or $G(P)$, that,

$$F(P, h) \ast G(P, h) = (FG)(P, h).$$

Finally, if $F$ is of order 0, and if $G$ is a c.a.s. defined in a neighborhood of
$\text{Sp}(f(p(x_0, \xi_0)))$, then we have,

$$G(F(P, h)) = (G \ast F)(P, h).$$

In fact, let $\alpha$, $\beta$ be contours around $\text{Sp}(f(p(x_0, \xi_0)))$ and $\text{Sp}(p(x_0, \xi_0))$
respectively, such that if $z \in \alpha$, then $f(z)$ is in the interior of $\alpha$. Then,

$$G(F(P)) = (2\pi i)^{-1} \int_{\alpha} G(w)(w-F(P))^{-1} dw =$$

$$(2\pi i)^{-2} \int_{\alpha} \int_{\beta} G(w)(w-F(z))^{-1} dw (z-P)^{-1} dz =$$

$$(2\pi i)^{-1} \int_{\beta} G(F(z)(z-P)^{-1} dz = (G \ast F)(P).$$

We finally notice that if the various symbols depend on parameters, but vary in
bounded sets, then the same is true for the resulting symbols.

The Weyl composition formula (a.1.13) implies that

$c = ab + (h/\iota)(a, b) + O(h^2)$, when $a$ and $b$ are of order 0. Using this, it is easy to
verify that the symbol of $(z-P)^{-1}$ is $(z-P(x, \xi))^{-1} + O(h^2)$, in the case when $P$
is a scalar symbol, so if $F$ is a symbol of order 0, then the Weyl symbol of
F(P, h) is $F(P(x, \xi), h) + O(h^2)$.

**a.3. Weyl quantization and conjugation by Fourier integral operators.**

The results of this subsection are valid either in the category of ordinary classical $C^\infty$ symbols on the real domain, when the corresponding phase functions are real, or in the category of analytic symbols, either on the real domain with real phase functions, or on the complex domain, with complex phases. The assumptions in each case will be local and the obvious ones. To fix the ideas we choose the real case (analytic or $C^\infty$). To start with, we examine the simple case of conjugation by an elliptic factor. Let $P(x, \xi)$ be defined near $(x_0, \xi_0) \in \mathbb{R}^{2n}$. (We could also let $P$ be a classical symbol of order 0). Let $\psi(x)$ be a real valued smooth function, defined near $x_0$, such that $\psi'(x_0) = \xi_0$. Let $P$ also denote the Weyl quantization of $P$. We are then interested in the Weyl symbol of the pseudodifferential operator $Q = e^{-i\psi(x)/h} \cdot P \cdot e^{i\psi(x)/h}$, which is defined near $(x_0, \xi_0 - \xi_0)$. We proceed formally, by first writing the integral kernel of $Q$ as,

$$K_Q(x, y) = \int e^{i((x-y)\theta - (\psi(x) - \psi(y))/h} P((x+y)/2, \theta) \, d\theta,$$

Here the $O$-term contributes to the Weyl symbol of $Q$ by $O(h^2)$ (i.e. by a classical symbol of order $-2$), so if we denote by $Q$ also the Weyl symbol, then we get,

$$Q(x, \xi) = P(x, \xi + \psi'(x)) + O(h^2).$$

If $a = a(x, h) \sim a_0(x) + a_1(x)h + \ldots$ is a classical symbol defined near $x_0$, then by formal stationary phase,

$$Q(a) \sim Q(x, 0)a(x) + (h/i)(Q'(x, 0)a'(x) + \frac{1}{2} \Sigma \partial_x q_j'(x, 0)a(x)) + O(h^2).$$

Assume now that $P$ is real valued, and that $P(x_0, \xi_0) = 0$, $P'(x_0, \xi_0) \neq 0$. Let $\varphi$ be real valued and satisfy the eiconal equation,

$$Q(x, \varphi'(x)) = 0.$$

We then want to construct a such that $e^{-i\varphi/h\partial_x} e^{i\varphi h/\partial_x} = 0$ (in the sense of formal classical symbols). Then combining (a.3.3) and (a.3.4), we get the leading transport equation,

$$q'(x, 0)a_0'(x) + \frac{1}{2} \Sigma \partial_x q_j'(x, 0)a_0(x) = 0,$$

where $q(x, \xi) = P(x, \varphi'(x) + \xi)$ is real valued. It follows that $a_0$ has constant argument along each bicharacteristic curve, associated to $P, \varphi$.

We now attack the more general question of conjugation with F.I.O.'s. Recall that if $A, B$ are classical pseudodifferential operators of order 0, defined microlocally near $(0, 0)$, and if $C = [A, B]$, then for the corresponding Weyl symbols (denoted by the same letters), we have,

$$C = (h/i)(A, B) + O(h^3).$$

We shall exploit this fact in order to give a refinement of Egorov's theorem.
Let $X$ be a canonical transformation from a neighborhood of $(0,0) \in \mathbb{R}^{2n}$ onto another neighborhood of the same point, and assume in order to fix the ideas that $X(0,0) = (0,0)$. Then, we can find a smooth 1-parameter family of canonical transformations, $X_t$, $t \in [0,1]$, with the same properties, such that $X_0 = \text{id}$ and $X_1 = X$. We have,

$$\partial_t X_t(x, \xi) = H_{Q_t}(X_t(x, \xi)),$$

where $Q_t$ is a smooth family of smooth real valued functions, defined in a neighborhood of $(0,0)$. (Here $H_Q$ denotes the Hamilton field of $Q$.) Let $Q_t$ also denote the corresponding Weyl quantization, so that $Q^*_t = Q_t$, where the formal adjoint is taken with respect to the standard inner product on $L^2(\mathbb{R}^n)$. We can then define a family of unitary Fourier integral operators, $U_t$ associated to $X_t$, defined microlocally near $(0,0)$, and given by,

$$hD_t U_t + Q_t U_t = 0, \quad U_0 = I, \quad \text{where } D_t = i^{-1} \partial_t.$$

Let $P$ be a pseudodifferential operator of order $0$, defined near $(0,0)$, and define $P_t$ by,

$$P_t U_t = U_t P.$$

For the corresponding Weyl symbols, we claim that,

$$P_t \star X_t = P + O(h^2).$$

The interesting fact here (as well as in (a.3.3)) is that we have $O(h^2)$ and not just $O(h)$.

To prove (a.3.11), we start by differentiating (a.3.10) with respect to $t$: $\partial_t P_t U_t + P_t (\partial_t U_t) = (\partial_t U_t) P_t$. Then (a.3.9) gives, $\partial_t P_t = (i/h)[P_t, Q_t]$, and using (a.3.7), we get for the corresponding symbols, $\partial_t P_t = (P_t, Q_t) + O(h^2)$, which we can also write as,

$$\partial_t P_t + H_{Q_t} P_t = O(h^2).$$

Since $\partial_t (P_t (X_t(x, \xi))) = (\partial_t P_t (X_t(x, \xi)) + (H_{Q_t} P_t)(X_t(x, \xi))$, we conclude that,

$$\partial_t P_t (X_t(x, \xi)) = O(h^2),$$

and after integration of this estimate, we get (a.3.11).

We now consider the converse question of associating a suitable canonical transformation to a given unitary Fourier integral operator. The basic idea is that if

$$P(x,h) = \int \int e^{i \varphi(x,y,\theta)/h} a(x,y,\theta, h) u(y) \, dy \, d\theta,$$

is such an operator, where $a$ is a classical symbol of order $m$, then we can always replace $\varphi$ by $\varphi + h \psi(x,y,\theta, h)$, where $\psi$ is a classical symbol of order $0$, since the new amplitude, $e^{-i \psi} a$ is again a classical symbol of order $0$. One then gets an associated canonical transformation, $X$ which depends on $h$ and satisfies $X = K_0 + O(h)$, where $K_0$ is the standard one. In the main text, the phase functions, will already depend on $h$ in some complicated way (but varying in a bounded set) so $X$ will be as natural as $K_0$.

Let $U$ be a unitary F.I.O. whose associated standard canonical transformation, $\tilde{K}$ maps $(0,0)$ to $(0,0)$. Taking an intermediate family of
canonical transformations, $\tilde{X}_t$ as above, we can then construct a smooth family of unitary F.I.O.'s, $U_t$ such that $U_0=1$, $U_1=U$. We then use (a.3.9) to define the pseudodifferential operators $Q_t$ of order 0. Then $Q_t=(h/i)(\partial_t U_t)U_t^*$, so $Q_t-Q_t^*=(h/i)((\partial_t U_t)U_t^*+U_t(\partial_t U_t^*))=(h/i)\partial_t(U_t^*U_t^*)=0$, so $Q_t$ is self-adjoint. We can then define $X_t$ by (a.3.8), with initial condition, $X_0=\text{id}$. Then we still have (a.3.11), if $P_t$ is given by (a.3.10), and for $t=1$, this shows that if we associate to $U$ the canonical transformation, $X=X_1$, then if $\tilde{P}$ and $P$ are two pseudodifferential operators of order 0, related by,

(a.3.15) $\tilde{P}U=UP$,  

then, for the Weyl symbols,

(a.3.16) $\tilde{P}\ast X=P+\mathcal{O}(h^2)$. 

By varying $\tilde{P}$, we see that $X$ is uniquely determined modulo $\mathcal{O}(h^2)$. (This means that in the correction of the phase $\psi$ in (a.3.14) it is enough to take $\psi$ independent of $h$.

It would be interesting to give a more explicit description of the relation between a unitary F.I.O. and the corresponding canonical transformation, $X$ given (modulo $\mathcal{O}(h^2)$) by the relations (a.3.15),(a.3.16). A reasonable guess would be that if we write a given unitary FIO in such a way that the leading amplitude has constant argument, (by adding a term $h^{x}(x,\theta)$ to the phase,) then the corresponding phase gives the more precise canonical transformation. We have not checked this however.

Let us here only investigate to what extent $U$ is uniquely determined by $X$. In the following we shall say that $X$ is associated to $U$, if (a.3.15),(a.3.16) hold. Let us first notice that if $X_j$ is associated to $U_j$ for $j=1,2$, then $U_1\ast U_2$ is associated to $X_1\ast X_2$. Now let $U$ and $V$ be associated to the same canonical transformation $X$. Then $A=U\ast V^{-1}$ is a unitary pseudodifferential operator, which is associated to the identity:

(a.3.17) $[P,A]=\mathcal{O}(h^2)$,  

for every pseudodifferential operator of order 0. This means on the symbol level, that $\nabla A=\mathcal{O}(h)$, so $A=\omega+\mathcal{O}(h)$, for some $\omega\in\mathcal{C}$ of modulus 1. Hence two unitary FIO are associated to the same canonical transformation (in the precise sense) iff there is a number $\omega\in\mathcal{C}$ with $|\omega|=1$, such that $U=AV$, where $A=\omega+\mathcal{O}(h)$.
b. **Normal forms for some scalar pseudodifferential operators.**

In this section, we shall give normal forms for some scalar, classical analytic pseudodifferential operators, valid near a critical point of the symbol. Contrary to the usual case in the theory of local solvability and hypoellipticity and so on, we do not (at least not to start with) admit multiplication by elliptic factors, but only conjugation by Fourier integral operators and passage to a function of the operator. Thus the general question here is: "Given a certain selfadjoint operator $P$, find a real function $f$ and a unitary operator $U$, such that $U^*f(P)U$ takes a simple form." There will be two cases (closely parallel): the case when the symbol has a saddle point, and the case when the symbol has a non-degenerate minimum. The second case has already been treated in [HS2], but here we give a second approach especially adapted to the analytic case.

We start with the saddle point (or "branching") case, and we consider first on the level of principal symbols, the model symbol $p_0 = x\xi$ on $\mathbb{C}^2$. The associated Hamilton field is $H_{p_0} = x\partial_x - \xi\partial_\xi$, and $\exp(tH_{p_0})(x,\xi) = (e^{t}x, e^{-t}\xi)$, $t \in \mathbb{C}$. The flow is periodic with period $T_0 = 2\pi i$. If $p_0(x,\xi) = E \neq 0$, then $(x,\xi) = \exp(tH_{p_0})(y, y)$, where $y^2 = E$ and $t = \log(x/y) + 2\pi ik$, $k \in \mathbb{Z}$. The complex curve $\Gamma_0: \xi = x$ has the property that $\exp(\frac{1}{2}T_0 H_{p_0})(\Gamma_0) = \Gamma_0$. and as we just saw, every point with $p_0(x,\xi) \neq 0$, is of the form $\exp(tH_{p_0})(y, y)$ for suitable $t$ and $y$. We also notice that $\Gamma_0$ is the complexification of its intersection with $\mathbb{R}^2$. Let now $p(x,\xi)$ be a real-valued analytic function defined in a neighborhood of $(0,0) \in \mathbb{R}^2$ and having a saddle point at $(0,0)$, with $p(0,0) = 0$. If we let $\Gamma_+$ and $\Gamma_-$ be the stable outgoing and incoming manifolds for the $H_p$ flow (see [AM]), we know (see for instance [S2]), that $\Gamma_+$ and $\Gamma_-$ are analytic curves, intersecting transversally at $(0,0)$, and after composing $p$ with a suitable real and analytic canonical transformation, we may assume that $\Gamma_+$ is the $x$-axis and that $\Gamma_-$ is the $\xi$-axis. Then, it is easy to see that we have reduced the problem to the case when

$$p(x,\xi) = a(x,\xi)x\xi,$$

where $a > 0$ is a real-valued analytic function. Replacing $p$ by $f(p)$, where $f(t) = Ct$, $C > 0$, we may assume that $a(0,0) = 1$. If we replace $x$ by $(\text{const.})a(x,\xi)x$, we get an analytic diffeomorphism, that transforms $p$ onto $(\text{const.})p_0$. In particular, we know that the surfaces $p = E_1$ and $p_0 = E_2$ have the same topological structure when intersected with suitable neighborhoods of $(0,0)$, provided that $E_1$ and $E_2$ are non-zero and small. In particular $p = E$ is a connected curve for $E \neq 0$, so if $(x_0,\xi_0)$ belongs to this curve and $\exp T H_p(x_0,\xi_0) = (x_0,\xi_0)$ for some $T \in \mathbb{C}$, then $\exp T H_p(x,\xi) = (x,\xi)$ for all $(x,\xi)$ on this curve, and $T = T(E)$ is a period for the $H_p$ flow, restricted to the energy curve $p = E$. To see that such non-trivial periods exist, let $E = \mu^2$ be small but non-vanishing and make the change of variables $(x,\xi) = (\mu y, \mu \eta)$. Put
$p_\mu = \mu^{-2}p(\mu y, \mu \eta)$. Then $(x, \xi)$ belongs to $p^{-1}(E)$ iff $(y, \eta)$ belongs to $p^{-1}(1)$.

Moreover, $H_p$ is the direct image under the map $(y, \eta) \rightarrow (x, \xi)$ of $H_{p_\mu}$, so an equivalent problem is then to study periods of the Hamilton flow of $p_\mu$ in the surface of energy 1. Since $p_\mu = p_0 + O(\mu)$ (uniformly in every fixed compact set) we conclude that the $H_p$-flow in $p^{-1}(E)$ has the period $T(E) = T_0 + O(E^{1/2})$, for small non-vanishing values of $E \in \mathbb{C}$. Since $T(E)$ is holomorphic in $E$ outside the origin, we have a removable singularity at $E = 0$. Hence $E$ is holomorphic in a full complex neighborhood of 0, and

$$(b.2) \quad T(E) = T_0 + O(E).$$

Since $p$ is real-valued, it is easy to see that $T(E) = -T_0 + O(E)$ is a period when $E$ is real. On the other hand $-T(E) = \overline{T(E)} + O(E)$ is also a period, and since two different periods cannot be too close to each other, they have to agree; $T(E) = -T(E)$, for $E$ real, so $T(E)$ is purely imaginary for real energies. By the unique continuation property we get more generally,

$$(b.3) \quad T(E) = -T(E),$$

where $E$ is complex. We now look for a real canonical transformation, that transforms $p$ into $p_0$. An obvious necessary condition is that $T(E) = T_0$, for all $E$, and we can easily reduce ourselves to this situation, by replacing $p$ by $f \circ p$, where $f$ is the real-valued analytic function given by,

$$(b.4) \quad \frac{df(E)}{dE} = \frac{T(E)}{T_0}, \quad f(0) = 0.$$

Hence, we shall assume from now on, that

$$(b.5) \quad T(E) = T_0.$$

We next study the $H_p$ trajectories for real times. For such a trajectory, we have $x' = (a(x,0) + O(\xi))x$, $\xi' = -(a(0,\xi) + O(x))\xi$, and we first see that the flow is expansive in $x$ and contractive in $\xi$, in the sense that for $t \geq s$:

$|x(t)| \geq e^{(t-s)/C} |x(s)|$, $|\xi(t)| \leq e^{-(t-s)/C} |\xi(s)|$. (Here, we restrict the attention to a trajectory which stays in a region: $|x| \leq \varepsilon_0$, $|\xi| \leq \varepsilon_0$, for some sufficiently small $\varepsilon_0$.) Consider now such a trajectory $(x(t), \xi(t))$, $0 \leq t \leq T$, with $|x(0)/\xi(0)| \in [1/2, 2]$. Then $|\xi(t)| \leq 2|x(t)|$, and we get,

$x' = (1 + O(x(t)))x(t)$. Using that $|x(t)| \leq \varepsilon_0 e^{-(T-t)/C}$, we can integrate the earlier relation, and obtain,

$x(t) = \exp(t + O(t)\varepsilon_0 \int_0^t e^{-(T-s)/C} ds) x(0),$

and hence,

$$(b.6) \quad x(t) = e^{(t + O(\varepsilon_0))} x(0).$$

Since the flow is $2\pi i$-periodic, this relation extends to the case when $t \in \mathbb{C}$, $0 \leq Re t \leq T$. We also have an analogous estimate for $\xi(t)$, when $Re t \leq 0$.

