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HANS JARCHOW

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ON SCHWARTZ SPACES AND MACKEY CONVERGENCE

Hans Jarchow

The purpose of this article is to give a survey on some of the recent results in the theory of Schwartz spaces and to show how they are connected with the theory of Mackey convergence of sequences in locally convex spaces. These results are scattered over various papers, see e.g. [1], [14], [19-22], [26], [27-30], [33], [34]. For some of them we will present here new and (or) simplified proofs.

There are at least two concepts which are highly appropriated to provide us with a satisfactory language for our purpose, namely the concept of spaces with bornology, see Hogbe-Nlend [13], and the concept of limit spaces, see Fischer [11]. It is not hard to see that in the case we are interested in the latter concept includes the first one in a natural way. Moreover, Buchwalter's theory on compactological spaces [4] appears as a special case of the theory of limit spaces. Since this theory is also important in our context, and since we are concerned here mainly with questions of convergence rather than of boundedness, we will work wholly within the category of limit spaces.

1. GENERALITIES ON LIMIT SPACES

We denote by $\mathbb{F}(M)$ the set of all filters on a given set M and by $\mathbb{P}(\mathbb{F}(M))$ the power set of $\mathbb{F}(M)$. Following Fischer [11] we call a map

$$\lambda : M \longrightarrow \mathbb{P}(\mathbb{F}(M))$$

a limit structure on M if it satisfies (L1) and (L2), for all $x \in M$:

(L1) The ultrafilter generated by $\{x\}$ belongs to $\lambda(x)$.

(L2) For all $\mathfrak{F}, \mathfrak{G} \in \mathbb{F}(M)$: $\mathfrak{F}, \mathfrak{G} \in \lambda(x) \Leftrightarrow \mathfrak{F} \cap \mathfrak{G} \in \lambda(x)$.

The ordered pair $[M, \lambda]$ is called a limit space.

Since each topology on a set determines canonically a unique limit structure on this set, we will consider topological spaces as special limit spaces throughout this paper. Generalizing the corresponding notations from topology, we are led to the following definitions:

Let $[M,\lambda]$ be a limit space. If $\mathcal{F}\in \mathbb{F}(M)$ satisfies $\mathcal{F}\in\lambda(x)$ for some $x\in M$, then $\mathcal{F}$ is said to converge to x. If, for all $x,y\in M$, $\lambda(x)\cap\lambda(y)$ is void unless $x=y$, then we say that $[M,\lambda]$ (or λ) is Hausdorff or separated.

Let $[N,\mu]$ be a second limit space. A map $f:M\longrightarrow N$ is called (λ,μ)--continuous, or simply continuous, if $f(\lambda(x))\subset\mu(f(x))$ holds for all $x\in M$. If λ and μ are limit structures on M such that the identity map of M is (λ,μ)-continuous, we say that λ is finer (or stronger) than μ, and that μ is coarser (or weaker) than λ, and we write $\lambda\geq\mu$ or $\mu\leq\lambda$.

Together with their continuous mappings the limit spaces form a category L containing the category TOP of all topological spaces as a full subcategory. L is complete and co-complete. Products, co-products, subobjects, quotients, and, more generally, projective and inductive limits in L are obtained in the usual way.

Let now $[M,\lambda]$ be the inductive limit in L of a family $([M_i,\lambda_i])_{i\in I}$ of limit spaces, with respect to the continuous mappings $f_{ij}:M_i\longrightarrow M_j$, $i\leq j$. Possibly after some elementary manipulations, we may assume that $M_i\subset M_j$ for $i\leq j$ and that f_{ij} is the corresponding inclusion mapping. We will reserve the notation

$$[M,\lambda]=\lim_{i\in I}\operatorname{ind}[M_i,\lambda_i]$$

exclusively for this special situation, and we will only consider inductive limits in L of this type.

