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COMPACTNESS IN $C_s(T)$ AND APPLICATIONS

Richard HAYDON (*)

1. - INTRODUCTION.

In this paper I look at some properties of compact subsets of $C_s(T)$ which have applications to the "more interesting" space $C_c(T)$. A little light is cast on the difficult problem of when $C_c(T)$ may be a Kelley space, the concept of infra-$k_R$-space is examined, and lastly I offer two generalizations of a theorem of BUCHWALTER concerning the repletion $UT$.

The notations throughout are "standard Lyon". The algebra $C(T)$ of all continuous real-valued functions on the completely regular space $T$ may be endowed with the topology either of simple, compact or bounded convergence on $T$ and is then denoted by $C_s(T)$, $C_c(T)$ or $C_b(T)$, respectively.

2. - ON KELLEY SPACES $C_c(T)$.

The characterization of $M(T)$ as the space $C_c(\emptyset T)' = M_c(\emptyset T)$, of all measures of compact support on the c-repletion $\emptyset T$, enables one to deduce ($\{B_1\}$ and $\{H_1\}$) that $C_c(\emptyset T)$ is always a Kelley space ($\{B_1\}$) and that, when $T$ is a $k_R$-space, $C_c(T)$ is Kelley if and only if $T$ is c-replete. Put into an attractively symmetric form:

$$C_c(T) \text{ is a complete Kelley space } \iff T \text{ is a c-replete } k_R \text{-space.}$$

One can, however, say more, namely that, when $T$ is a $k_R$-space, $C_c(\emptyset T)$ is the Kelleyifié $\bar{K} C_c(T) \left( \{B_1\} \right)$ of $C_c(T)$.

But what can we say if we do not assume $T$ to be a $k_R$-space? We can note first that the property used to prove the above results is not the full strength

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of being a $k_R$-space, but only that the compact discs of $C_c(T)$ should be equi-
continuous. H. BUCHWALTER has introduced the definition of a property inter-
mediate between these:

(2.1) DEFINITION. - $T$ is said to be an infra-$k_R$-space if every precompact
subset of $C_c(T)$ is equicontinuous.

If $T$ is the space of $(H_2)$, $\theta T$ is infra-$k_R$ and not $k_R$. Evidently, when $T$
is an infra-$k_R$-space, $\theta T$ is also infra-$k_R$ and we have $C_c(\theta T) = \overline{k} C_c(T)$.

But this last equality does not hold for arbitrary $T$, as has been pointed
out in $(H_1)$. I want to consider here the problem posed at the end of that Note:

If $C_c(T)$ is Kelley, need $T$ be $c$-replete?

This question remains open still, but I am able to give some partial results and
to show how it is linked to properties of compactness in $C_s(T)$.

(2.2) PROPOSITION. - Let $T$ be non-$c$-replete and suppose that $C_c(T)$ is a Kelley
space. Then there is a compact disc in $C_c(T)$ that is not compact in $C_s(\theta T)$.

Proof. - The continuous characters of the algebra $C_c(T)$ are the evaluations
$\delta_t(t \in T)$. If $u \in \theta T \setminus T$, $u$ is not continuous on $C_c(T)$ and, since $C_c(T)$ is Kelley,
not continuous on some compact disc of $C_c(T)$. This disc is not compact in $C_s(\theta T)$.

I know of no example of a space $T$ for which some compact subset of $C_c(T)$,
even of $C_s(T)$, fails to be compact in $C_s(\theta T)$. Propositions (2.4) and (2.8)
suggest that such a space (if one exists!) would be difficult to construct.

Let us denote by $R(T)$ the set of all closures in $T$ of $K_C$ subsets of $T$
and consider the property:

(A) Every function $\psi \in R^T$ which coincides on each $C \in R$ with a suitable
$f \in C(T)$ is itself in $C(T)$.

This property was introduced by J.D. PRYCE who proved:

(2.3) THEOREME ([P], Theorem 2.4). - If $T$ has property (A) then every relati-
vely countably compact (rcc) subset of $C_s(T)$ is relatively compact (rc)
in $C_s(T)$. 

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(2.4) PROPOSITION. - When $\theta T$ has property (A) the compact subsets of $C_s(T)$ are compact in $C_s(\theta T)$.