We next observe that, locally near $(0,0)$,

$$(b.7) \quad \exp\left(\pm T_0 H_p\right) (\mathbb{R}^2) = \mathbb{R}^2.$$

In fact, the set of conjugate points of the left hand side is equal to $\exp\left(-\frac{1}{2} T_0 H_p\right) (\mathbb{R}^2) = \exp\left(\frac{1}{2} T_0 H_p\right) (\mathbb{R}^2)$, since $T_0$ is a period. Hence the left hand side of (b.7) is invariant under complex conjugation. Since it is a
twodimensional surface, close to $\mathbb{R}^2$ in the $C^1$ topology, the only possibility is
that we have equality in (b.7).

Let $\Gamma_0=\{(x,\xi); x=\xi\}$ and recall that $\exp(\frac{1}{2}T_0H_0)(\Gamma_0)=\Gamma_0$. Let $f$ be a
real valued analytic function vanishing on $\Gamma_0$, with $df(0,0)\neq 0$, for instance
$f=\xi-x$. Then $g=f\cdot\exp(\frac{1}{2}T_0H_0)$ satisfies $dg(0,0)=-df(0,0)$, and if we put
$h=f-g$, then $dh(0,0)=2f(0,0)\neq 0$, and $h\cdot\exp(\frac{1}{2}T_0H_0)=-h$. Hence $\Gamma=\Gamma^{-1}(0)$ is
a complex curve, equal to the complexification of a real analytic curve, tangent
to $\Gamma_0$ at $(0,0)$. We have,
\begin{equation}
(b.8) \quad p|\Gamma=q^2, \quad p_0|\Gamma_0=(q_0)^2,\nonumber
\end{equation}
where $q, q_0$ are holomorphic on $\Gamma$ and $\Gamma_0$ respectively, satisfying
dq(0,0)=dq_0(0,0), \quad q_0(x,\xi)=x.

Let $U_0=((x,\xi)\in \mathbb{C}^2; |x|,|\xi|\leq \varepsilon_0)$ for some sufficiently small $\varepsilon_0>0$.
Then every $(x,\xi)\in U_0\setminus p_0^{-1}(0)$ can be written $(x,\xi)=\exp(tH_0\rho)(\rho)$,
$\rho\in \Gamma_0\cap U_0$. Here $\rho$ is well-defined up to a choice of sign; we can replace $\rho$ by
$-\rho$ if we change $t$ to $t+\pi i$. Once the choice of $\rho$ is fixed, then $t$ is well defined
up to a multiple of $2\pi i$. We now define the map $\kappa: U_0\setminus p_0^{-1}(0) \to U\setminus p^{-1}(0)$,
where $U$ is a suitable small neighborhood of $(0,0)$, by:
\begin{enumerate}
\item If $\rho \in \Gamma_0$, then $\kappa(\rho)\in \Gamma$ is given by $q(\kappa(\rho))=q_0(\rho)$.
\item If $(x,\xi)=\exp(tH_0\rho)(\rho)$, as above, we put $\kappa(x,\xi)=\exp(tH_0)(\kappa(\rho))$.
\end{enumerate}
Since $|\text{Re} t|$ may be very large here, we have to verify (A); that $\exp(tH_0)(\rho)$ is
well-defined and belongs to a small neighborhood of $(0,0)$, and (B); that the
definition of $\kappa(x,\xi)$ does not depend on the choice of $\rho, t$ in the representation
of $(x,\xi)$. Assuming (A), the verification of (B) is immediate, using that
$\exp(\frac{1}{2}T_0H_0)(\Gamma)=\Gamma$, and that $T_0$ is a period for the $H_0$ flow. (A) follows from
(b.6) and its analogue for $\xi(t)$, which imply that $\kappa(x,\xi)$ is well-defined, and
even that $|\kappa(x,\xi)|\leq \varepsilon^0(\mathcal{C}^0)(x,\xi)$ (if we use $\varepsilon^0$-norms). We have
constructed a single-valued bounded holomorphic map $\kappa$:
\begin{equation}
U_0\setminus p_0^{-1}(0) \to U\setminus p^{-1}(0), \nonumber
\end{equation}
which then clearly has a holomorphic extension to:
\begin{equation}
U_0 \to U. \nonumber
\end{equation}
Moreover, by construction,
\begin{equation}
(b.9) \quad p\circ \kappa = p_0, \nonumber
\end{equation}
and $\kappa$ is symplectic, mapping the real domain into the real domain, and we
have $d\kappa(0,0)=id$.

Let now $P(x,\xi,h)\sim \Sigma_{j\geq 0} p_j h^j$ denote a formal classical analytic symbol,
defined in a neighborhood of $(0,0)$, with leading part $p_0=p$, and assume that
the corresponding formal pseudo-differential operator $P(x,hD,h)$ is formally
self-adjoint. (We here take the classical quantization, but nothing changes in
the arguments, if instead we take the $h$-Weyl quantization.)

From what we have done so far, it follows that there is a formal unitary
Fourier integral operator, associated to the canonical transformation $\kappa$, and a
real-valued analytic function $f(t)$, with $f(0)=0$, $f'(0)>0$, such that $U^{-1}f(P)U$
has the principal symbol $\xi^2$. From now on, we assume that already $p=p_0=\xi^2$.

Since the $H_p$-flow is $T_0$-periodic, we find formally that,

\begin{equation}
(b.10) \quad e^{-iT_0p/\hbar} = R,
\end{equation}

where $R$ is a formal analytic elliptic pseudodifferential operator of order 0. Since $T_0=2\pi i$ is non-real, we have to be a little careful, in giving a sense to $(b.10)$. We do this in the following way. Let $\Phi_0$ be a strictly subharmonic quadratic form on $\mathbb{C}$, close to 0, and define (as in [S1]):

\[ \Lambda_{\Phi_0} = ((x, (2/\iota) \partial \Phi_0/\partial x); x \in \mathbb{C}). \]

Then for $t$ imaginary, we can define $\Lambda_{\Phi_t} = \exp(iH_p)(\Lambda_{\Phi_0})$, and $\Phi_t$ is again a strictly subharmonic quadratic form close to 0. Using the theory developed in the first sections in [S1], we can then define unambiguously a classical Fourier integral operator $U_t = e^{-it\Phi/\hbar}$, that can be realized as an operator from $H_{\Phi_0}^{\text{loc}}(\Omega_0)$ to $H_{\Phi_t}^{\text{loc}}(\Omega_t)$, where $\Omega_0$ and $\Omega_t$ are suitable complex neighborhoods of 0. As a matter of fact, this is just the standard construction, except for the fact, that due to an embarrassing imaginary part of the phase-function, there is no reasonable way to define this operator acting microlocally in the usual $L^2$ space. Nevertheless we have $\Phi_{T_0} = \Phi_0$, and we see that $R$ is a pseudodifferential operator. If $\alpha = (\alpha_x, \alpha_\xi) \in \mathbb{R}^2$ belongs to a small neighborhood of the origin, we put $u_{\alpha}(x, \hbar) = e^{i((x-\alpha_x)\alpha_\xi + i(x-\alpha_x)^2/2)/\hbar}$. Then for $\alpha, \beta$ sufficiently close to $(0, 0)$, the scalar products $(U_t u_\alpha \mid U_t u_\beta)_{L^2}$ are well-defined up to exponentially small terms, and up to such a term, we have $(U_t u_\alpha \mid U_t u_\beta)_{L^2} = (u_\alpha \mid U_t u_\beta)_{L^2}$, as we can see, using the selfadjointness of $P$ and the fact that we now restrict the attention to imaginary values of $t$. Since the functions $u_\alpha$ fill up a microlocal neighborhood of $(0, 0)$ (in a sense that can be made precise by using an FBI-transform), it follows that $R$ is formally self-adjoint, so the principal symbol, $r$ is either $>0$ or $<0$ on the real domain. As already mentioned, $U_t$ is formally obtained by the standard WKB-procedure, which in this case gives us,

\begin{equation}
(b.11) \quad U_t u(x) = (2\pi \hbar)^{-1} \int e^{i\tilde{\psi}(t, x, \eta) - iy\eta}/\hbar a(t, x, \eta, \hbar) u(y) \, dy \, d\eta,
\end{equation}

where $\tilde{\psi}(t, x, \eta) = e^{-t} x \eta$, and where $a$ is a classical analytic symbol of order 0, determined by the standard transport equations. Notice that $\tilde{\psi}(T_0, x, \eta) = x \eta$ as we could expect. Analyzing the first transport equation (as we shall do below), we see that the leading part, $a_0$ of $a$, satisfies,

\[ a_0(T_0, 0) < 0. \]

This supports the belief that $r < 0$, but we have to be careful, since the realization of the operators $U_t$ involves the choice of certain contours, and this may be responsible for an additional factor $-1$. However, we can deform $P$ continuously into $P_0 = \frac{1}{2}((xhD + hDx)xhD + h)/2i$, and in the case of $P_0$ the corresponding group is given by $U_t u(x) = e^{-t/2} u(e^{-t} x)$, so we see in that case that $R = -id$. Hence, in general,

\begin{equation}
(b.12) \quad r < 0.
\end{equation}
It is also easy to check that \([P,R]=0\). Microlocally, where \(P\) is of principal type (and hence reducible to \(hD\)), it is easy to see that \(R=F(P,h)\), where \(F\) is an analytic symbol of order 0, elliptic and real valued on the real domain. Since \(R\) and \(F(P,h)\) are both well defined analytic symbols on a neighborhood of \((0,0)\), which agree on some open set, they have to agree on a full neighborhood of \((0,0)\), and hence,
\[
(P,R)=0.
\]
Since \(F\) is negative on the real domain, there is a real valued analytic symbol \(G(t,h)\), of order 0, such that,
\[
\exp(iT_0 G(t,h))=-F(t,h),
\]
and hence,
\[
\exp(iT_0 (P+hG(P,h))/h)=-1.
\]
Replacing \(P\) by \(P+hG(P,h)\), we have now reduced the problem of normal forms to the case when,
\[
\exp(iT_0 P/h)=-1.
\]
Solving the first transport equation for \(a_0\) (cf (b.11)),
\[
\partial_t a_0 + x\partial_x a_0 + i\partial_{-1}(x,\partial_x) a_0 = 0, \quad a_0(0)=1,
\]
we obtain, since we now know that \(a_0(T_0,x)=-1:\)
\[
\exp(\int_0^{T_0} p_{-1}(x(t),\xi(t))dt)=-1,
\]
or equivalently,
\[
\int_0^{T_0} p_{-1}(x(t),\xi(t))dt=(2k+1)\pi, \quad k \in \mathbb{Z}.
\]
In the case of \(P_0\), we have \(p_{-1}=1/2i\), and \(k=0\) in (b.19). This is then also true in the general case, since by self-adjointness the subprincipal symbol \(p_{-1}-(1/2i)\partial^2 P_0/\partial x\partial x= p_{-1}-1/2i\), is real, and \(T_0\) is imaginary. Hence in general,
\[
\int_0^{T_0} p_{-1}(x(t),\xi(t))dt=\pi,
\]
and in particular \(p_{-1}(0,0)=1/2i\).

We next look for an elliptic pseudodifferential operator \(R_0\) of order 0, such that,
\[
P_0 R_0 = R_0 P\text{ modulo an operator of order } -2.
\]
For the principal symbol \(r_0\), we then obtain the condition,
\[
i^{-1/2} H_{p_0}(r_0) - (p_{-1}-1/2i)r_0 = 0,
\]
so we can take
\[
r_0 = e^{ig},
\]
where \(g\) solves,
\[
H_{p_0}(g) = (p_{-1}-1/2i).
\]
Here we make a general remark on the equation \(H_{p_0}(a)=b\). If \(b\) is given, we put \(a = -\int_0^{T_0} (1-t/T_0) b\cdot \exp H_{p_0} dt\), then
\[
H_{p_0}(a) = -\int_0^{T_0} (1-t/T_0)(d/dt)(b\cdot \exp H_{p_0}) dt - b - T_0^{-1}\int_0^{T_0} b\cdot \exp H_{p_0} dt.
\]
We have then a solution, if the last integral vanishes. The vanishing of this integral is also obviously a necessary condition for solvability. In the case of
(b.24) the condition is fulfilled, in view of (b.20), so a solution \( q \) exists. Moreover, since \( p_{-1-1/2i} \) is real valued, we may choose \( q \) real. And this means that we may choose \( R_0 \) unitary in (b.21). We are now reduced to the case when \( P-P_0 \) is of order \(-2\), and (b.16) holds. We then look for a pseudodifferential operator \( R \) of order \(-1\), such that,

\[
P_0(I+R)=(I+R)P.
\]

We rewrite this equation as

\[
\mathcal{I}R=P-P_0,
\]

where \( \mathcal{I}R=P_0R-PR \). By a formal WKB construction, we can define \( e^{-it\mathcal{I}/h} \), acting on formal analytic pseudodifferential operators, and since,

\[
e^{-it\mathcal{I}/h}R=e^{-itP_0/h}e^{itP/h},
\]

we see that \( e^{-it\mathcal{I}/h}R \) is again a classical analytic pseudodifferential operator, and that

\[
e^{-iT_0\mathcal{I}/h}=id.
\]

If \( B \) is an analytic pseudodifferential operator of order \( k \), then,

\[
A=(i/h)\int_0^{T_0}(1-t/T_0)e^{-it\mathcal{I}/h}B\,dt
\]

is an analytic pseudodifferential operator of order \( k+1 \), and we have,

\[
\mathcal{I}A=B-(i/T_0)\int e^{-it\mathcal{I}/h}B\,dt.
\]

As before, a necessary and sufficient condition, in order to solve \( \mathcal{I}A=B \), is that the integral in (b.30) vanishes. Let us verify this condition in the case of (b.26):

\[
\int_0^{T_0}e^{-it\mathcal{I}/h}(P-P_0)dt=\int_0^{T_0}e^{-itP_0/h}(P-P_0)e^{itP/h}dt=\int_0^{T_0}(d/dt)(e^{-itP_0/h}e^{itP/h})dt=0,
\]

where the last equality follows from (b.16), which is also verified by \( P_0 \).