The behavior of this "lim ind" is rather good. We start with the following proposition which is mainly due to Fischer [11] (see also [36], [23]):

(1.1) A filter $\mathcal{F}$ in $[M,\lambda]=\lim_{i\in I}\operatorname{ind}[M_i,\lambda_i]$ converges to $x\in M$ if and only if we have $x\in M_i$, $M_i\in\mathcal{F}$ and $\mathcal{F}_i\in\lambda_i(x)$ for some $i\in I$, where $\mathcal{F}_i$ is the trace of $\mathcal{F}$ on M_i.

It follows that $[M,\lambda]$ is Hausdorff iff this is true for all $[M_i,\lambda_i]$.

Another important property of L, namely to be a cartesian closed category, is expressed in the following theorem, the proof of which can be found in [7] and [2]:

(1.2) <u>Let $[M,\lambda]$ and $[N,\mu]$ be limit spaces. Let $\mathscr{C}(M,N)$ denote the set of all (λ,μ)-continuous maps from M into N. Among all limit structures Λ on $\mathscr{C}(M,N)$ such that the evaluation map</u>

$$\omega: [\mathscr{C}(M,N),\Lambda] \times [M,\lambda] \longrightarrow [N,\mu] : (f,x) \longmapsto f(x)$$

<u>is continuous, there is a coarsest one which we denote by Λ^c.</u>

Λ^c is called the <u>limit structure of continuous convergence</u>. It is Hausdorff iff μ has this property. A filter Φ in $\mathscr{C}(M,N)$ belongs to $\Lambda^c(f)$ for some $f \in \mathscr{C}(M,N)$ if and only if, for every $x \in M$ and every $\mathfrak{F} \in \lambda(x)$, we have $\omega(\Phi \times \mathfrak{F}) \in \mu(f(x))$.

The limit structure induced from Λ^c on $H \subset \mathscr{C}(M,N)$ will be denoted by Λ^c again. Sometimes we will write H_c instead of $[H,\Lambda^c]$, and $\mathscr{C}_c(M,N)$ instead of $[\mathscr{C}(M,N),\Lambda^c]$. Also, if $[M,\lambda]$ is any limit space, and if no confusion is to be feared, we sometimes will omit the λ and write M instead of $[M,\lambda]$.

For limit spaces M and N, (1.2) yields the following universal property of the spaces H_c for $H \subset \mathscr{C}(M,N)$:

(1.3) <u>If Z is any limit space, then a map $f: Z \longrightarrow H_c$ is continuous if and only if $\tilde{f}: Z \times M \longrightarrow N: (z,x) \longmapsto f(z)(x)$ is continuous.</u>

It is clear that (1.3) actually characterizes the limit structure Λ^c on H.

We take the occasion to establish the connection with the compactological spaces of Buchwalter [4]:

A <u>compactological space</u> is an ordered pair $(M,\mathfrak{K})$, where M is a set and $\mathfrak{K}$, the <u>compactology</u> of $(M,\mathfrak{K})$, is an upwards directed covering of M, each of its members K being supplied with a fixed Hausdorff compact topology τ_K such that (C1) and (C2) are satisfied:

(C1) If $K, L \in \mathfrak{K}$ and $K \subset L$, then $[K,\tau_K]$ is a subspace of $[L,\tau_L]$.

(C2) If $K \in \mathfrak{K}$ and $H \subset K$, then the τ_K-closure $\bar{H}$ of H, supplied with the topology induced from τ_K, belongs to $\mathfrak{K}$.

We will call the compactology $\mathfrak{K}$ of $(M,\mathfrak{K})$ <u>maximal</u> if there exists a limit structure μ on M such that all compact subsets of $[M,\mu]$ are topological spaces under the induced limit structure, and $\mathcal{K}$ consists exactly of these compact sets. A subset S in $[M,\mu]$ is said to be compact if every ultrafilter in M containing S converges to some $x \in S$.

Let $(M,\mathcal{K})$ and $(M',\mathcal{K}')$ be compactological spaces. A map $f:M \longrightarrow M'$ is called compactological if it maps continuously each member of $\mathcal{K}$ into some member of $\mathcal{K}'$. In this case, f(K) is an element of $\mathcal{K}'$ under the final topology of $K \rightarrow f(K)$, for every $K \in \mathcal{K}$.