Proof. - When $(f_n)$ is a sequence in $C(T)$ and $u \in \theta T$, there exists $t \in T$ such that $f_n(t) = f_n(u)$ for every integer $n$. It follows at once from this that the rc subsets of $C_s(T)$ and $C_s(\theta T)$ (hence also of $C_s(\theta T)$) are the same. Thus an rc subset of $C_s(T)$ is rc in $C_s(\theta T)$ and, by the theorem of Pryce, rc in $C_s(\theta T)$. If a subset is compact in $C_s(T)$ it is rc and closed, hence compact, in $C_s(\theta T)$.

We can note that $\theta T$ satisfies (A) if $\theta T$ is $k_\infty$-closed or if there is a dense $k_0$ subset of $\theta T$, in particular if $T$ is pseudocompact or has a dense $B_\infty$ ($\sigma$-bounded) subset.

Write $R'(T)$ for the set of all closures in $T$ of $B_\infty$ subsets of $T$. Pryce's theorem allows the generalization below.

(2.5) PROPOSITION. - Let $T$ be a completely regular space that satisfies:

(A') Every $\psi \in R^T$ which coincides on each $C \in R'$ with a suitable $f \in C(T)$ is itself in $C(T)$.

Then every rc subset of $C_s(T)$ is rc in $C_s(T)$.

Proof. - Suppose first that $T$ satisfies (A'). I shall show that the bidual $T''$ satisfies (A). Recall that the bidual of $T$ is defined ($\left(B_2\right)$) as the space $T''$ of all continuous characters of the algebra $C_b(T)$, embedded as a subspace of $\theta T$.

Let $\psi$ be a real-valued function on $T''$ and suppose that for all $C \in R(T'')$ there is an $f \in C(T'')$ with $f|C = \psi|C$. Now if $B$ is a bounded subset of $T$, $B$, taken in $T''$, is compact, so that the $T''$ closure $\overline{B}$ of any $D \in R'(T)$ is in $R(T'')$. Thus, for every such $D$, there is a $g \in C(T)$ such that $g|D = \psi|D$. Applying (A'), we see that $\psi|T \in C(T)$. Let us denote by $\phi$ the continuous extension of $\psi|T$ to $T''$. It will be enough to prove that $\phi = \psi$. If $B$ is bounded in $T$, $\overline{B}$ is compact in $T''$; $\phi$ and $\psi$ are both continuous on $\overline{B}$ and coincide on $B$. Hence $\phi$ and $\psi$ coincide on $\overline{B}$. But by proposition 2 of ($B_2$) we know that $T'' = \bigcup \{\overline{B} : B$ bounded in $T\}$ and we can deduce that $\phi$ and $\psi$ coincide on $T''$.

If now $A$ is rc in $C_s(T)$, $A$ is rc in $C_s(T'')$ by the same reasoning as was used in proposition (2.4). $A$ is therefore rc in $C_s(T'')$ and so certainly rc in
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$C_s(T)$.

(2.6) **DEFINITION.** - A space $X$ is said to be angelic ([P], p. 534) if

(i) $roc \Rightarrow rc$ for the subsets of $X$, and

(ii) every element of the closure of an rc subset $A$ of $X$ is the limit of some sequence in $A$.

If $T \in R'(T)$, we know already by the first part that (i) is satisfied.

PRYCE showed that $C_s(T)$ is angelic if $T \in R(T)$ ([P], theorem 2.5). Therefore $C_s(T'')$ is angelic. If $A$ is rc in $C_s(T)$ (and hence also in $C_s(T'')$) and $f \in \overline{A}$ (the closure being the same in the two topologies), there is a sequence in $A$ that converges to $f$ in $C_s(T'')$, and which converges to $f$, a fortiori, in $C_s(T)$. Then:

(2.7) **PROPOSITION.** - If $T \in R'(T)$ (particularly if $T$ is pseudocompact), $C_s(T)$ is angelic.

(2.8) **PROPOSITION.** - Let $T$ be a (completely regular) space in which all closed and discrete subspaces are $C_\infty$-embedded (particularly if $T$ is normal or countably compact) and that satisfies:

(B) For every $u \in \theta T \setminus T$ there is a base $U$ of neighbourhoods of $u$ in $\theta T$ such that, whenever $V \subseteq U$ and the cardinality of $V$ is strictly less than that of $U$, then $T \cap (\bigcap V)$ is nonempty.

Then the compact subsets of $C_s(T)$ are compact in $C_s(\theta T)$.