Hence, it is clear that we can find \( R \) of order \(-1\), solving (b.26). Changing notations, we have now found an elliptic analytic pseudo-differential operator \( R \) of order 0, such that,

\[
RP=P_0R.
\]

It remains to see that we can take \( R \) to be unitary. If we let \( R^* \) denote the complex adjoint of \( R \), then, since \( P \) and \( P_0 \) are formally selfadjoint, we get from (b.31),

\[
PR^*=R^*P_0,
\]

and hence,

\[
(R^*R)P=P(R^*R).
\]

The operators \( (R^*R)^{\pm1/2} \), defined by the functional calculus of appendix a, then commute with \( P \), and we put \( U=R(R^*R)^{-1/2} \). Then,

\[
U^*U=(R^*R)^{-1/2}R^*R(R^*R)^{-1/2}=I,
\]

so \( U \) is unitary, and an easy computation gives,

\[
P_0U=UP.
\]

Let us sum up what we have proved so far:
**Theorem b.1.** Let $P(x, hD, h)$ be a formal classical analytic pseudodifferential operator, of order 0, formally selfadjoint, with symbol defined in a neighborhood of $(0, 0)$. Let $p$ be the principal symbol, and assume that $p$ has a non-degenerate saddle point at $(0, 0)$ with critical value 0. Then there is a real-valued analytic symbol; $F(t, h) \sim \Sigma_0 \infty f_j(t) h^j$, defined for $t$ in a neighborhood of 0, and a formal unitary analytic Fourier integral operator, whose associated canonical transformation (in the classical sense) is defined in a neighborhood of $(0, 0)$, and maps this point onto itself, such that

$$(b.36) \quad U^* F(P, h) U = P_0 = \frac{1}{2}(xhD + hDx).$$

To end this section, we study the case when $P$ has additional symmetries. We start with the most important case, when $P$ commutes with $C_0 = y^2$, which is a Fourier integral operator, whose associated canonical transformation is given by $C_0(x, \xi) = (-x, -\xi)$, and which satisfies $C_0^2 = I$. We notice that $P_0$ also commutes with $C_0$, and the natural problem is then to choose $U$ in (b.36) so that $U$ commutes with $C_0$. Examining the proof of the theorem, we first observe that it is possible to choose $U$, such that the associated canonical transformation, $\Xi_U$ commutes with $C_0$. We now fix such a canonical transformation $\Xi_0$ and we let $\mathcal{U}(\Xi_0)$ be the set of unitary Fourier integral operators satisfying (b.36) (with a fixed $F$) with $\Xi_0$ as their associated canonical transformation. If $U, V \in \mathcal{U}(\Xi_0)$, then $U^{-1}V$ is a pseudodifferential operator which commutes with $P_0$, so by an argument already given above,

$$(b.37) \quad V = U e^{i\mathcal{G}(P_0, h)},$$

where $\mathcal{G}$ is a real valued analytic symbol of order 0. (Here we work with associated canonical transformations in the classical sense. We could also have chosen the more precise correspondence of appendix $a$. The we would have $\mathcal{G} = \text{const.} + (\text{operator of order } \leq -1).$) Conversely, if $U \in \mathcal{U}(\Xi_0)$, and $V$ is given by (b.37) with $\mathcal{G}$ as above, then $V \in \mathcal{U}(\Xi_0)$.

We now fix $U \in \mathcal{U}(\Xi_0)$. Then $C_0 UC_0 \in \mathcal{U}(\Xi_0)$, and hence $C_0 UC_0 = U e^{i\mathcal{G}(P_0, h)}$, with $\mathcal{G}$ as above. Iterating this relation and using that $C_0^2 = I$, we get $U = C_0 UC_0 e^{i\mathcal{G}(P_0, h)} = U e^{i2\mathcal{G}(P_0, h)}$, and hence there are only two possibilities, either $\mathcal{G} = 0$, or $\mathcal{G} = \pi$ modulo a multiple of $2\pi$. In the first case, we get $C_0 UC_0 = U$, or rather $C_0 U = UC_0$, which is the desired commutation relation. In the second case, we obtain $C_0 U = -UC_0$. Let us apply this equation to the function $u_0(x) = e^{-x^2/2h}$, noticing that $C_0 u_0(x) = u_0(-x)$. We then have $C_0 u_0 = u_0$, and $u_0 = a(x, h) e^{i\mathcal{G}(x)/h}$, where $a$ is an elliptic symbol of order 0, and $\mathcal{G}$ is analytic with $\text{Im}(\mathcal{G}) \geq 0$. We then get,

$a(-x, h) e^{i\mathcal{G}(-x)/h} = -a(x, h) e^{i\mathcal{G}(x)/h}$, which for $x = 0$ contradicts the ellipticity of $a$. Hence only the first case can appear, and we have proved.
**Proposition b.2.** In addition to the assumptions of the preceding theorem, we also assume that $P$ commutes with $C_0 = \mathcal{F}^2$. Then it is possible to find $U$ as in Theorem b.1, which commutes with $C_0$.

At least in the case of the first step in our general iteration procedure, the operator $P$ will satisfy more elaborate symmetry relations, that we also want $U$ to respect. We shall now assume in addition to the assumptions of Theorem b.1., that

\[ \Delta P = -P \Delta, \quad \Gamma P = -P \Gamma. \]

where $\Gamma = \Gamma_0$ denotes the operator of complex conjugation. This assumption implies that the principal symbol $p$ changes sign after composition by $\kappa = \kappa - \pi/2$ or by reflection in one of the axes. It follows that $p$ is invariant under reflection in the line $x = \xi$, that is: $p \ast \Delta = p$, where $\Delta(x, \xi) = (\xi, x)$. Using that $p \ast \kappa = -p$, it is easy to see that the period $T(E)$, introduced in the beginning of this section, is even, and hence the function $f$, such that the period of $f(p)$ is constant, is odd. This implies that the symmetries for $p$ deduced from (b.38), are also valid for $f(p)$. In the earlier construction of $K = K_0$, we can then take the curve $\Gamma$ to be the curve $\Gamma_0: x = \xi$. It then becomes clear that $K_0$ commutes with reflection in $x = \xi$ and with $C_0 = \mathcal{F}^2$. If we also impose the condition that $dK_0(0,0) = i$, then we see that $K_0$ is the unique canonical transformation determined by this property and $f(p) \ast K_0 = p_0$, $K_0 \ast \Delta = \Delta \ast K_0$. Now write: $f(p) \ast K_0 = -f(p) \ast K_0 \ast \kappa = -p_0 \ast \kappa = p_0$, $K_0 \ast \Delta = \kappa_0 \ast \Delta \ast K_0 \ast \kappa_0 = \Delta \ast p_0$, $\kappa_0 \ast \Delta = \kappa_0 \ast \kappa \ast \Delta = \kappa_0 \ast \kappa \ast \Delta = \kappa_0 \ast \kappa \ast \Delta$, which shows that $\kappa_0 \ast \kappa$ has the same properties as $K_0$, and hence is equal to $K_0$. In other words,

\[ \kappa_0 \ast \kappa = \kappa \ast \kappa_0. \]

We next reexamine the construction of the real valued analytic symbol $F(t, h)$ with leading part $f$, such that $\exp(-it_0 F(P, h)/h) = I$. In fact, the requirement that $F$ be real-valued, implies that $F$ is uniquely determined with this leading part. If we suppress the $-\ast$ sign in the last relation, and conjugate with $\mathcal{F}$, we get in view of (b.38): $\exp(-it_0 (-F(-P_0, h))/h) = I$, and knowing that $f$ is odd, the earlier remark on the uniqueness of $F$, shows that $F(-E, h) = -F(E, h)$. It follows that $F(P, h)$ satisfies (b.38). In the following, we may then assume that $F(E, h) = E$, and work directly with $F$ instead of $F(P, h)$. We fix $K_0 = K_0$ as above, and define $U(K_0)$ to be the set of unitary Fourier integral operators $U$ associated to $K_0$, and such that $P \ast U = U \ast P$. In view of the properties of $K_0$, we see that if $U \in U(K_0)$, then $\mathcal{F} U \mathcal{F}^{-1} \in U(K_0)$, so by an earlier argument,

\[ \mathcal{F} U \mathcal{F}^{-1} = U \ast G(P_0, h), \]

where $G$ is real and of order 0. Testing this relation on the same function $u_0$ as before, we see that the leading part $G_0(E)$ of $G$ satisfies $G_0(0) = 0$. Iterating (b.40), and using that $P_0$ satisfies (b.38), we get
\[ \psi^2 U \psi^{-2} = U e^{i\theta_0} G(P_0, h) \psi^{-1} = \psi e^{i\theta} G(-P_0, h) = U e^{i(G(P_0, h) + G(-P_0, h))} \]

Conjugating this with \( \psi^2 \) and using that \( \psi^4 = 1 \), we get

\[ U = U e^{i(G(P_0, h) + G(-P_0, h))}. \]

Hence \( G(P_0, h) + G(-P_0, h) = \pi k, k \in \mathbb{Z}. \) Since \( G(0, 0) = 0 \), we have necessarily that \( k = 0 \), so

(b.41) \[ G(-t, h) = -G(t, h). \]

We now look for \( V = U e^{iJ(P_0, h)} e U(\psi_0) \) which commutes with \( \psi \). A straightforward calculation shows that \( \psi V \psi^{-1} = V \) if

\[ G(t, h) = J(t, h) - J(-t, h) + 2\pi n \]

for some \( n \in \mathbb{Z} \). In view of (b.41), we can take

\[ J = \frac{1}{2} G \]

and \( k = 0 \). Thus, \( V \) commutes with \( \psi \) and in order to save notations, we shall assume that already \( U \) commutes with \( \psi \). Let \( V = U e^{iJ(P_0, h)} e U(\psi_0) \) be a second element which commutes with \( \psi \). Then a simple computation gives the sufficient condition on \( J: 0 = J(t, h) - J(-t, h) + 2\pi k \)

for some \( k \in \mathbb{Z} \). Taking \( t = 0 \), we see that \( k \) must be \( 0 \), so the condition on \( J \) is simply that \( J \) be even. Now look for \( V = U e^{iJ(P_0, h)} \), with \( J \) even, such that

(b.42) \[ \Gamma V \Gamma = V. \]

First, we notice that \( \Gamma U \Gamma e U(\psi_0) \), so \( \Gamma U \Gamma = U e^{iK(P_0, h)} \) with \( K \) real. Iterating this and using that \( \Gamma^2 = I \), we get \( K(t, h) - K(-t, h) = 2\pi n \). Again \( t = 0 \) implies that \( n = 0 \), so \( K \) is even. A simple computation shows that (b.42) holds if

\[ K(t, h) = J(t, h) + J(-t, h) + 2\pi n \]

for some \( n \in \mathbb{Z} \). Since \( K \) is even, we can take \( J = \frac{1}{2} K \). We have proved,

**Proposition b.3.** In addition to the assumptions of Theorem b.1, we assume (b.38). Then in the theorem it is possible to take \( F \) to be odd, and \( U \) to commute with \( \psi \) and \( \Gamma \).

**Remark b.4.** If we assume that \( P = P_\mu \) depends on the real parameter \( \mu \in \text{neighborhood of } 0 \), in such a way that \( P_\mu(x, \xi, h) \) is an analytic symbol in \( \mu, x, \xi \). Then, possibly after shrinking the neighborhood in \( \mu \), we can construct \( U, F \) depending analytically on \( \mu \) in the sense that all phase-functions and canonical transformations are analytic in \( \mu \), and all symbols remain analytic symbols after addition of \( \mu \) as an independent variable. This is valid for all the results above. Moreover, if \( P \) depends on additional parameters \( \alpha \), in such a way that \( P \) or \( P_\mu \) varies in a bounded set of analytic symbols, then all symbols, phases, and transformations, will also vary in bounded sets.

**Remark b.5.** Assume that \( P = P_\mu \) depends analytically on \( \mu \) as in the preceding remark, that \( P_\mu \) is selfadjoint for real \( \mu \), but that the assumptions of the theorem are satisfied only for \( \mu = 0 \). Let \( p_\mu \) denote the principal symbol of \( P_\mu \). Then \( p_\mu \) has a saddle point \((x_\mu, \xi_\mu)\) depending analytically on \( \mu \), and after conjugation by a unitary Fourier integral operator, we may assume that \((x_\mu, \xi_\mu) = (0, 0)\). Then \( P_\mu - p_\mu(0, 0) \) satisfies all the assumptions of the theorem, so there is a real-valued analytic symbol \( F(\mu, t, h) \) and unitary operators \( U_\mu \) depending analytically on \( \mu \), such that

\[ U_\mu^{-1} F(\mu, P_\mu - p_\mu(x_\mu, \xi_\mu), h) U_\mu = P_0. \]

If \( u = U_\mu v \) is a microlocal solution of the
equation \( P_{\mu u} = 0 \), then \( P_0 v = F(\mu, -P_{\mu}(x_\mu, \xi_\mu), h)v \), and conversely.

We shall now look at the second case when \((0,0)\) is a non-degenerate minimum, so we keep all the assumptions of Theorem b.1, except that we replace the assumption that \((0,0)\) be a (non degenerate) saddle point by the assumption that \((0,0)\) is a non-degenerate minimum for \(p\), again with critical value 0. Consider first the symbol \( p_0 = \frac{1}{2}(\xi^2 + x^2) \), \( H_{p_0} = \xi \partial_x - x \partial_x \). Then the Hamilton flow is periodic with period \( T_0 = 2\pi \). If \( \Gamma_0 = \{(x,x)\} \), then every real point \((x,\xi)\) can be written \((x,\xi) = \exp(tH_{p_0})(y,y)\), for some real \((y,y)\) and some \( t \in \mathbb{R} \). Here \( y \) is unique up to the sign and once \( y \) has been fixed, \( t \) is unique up to a multiple of \( 2\pi \). If we take \((x,\xi)\) complex, we have the analogous result, provided that \( p_0(x,\xi) \neq 0 \). Also notice that \( \exp(\frac{1}{2}T_0 H_{p_0})(\Gamma_0) = \Gamma_0 \).

Returning to the symbol \( p \), we see as in the beginning of this section, that there is a real and analytic function \( T(E) \), such that \( T(E) \) is a period for the \( H_p \) flow restricted to the energy surface \( p^{-1}(E) \), and we may assume that \( T(E) = T_0 + O(E) \). Replacing \( p \) by \( f(p) \), where \( f' = Df(L) \), we may assume

\[
(b.43) \quad T(E) = T_0.
\]

As before, we construct an analytic curve \( \Gamma \) tangent to \( \Gamma_0 \) at \((0,0)\), such that \( \exp(\frac{1}{2}T_0 H_p)(\Gamma) = \Gamma \), and we write,

\[
(b.44) \quad p|_{\Gamma} = q^2, \quad p_0|_{\Gamma_0} = q^2_0,
\]

where \( q \) and \( q_0 \) are analytic on \( \Gamma \) and \( \Gamma_0 \) respectively, and \( dq(0,0) = dq_0(0,0), \)
\( q_0(x,x) = 2^{-\frac{1}{2}}x \). Using this, we can proceed as before, and construct a real valued analytic canonical transformation \( \chi \), such that,

\[
(b.45) \quad p \star \chi = p_0, \quad d\chi(0,0) = id.
\]

(In the estimates of the flows, the roles of \( \text{Re}(t) \) and \( \text{Im}(t) \) are now permuted, and the stable manifolds for the \( H_{p_0} \) flow are now given by \( \xi \pm ix = 0 \).) This means that we may assume that \( p = p_0 \).