If $(M,\mathcal{K})$ is a compactological space, we can introduce a limit structure $\lambda_{\mathcal{K}}$ on M by means of

$$[M,\lambda_{\mathcal{K}}] = \lim_{\substack{\longrightarrow \\ K \in \mathcal{K}}} \text{ind}\, [K,\tau_K] .$$

If $(M',\mathcal{K}')$ is another compactological space and $f:M \longrightarrow M'$ is a compactological map, then f is clearly $(\lambda_{\mathcal{K}},\lambda_{\mathcal{K}'})$-continuous. Conversely, if f is $(\lambda_{\mathcal{K}},\lambda_{\mathcal{K}'})$-continuous and if $\mathcal{K}'$ is maximal, then it is easy to see that f is also compactological. In this case, $\mathcal{K}'$ consists exactly of the compact subsets of $[M',\lambda_{\mathcal{K}'}]$. So we have:

(1.4) The category of all maximal compactological spaces (compactological maps) can be considered as a full subcategory of $\underline{L}$.

2. LIMIT STRUCTURES ON DUAL SPACES

A limit vector space is a limit space $[E,\lambda]$ consisting of a (real or complex) vector space E and a limit structure λ on E such that addition $E \times E \longrightarrow E$ and scalar multiplication $\mathbb{K} \times E \longrightarrow E$ are continuous for the respective product limit structures. Here $\mathbb{K}$ denotes the field of real or complex numbers, with its euclidean topology.

If $[E,\lambda]$ is a limit vector space, then, for every $x \in E$, $\lambda(x)$ is obtained from $\lambda(o)$ by $\lambda(x) = x+\lambda(o)$. Therefore a linear map $T:E \longrightarrow F$, $[E,\lambda]$ and $[F,\mu]$ limit vector spaces, is (λ,μ)-continuous iff it satisfies $T(\lambda(o)) \subset \mu(o)$.

Again, the limit vector spaces form, together with their continuous linear maps, a category $\underline{LVR}$ having arbitrary limits and co-limits. For details we refer to [11] , [36] , [23] , [16] , and others.

If $[E,\lambda]$ is a limit vector space, then the family of all continuous semi-norms on $[E,\lambda]$ defines a locally convex topology $\tau(\lambda)$ on E which is actually the finest among all locally convex topologies on E majorized by λ. This topology, however, need not be separated, even if λ is. We will not consider the obvious functorial interpretation of this construction here.

Our interest mainly goes into the properties of special limit structures on the dual E' of a given locally convex space E, the latter being tacitly assumed to be Hausdorff throughout this note. By $\mathfrak{U}_E$ we always will denote a given neighborhood basis of zero in E.

Firstly, we examine some of the properties of the limit structure of continuous convergence on E'. We start with the following general result the easy proof of which may be found in [2]:

(2.1) Let M be a limit space and F a limit vector space. For every linear subspace H of $\mathcal{C}(M,F)$, H_c is a limit vector space.

In our situation, where M is the locally convex space E and F is the scalar field $\mathbb{K}$, we take for H the dual E' of E. The canonical bilinear form $\omega: E'_c \times E \longrightarrow \mathbb{K}$ is continuous, and Λ^c is even the coarsest limit structure on E' having this property.

We will give a concrete representation of E'_c as a maximal compactological space now. While our proof given in [19] referred to the general theory of locally compact limit spaces of Schroder [31], we now give a direct proof using only some well-known facts on topological vector spaces:

(2.2) Let E be a locally convex space. For each $U \in \mathfrak{U}_E$, let us denote by σ_{U^o} the (compact) topology induced from the weak topology $\sigma(E',E)$ on the polar U^o of U. Then we have

$$[E',\Lambda^c] = \lim_{U\in\mathfrak{U}_E} \operatorname{ind}\, [U^o,\sigma_{U^o}]$$

(independently from the special choice of $\mathfrak{U}_E$, of course).

Proof. The inductive limit occurring on the right hand side of the above equation clearly exists. Let us denote the corresponding limit structure on E' by Λ. It is easy to see that $[E',\Lambda]$ is a limit vector space.