**Proof.** - It is enough to show that every character $u \in \theta T$ is continuous on each compact $A \subseteq C_s(T)$. Suppose then that $u$ is not continuous on such an $A$; there is a net $(f_\alpha)$ in $A$ such that $f_\alpha \to f$ in $C_s(T)$ while $f_\alpha^u(u) \neq f^u(u)$. We can assume that the $f_\alpha$ are uniformly bounded by 1, that $f_\alpha \to 0$ in $C_s(T)$ and that $f_\alpha^u(u) = 1$ for all $\alpha$.

Let $U$ be a base of neighbourhoods of $u$ in $\theta T$ with the property of (B). Then if $B \subseteq \theta C(T)$, $V \subseteq U$ and card $B$, card $V$ are strictly less than card $U$, there exists $t \in T$ such that $t \in \bigcap V$ and $f(t) = f^u(t)$ for every $f \in B$. Let us denote by $\Omega$ the first ordinal of cardinality card $U$ and index $\xi$ as $(U_\xi)_{\xi < \Omega}$. I shall define, by transfinite induction, families $(x_\xi)$ in $T$ and $(g_\xi)$ in $A$ with the properties:
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(a) $g_\xi(x_\xi) \leq 1/2$ ($\xi \geq \xi$),
(b) $g_\xi(x_\xi) = 1 = g_\xi^\theta(u)$ ($\xi < \xi$),
(c) $x_\xi + u$ in $\mathcal{E}T$.

Let $x_o$ be an arbitrary point of $T$ and choose $\alpha_o$ such that $f_{\alpha_o}(x_o) \leq 1/2$. Put $g_o = f_{\alpha_o}$. Suppose that $x_\xi$ and $g_\xi$ have been defined for all $\xi$ less than some $\eta < \Omega$ and that (a) and (b) are satisfied. Since the cardinality of $(0, \eta]$ is less than $\text{card } U$, there exists $x_\eta \in T \cap \bigcap_{\xi < \eta} U_\xi$ such that

$g_\xi(x_\eta) = g_\xi^\theta(u) = 1$ ($\xi < \eta$).

Let us now choose, for each finite subset $S$ of $(0, \eta]$, an $\alpha_S$ such that $f_{\alpha_S}(x_\xi) \leq 1/2$ ($\xi \in S$). Let $g_\eta$ be a cluster point of the net $(f_{\alpha_S})$, directed by the upward filtering set of finite subsets of $(0, \eta]$. Then we have

$g_\eta(x_\xi) \leq 1/2$ ($\xi < \eta$) and

$g_\eta(u) = 1$ (because there is $t \in T$ with $g_\eta(t) = g_\eta^\theta(u)$ and $f_{\alpha_S}(t) = f_{\alpha_S}^\theta(u)$ for every finite set $S \subset (0, \eta]$).

Since, by construction, each $x_\eta$ is in $\bigcap_{\xi < \eta} U_\xi$, we see that $x_\eta + u$ in $\mathcal{E}T$.

I shall now show that $\{x_\eta ; \eta < \Omega\}$ is a closed discrete subspace of $T$. If not, there is $\xi < \Omega$ such that $\{x_\eta ; \eta < \xi\}$ has an accumulation point $x$ in $T$. Choose to be the least such ordinal; then $x$ is in the closure of $\{x_\eta ; \eta < \xi\}$ for each $\xi < \xi$. Hence $g_\xi(x) = 1$ for every $\xi < \xi$. Let $g$ be a cluster point of the net

$(g_\xi)_{\xi < \xi}$. Then

$g(x) = 1$, but

$g(x_\xi) \leq 1/2$ ($\eta < \xi$), since

$g_\xi(x_\xi) \leq 1/2$ ($\eta < \xi < \xi$). This contradicts the continuity of $g$ at $x$.

Since $\{x_\xi ; \xi < \Omega\}$ is a closed discrete subspace of the space $T$, there is a continuous function $f \in \mathcal{C}(T)$ with $f(x_\xi) = 0$ ($\xi$ an isolated ordinal)

$f(x_\xi) = 1$ ($\xi$ a limit ordinal).

But such an $f$ can have no extension that is continuous on $\mathcal{E}T$, and this contradiction ends the proof.

Proposition (2.8) applies in particular to the non-$c$-replete $P$-space of $((GJ), 9.L)$. In this case there exists, for every $\psi \in \mathcal{R}(\mathcal{E}T)$ and every $C \in \mathcal{R}(\mathcal{E}T)$,
a function \( f \in C(\theta \mathcal{T}) \) with \( f|c = \psi|c \); a situation very different from that considered in proposition (2.4).