Consider now the full operators. As a model operator, we take

\[
P_0 = \frac{1}{2}((h\partial)^2 + x^2 - h),
\]

and we notice that,

\[
(b.46) \quad e^{-iT_0 p_0}/h = I,
\]

where we no more have to use the \( H_p \) spaces to justify our arguments. As before, we can construct a real valued analytic symbol \( G(t,h) \) of order 0, such that,

\[
(b.47) \quad e^{-iT_0 (P + hG(P,h))}/h = I.
\]

Contrary to the earlier case, \( G \) is uniquely determined only up to an integer. Replacing \( P \) by \( P + hG(P,h) \), we have then reduced the problem to the case when,

\[
(b.48) \quad e^{-iT_0 P}/h = I.
\]

Comparing the first transport equations, given by (b.46) and (b.47), we get,

\[
(b.49) \quad \int_0^{T_0} p_{-1}(x(t), \xi(t)) dt - \int_0^{T_0} \text{int}_{-1/2} dt = 2\pi k,
\]

for some \( k \in \mathbb{Z} \). After modifying \( G \) by an integer, we can assume that,
We can then repeat the earlier arguments with almost no changes, and obtain,

**Theorem b.6.** Let $P(x, hD, h)$ be a formal classical analytic pseudodifferential operator, of order 0, formally selfadjoint, with symbol defined in a neighborhood of $(0, 0)$. Let $p$ be the principal symbol, and assume that $p$ has a non-degenerate minimum at $(0, 0)$ with critical value 0. Then there is a real-valued analytic symbol $F(t, h) \sim \sum_0^\infty f_j(t) h^j$, defined for $t$ in a neighborhood of 0, and a formal analytic Fourier integral operator, which is unitary, and whose associated canonical transformation is defined in a neighborhood of $(0, 0)$, and maps this point onto itself, such that

$$U^* F(P, h) U = P_0 = \frac{1}{2} ((hD)^2 + x^2) - h.$$
For typographical reasons, we denote by \([A]_{j,k}, j,k=1,2\), the 2x2-matrix with \(a_{\alpha,\beta}=1 \text{ if } (\alpha,\beta)=(j,k)\), and \(a_{\alpha,\beta}=0\) otherwise. We shall first discuss reductions to normal forms on the level of principal symbols, and we start with the case of an analytic hermitian 2x2-matrix \(p(x,\xi)\), defined for \((x,\xi)\) in a neighborhood of \((0,0)\) in \(\mathbb{R}^2\), which satisfies
\[
-\det(p(x,\xi)) \sim x^2 + \xi^2, \quad p(0,0)=0.
\]
Here \(\sim\) denotes "of the same order of magnitude as". On the principal symbol level, we allow transformations \(p \rightarrow a^*pa\), where \(a\) is invertible and analytic, as well as real, analytic canonical changes of variables which preserve the origin. Applying the reductions of [HS2], we then have a first reduction to the case when \(p=[\xi_{1,2}+\xi_{2,1}]\), \(\xi=\xi+i\eta\). After replacing \(\xi\) by \(\xi+O((x,\xi)^2)\), we can then write,
\[
\begin{align*}
(p(x,\xi))&=[c_1(x,\xi)]_{1,1}+[c_2(x,\xi)]_{1,2}+[\xi_{1,2}+\xi_{2,1}],
\end{align*}
\]
with \(c_j=O((x,\xi)^2)\). We then look for \(a_1, a_2=O((x,\xi))\), such that,
\[
(1+[a_1]_{1,2}+[a_2]_{2,1})p(1+[a_1]_{1,2}+[a_2]_{2,1})^*=[\xi']_{1,2}+[\xi'_{2,1}],
\]
with \(\xi'=\xi+O((x,\xi)^3)\). To obtain (c.3), it is enough to solve the system,
\[
\begin{align*}
c_1+a_1\xi+a_2\xi+|a_1|^2c_2&=0, \\
-c_2+a_2\xi+a_1\xi+|a_2|^2c_1&=0,
\end{align*}
\]
and we will then have \(\xi' = \xi + a_1c_2 + a_1\bar{a}_2\xi + c_1\bar{a}_2\). The equations in (c.4) are independent, so we may concentrate on the first one. We write,
\[
(1+c_1)\xi+2g\xi = a_1h+\bar{k}\xi, \quad a_1h+\bar{k}\xi = 0.
\]
It is then enough to find analytic function \(h, k\) such that,
\[
\begin{align*}
f+k+c_2h &= 0, \\
2g+h+\bar{h}+c_2(h\bar{h}+k\bar{k}) &= 0.
\end{align*}
\]
Here \(g\) and \(c_2\) are real and since \(c_2\) is very small, if we shrink the neighborhood of \((0,0)\) under consideration, the implicit function theorem gives a unique analytic solution of (c.6), if we impose the additional assumption that \(h\) should be real.

Thus, a symbol \(p\) satisfying (c.1) can be reduced to,
\[
(p_0(x,\xi))=[\xi_{1,2}+\xi_{2,1}]_{1,1}, \quad \xi=\xi+i\eta+O((x,\xi)^2).
\]
We now consider a new selfadjoint analytic symbol \(p=p_0+O(\epsilon)\), where \(O(\epsilon)\) refers to a perturbation whose \(L^\infty\) norm over a fixed complex neighborhood of \((0,0)\) is \(O(\epsilon)\). Changing \(\xi\) by \(O(\epsilon)\), we can then write,
\[
(p(x,\xi))=[c_1(x,\xi)]_{1,1}+[c_2(x,\xi)]_{1,2}+[\xi_{1,2}+\xi_{2,1}], \quad c_j=O(\epsilon),
\]
and we look for \(a_j=O(\epsilon)\) such that,
\[
(1+[a_1]_{1,2}+[a_2]_{2,1})p(1+[a_1]_{1,2}+[a_2]_{2,1})^*=[c_1']_{1,1}+[c_2']_{1,2}+[\xi']_{1,2}+[\xi'_{2,1}],
\]
with \(\xi-\xi'=O(\epsilon^2), c_j-c_j'=O(\epsilon),\) and \(c_j'\) is constant. More explicitly, we have
to find \( a_j = \Theta(\varepsilon) \), such that,

\[
\begin{align*}
(c.10) & & c_1 + a_1 \xi + \bar{a}_1 \xi + |a_1|^2 c_2 = c_1' = \text{const.} \\
& & c_2 + a_2 \xi + \bar{a}_2 \xi + |a_2|^2 c_1 = c_2' = \text{const.}
\end{align*}
\]

and again we may concentrate on one of the equations, for instance the first one. Write \( \xi = x + iy \), and notice that \( x, y \) can be used as local coordinates. If we write \( a_1 = f(x, y) + ig(x, y) \), then \( a_1 \xi + \bar{a}_1 \xi = 2(f(x + iy), \) and the first equation in (c.10) takes the form,

\[
(c.11) \quad c_1(x, y) + 2(f(x, y)x + g(x, y)y) + c_2(f^2 + g^2) = d_1 = \text{const.}
\]

Let us assume for simplicity, that \( c_j \) are holomorphic and bounded in

\( D = ((x, y) \in \mathbb{C}^2; |x| < 1, |y| < 1) \), and that \( \|c_j\|_{L^\infty(D)} \leq \varepsilon \). If \( F \) is a bounded holomorphic function on \( D \), we obtain by successive divisions and the maximum principle that \( F(x, y) - F(0, 0) = A(x, y)y + B(x, y)x \), where

\( \|A\|_{L^\infty(D)} \leq \|F\|_{L^\infty(D)} \), \( \|B\|_{L^\infty(D)} \leq \|F\|_{L^\infty(D)} \). As a first approximate solution of (c.11), we can take \( f_1, g_1 \) holomorphic on \( D \), with

\( 2\|f_1\|, 2\|g_1\| \leq \|c_1(0, 0) - c_1(0, 0)\| \leq 2\varepsilon \), such that,

\[
(c.12) \quad c_1(x, y) + 2(f_1(x, y)x + g_1(x, y)y) = c_1(0, 0).
\]

Then,

\[
(c.13) \quad c_1 + 2(f_1x + g_1y) + c_2(f_1^2 + g_2^2) = c_1(0, 0) + c_2(f_1^2 + g_2^2), \quad \text{with}
\]

\( \|c_2(f_1^2 + g_2^2)\| \leq 2\varepsilon^3 \). Assume by induction, that we have found \( f_k, g_k \), with

\[
(c.14) \quad c_1 + 2(f_kx + g_ky) + c_2(f_k^2 + g_k^2) = d_k + r_k,
\]

where \( d_k \) is constant (and real), \( 2\|f_k\|, 2\|g_k\| \leq M_k, \|r_k\| \leq m_k \). Then we choose \( f_{k+1}, g_{k+1} \) real with

\[
(c.15) \quad 2((f_{k+1} - f_k)x + (g_{k+1} - g_k)y) = -(r_{k+1} - r_k),
\]

\[
(c.16) \quad 2\|f_{k+1} - f_k\|, 2\|g_{k+1} - g_k\| \leq \|r_k\| \leq 2m_k,
\]

so \( \|f_{k+1}\|, \|g_{k+1}\| \leq M_k + m_k = \text{def.} M_{k+1} \). Then,

\[
(c.17) \quad c_1 + 2((f_{k+1}x + g_{k+1}y) + c_2((f_{k+1}^2 + (g_{k+1})^2) = d_{k+1} + r_{k+1},
\]

where \( d_{k+1} = d_k + r_k(0, 0), r_{k+1} = c_2((f_{k+1}^2 - f_k^2 + g_{k+1}^2 - g_k^2) \). Here,

\( \|f_{k+1} - f_k\| \leq (\|f_{k+1}\| + \|f_k\|) \|f_{k+1} - f_k\| \leq 2(M_k + m_k)m_k \), and similarly for \( g_{k+1} - g_k \) and hence, \( \|r_{k+1}\| \leq 4\varepsilon(M_k + m_k)m_k = \text{def.} m_{k+1} \). With the initial choice \( f_0 = g_0 = 0 \), we have \( M_0 = 0, m_0 = 2\varepsilon \), and we are led to study the recurrence relations,

\[
(c.18) \quad m_{k+1} = 4\varepsilon(M_k + m_k)m_k,
\]

\[
M_{k+1} = M_k + m_k.
\]

Assume by induction that \( m_j \leq 2^{1-j}\varepsilon, 0 \leq j \leq k \), which is certainly true for \( k = 0 \). Then \( M_k = m_0 + \ldots + m_k \leq 4\varepsilon \), and

\( m_{k+1} \leq 4\varepsilon \varepsilon 2^{1-k}\varepsilon = (4\varepsilon^2)2^{1-(k+1)}\varepsilon \leq 2^{1-(k+1)}\varepsilon \), if \( 4\varepsilon^2 \leq 1 \), so the induction procedure goes through, if \( \varepsilon > 0 \) is small enough, and \( f_k, g_k \) converge to \( f, g \); analytic solutions of the problem (c.11) with \( f, g = \Theta(1)(\|c_1\| + \|c_2\|) \).
This implies a solution of the problem (c.10).

Now let $P$ be a formally selfadjoint classical $2 \times 2$ pseudodifferential operator of order 0. We assume that the principal symbol $p$ satisfies $|p-p_0| \leq \varepsilon$ in a fixed complex neighborhood of $(0,0)$, where $p_0$ is hermitian, and satisfies (c.1). If $\varepsilon > 0$ is sufficiently small, the preceding discussion implies that there is an elliptic scalar Fourier integral operator $U$ of order 0 and an elliptic $2 \times 2$-system of zero order pseudodifferential operators, $A$, such that $(AU)^*P(AU)$ has the principal symbol

\[
(c.19) \quad |c_{11}|_1 + |c_{22}|_2 + |\zeta|_1 + |\bar{\zeta}|_2 , \quad \text{where } c_j = O(\varepsilon) \text{ are constants, and } \\
\zeta = \xi + ix + O((x, \xi)^2).
\]

By further conjugations, we shall now see that we can arrange so that the full symbol also has diagonal elements which are independent of $(x, \xi)$. We may assume that already $P$ has the principal symbol (c.19). Then, $(I + [A_1]_1 + [A_2]_2)_1P(I + [A_1]_1 + [A_2]_2)_1^*$ will have this property, if we can find classical analytic pseudodifferential operators, $A_j$ of order $-1$, such that

\[
(c.20) \quad C_1 + A_1Z^* + ZA_1^* + A_1C_2A_1^* = \text{const.}, \\
C_2 + A_2Z + ZA_2^* + A_2C_1A_2^* = \text{const.}.
\]

Here, we have written $P = [C_1]_1 + [C_2]_2 + [Z]_1 + [Z^*]_2$, so the principal symbol $c_j$ of $C_j$ is constant, and $Z$ has the principal symbol $\zeta$. Again the two equations can be treated separately, and it is quite easy to see that there are classical pseudodifferential operators, $A_j$ which satisfy these equations. The difficulty, is to verify that $A_j$ are analytic pseudodifferential operators, i.e. we have to show the usual growth conditions on the asymptotic expansions of $A_j$.

Let $\Omega_t$, $0 \leq t \leq t_1$ be an increasing family of relatively compact open sets in $\mathbb{C}^2$, such that dist$(\Omega_t, \mathbb{C} \setminus \Omega_s) \geq s-t$, for $s \geq t$. If $a(x, \xi, h) = \Sigma_0^{\infty} a_k(x, \xi) h^k$ is an analytic symbol defined on $\Omega_t$, we put $A(x, \xi, D_x, h) = a(x, \xi + hD_x, h) = \Sigma_0^{\infty} h^k A_k(x, \xi, D_x)$ (as in [S1]), so that $A_k$ is a differential operator of order $\leq k$. Let $f_k(A) \geq 0$, be the smallest constant such that $\|A_k\|_{t, s} \leq f_k(A)h^k (s-t)^k$, $0 \leq t < s \leq t_1$, where $\|A_k\|_{t, s}$ denotes the operator norm of $A_k$: $\text{Hol}(\Omega_s) \cap L^\infty(\Omega_s) \rightarrow \text{Hol}(\Omega_t) \cap L^\infty(\Omega_t)$, and $\text{Hol}(\Omega)$ denotes the space of holomorphic functions on $\Omega$. If $a$ is an analytic function defined in a neighborhood of $\Omega_t$, then for $p > 0$ sufficiently small, $\|a\|_p = \defin \Sigma f_k(A) h^k$ is finite, and conversely, if the $a_k$ are all holomorphic in $\Omega_t$, then $a$ is an analytic symbol in $\Omega_t$. We recall from [S1], that if $a$ and $b$ are analytic symbols, then $\|a \cdot b\|_p \leq \|a\|_p \|b\|_p$. Here $a \cdot b$ is the symbol of the composition of the corresponding pseudodifferential operators. (In this section we do not use the Weyl quantization, but rather the "classical" one.)

Before attacking (c.20), we analyze two simpler division problems. Let $\Omega_t = \{x; |x| < t\} \times (\xi; |\xi| < t)$. We start by looking at division to the left by $x$. If
\(a(0, \xi, h) = 0\), we can write \(a(x, \xi, h) = xb(x, \xi, h)\), where for the corresponding operators \(A, B\), we have \(A_k = xB_k\). By the maximum principle, if \(u \in H^1(\Omega_0) \cap L^\infty\), and \(\|\cdot\|_t\) denotes the corresponding \(L^\infty\) norm, then
\[
\|A_ku\|_t = t \|B_ku\|_t,
\]
which implies that \(\|B_k\|_{t, s} = t^{-1} \|A_k\|_{t, s} \leq \|A_k\|_{t, s}\), and hence,
\[
(c.21) \quad \|b\|_p \leq t_0^{-1} \|a\|_p, \text{ when } a = xb.
\]

Next we look at division to the right by \(hD\). Assume that \(a(x, 0, h) = 0\). Then we can write; \(a(x, hD, h) = b(x, hD, h)hD\), or simply, \(a(x, \xi, h) = b(x, \xi, h)\xi\), which gives, \(A = B \ast (\xi + hD_x) = \Sigma_0 \ast \infty h_k B_k(x, \xi, hD_x)\xi = \Sigma_0 \ast \infty h_k (\xi B_k + B_k^{-1} D_x)\), with the convention that \(B_{-1} = 0\). In other words, \(A_k = \xi B_k + B_k^{-1} D_x\), or rather, \(\xi B_k = A_k - B_k^{-1} D_x\), and as in the case of division by \(x\), we obtain,
\[
\|B_k\|_{t, s} = \|B_k\|_{t, s} = f_k^{-1}(B)(-1)^{k-1}((r-t)-(s-t))^{-1}, \text{ for } t < r < s.
\]
Choosing \(r\) such that \(r-t=(k-1)(s-t)/k\), we get \(\|B_k\|_{t, s} \leq f_k^{-1}(B)k/(s-t)^k\), so \(\|b\|_p \leq \|a\|_p + \|b\|_p\), and hence,
\[
(c.22) \quad \|b\|_p \leq (t_0 - p)^{-1} \|a\|_p, \text{ if } p < t_0 \text{ and } a = b\xi.
\]

**Lemma c.1.** If \(a\) is an analytic symbol on \(\Omega_{t_1}\) of order 0, with \(\|a\|_p < \infty\), for some sufficiently small \(p > 0\), then we have a decomposition,
\[
(c.23) \quad a = a(0, 0, h) + xc + b\xi,
\]
where \(b\) and \(c\) are analytic symbols of order 0, with \(\|b\|_p \leq 2(t_0 - p)^{-1} \|a\|_p\), \(\|c\|_p \leq 2t_0^{-1} \|a\|_p\). We also have \(\|a(0, 0, h)\|_p = \|a\|_p\), and the choice of \(b, c\) is unique if we impose the additional assumption, that \(c = c(x, h)\) is independent of \(\xi\).

**Proof.** The estimate on \(\|a(0, 0, h)\|_p\) follows easily from the definitions. We also have \(\|a(x, 0, h)\|_p \leq \|a\|_p\). With the additional condition that \(c\) is independent of \(\xi\), it is clear that the unique choice of \(c\), \(b\) is given by,
\[
(c.24) \quad xc(x, h) = a(x, 0, h) - a(0, 0, h), \quad a(x, \xi, h) - a(x, 0, h) = b(x, \xi, h)\xi.
\]
Again, it follows from the definitions that \(\|c\|_p \leq 2t_0^{-1} \|a\|_p\). (Notice that the operator \(C\) associated to \(c\) is simply multiplication by \((a(x, 0, h) - a(0, 0, h))/h\), so \(C_k = a_k(x, 0) = A_k(1)(x, 0)\). Hence \(\|C_k\|_{t, s} = \|a_k(x, 0)\|_t \leq \|a_k\|_t = \|A_k(1)\|_t \leq \|A_k\|_{t, s}\).) The estimate of \(\|b\|_p\) follows from this and from (c.22).