If $\mathfrak{F}$ is in $\Lambda^c(o)$, then the filter (generated by) $\omega(\mathfrak{F} \times \mathfrak{U}_E)$ converges to zero in $\mathbb{K}$. Thus we have $U^o \in \mathfrak{F}$ for some $U \in \mathfrak{U}_E$. Since $\sigma(E',E)$ is obviously coarser than Λ^c it follows that the trace of $\mathfrak{F}$ on U^o converges to zero in $[U^o, \sigma_{U^o}]$. Hence we have, by continuity, $\mathfrak{F} \in \Lambda(o)$ which proves $\Lambda^c \geq \Lambda$.

On the other hand, $\omega : [E', \sigma(E',E)] \times E \longrightarrow \mathbb{K}$ is $\mathfrak{B}_e$-hypocontinuous, $\mathfrak{B}_e$ the family of all equicontinuous subsets of E'. Therefore the restriction of ω to each of the spaces $[U^o, \sigma_{U^o}] \times E$ is continuous, cf. [3]. From this we get continuity of $\omega : [E', \Lambda] \times E \longrightarrow \mathbb{K}$ and thus $\Lambda \geq \Lambda^c$ from the universal property of Λ^c .

As a consequence we have:

(2.3) Λ^c is the finest limit structure on E' inducing on every equicontinuous subset of E' the same topology as $\sigma(E',E)$.

The topology $\varkappa(\Lambda^c)$ is therefore the finest locally convex topology on E' which induces on every equicontinuous subset of E' the weak topology. Hence we get from the Grothendieck construction of the completion of a locally convex space (cf. [24]):

(2.4) The dual of E'_c is (algebraically isomorphic with) the completion $\tilde{E}$ of E.

Moreover, the dual of E'_c, supplied with the corresponding limit structure of continuous convergence, is a locally convex space which is even topologically isomorphic with $\tilde{E}$; cf. Butzmann [6].

The family $\mathfrak{C}_e$ of all $\sigma(E',E)$-closed equicontinuous subsets of E' is a compactology on E'. From a theorem of Cook-Fischer [7] it follows that $\mathfrak{C}_e$ consists exactly of the compact subsets of E'_c. So we have:

(2.5) $(E', \mathfrak{C}_e)$ is a maximal compactological space, and $\lambda_{\mathfrak{C}_e}$ is the limit structure Λ^c of continuous convergence.

We now will introduce a second canonical limit structure on the dual E' of a locally convex space E.

To begin with, let $[F, \lambda]$ be any limit vector space. We will say that $[F, \lambda]$ is a convex limit space, and λ is a convex limit structure on F

(Marinescu space and Marinescu limit structure in [16]), if it has a representation of the form

$$[F,\lambda] = \lim_{i \in I} \operatorname{ind} [F_i, \tau_i],$$

where the $[F_i, \tau_i]$ are locally convex spaces (not necessarily Hausdorff). Of course, we require here the linearity of all continuous embedaings $F_i \longrightarrow F_j$, $i \leq j$.

The convex limit spaces form a full subcategory CLVR of LVR. In CLVR, we have arbitrary co-limits and limits again, the latter, however, not being necessarily identical with the corresponding ones formed in LVR, cf. [16]. In general, the limit structure of continuous convergence on spaces of continuous linear functions between convex limit spaces is not a convex limit structure, see below. There is, however, a substitute for this limit structure in CLVR. For the construction which is based on ideas of Marinescu [25] see [16], [23]. Here we only will give an explicit formula for this limit structure on the dual E' of a locally convex space E.

Before doing this, we will establish the connection with the work of Hogbe-Nlend [13] on bornological vector spaces.

Let E be a $\mathbb{K}$-vector space, and let $\mathfrak{B}$ be a convex bornology on E. This means that we are given an upwards directed covering $\mathfrak{B}$ of E, stable under the operations of forming subsets, vectorial sums, homothetic images and absolutely convex hulls of its members. The pair $(E,\mathfrak{B})$ is called a convex bornological (vector) space.