For the last result in this paragraph, we return to the methods of propositions (2.4) and (2.5).

(2.9) **PROPOSITION.** - A compact subset of \( C_s(\mathcal{T}) \) remains compact in \( C_s(\mu \mathcal{T}) \).

*Proof.* - By the characterization of \( \mu \mathcal{T} \) as the space obtained by transfinite iteration of the bidual operation \( (\mathcal{B}_2, \text{théorème } 2) \), it is enough to prove that a compact subset \( A \) of \( C_s(\mathcal{T}) \) is compact in \( C_s(\mathcal{T}''') \). Such an \( A \) is countably compact in \( C_s(\mathcal{T}''') \) and hence, for each bounded \( B \subset \mathcal{T} \), \( A|B \) is countably compact in \( C_s(B) \). But countable compactness and compactness coincide in this space, since \( B \) is compact. Thus, for all characters \( u \in B \), \( u|A \) is continuous for the topology of pointwise convergence on \( B \), and we deduce that \( u|A \) is \( C_s(\mathcal{T}) \)-continuous for every \( u \in \mathcal{T}'''' \).

(2.10) **COROLLARY.** - If \( C_c(\mathcal{T}) \) is a Kelley space then \( \mathcal{T} \) is a \( \mu \)-space, i.e. \( C_c(\mathcal{T}) \) cannot be Kelley without being barrelled.

3. - **INFRA-\( k_R \)-SPACES.**

The space \( \mathcal{T} \) of \( \mathcal{H}_2 \) has given us an example of a complete lcs \( E = C_c(\mathcal{T}) \), the Kelleyfié of which, \( F = \mathcal{K}E = C_c(\theta \mathcal{T}) \), is not quasi-complete. \( F \) is, however, a \( p \)-semi-reflexive space \( ([DJ]) \), that is to say, every precompact subset is relatively compact. In this example \( \theta \mathcal{T} \) happens to be an infra-\( k_R \)-space, but it would seem, a priori, that the property "every precompact set is relatively compact" was a good deal weaker than the infra-\( k_R \)-property, "every precompact set is equicontinuous". But it turns out that this is not the case.

(3.1) **THEOREM.** - \( \mathcal{T} \) is an infra-\( k_R \)-space if and only if every precompact subset of \( C_c(\mathcal{T}) \) is relatively compact in \( C_s(\mathcal{T}) \).

(3.2) **COROLLARY.** - \( \mathcal{T} \) is an infra-\( k_R \)-space if and only if \( C_c(\mathcal{T}) \) is \( p \)-semi-reflexive.

We shall need a definition and two preliminary results.
(3.3) **DEFINITION.** - Let us say that a subset $H$ of $C(T)$ is closed under lattice operations (or, more simply, lattice-closed) if $f \lor g \in H$ and $f \land g \in H$ whenever $f, g \in H$. If $H \subseteq C(T)$, define the lattice-closed hull $\Lambda H$ of $H$ to be the smallest lattice closed set that contains $H$.

(3.4) **LEMMA.** - For a subset $H$ of $C(T)$ the following are equivalent:

- (a) $H$ is precompact in $C_c(T)$;
- (a') for every compact $K \subseteq T$, $H|_K$ is bounded and equicontinuous in $C(K)$ (i.e. $H|_K \in H(K)$);
- (b) $\Lambda H$ is precompact in $C_c(T)$;
- (b') for every compact $K \subseteq T$, $\Lambda H|_K \in H(K)$.

**Proof.** - The equivalences $(a) \iff (a')$ and $(b) \iff (b')$ are consequences of ASCOLI's theorem. $(a')$ is equivalent to $(b')$ since the lattice-closed hull of an equicontinuous set is equicontinuous.

(3.5) **PROPOSITION.** - A lattice-closed, relatively compact subset of $C_s(T)$ is equicontinuous.