As a preparation for the handling of complex adjoints, we also need to study the behaviour of the "\(p\)-norms", when we take commutators with \(x\) and with \(hD_x\). Let us first estimate \(b = [x, \alpha]\), where the composition is that of the corresponding operators, and where \(\alpha\) is an analytic symbol of order 0 with \(\|\alpha\|_p < \infty\) for some \(p > 0\). Let \(A = \Sigma h_k A_k, B = \Sigma h_k B_k\), be the corresponding operators introduced before. Using the explicit form,
\[
A_k = \Sigma_j + \alpha_k = k(\alpha!)^{-1} \partial_\xi \alpha_j(x, \xi) D_x \alpha_j, \text{ and similarly for } B, \text{ one finds that}
\]
Using that \[ \parallel B \parallel \leq 2u^{-1}f_{k-1}(A)(k-1)^{-1}v_{k-1}(k-1), \] for all \( u, v > 0 \) with \( u + v = s - t \). Choosing \( u = (s-t)/k, v = (k-1)(s-t)/k \), we obtain \( f_k(B) \leq 2f_{k-1}(A) \). (Notice that \( B_0 = 0 \), since \( b \) is of order \(-1\).) This implies that,

\[ ||B(u,v)||_p^2 \leq 2p ||a||_p. \]

The estimate of \( b = [hD,a] \), works the same way, since for the corresponding operators \( A, B \), we have, \( B_k = [D_k, A_{k-1}] \), and we obtain,

\[ ||[hD,a]||_p^2 \leq 2p ||a||_p. \]

Now return to the decomposition (c.23). We write \( a = a(0,0,h) + xc + hDb - [hD,b] \), where \( [hD,b] \) is of order \(-1\) and \( ||[hD,b]||_p^2 \leq 2p ||b||_p^2 \leq 4(p/(t_0-p)) ||a||_p \). Taking \( p \) so small that \( 4p/(t_0-p) = 1/2 \), we redecompose \( [hD,b] \) as in the lemma, et c. Eventually we then obtain the general decomposition,

\[ a = d(h) + xc'(x,h) + hDob'(x,hD,h), \]

where \( d, c', b' \) are analytic symbols of order 0, such that,

\[ ||d||_p, ||c'||_p, ||b'||_p \leq C_0 ||a||_p \]

Here \( C_0 > 0 \) only depends on \( t_0 \). Again, it is easy to see that \( d, c', b' \) are unique (with the requirement that \( c' \) should be independent of \( \xi \)).

We can now combine decompositions and adjoints. If \( a \) is an analytic symbol, we let \( a^* \) denote the symbol corresponding to the complex adjoint, and we put, \( ||a||_{p'} = ||a||_p + ||a^*||_p \). Notice the general inequality:

\[ ||a \cdot b||_p \leq ||a||_p ||b||_p. \]

Taking the adjoints of the decomposition (c.23), we get \( a^* = \bar{a}(0,0,h) + xc^*(x,h) + hD^*b^*(x,hD,h) \), which is of the form (c.28), and hence we get for the decomposition (c.23):

\[ ||a(0,0,h)||_{p'} + ||b||_{p'} + ||c||_{p'} \leq C_0 ||a||_{p'}. \]

We now return to the problem (c.20). There are analytic pseudodifferential operators \( F, G \) of order 0 and \( F_{-1}, G_{-1} \) of order \(-1\), such that,

\[ hD = FZ + Z^*F^* + F_{-1}, \quad x = GZ + Z^*G^* + G_{-1}. \]

Moreover \( F_{-1} \) and \( G_{-1} \) are selfadjoint. Let \( a \) be an analytic symbol of order 0, and let \( b, c \) be as in the decomposition (c.23). Substituting (c.30) into (c.23), we get,

\[ a - \text{const.} = (bF + Gc)Z + Z^*(bF + Gc)^* + R(a), \]

where,

\[ R(a) = G[Z,c] + [b,Z^*]F^* + bF_{-1} + G_{-1}c + Z*[b,F^*] + Z^*[b-b^*] + Z^*[G^*,c] + Z^*[c-c^*]G^*. \]

Now assume that \( a \) is selfadjoint. Then \( b - b^* \) and \( c - c^* \) are of order \(-1\), and hence \( R(a) \) is of order \(-1\) and we obtain,

**Lemma c.2.** Let \( a \) be an analytic symbol of order \( m \leq 0 \), defined in \( \Omega_t \), with \( ||a||_{p'} < \infty \) for \( p < p_0 \), where \( p_0 > 0 \). If \( a \) is selfadjoint, we can find analytic symbols \( A \) of order \( \leq m \) and \( R(a) \) of order \( \leq m - 1 \), such that
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\[(c.33) \quad a = AZ + ZA^* + R(a) + \text{const.},\]
and \[\left\| A \right\|_p^' + \left\| R(a) \right\|_p \leq C_0 \left\| \text{a} \right\|_p, \text{ for } p < \min(p_0, 1/C_0).\] Here \(C_0\) is independent of \(a, m, p\).

If \(C, D\) are self-adjoint of order \(0\), and \(D\) has a constant principal part, we next want to find an analytic symbol \(A\) of order \(-1\), such that

\[(c.34) \quad AZ + ZA^* + ACA^* = D + \text{const.},\]
which is nothing but the second equation of \((c.20)\) (with \(A = A_2, D = -C_2, C = C_1\)). Trying \(A = A_{-1} + A_{-2} + \ldots\) with \(A_{-j}\) of order \(-j\), we get by successive use of Lemma \(c.2\) the following recursive system:

\[A_{-1}Z + ZA_{-1} = D_{-1} + R(D_{-1}) + \text{const.}, \text{ (where } D = D_{-1} + \text{const.}),\]
\[A_{-2}Z + ZA_{-2} = -R(D_{-1}) - A_{-1}CA_{-1} - R(R(D) + A_{-1}CA_{-1}) + \text{const.},\]

....

\[A_{-N}Z + ZA_{-N} = D_{-N} + R(D_{-N}) + \text{const.},\]

where,

\[D_{-N} = -R(D_{-N+1}) - (A_{-1}CA_{-N+1} + A_{-2}CA_{-N+2} + \ldots + A_{-N+1}CA_{-1}).\]

Put \(m(n, p) = \left\| D_{-n} \right\|_p^'\). Then for \(p \leq 1/C_0\): \[\left\| A_{-n} \right\|_p^' \leq C_1 m(n, p),\]

\[(*) \quad m(n, p) \leq C_0 m(n-1, p) + C_0 \left\| C \left\|_p^' \right\| (m(1, p)m(n-1, p) + m(2, p)m(n-2, p) + \ldots + m(n-1, p)m(1, p)),\]

and we make the induction hypothesis,

\[(N-1) \quad m(k, p) \leq E k^{-2} F^k, \quad k \leq N-1.\]

This hypothesis is fulfilled for \(N-1 = 1\), if \(EF \geq \left\| D_{-1} \right\|_p^'\).

Substitution into (*) gives,

\[m(n, p) \leq C_0 \left\| C \left\|_p^' \right\| E^2 F^n (\sum_{1 \leq k \leq N-1} k^{-2}(N-k)^{-2}) + E(N-1)^{-2} F^N \right\| \leq C_0 C_1 \left\| C \left\|_p^' \right\| E^2 F^N / N^{2} + C_0 C_1 E F^{N-1}(N-1)^{-2},\]

where \(C_1\) is a universal constant. The hypothesis \((N)\) will then be satisfied, if \(C_0 C_1 E \left\| C \left\|_p^' \right\| < 1\). This can be achieved, if we first fix \(E\) sufficiently small so that \(C_0 C_1 E \left\| C \left\|_p^' \right\| \leq \frac{1}{2}\), and then choose \(F\) sufficiently large. (We then also get the induction hypothesis \((1)\).)

Hence there is a choice of \(E, F\), such that \(m(k, p) \leq E k^{-2} F^k\) for all \(k \geq 1\). It follows that \(A = A_{-1} + A_{-2} + \ldots\) is a well defined analytic symbol satisfying \((c.34)\). We have thus solved the second equation of \((c.20)\). The first equation can be handled the same way, and this completes the proof of:

**Proposition c.3.** Let \(p_0\) be hermitian and satisfy \((c.1)\). Let \(P\) be a formally self-adjoint 2 × 2 system of classical analytic pseudodifferential operators, whose complete symbol is defined in some fixed complex neighborhood of \((0,0)\) in \(\mathbb{R}^2\). Assume further that the principal symbol \(p\) satisfies, \(\left\| p - p_0 \right\| \leq \varepsilon,\) in that neighborhood. Then if \(\varepsilon > 0\) is sufficiently small, there exist an elliptic 2 × 2 system, \(A\) of classical analytic pseudodifferential operators of order 0, (with symbol defined in a fixed \(\varepsilon\)-independent neighborhood of \((0,0)\)), and an elliptic scalar analytic Fourier integral operator of order 0, (whose associated
canonical graph is closed in an $\varepsilon$-independent neighborhood of $((0,0),(0,0))$, such that,
\begin{equation}
(AU)^*P(AU) = [C_1]_{1,1} + [C_2]_{2,2} + [Z]_{1,2} + [Z^*]_{2,1},
\end{equation}
where $C_1$ and $C_2$ are analytic symbols of order 0, independent of $(x, \xi)$, and $Z$ is of order 0 with principal symbol $\xi = \xi + ix + O((x, \xi)^2)$. 

We shall first develop a simple theory of WF and WF' in the microlocal setting of section a.1. Most of this material is already in [S1] and in [GrS]. If \( \Omega \subset \mathbb{C}^n \) is open, and \( \Phi: \Omega \to \mathbb{R} \) is continuous, we slightly modify our earlier definition, by letting \( H^\phi_{10C}(\Omega) \) denote the space of functions, \( u(x,h) \) on \( \Omega \times [0,2\pi] \), such that,

1° \( u \) is of class \( C^\infty \) with respect to \( x \), and for every compact set, \( K \subset \Omega \),

\[
\text{there exist } \varepsilon = \varepsilon_K > 0, \text{ and } C = C_K > 0 \text{ such that } |\overline{\partial} u| \leq C \frac{e(\Phi(x) - \varepsilon)}{h} \text{ on } K \times [0,2\pi].
\]

2° For every compact set \( K \subset \Omega \) and all \( \varepsilon > 0 \), and \( \alpha, \beta \in \mathbb{N}_n \), there exists a constant \( C = C_{\varepsilon, \alpha, \beta} > 0 \) such that,

\[
|\partial^\alpha \overline{\partial}^\beta u| \leq C \frac{e(\Phi(x) + \varepsilon)}{h} \text{ on } K \times [0,2\pi].
\]

We also say that two elements \( u_1, u_2 \in H^\phi_{10C}(\Omega) \) are equivalent, and write \( u_1 = u_2 \), if for every compact set \( K \subset \Omega \), there exist \( C, \varepsilon > 0 \), such that,

\[
|u_1 - u_2| \leq C \frac{e(\Phi(x) - \varepsilon)}{h} \text{ on } K \times [0,2\pi].
\]

If \( \Phi \) is strictly plurisubharmonic, and if \( \Omega' \subset \Omega \) is strictly pseudoconvex, then for every \( u \in H^\phi_{10C}(\Omega) \), there exists \( u' \in H^\phi_{10C}(\Omega') \), holomorphic in \( x \), such that \( u = u' \) in \( H^\phi_{10C}(\Omega') \). A family \( \{u_\alpha\} \) of elements in \( H^\phi_{10C}(\Omega) \) is said to be bounded, if for \( u = u_\alpha \), we can choose all constants in 1° and 2° independent of \( \alpha \). Similarly, we define the notion of equivalent families.

If \( \{u_\alpha\} \) is such a bounded family, we define \( \Omega(\{u_\alpha\}) \), to be the largest open subset \( \Omega' \subset \Omega \) such that \( \{u_\alpha\} = 0 \) in \( H^\phi_{10C}(\Omega') \), and we put \( WF(\{u_\alpha\}) = \Omega \setminus \Omega(\{u_\alpha\}) \). Restricting to a single element, we also get a definition of \( WF(u) \), when \( u \in H^\phi_{10C}(\Omega) \).

We next extend these notions to the case of kernels, and for simplicity, we only discuss the case of single elements, the extension to the case of families being immediate. Let \( \Phi_j \in C(\Omega_j, \mathbb{R}) \), \( j = 1,2 \), where \( \Omega_j \) is an open subset of \( \mathbb{C}^n \). Let \( \rho \) denote the map \( y \mapsto \overline{y} \), and put \( (\Phi_1 + \Phi_2) \rho(x,y) = \Phi_1(x) + \Phi_2 \rho(y) \). If \( K(x,y,h) \in H^\phi_{10C}(\Omega_1 \times \Omega_2) \), then we put \( WF'(K) = (x, y) \in WF(K) \). \( WF'(K) \) is the smallest closed set in \( \Omega_1 \times \Omega_2 \), such that \( e^{-\Phi_1(x) - \Phi_2(y)} / h K(x, y, h) \) is locally uniformly of exponential decrease in its complement. We can associate to \( A \) a formal integral operator,

\[
Au(x,h) = \int K(x, y, h) u(y,h) e^{-2\Phi_2(y)} / hL(dy).
\]

We shall write \( K = K_A \) and \( WF'(A) = WF'(K) \). If \( u \in H^\phi_{10C}(\Omega_1) \) and if the projection \( WF'(A) \cap (\Omega_1 \times WF(u)) = (x,y) \mapsto x \in \Omega_1 \) is proper, then \( Au \) is well defined up to equivalence in \( H^\phi_{10C}(\Omega_1) \), and \( WF(A) \subset WF'(A)(WF(u)) \), where \( WF'(A) \) is interpreted as a relation. The equivalence class of \( Au \) only depends on the equivalence classes of \( A \) and of \( u \). Similarly, if \( B \) is given by a kernel \( K = K_B \in H^\phi_{10C}(\Omega_2) \), and if the projection,
(WF'(A)×WF'(B))∩Ω₁×diag(Ω₂×Ω₂)×Ω₃→Ω₁×Ω₃ is proper, then we can
define the composed kernel,
(d.3) \[ K_{A+B}(x,\bar{z},h) = \int K_A(x,\bar{y},h)K_B(y,\bar{z},h)e^{-2\Phi_2(y)/h}dy, \]
and we get WF'(A+B)⊂WF'(A)×WF'(B).

From now on, we assume that all domains are bounded, and that the
weight functions are analytic and strictly plurisubharmonic in a neighborhood
of the closure of the corresponding domains. For such a couple, \( (\Omega,\Phi) \), it is
well known that there exists a unique holomorphic function \( \Psi(x,y) \), defined in
a neighborhood of \( \{(x,\bar{x}); x\in \Omega \} \), such that \( \Psi(x,\bar{x})=\Phi(x) \) and that this function
has the property that,
(d.4) \[ \frac{1}{2}(\Phi(x)+\Phi(y))-\text{Re}\,\Psi(x,\bar{y}) \sim |x-y|^2. \]
A classical analytic pseudodifferential operator \( A \) of order \( m \) with symbol
defined in a neighborhood of \( \Lambda_{\Phi} \), can be realized by a kernel of the form,
(d.5) \[ K_A(x,\bar{y},h)=a(x,\bar{y},h)e^{2\Psi(x,\bar{y})/h}\chi(x,y), \]
where \( a \) is a classical analytic symbol of order \( m-\frac{1}{2}n \), defined in a
neighborhood of \( \{(x,\bar{x}); x\in \Omega \} \). Here \( \chi\in C^\infty(\mathbb{C}^2n) \) is equal to one near the set,
\( \{(x,\bar{x}); x\in \Omega \} \). This is obtained (see [S1]) by representing \( A \) as a formal Fourier
integral operator with a suitable phase function, and then choosing a suitable
integration contour. In particular, when \( A=I \), (d.5) becomes an approximate
Bergmann kernel. Using such an approximate Bergmann kernel, we can extend
the result above to the case when \( A \) is a Fourier integral operator of the type
considered in section a.1, with an associated canonical transformation \( \kappa \)
mapping \( \Lambda_{\Phi_2} \) to \( \Lambda_{\Phi_1} \) and such that \( \Phi_1(x_0)=1-m\Phi(x_0,y_0,\Theta_0)+\Phi_2(y_0) \) if
\( \kappa(y_0,\eta_0)=(x_0,\xi_0) \), and if \( \Phi \) is the phase used in the description of \( A \). If \( \Omega_1 \)
and \( \Omega_2 \) are suitable neighborhoods of \( x_0 \) and \( y_0 \) respectively, then we can
realize \( A \) by a kernel of the form (d.5), where now \( \Psi \) and \( a \) are defined in a
neighborhood of \( \Gamma_{\kappa} \), which by definition is the \( (x,y) \)-projection of the graph
of the restriction of \( \kappa \) to \( \Lambda_{\Phi_1} \), where \( \chi \) is equal to 1 near \( \Gamma_{\kappa} \) and where
\[ \frac{1}{2}(\Phi_1(x)+\Phi_2(y))-\text{Re}\,\Psi(x,\bar{y}) \sim \text{dist}((x,y),\Gamma_{\kappa})^2. \] In particular, we see that
WF'(A)⊂Γ_{\kappa}.