For each $B \in \mathfrak{B}$, we denote by E_B the linear space generated by B. This becomes a semi-normed locally convex space by means of the topology τ_B generated by the gauge of (the absolutely convex cover of) B. With $\mathfrak{B}$, we may thus associate the convex limit structure $\lambda_{\mathfrak{B}}$ on E defined by

$$[E,\lambda_{\mathfrak{B}}] = \lim_{B \in \mathfrak{B}} \operatorname{ind} [E_B, \tau_B].$$

Let $(F,\mathfrak{B}')$ be a second convex bornological space. A linear map $f:E \longrightarrow F$ is called bounded if it satisfies $f(\mathfrak{B}) \subset \mathfrak{B}'$. Using (1.1) one proofs that this is the case if and only if f is $(\lambda_{\mathfrak{B}},\lambda_{\mathfrak{B}'})$-continuous, cf. [23]. If a convex limit space $[E,\lambda]$ is representable in the form $[E,\lambda] = \lim_{j \in J} \operatorname{ind} [E_j, \tau_j]$ with semi-normed spaces $[E_j, \tau_j]$, then the system

$\mathfrak{B}$ of all subsets of E which are contained and bounded in some $[E_j, \tau_j]$ is a convex bornology on E. Since $\lambda = \lambda_{\mathfrak{B}}$ holds we have:

(2.6) <u>The category of all convex bornological spaces (bounded linear maps) can be identified with the full subcategory of CLVR consisting of those convex limit spaces which have a representation as inductive limits of semi-normed locally convex spaces.</u>

We will name the objects in this subcategory <u>bornological</u> (convex) <u>limit spaces</u>.

It is clear, that the term "Mackey convergence" for convex bornological spaces (cf.[13]) means just convergence in the corresponding bornological limit space.

Let now E be a locally convex space again, and let $\mathfrak{U}_E$ be a basis for the neighborhood filter of zero in E. We prove as promised the "bornological" analogue of (2.2):

(2.7) <u>There exists a coarsest among all convex limit structures</u> Λ <u>on</u> E' <u>such that</u> $\omega: [E', \Lambda] \times E \longrightarrow \mathbb{K}$ <u>is continuous</u>. <u>We denote this limit structure by</u> Λ^m.
$[E', \Lambda^m]$ <u>is bornological and may be represented in the form</u>

$$[E', \Lambda^m] = \lim_{\substack{\longrightarrow \\ U \in \mathfrak{U}_E}} \text{ind}\, [E'_{U^o}, \tau_{U^o}]$$

(the representation being clearly independent from the special choice of $\mathfrak{U}_E$ again).

By means of (2.6) we may therefore identify Λ^m with the equicontinuous bornology $\mathfrak{B}_e$ on E', cf. [13].

<u>Proof.</u> Define $\lambda\, (= \lambda_{\mathfrak{B}_e})$ by $[E', \lambda] = \lim\limits_{\substack{\longrightarrow \\ U \in \mathfrak{U}_E}} \text{ind}\, [E'_{U^o}, \tau_{U^o}]$. We have to show, that λ has the required universal property.

$\lambda \geq \Lambda^c$ is easily seen, hence $\omega: [E', \lambda] \times E \longrightarrow \mathbb{K}$ is continuous. To see that λ is the coarsest convex limit structure on E' having this property, we prove a little lemma (see [23] and [16] for generalizations).

Let F be a topological vector space, and let $T: F \longrightarrow E'_c$ be linear and continuous. Then $T: F \longrightarrow [E', \lambda]$ is continuous.

Let us denote by $\mathfrak{U}_F$ the neighborhood filter of zero (or a basis of it) in F. For the filter (generated by) $T(\mathfrak{U}_F)$ we have $T(\mathfrak{U}_F)\in\Lambda^c(o)$. This yields the existence of some $U\in\mathfrak{U}_E$ such that $U^o\in T(\mathfrak{U}_F)$, or $T(V)\subset U^o$ for some $V\in\mathfrak{U}_F$. Hence T maps F continuously into $[E'_{U^o},\tau_{U^o}]$ which means that $T:F \longrightarrow [E',\lambda]$ is continuous.