**Proof.** - Let $H$ be such a set and suppose, if possible, that $H$ is not equicontinuous at some $t \in T$. We can assume that, for some $\varepsilon > 0$, there are, for each neighbourhood $U$ of $t$, a function $h \in H$ and a point $t_U \in U$ such that

$$h(t_U) \geq h(t) + \varepsilon.$$

Now the set $\{h(t) ; h \in H\}$ is bounded in $\mathbb{R}$ and there exists a subnet of $(h_t(t))$ convergent to some $a \in \mathbb{R}$. That is to say that there is a base $U$ of neighbourhoods of $t$ such that $h_t(t) \to a$ as $U$ decreases through $U$. We can suppose that

$$|h_t(t) - a| \leq \varepsilon/3 \quad (U \in U),$$

so that $h_t(t) \leq a + \varepsilon/3$ and $h_t(t_U) \geq a + 2\varepsilon/3$ for all $U \in U$.

Now let us define, for each finite subset $F = \{U_1, \ldots, U_n\}$ of $U$, $g_F = h_{U_1} \lor \ldots \lor h_{U_n}$ and note that $g_F(t) \leq a + \varepsilon/3$ for all $F$, and $g_F(t_U) \geq a + 2\varepsilon/3$ whenever $U \in F$.

Each $g_F$ is in $H$ and so there is a subnet of $(g_F)$ convergent in $C_s(T)$ to some $g$ (in fact, to $g = \text{Sup } h_U$) and we see that $g(t) \leq a + \varepsilon/3$ while $g(t_U) \geq a + 2\varepsilon/3$ ($U \in U$). This contradicts the continuity of $g$ at $t$. 

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Proof of theorem (3.1). - The necessity of the condition comes from the fact that a pointwise bounded equicontinuous subset of $C(T)$ is relatively compact in $C_s(T)$.

Suppose now that the condition is satisfied and that $H$ is a precompact subset of $C_c(T)$. By lemma (3.4), $\Lambda H$ is precompact in $C_c(T)$, and hence relatively compact in $C_s(T)$. But now, by proposition (3.5), we deduce that $\Lambda H$ is equicontinuous.

4. - TWO GENERALIZATIONS OF A THEOREM OF BUCHWALTER.

H. BUCHWALTER has shown that, if $\mathcal{U}T$ is a $k_*^R$-space, then necessarily $\mathcal{U}T = \emptyset T$. There follow two generalizations of this result.

(4.1) **Lemma.** - If $H \in H(T)$ and card $H$ is non-measurable, then the metrizable space $T_H$ is replete and $H \cup \in H(\mathcal{U}T)$.

**Proof.** - Recall that $T_H$ is defined to be the Hausdorff quotient of $T$ endowed with the pseudometric $d(s,t) = \sup_{h \in H} |h(s) - h(t)|$. There is an injection $T_H \to \mathcal{R} H$ so that card $T_H \leq \text{card } H$. Now if $m,n$ are non-measurable cardinals, so is $m^n$ ([1], p. 128) and it follows that $T_H$ is replete.

$H$ factors through the quotient mapping $\pi_H: T \to T_H$, as $H = H_1 \circ \pi_H$ where $H_1 \in H(T_H)$. Since $T_H$ is replete, $\pi_H$ extends to $\pi_H^\cup: \mathcal{U}T \to T_H$ and $H^\cup = H_1 \circ \pi_H^\cup \in H(\mathcal{U}T)$.

(4.2) **Theorem.** - Let $T$ be a completely regular space and suppose either:

(a) $\mathcal{U}T$ has property (A), or
(b) $\mathcal{U}T$ is an infra-$k_*^R$-space.

Then $\mathcal{U}T = \emptyset T$.

**Proof:**

(a) Let $H \in H(T)$. $H$ is relatively compact in $C_s(T)$ and hence relatively countably compact in $C_s(\mathcal{U}T)$. By the theorem of PRYCE, $H$ is relatively compact in $C_s(\mathcal{U}T)$. We can deduce that the topologies of $C_s(T)$ and of $C_s(\mathcal{U}T)$ coincide on $H$ and hence that, for any $u \in \mathcal{U}T$, $u|H$ is continuous for the topology of simple convergence on $T$. But this is exactly the condition for a character $u$ to be in $\emptyset T$. 

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(b) Again suppose $H \in H(\mathcal{T})$. As above, it will be enough to show that $H^U$ is relatively compact in $C_s(\mathcal{U}T)$ and hence enough to show that $H^U$ is precompact in $C_c(\mathcal{U}T)$. This will be true provided that $J^U$ is precompact in $C_c(\mathcal{U}T)$ for each countable $J \subset H$. But, by lemma (4.1), we know that each $J^U$ is even in $H(\mathcal{U}T)$.

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