Using this observation, and the fact that different representatives of a
microlocally defined function (as in section a.1) are related by elliptic Fourier
integral operators, we see that if \( u=u_\hbar \) is a function, which is defined
microlocally in an open set \( V\subset\mathbb{R}^{2n} \), then we can define in a natural way, its
wavefront set WF(u) as a closed subset of \( V \). In the case when \( u \) is a
distribution independent of \( \hbar \), defined in some open set \( X\subset\mathbb{R}^n \), and we take \( V \)
to be \( X\times\mathbb{R}^n \), then WF(u)=\( \text{Supp}(u)\times\{0\}\cup WF_A(u) \), where \( WF_A \) is the classical
analytic wavefront set. (See [S1].) Similarly, let \( V_j\subset\mathbb{R}^{2nj} \) be open and let \( T_j \)
be FBI-transforms, permitting to represent microlocally defined functions in
\( \Omega_j \) as \( T_j u_j \) in \( H^{1/2}_\partial \cap_c (\Omega_j) \). If \( K(x,\bar{y},h) \) is a kernel as above, then we can think of
\( K \) as a kernel of an operator, \( A \), which takes certain functions, with compact
wavefront set in $V_2$ into functions, defined microlocally in $V_1$, and we get a corresponding definition of $WF'(A)$ in $V_1 \times V_2$. Using the natural change of $K$ under changes of FBI-transforms, we see that this notion is independent of the choice of such transforms, and concerning the possibility of defining $A_\mu$ or $A_\mu B$ we have the obvious analogous results to the one stated above for kernels and in the frame work of the $H_{10}$ spaces.

As an example, we shall compute $WF'(E_{\mu'})$, where $E=E_{\mu'}$ is the parametrix of $P_{0}-\mu'$, given by (4.21). Put,

\[(d.6) \quad U_0, t u(x) = e^{-t/2} u(e^{-t}x),\]

so that $U_0, t$ is unitary, and $U_t = e^{i\mu' t / h} U_0, t = \text{def.} U_{\mu', t}$. We choose the global unitary FBI-transform,

\[(d.7) \quad T u(x, h) = C_0 h^{-3/4} \int e^{-(x-y)^2/2h} u(y) dy,\]

where $C_0 > 0$ is a suitable normalization constant, and where the associated weight is,

\[(d.8) \quad \Phi_0(x) = \frac{1}{2} (|\text{Im} x|)^2,\]

and the associated canonical transformation is,

\[(d.9) \quad \kappa_T : (y, \eta) \mapsto (y + i\eta, \eta).\]

The adjoint of $T$ is then,

\[(d.10) \quad T^* v(y, h) = C_0 h^{-3/4} \int e^{-(\overline{z}-y)^2/2h} v(z) e^{-2\Phi_0(z)/h} L(dz),\]

so the kernel of $TU_0, t T^*$ is,

\[(d.11) \quad K_t(x, \overline{z}, h) = |C_0|^2 e^{-t/2} \int e^{-F(x, \overline{z}, y)/h} dy,\]

where,

\[(d.12) \quad F(x, \overline{z}, y) = \frac{1}{2} ((x-y)^2 + (\overline{z}-e^{-t}y)^2).\]

The gaussian integral can be computed, and we get,

\[(d.13) \quad K_t(x, \overline{z}, h) = C_1 h^{-1/2} e^{-t/2} (1 + e^{-2t})^{-1/2} e^{-G(t, x, \overline{z})/h},\]

where,

\[(d.14) \quad G(t, x, \overline{z}) = \frac{1}{2} (1 + e^{-2t})^{-1} (\overline{z}-e^{-t}x)^2.\]

We get,

\[(d.15) \quad \Phi_0(x) + \Phi_0(z) + \text{Re}(G(t, x, \overline{z}) = \frac{1}{2} (1 + e^{-2t})^{-1} (e^{-t}\text{Re} - \text{Re} z)^2 + (\text{Im} x - e^{-t} \text{Im} z)^2),\]

from which it follows that,

\[(d.16) \quad WF'(TU_0, t T^*) \subset \{(x, z); \text{Re} x = e^t \text{Re} z, \text{Im} x = e^{-t} \text{Im} z\}.\]

Using (d.9), we conclude that,

\[(d.17) \quad WF'(U_0, t) \subset \{(x, \xi; y, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2; x = e^t y, \xi = e^{-t} \eta\}.\]

Combining (d.13),(d.15), we also get,

\[(d.18) \quad WF'(\mathbb{C}_{\mu'}) \subset \text{Closure of } \{(x, z) \in \mathbb{C}^2; \text{there exists } t \geq 0 \text{ with } \text{Re} x = e^t \text{Re} z, \text{Im} x = e^{-t} \text{Im} z\},\]

where, $\mathbb{C}_{\mu'}$ is defined to be the kernel of,

\[(d.19) \quad (i/h) \int_0^\infty TU_{\mu'}, t T^* dt.\]

By contour deformation, we see that if
\(\text{(d.20)}\) \(\text{Re}x_0 = e^{t_0} \text{Re}z_0, \text{Im}x_0 = e^{-t_0} \text{Im}z_0,\) for some \(t_0 > 0,\) and if at that point,
\(\text{(d.21)}\)
then \((x_0, z_0) \notin WF'(\mathcal{C}(\mu'))\). In order to study this condition, we first notice that the map \(t \mapsto \text{Re}(G(t, x_0, z_0))\) has a critical point at \(t_0\), so it is enough to study \(\partial_t \text{Im}(G)\) at \((t_0, x_0, z_0)\). Making use of the relations \(\text{(d.20)}\), we get,
\(\text{(d.22)}\)
and for \(\mu' = 0\), we get,
\(\text{(d.23)}\)
so \((x_0, z_0) \notin WF'(\mathcal{C}(\mu'))\) if \(\mu' \neq -(\text{Im}x_0)(\text{Re}x_0)\). Similarly, for \(\mu' \neq 0\), we can eliminate any point \((x_0, z_0)\) outside the diagonal with \((\text{Im}x_0)(\text{Re}x_0) = (\text{Im}z_0)(\text{Re}z_0) = 0\), by contour deformation near \(\infty\). For \(\mu' \neq 0\) \(\text{(d.18)}\) improves to,
\(\text{(d.22)}\)
so \((x_0, z_0) \notin WF'(\mathcal{C}(\mu'))\) if \(\mu' \neq -(\text{Im}x_0)(\text{Re}x_0)\). Similarly, for \(\mu' \neq 0\), we can eliminate any point \((x_0, z_0)\) outside the diagonal with \((\text{Im}x_0)(\text{Re}x_0) = (\text{Im}z_0)(\text{Re}z_0) = 0\), by contour deformation near \(\infty\). For \(\mu' \neq 0\) \(\text{(d.18)}\) improves to,
\(\text{(d.23)}\)
and for \(\mu' = 0\), we get,
\(\text{(d.24)}\)
Since \(\kappa_T\) maps \(y \eta \in \mathbb{R}^2\) to \(-(\text{Im}x_0)(\text{Re}x_0)\), we finally get for \(\mu' \neq 0,\)
\(\text{(d.25)}\)
here \(p_0(x, \xi) = x \xi\). For \(\mu' = 0,\) we get,
\(\text{(d.26)}\)
Denote the right hand sides of \(\text{(d.25)}, \text{(d.26)}\) for \(\mu' \neq 0\) and \(\mu' = 0\) respectively by \(\Gamma'(\mu')\). If we use the notion of \(WF'\) for bounded families, we get,
\(\text{(d.27)}\)
f for every compact interval \(I \in \mathbb{R}\). These results can easily be transported by Fourier integral operators, to give (unique) microlocal parametrices for \(P_1^*P\) and \(PP_2^*\) in section 4, near the branching point \((0, \pi)\) (as well as the other branching points).

As a second preparation, we consider a formally self adjoint analytic classical pseudodifferential operator, \(P\) of order 0, defined microlocally in an open set \(\Omega \subset \mathbb{R}^2\), and such that the real principal symbol, \(p\) has the property that \(p^{-1}(0) = \gamma([a, b])\), is a bicharacteristic strip. Microlocally in \(\Omega \times \Omega\), we can then define the forward and the backward parametrices, by,
\(\text{(d.28)}\)
\(\text{(d.29)}\)
We also put,
\(\text{(d.30)}\)
Here \(e^{-itP/h}\) is a microlocally defined unitary Fourier integral operator,
associated to the canonical transformation $\chi_t=\exp(tH_0)$. Working on the FBI-side, it is also easy to see (and well known) that $G$ is a Fourier integral operator, of the form,

\[(d.31)\quad G\chi(x,h)=\int a(x,y,h)e^{i(\varphi(x)-\varphi(y))/h} u(y)dy,
\]

where the phase $\varphi$ satisfies $p(x,\varphi'(x))=0$. The relations $PG=0, GP=0$ give two transport equations for $a$, and we conclude that $a(x,y,h)=b(x,h)c(y,h)$, where $b,c$ are classical analytic symbols of order $\frac{1}{h}$, satisfying $P(\varphi e^{h\varphi'/h})=0$, $tP(be^{-h\varphi'/h})=0$. Undoing the FBI-transform, we see that $G$ is a Fourier integral operator associated to the canonical relation,

\[
\Gamma=\{(\exp(tH_0)(\rho),\rho)\in\Omega\times\Omega; \rho(\rho)=0\},
\]

and,

\[(d.32)\quad G\chi=iC(\chi|\chi_0)\chi_0,
\]

where $\chi_0$ is a microlocal normalized solution to $Pu_0=0$, and $C=C(h)$ is an elliptic c.a.s. of order 0. It is easy to see that $G^*=\bar{G}$, so it follows that $C$ is real valued.

Writing, $\Gamma=\Gamma_f\cup\Gamma_b$, where $\Gamma_f$ corresponds to $t\geq0$ and $\Gamma_b$ corresponds to $t\leq0$, we also have,

\[(d.33)\quad WF(E_f)\subset\text{diag}(\mathbb{R}^2\times\mathbb{R}^2)\cup\Gamma_f, \quad WF(E_b)\subset\text{diag}(\mathbb{R}^2\times\mathbb{R}^2)\cup\Gamma_b.
\]

If $\rho_0=\exp(t_0H_0)(\mu_0)$, with $p(\mu_0)=0$ and $t_0>0$, this implies that $E_f=0$ and $E_b=0$ microlocally, near $(\rho_0,\mu_0)$. In order to determine $C$, we let $\chi$ be a pseudodifferential operator of order 0, such the symbol is equal to 1 in a neighborhood of $\rho_0$ and equal to 0 near $\mu_0$. (We may work with a gaussian quantization of this symbol, so that $\chi u$ is well defined microlocally, and $WF(\chi u)\subset WF(u)$ for every microlocally defined function, $u$.) Put,

\[
\tilde{E}_f=(1-\chi)E_f+E_f[P,\chi]E_f. \quad \text{If} \ WF(u)\subset\Omega, \ \text{we know that} \ E_f \ \text{can be applied to} \ u,
\]

and $WF(\tilde{E}_f u)\subset WF(u)\cup\Gamma_f(WF(u))$. We also know that $E_f P$ and $P E_f$ reduce to the identity on such functions. Hence, $\tilde{E}_f u=E_f P E_f u$. Since

\[
P \tilde{E}_f=(1-\chi)-P, \chi E_f+[P,\chi]E_f=(1-\chi), \ \text{we get,} \ \tilde{E}_f=E_f(1-\chi). \ \text{Hence microlocally near} \ (\rho_0,\mu_0), \ \text{we have} \ \tilde{E}_f=E_f=G. \ \text{On the other hand we see from the definition of} \ \tilde{E}_f, \ \text{that} \ \tilde{E}_f=G[P,\chi]G \ \text{nearly the same point, so there we have} \ G=G[P,\chi]G.
\]

This means that if $WF(u)$ is close to $\mu_0$, then near $\rho_0$, we have,

\[
iC(u|u_0)u_0=(iC)^2(u|u_0)(iP,\chi)u_0 |u_0 u_0 |u_0. \ \text{This means that} \ iC=iC^2(iP,\chi)u_0 |u_0, \ \text{and since} \ u_0 \ \text{is normalized,} \ (iP,\chi)u_0 |u_0 |u_0=1. \ \text{Hence} \ C=C^2, \ \text{and since} \ C \ \text{is elliptic, we must have} \ C=1.
\]

Hence,

\[(d.34)\quad G\chi=i(u|u_0)u_0,
\]

if $u_0$ is a normalized solution of $Pu_0=0$.

We shall now establish the wellposedness of the Grushin problem of section 4, by using a priori estimates. The first step will be to establish microlocal estimates near $\text{Int}(s(0,1))$, then near $(0,\pi)$, then near $\text{Int}(s(0,2))$. Patching together these estimates, we get an estimate in a neighborhood of the closed square with corners, $(-\pi/2\pm\pi,\pi/2\pm\pi)$, and combining translates.
of this we finally get a global a priori estimate.

Let $D(r) = D((\pi/2, \pi/2), r)$ be the open disc with center $(\pi/2, \pi/2)$ and radius $r < \pi/2$, sufficiently large so that $WF(f_{0,1}) \subset D(r)$. In $D(r)$ we then know that $H_j$, $j=1,2$, are of principal type and as above we can define the corresponding forward and backward parametrices, $E^+_f(j)$, $E^-_b(j)$, $j=1,2$. Also put $G(j) = E^+_f(j) - E^-_b(j)$. Once and for all, we fix a global FBI-transform, say (d.7) and let $\Phi_0$ be the corresponding weight function. For simplicity, we write $H(\Omega)$ instead of $H_{\Phi_0}(\Omega)$ and the corresponding norm will be denoted simply by $\| \|_\Omega$. By means of $\pi_{X^* \Phi T}$, (where $\pi_{X^*} : C^2(\nu X, \xi) \to C_X$ is the natural projection,) we shall identify sets in $\mathbb{R}^2$ with sets in $\mathbb{C}$. For simplicity, we shall identify operators, $(A)$ with their conjugates under $T$, $(TAT^{-1})$. In the case of pseudodifferential operators, if $\Omega'' \subset \subset \Omega'$ are open with smooth boundary, (and automatically pseudoconvex since we work in one complex dimension,) and $Q$ is a formal analytic pseudodifferential operator with Weyl symbol defined in a neighborhood of $\bar{\Omega}'$ (i.e. near $\Lambda_{\Phi_0} \subset \bar{\Omega}'$ after conjugation by $T$), then we can find a natural realization, $H(\Omega') \to H(\Omega'')$, (see [S1],[GrS],) and two such realizations will differ by a term which is $O(e^{-1/Ch})$ in norm for some constant $C>0$. Such a natural realization will always (tacitly) be chosen and will be denoted by the same letter. At a later stage in our estimates, we will have to specify the relation between the globally defined operator, $P$ and its local realizations.

As a microlocal approximation near $Int(s(0,1))$ of the full Grushin problem, we consider the problem,

\[ \begin{align*}
& Pu = v, \quad (u|f_{0,1}) = v_+,
\end{align*} \]

where $u \in H(D(r))$, $v \in H(D'(r'))$, $v \in \mathbb{C}$ and $r' \approx r$ with $r-r'$ small. Composing the first equation by $P_i^*$, we get with a new slightly smaller $r'$:

\[ \begin{align*}
& H(\nu - v + u_{0,1}) = P^* i v + O(1v + 1e^{-1/Ch}) \quad \text{in } H(D(r')),
& (u - v + u_{0,1} | f_{0,1}) = 0.
\end{align*} \]

Now $H(\nu X_{0,1} (\nu - v + u_{0,1}) = X_{0,1} P^* i v + [H, X_{0,1}] (\nu - v + u_{0,1}) + O(1v + 1e^{-1/Ch})$, (where we work with the Gaussian realization of $X_{0,1}$, which has a natural local realization,) and in in view of the properties of $WF(X_{0,1}(\nu - v + u_{0,1}))$, we can apply $E_f = E_f(1)$ to the last equation, and get,

\[ \begin{align*}
& X_{0,1} (\nu - v + u_{0,1}) = E_f X_{0,1} P^* i v + E_f [H, X_{0,1}] (\nu - v + u_{0,1}) + O(1v + 1e^{-1/Ch})(\|u\|_{D(r')} + 1v + 1),
\end{align*} \]

in $H(D'(r'))$, with a new slightly smaller $r'$. Here we notice that,

\[ \begin{align*}
& E_f [H, X_{0,1}] (\nu - v + u_{0,1}) = \Omega [H, X_{0,1}] (\nu - v + u_{0,1}),
\end{align*} \]

modulo the same error as in (d.37), in any subdomain of $D'(r')$ whose closure is compact and disjoint from $\Gamma_1([H_{10}, \nu X_{0,1}])$, where the last set by definition is the union of $WF([H_{10}, \nu X_{0,1}])$ and the largest bicharacteristic segment of $H_1$ with both end points in $WF([H_{10}, \nu X_{0,1}])$. Here the last set is defined by:
WF'([H_1, X_0, 1]) = \{(x, x) : x \in WF([H_1, X_0, 1])\}. Now modulo the same type of errors, the right hand side of (d.38) is,

\[ (i[H_1, X_0, 1])(u - v + u_0, 1) | u_0, 1) u_0, 1 = (u - v + u_0, 1) f_0, 1) = 0, \]
so in any compact set disjoint from \( \Gamma_1([H_1, X_0, 1]) \), we get,

(d.39) \[ X_0, 1(u - v + u_0, 1) = E_f(1) X_0, 1 P_1 \ast v. \]

Similarly,

(d.39') \[ X_0, 1(u - v + u_0, 1) = E_b(1) X_0, 1 P_1 \ast v, \]
in the same set. Since \( E_f(1), E_b(1) \) (when suitably realized) have norms, \( O(1/h) \), we get after combining (d.39), (d.39') with simple elliptic estimates outside the characteristics of \( H_1 \), that

(d.40) \[ \| u - v + u_0, 1 \|_{D(r''')} \leq (C/h)(\| P u \|_{D(r')} + 1/Ch \| u \|_{D(r')}), \]

for all solutions of (d.35), if \( r''' < r'' < r \) and \( r''' \) is sufficiently large. Since \( \| u_0, 1 \| = O(h^{-1/2}) \), we conclude that,

(d.41) \[ \| u \|_{D(r''')} \leq C(h^{-1/2} \| P u \|_{D(r')} + 1/Ch \| u \|_{D(r')}). \]

Controlling \( u \) in the annulus, we can apply a cut off operator \( \chi \) with support in \( D(r''') \), equal to 1 near the closure of \( D(r''') \):

(d.42) \[ \| [H, \chi] u \|_{D(r''')} \leq C(h \| u \|_{D(r''')} + 1/Ch \| u \|_{D(r')}), \]

and if we write,

\[ H_1 \chi u = X_1 u + [H_1, \chi] u, \]

we deduce after applying \( E_f \) or \( E_b \):

\[ \| [H, \chi] u \|_{D(r''')} \leq C(h^{-1/2} \| P u \|_{D(r')} + 1/Ch \| u \|_{D(r')}), \]

for all solutions of (d.35), when \( r'' \) is large enough and \( 0 < r'' < r < \pi/2, r'' \)

\( D(r') = D((-\pi/2, \pi/2), r) \).

We next write down an easy estimate near the branching point, which follows from the fact that \( H_1 \) and hence \( P \) has a left parametrix of norm \( O(h^{-3/2}) \) whose \( WF' \) can be obtained from the \( WF' \) of the parametrix of \( P_0 - \mu' \) above, by applying the canonical transformation of \( U_1 \). (Here the estimate on the norm follows from (4.25).) We now let \( D(r') = D((-\pi, \pi), r), 0 < r < 2\pi \). If \( r' \) is slightly smaller than \( r \), we let \( P \) denote a realization, \( H(D(r') \rightarrow H(D(r'))) \). Let \( W \) be the intersection of \( D(r') \setminus D(r) \) and a small neighborhood of \( s(0, 1) \cup s((-1/2, 0, 1), 3) \). Then if \( r'' < r' \), we easily get,

(d.44) \[ \| u \|_{D(r'')} \leq C(h^{-3/2} \| P u \|_{D(r')} + 1/Ch \| u \|_{D(r')}), \]

for all solutions of (d.35), when \( r'' \) is large enough and \( 0 < r'' < r < \pi/2, r'' \)
\( D(r) = D((-\pi/2, \pi/2), r) \).

We next move to \( Int(s(0, 2)) \), so we now put \( D(r) = D((-\pi/2, \pi/2), r) \), with \( 0 < r < \pi/2, r \) sufficiently large so that \( WF(f_{0, 1}) \) is contained in \( D(r) \). If \( r' \) is slightly smaller than \( r \), we consider the problem,

(d.45) \[ P u + u - f_{0, 2} = v \text{ in } D(r'), u \in H(D(r)), u \in C. \]

Let \( \chi \) be a cut off operator with support in \( D(r'') \) and equal to 1 near \( D(r'') \), where \( r''' < r'' < r < r \) and \( r'' \) is only slightly smaller than \( r \). From (d.45) we get,

(d.46) \[ (P u \chi_{0, 2} + u - f_{0, 2} \chi_{0, 2}) = (v \chi_{0, 2}), \]
where \((f_0,2|Xu_0,2) = 1 + O(e^{-1/\text{Ch}})\), and

\[
(Pu|Xu_0,2) = (u| [P^*, X] u_0,2) + O(e^{-1/\text{Ch}}) \|u\|_{D(r)}.
\]

Hence,

\[
|\langle Pu | Xu_0,2 \rangle| \leq C(h \frac{1}{2} \|u\|_{D(r')} \|D(r'')\| + e^{-1/\text{Ch}} \|u\|_{D(r)}).
\]

Using also that, \(|\langle v | Xu_0,2 \rangle| \leq Ch^{-\frac{1}{2}} \|v\|_{D(r')}\), we get from (d.46),

\[
|u^-| \leq C(h^{-\frac{1}{2}} \|v\|_{D(r')} + h^{\frac{1}{2}} \|u\|_{D(r')} \|D(r'')\| + e^{-1/\text{Ch}} \|u\|_{D(r')}).
\]

Using this estimate and the fact that \(\|f_0,2\| = O(h^\frac{3}{2})\), we can return to (d.45), truncate and apply \(E_f P^*\) or \(E_{\delta} P^*\), which gives,

\[
\|u\|_{D(r')} + h^{-\frac{1}{2}} |u^-| \leq C(h^{-1} \|v\|_{D(r)} + h^{-\frac{3}{2}} \|u\|_{D(r') \|D(r'')\| + e^{-1/\text{Ch}} \|u\|_{D(r')}).
\]

for the solution \((u, u^-)\) of the problem (d.45).

Let \(\Omega\) be the open square with corners at \((-\pi/2 \pm (\pi + \delta), \pi/2 \pm (\pi + \delta))\), where \(\delta > 0\) is so large that \(WF(f^\alpha_j) \subset \Omega\), when \((\alpha, j) = (0,1), (0,3), ((0,1),1), ((0,1),3)\), but so small that \(WF(f^\alpha_j) \cap \bar{\Omega} = \emptyset\), for all other values of \((\alpha, j)\) with \(j\) odd, and so that \((0, \pi)\) and \((-\pi, 0)\) are the only branching points in \(\bar{\Omega}\). Combining the three a priori estimates (d.43), (d.44), (d.48) with simpler estimates in the elliptic region, we see that if \(\Omega'' \subset \Omega' \subset \Omega\) are slightly smaller squares, and if \(P\) is realized as an operator \(H(\Omega) \rightarrow H(\Omega')\), then if \(u \in H(\Omega)\) and \(v^+((0,1), v^+((0,3), v^+((0,1),1), v^+((0,1),3) \in \mathbb{C}\), \(u^-((0,2) \in \mathbb{C}\), and

\[
(Pu + u^-(0,2))f_0,2 = v \text{ in } \Omega', (u | f^\alpha_j) = v^+((\alpha, j)) \text{ for } (\alpha, j) = (0,1), (0,3), ((0,1),1), ((0,1),3),
\]

then we have the a priori estimate,

\[
\|u\|_{\Omega''} + h^{-\frac{1}{2}} |u^-((0,2))| \leq C(h^{-3/2} \|v\|_{\Omega'} + h^{-1} \|\Delta v^+((\alpha, j))\| + O(e^{-1/\text{Ch}}) \|u\|_{\Omega}).
\]

If we now consider the full Grushin problem,

\[
Pu + R^- u^- = v, \quad R^+ u = v^+,
\]

and make an FBI-transform and restrict to \(\Omega'\), we have to take into account two facts:

1° The full Weyl quantization of \(P\) can be realized first by taking \(\Lambda_{\Phi_0}\) as an integration contour, then using the fact that the symbol of \(P\) is holomorphic in a neighborhood of \(\Lambda_{\Phi_0}\), we see that if \(P_{\Omega', \Omega}\) is the realization used in (d.49), then we get,

\[
\|P_{\Omega', \Omega}(\Upsilon|_{\Omega}) - Pu\|_{\Omega} \leq C e^{-1/\text{Ch}} \|u\|_{\Phi_0} + \text{dist}(\Omega, \Omega)/C,
\]

where in general, \(\|u\|_f\) (when \(f\) is a function) will denote the \(L^2\)-norm over \(\mathbb{C}\), with respect to the measure \(e^{-2i/\text{Ch}} L(dx)\). (It will be clear from the context whether the subscript ... in \(\|\| \...) denotes a domain or a function.

2° The terms \(u^-(\alpha, j)f^\alpha_j\) with \(j\) even \((\alpha, j) \neq (0,2)\) will have some
exponentially small influence in \( \Omega' \).

We then get,

\begin{equation}
\begin{aligned}
P_{\Omega'} \Omega u + u^-(0,2) f_{0,2} &= v + w \text{ in } \Omega', \\
(u|f_{\alpha,j}) &= v^+(\alpha,j) + w^+(\alpha,j), \quad (\alpha,j) = (0,1), (0,3), ((0,1),1), ((0,1),3),
\end{aligned}
\end{equation}

where,

\[ \|w\|_{\Omega'} \leq C e^{-1/Ch} (\|u\|_{\Phi_0 + \text{dist}(.,\Omega)} / C + \|u^-\|^2, |\alpha| / C), \]

\[ |w|^2 \leq C e^{-1/Ch} (\|u\|_{\Phi_0 + \text{dist}(.,\Omega)} / C). \]

Here we write, \((\|u^-\|^2, p)^2 = \sum_{\alpha,j} e^{-2f/h} |u^-|_{\alpha,j}^2 \). Applying (d.50) with \((v, v^+)\) replaced by \((v + w, v^+ + w^+)\) gives,

\begin{equation}
\begin{aligned}
\|u\|_{\Omega''} + h^{-1/2} \|u^-\|_{(0,2)} &\leq C (e^{-1/Ch}(\|u\|_{\Phi_0 + \text{dist}(.,\Omega)} + \|u^-\|^2, |\alpha| / C)) \\
(\|u\|_{\Phi_0 + \text{dist}(.,\Omega)} + \|u^-\|^2, |\alpha| / C) \\
\end{aligned}
\end{equation}

Now write, \( \Omega = \Omega(0,2) \) and similarly for \( \Omega', \Omega'' \). Let \( m(\alpha,j) \) denote the middle point of \( \text{dist}(., \alpha,j) \). For \( j \) even we put, \( \Omega(\alpha,j) = \Omega(0,2) + (m(\alpha,j) - m(0,2)) \), and similarly for \( \Omega', \Omega'' \). The estimate (d.54) remains valid, if we replace \( \Omega, \Omega', \Omega'' \) by \( \Omega(\beta, k), \Omega'(\beta, k), \Omega''(\beta, k) \), and then the norm, \( \|u - \beta 1/C \) of \( u^- \) to the right. Squaring all these inequalities and summing with respect to \( (\beta, k) \), we get, (since the \( \Omega''(\beta, k) \) will cover \( C, \)

\begin{equation}
\begin{aligned}
\|u\|^2 + h^{-1} \|u^-\|^2 &\leq C (e^{-1/Ch}(\|u\|_{\Phi_0 + \text{dist}(.,\Omega)} + \|u^-\|^2, |\alpha| / C) \\
(\|u\|_{\Phi_0 + \text{dist}(.,\Omega)} + \|u^-\|^2, |\alpha| / C) \\
\end{aligned}
\end{equation}

where the norms are now the standard \( L^2 \) and \( L^2 \)-norms over \( \mathbb{R} \) and \( \mathbb{Z}^2 \) respectively. When \( h > 0 \) is sufficiently small, we can absorb the last two terms to the right, and get,

\begin{equation}
\begin{aligned}
\|u\|^2 + h^{-1} \|u^-\|^2 &\leq C (e^{-1/Ch}(\|u\|^2 + \|u^-\|^2) \\
(\|u\|^2 + \|u^-\|^2) \\
\end{aligned}
\end{equation}

This shows that \( \mathcal{P} \) is injective: \( L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) with closed range. Since all our estimates work equally well for \( \mathcal{P}^* \) which has the same structure, and since \( \text{Ker} \mathcal{P}^* = (\text{Im} \mathcal{P})^\perp = \{0\} \), we conclude that \( \mathcal{P} \) is bijective with bounded inverse,

\[ \mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_- \end{pmatrix}, \]

satisfying,

\begin{equation}
\begin{aligned}
\|E\| &= O(1), \quad \|E_+\| = O(h^{-1/2}), \quad \|E_-\| = O(h^{-1}), \quad \|E_- \| = O(h^{-1/2}).
\end{aligned}
\end{equation}

Before Proposition 4.1, we constructed an approximate solution of \( Pu + R_+ u^- = 0, R_+ u^- = \delta_{0,1} \), satisfying these equations with exponentially small errors. In view of (d.57), the approximate solution differs from the exact one with exponentially small errors, in particular the computation of \( E_+(\alpha,j; 0,1) \) for \( (\alpha,j) = (0,2), ((0,1),4) \) is correct up to an exponentially small error.

It remains to establish the exponential decrease estimates for \( E(\alpha,j; \beta, k) \) in Proposition 4.1, when \( s(\alpha,j) \cap s(\beta, k) = \emptyset \). Let \( (u, u^-) \) be a solution of,

\begin{equation}
\begin{aligned}
Pu + R_- u^- = 0, \quad R_+ u^- = \delta_{0,1}.
\end{aligned}
\end{equation}

If \( \Omega(\beta, k) \) does not contain \( m(0,1) \), we get from (d.54):
\begin{align*}
\text{(d.59)} \quad & \|u\|^2 \Omega''(\beta, k) + h^{-1} |u^{-}(\beta, k)|^2 \\
& \quad \leq Ce^{-1/Ch} \|u\|^2 \Phi_0 + \text{dist}(. , \Omega(\beta, k))/C + \|u^{-} \|^2 |2., - \beta|/C),
\end{align*}
so after increasing C, we get,
\begin{align*}
\text{(d.60)} \quad & \|u\|^2 \Omega''(\beta, k) + h^{-1} |u^{-}(\beta, k)|^2 \\
& \quad \leq Ce^{-1/Ch} \sum \alpha \jmath e^{-1/(\beta, k) - (\alpha, j)}/C \|u\|^2 \Omega''(\alpha, j) + |u^{-}(\alpha, j)|^2).
\end{align*}
Write, $\alpha = (\alpha, j)$, $\beta = (\beta, k)$, $f(\alpha) = \|u\|^2 \Omega''(\alpha) + |u^{-}(\alpha)|^2$, $g(\alpha) = f(\alpha)$, if $(0,1) \in \Omega(\alpha)$ and = 0 otherwise. Then (d.60) gives,
\begin{align*}
\text{(d.61)} \quad & f(\beta) \leq g(\beta) + \mathcal{K} f(\beta),
\end{align*}
where $\mathcal{K}$ is the positivity preserving linear operator,
\begin{align*}
\text{(d.62)} \quad & \mathcal{K} u(\beta) = Ce^{-1/Ch} \sum \alpha e^{-1/\beta - \alpha|/Ch u(\alpha)}.
\end{align*}
Since $\|\mathcal{K}\|_{L(1,1)} \leq C_1 e^{-1/Ch}$, we get for $h>0$ sufficiently small,
\begin{align*}
\text{(d.63)} \quad & f \leq g + \mathcal{K} g + \mathcal{K}^2 g + ...,
\end{align*}
and in particular,
\begin{align*}
\text{(d.64)} \quad & \|f - g\|_1 \leq C_2 e^{-1/Ch} \|g\|_1. \text{ If } \gamma(\alpha) \text{ is a function such that } \|\gamma(\alpha) - \gamma(\beta)| \leq |\alpha - \beta|/2C, \text{ then we also get,}
\text{(d.65)} \quad & \|f - g\|_1 \leq C_3 e^{-1/Ch} \|g\|_1, \gamma.
\end{align*}
Recalling that $E_{-}(\beta; 0,1) = u^{-}(\beta)$, we get the required exponential decrease estimates from (d.64), (d.65). This completes the proof of Proposition 4.1.
Application to the magnetic Schrödinger operator.

The main results of this article apply to h-Weyl quantizations of symbols, \( P(x, \xi) \) which are \( 2\pi \)-periodic in \( x \) and \( \xi \), close to \( \cos(\xi) + \cos(x) \) (in the sense of strong type 1 operators) and which satisfy the following invariance properties,

1. \( P \) is real valued (so that the corresponding operator, \( P \) is self adjoint).
2. \( P(x, \xi) = P(\xi, -x) \) (so that the operator \( P \) commutes with the Fourier transform, \( \mathcal{F} \)).
3. \( P(x, -\xi) = P(x, \xi) \) (so that the operator \( P \) commutes with \( \Gamma \), where \( \Gamma u(x) = \overline{u(x)} \)).

In [HS1], §9.4 we saw that the study of the spectrum of the Schrödinger operator with periodic electric potential, \( V \) and periodic magnetic field, \( B \) could be reduced to the study of an operator \( P \) satisfying (e.1) and (e.2) under the following assumptions,

1. \( V(x + aj_1e_1) = V(x), B(x + aj_1e_1) = B(x), e_1 = (1,0), e_2 = (0,1), aj \in \mathbb{R}. \)
2. \( \alpha^*V = V, \)
3. \( \alpha^*B = B, \)

where \( \alpha(x_1, x_2) = (x_2, -x_1) \) and \( \alpha^*u = u \cdot \alpha^{-1} \). A more intrinsic formulation of (e.4.b) is:

(e.4.b)' \( \alpha^*\sigma_B = \sigma_B, \)

where \( \sigma_B \) is the 2-form, \( B(x_1, x_2)dx_1 \wedge dx_2 \), and \( \alpha^* = (\alpha^*)^{-1} \), where \( * \) denotes the standard pull-back operation. If \( f \) is a function such that \( \alpha^*\omega_A - \omega_A = df \), where \( \omega_A = A_1dx_1 + A_2dx_2 \), then we saw in [HS1], that the magnetic Schrödinger operator \( P_A(h) = \sum_{j=1,2} (\hbar D_{x_j} - A_j)^2 + V(x) \) commutes with the operator,

(e.5) \( \mathcal{F} = e^{i\Phi}/\hbar \alpha^* \).

We also saw that \( P_A \) commutes with the two "translation operators", \( T_1 \) and \( T_2 \), given by,

(e.6) \( T_j = e^{i\mathcal{F}/\hbar \tau_j}, j = 1, 2, \)

where,

(e.7) \( \tau_j g(x) = g(x - aj_1e_1), \tau_j(x) = x + aj_1e_1 \)

(e.8) \( d\Psi_j = (\tau_j)^*\omega_A - \omega_A, \)

and we also had,

(e.9) \( T_1 T_2 = e^{i\Phi}/\hbar T_2 T_1. \)

Here \( \Phi \) is the magnetic flux through a base cell. After a modification of \( \Psi_j \) and \( f \) by adding suitable constants, we obtained the following properties,

(e.10) (a) \( [T_j, P_A] = 0, \)

(b) \( [\mathcal{F}, P_A] = 0, \)

(c) \( (P_A)^* = P_A \) (expressing the self-adjointness of \( P_A \)),

Here \( \Phi \) is the magnetic flux through a base cell. After a modification of \( \Psi_j \) and \( f \) by adding suitable constants, we obtained the following properties,
(d) \( T_1 T_2 = e^{ih'} T_2 T_1, \ h' = \Phi/h \mod(2\pi \mathbb{Z}) \),
(e) \( F^4 = 1 \),
(f) \( T_2^{-1} F = FT_1, \ T_1 F = FT_2 \).

As in [HS1], we concentrate on a suitable interval containing an isolated piece of the spectrum of \( P_A \), and we let \( \Pi \) denote the corresponding spectral projection. We then constructed an orthonormal basis in \( F = \Pi(L^2) \) of the form \( \{ \Pi(\psi_\alpha) \} \), where \( \psi_0 \) is a suitable (approximate eigen-) function satisfying

(e.11) \( F \psi_0 = \omega \psi_0 \), with \( |\omega| = 1 \), and,

(e.12) \( \psi_\alpha = T^\alpha \psi_0 \), where \( T^\alpha = T_1^{\alpha_1} T_2^{\alpha_2} \).

The matrix of \( P_A \) restricted to \( F \) is then of the form,

\[
\begin{pmatrix}
\begin{array}{cc}
 m_{\alpha,\beta} & w_{\alpha,\beta} \\
 w_{\alpha,\beta}^* & \tilde{m}_{\alpha,\beta}
\end{array}
\end{pmatrix}
\]

already been treated in [HS1], and we shall here mainly discuss the additional symmetries that will permit us to obtain (e.3). It follows from (e.10)–(e.12), that \( m_{\alpha,\beta} \) and \( w_{\alpha,\beta} \) satisfy,

(e.13) (a) \( m_{\alpha,\beta} = \overline{m}_{\beta,\alpha} \) (by (e.10.c)),
(b) \( w_{\alpha,\beta} = e^{ih'} \delta_{2}(\beta_1 - \alpha_1) w_{\alpha',\beta'} + x_{\alpha,\beta} \) (cf (4.26), (4.27) in [HS1]).
(c) \( w_{\alpha,\beta} = e^{ih'(\alpha_1 \alpha_2 - \beta_1 \beta_2)} W(\alpha, \beta), \) where \( W(\alpha) = (\alpha_2, -\alpha_1) \) (consequence of (e.10,b,f)).

We conclude that,

(e.14) \( w_{\alpha,\beta} = e^{ih'} \beta_2 (\alpha_1 - \beta_1) f(\alpha - \beta) \),

with,

(e.15) (a) \( f(-j, -k) = f(j, k) e^{ijklh'} \),
(b) \( f(\alpha) = e^{ih' \alpha_1 \alpha_2 f(\alpha)} \).

Iterating (b) we get \( f(-\alpha) = f(\alpha) \) and consequently, \( f(\alpha) = f(\alpha) e^{ih' \alpha_1 \alpha_2} \). Finally, we saw that the operator \( T \) is isospectral to \( P \), the \( h' \)-Weyl quantization of the symbol \( P(x, \xi) \) defined by,

(e.16) \( P(x, \xi) = \Sigma \Sigma f(j, k) e^{-i jkh'/2} e^{-i(kx + j\xi)}. \)

In addition to the \( 2\pi \)-periodicity of \( P(x, \xi) \), we get from (e.15), that \( P \) satisfies (e.1), (e.2). The purpose of this appendix is to add a natural symmetry assumption on \( B \) and \( V \) which will imply that \( P \) satisfies (e.3). The natural idea is to find a suitable antilinear quantization of the map,

(e.17) \( \chi: (x_1, x_2) \rightarrow (x_1, -x_2) \).

\( e = \text{Id}, \ \chi \) generate a finite subgroup, \( G \) of \( O_2(\mathbb{R}) \) (the orthogonal \( 2 \times 2 \)-matrices) of 8 elements: \( \alpha^k \chi^l, \ 0 \leq k \leq 3, \ 0 \leq l \leq 1 \). In addition to (e.4) we shall now assume,

(e.18) (a) \( \chi \cdot V = V \), (b) \( \chi \cdot B = B \).

The intrinsic formulation of (b) is,

(e.18) (b') \( \chi \cdot \sigma B = -\sigma B \).

The assumptions on \( \alpha, \chi \) in (e.4), (e.18) can be reformulated, by using the
following two representations of G in $O_2(\mathbb{R})$,

\[(e.19) \quad g \mapsto M_0(g) = g, \quad g \mapsto M_1(g) = (-1)^k(g)M_0(g),\]

where $k(g)$ is defined modulo $2\mathbb{Z}$ by $g = \alpha_1^*k(g)$. The reformulation is then that,

\[(e.20) \quad M_0(g) \circ V = V, \quad M_1(g) \circ \sigma_B = \sigma_B, \text{ for all } g \in G.\]

The aim is then to find a representation, $\pi_1(g)$ of $G$ on $L^2(\mathbb{R}^2)$, which is "pseudo linear" in the sense that $\pi_1(g)$ is linear when $k(g)$ is even and antilinear when $k(g)$ is odd, and such that all the $\pi_1(g)$ commute with $P_A$ and satisfy suitable commutation relations with $T_1$ and $T_2$. After a gauge transform we can assume from now on that:

\[(e.21) \quad M_1(g) \circ \omega_A = \omega_A.\]

We put $\pi_1(\alpha) = \Phi, \pi_1(\gamma) = \Gamma$, where,

\[(e.22) \quad \Phi u(x_1, x_2) = u(-x_2, x_1),\]

\[(e.23) \quad \Gamma u(x_1, x_2) = \overline{u(x_1, -x_2)}.\]

Also define $T_j$ by (e.6), (e.7), where $\varphi_j$ is the unique solution of (e.8) with

\[(e.24) \quad \varphi_j(0) = 0.\]

This is a natural normalization in view of the fact that $0$ is a fixed point for $\pi_1(G)$. Essentially as in [HS1], (9.4.24), we verify that,

\[(e.25) \quad \varphi_2 = \alpha_1^* \varphi_1, \varphi_1 = -\tau_1 * \alpha_1^* \varphi_2, \varphi_2 = \tau_2 * \alpha_1^* \varphi_2,\]

and this with (e.20) shows that we have (e.10) and,

\[(e.25) \quad (a) \quad [T_1, P_A] = 0, \quad (b) \quad \Gamma T_1 = T_1 \Gamma, \quad \Gamma T_2 = T_2^{-1} \Gamma, \quad (c) \quad \Gamma P = P \Gamma^{-1}.\]

We recall how $\varphi_0$ was constructed in [HS1]. With a suitable function $\varphi_0 = \varphi(0, 0)$ associated to a potential well, $U_0$ we put $u_0 = \pi \varphi_0$ (and by the choice of $\varphi_0$, that we do not recall here, we know that $u_0$ is very close to $\varphi_0$ and of exponential decrease outside $U_0$). Putting $u_\alpha = T^\alpha u_0$, where

\[T^\alpha = T_1^\alpha T_2^\alpha \text{ we then obtained } (\varphi_\alpha) \text{ as the orthonormalization of the basis } (u_\alpha) \text{ in the image of } \Pi.\]

We then had $\varphi_\alpha = T^\alpha \varphi_0$. Now we may assume that

\[\Gamma \varphi_0 = \omega_1 \varphi_0, \quad |\omega_1| = 1, \text{ since } P_A \text{ commutes with } \Gamma \text{ and since we may choose the reference operator } (\text{in [HS1]} \text{)} \text{ with only the well } U_0, \text{ having the same property. As in [HS1] we then have } \Gamma \varphi_0 = \omega_1 \varphi_0 \text{ and in view of the antilinearity of } \Gamma \text{ we may assume that},\]

\[(e.26) \quad \Gamma \varphi_0 = \varphi_0,\]

without destroying the properties (e.11), (e.12). Using (e.25,b), we get,

\[(e.27) \quad \Gamma \psi_\alpha = \psi_{\gamma(\alpha)}.\]

Now observe that for $u = \Sigma z_\alpha^* \psi_\alpha \in \text{Im}(\Pi)$, we have,

\[P_A \Gamma u = \Sigma m_\beta \alpha \overline{z}_\beta \psi_\beta, \quad \Gamma P_A u = \Sigma \overline{m}_\beta \alpha \overline{z}_\beta \psi_\beta, \quad \gamma(\alpha) \overline{z}_\beta \psi_\beta, \quad \text{ and from this and (e.25,a) we deduce,}\]

\[(e.28) \quad \overline{m}_\beta (\alpha, \gamma(\beta)) = -m_\alpha, \beta.\]
Putting $\beta = 0$ and combining with (e.14), we get,
(e.29) \[ f = \tilde{f} \gamma, \]
This is exactly the condition that implies (e.3) in view of (e.16). We have then proved,

**Proposition e.1.** Under the assumptions (e.4), (e.18), (as well as the other technical assumptions of Theorem 9.4.1 in [HS1]), the study of the spectrum of $P_A$ in a suitable interval containing the ground level of the modified 1-well operator can be reduced by an affine transformation to the study of the spectrum of a strong type 1 self adjoint $h'$-pseudodifferential operator, $P$ with $\varepsilon(P) \to 0$, when $h \to 0$.

This means that the results of the present paper are applicable, and if $h > 0$ is sufficiently small, then the spectrum of $P_A$ near the ground level is a Cantor set of Lebesgue measure 0. For the sake of completeness, we formulate this as a theorem,

**Theorem e.2.** Let $V, A_1, A_2 \in C^\infty(\mathbb{R};\mathbb{R})$ satisfy:

(H.1) \[ V(x + a e_1) = V(x), \ V(-x_1, x) = V(x_1, x) \text{ where } a > 0 \text{ is fixed and } e_1, e_2 \text{ is the canonical basis in } \mathbb{R}^2, \]
(H.2) The same relations for $B = \partial x_1 A_2 - \partial x_2 A_1$.
(H.3) $V$ has only one minimum mod$(a \mathbb{Z}^2)$, namely 0, and this minimum is non-degenerate.

Without loss of generality, we may assume that $V(0) = 0$.

(H.4) Let $d^2_V$ be the Agmon distance associated to $Vdx^2$. Then the points $a \alpha$ in $a \mathbb{Z}^2 \setminus \{0\}$, which are closest to 0 with respect to this distance, are precisely the ones with $|\alpha| = |\alpha_1| + |\alpha_2| = 1$.

Moreover, between 0 and each such point, there is only one minimal geodesic $\gamma = \gamma_\alpha$, which is non-degenerate, and near this geodesic, $V$ and $B$ are analytic.

Let $\Phi$ denote the flux of the 2-form $\sigma_B$ through the cell $[0, a]^2$, and let $P_{tA}(h) = (hD_{x_1} - tA_1)^2 + (hD_{x_2} - tA_2)^2 + V$ with the magnetic field $tB$. Let $\lambda_t(h)$ be the first eigenvalue of the harmonic oscillator (approximating $P_{tA}$ at 0),

\[ (hD_{x_1} - t\langle A_1(0), x \rangle)^2 + (hD_{x_2} - t\langle A_2(0), x \rangle)^2 + \frac{1}{2} \langle V''(0)x, x \rangle. \]

If $\varepsilon_0 > 0$ is sufficiently small, and $|t| \leq \varepsilon_0, |h| \leq h_0$ (with $h_0 > 0$ sufficiently small), then the the spectrum of $P_{tA}(h)$ in the interval $(E; |E - \lambda_t(h)| \leq h^{3/2}/\varepsilon_0)$ can after an affine transformation be identified with the spectrum of a strong type 1 self adjoint $h'$-pseudodifferential operator, $P$, with $h' = -t\Phi/h \mod(2\pi \mathbb{Z})$. In particular, the results of this work apply, so if $h'/2\pi$ is irrational and has an expansion as in (0.3), (0.4), with $C_0 > 0$ sufficiently large, then the piece of the spectrum under consideration is a Cantor set of Lebesgue measure 0.

**Remark e.3.** The study of symmetries in a closely related setting appears in the work of Wilkinson [W2–4].
Spectrum near $h=0$. For each $h$ with $h/2\pi \in Q$, $h/2\pi = p/q : q \leq 90$, $h/2\pi \in ]0,0.0625[$, we present in $A$ the spectrum of $P_o = \cos hD_x + \cos x$ in $]-0.5,0.5[$ on a horizontal line of ordinate $h/2\pi$. This is a part of the celebrated Hofstadter's butterfly (presented in $B$). The length of the bands in the middle do not decrease exponentially rapidly with $h$ due to a strong tunneling effect.
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