The lemma remains clearly true if we replace F by any convex limit space, and from this follows our assertion.

As a consequence we get (cf. [23]) that Λ^m is a topology if and only if E is a normed space, in which case we have $\Lambda^m=\beta(E',E)$. If E is not normed, then there is no vector space topology ϱ on E' such that $\omega:[E',\varrho]\times E \longrightarrow \mathbb{K}$ is continuous.

We give an application to Schwartz spaces. For our purpose it is convenient to choose the following definition ([15],[34]): A locally convex space E is called a <u>Schwartz space</u> if for every $U\in\mathfrak{U}_E$ there is a $V\in\mathfrak{U}_E$ such that U^o is compact in $[E'_{V^o},\tau_{V^o}]$.

(2.8) <u>For every locally convex space</u> E <u>the following statements are equivalent</u>:

(a) E <u>is a Schwartz space</u>.

(b) $[E',\Lambda^c]$ <u>is convex</u>.

(c) $[E',\Lambda^c]=[E',\Lambda^m]$.

The implications (a)⇒(b)⇔(c) are easy to prove. (c)⇒(a) is a consequence of (1.1) and a lemma of Grothendieck (cf.§21,6.(5) in [24]). For a detailed proof see [19].

As a corollary we get:

(2.9) <u>Let</u> E <u>be a locally convex space</u>. E'_c <u>is a topological space if and only if</u> E <u>is finite dimensional</u>.

This follows from the fact that Schwartz space which is normed is of finite dimension.

3. NULL SEQUENCES IN THE DUAL

It is well known, cf. [15], [24], that in every metrizable locally convex space E a fundamental system of precompact subsets is given by the closed absolutely convex covers of the null sequences in E. Using this together with the description (1.1) of convergence in convex limit spaces, we are led to the following characterization of Schwartz spaces ([34], [19], [14]):

(3.1) A locally convex space E is a Schwartz space if and only if its topology is that of uniform convergence on the null sequences in $[E', \Lambda^m]$.

If one replaces here Λ^m by Λ^c then one obtains exactly Buchwalter's definition of an "espace semi-faible", cf. [5] (in this lecture note). From (2.8) it follows also that every Schwartz space is semi-faible.

A separable Fréchet space is a Schwartz space if and only if the null sequences in $[E', \sigma(E', E)]$ and in $[E', \Lambda^m]$ are the same, see [12], [19]. Hence we have:

(3.2) If E is a Fréchet-Schwartz space, then it has the topology of uniform convergence on the null sequences in $[E', \sigma(E', E)]$.

In this case we may even choose a countable family $\{(f_n^k)_{n\in\mathbb{N}} \mid k=1,2,..\}$ of null sequences in $[E', \sigma(E', E)]$ such that $\{f_n^k \mid n\in\mathbb{N}\} \subset \{f_n^{k+1} \mid n\in\mathbb{N}\}$ holds for every k and the $U_k = \{f_n^k \mid n\in\mathbb{N}\}^\circ$ form a neighborhood basis of zero in E. The locally convex inductive limit of the Banach spaces $[E'_{U_k^o}, \tau_{U_k^o}]$ is $[E', \beta(E', E)]$. Denoting by $[F, \tau]$ the locally convex inductive limit of the linear hulls F_k of the $\{f_n^k \mid n\in\mathbb{N}\}$, supplied with the topology induced from $\tau_{U_k^o}$, and extending slightly an argument of Köthe, cf. §31, 6.(1) in [24], one finds that $[F, \tau]$ is a sequentially dense subspace of $[E', \beta(E', E)]$. Since all F_k have countable (algebraic) dimension, one obtains:

(3.3) Every Fréchet-Schwartz space is the strong dual of a bornological (DF)-Schwartz space of countable dimension.

For details we refer to [20].

Let now $[E,\tau]$ be an arbitrary (Hausdorff) locally convex space. The family $\mathcal{S}$ of all Schwartz topologies π on E satisfying $\pi \leq \tau$ contains $\sigma(E,E')$, hence it is non-empty. The supremum τ_o of $\mathcal{S}$ belongs to $\mathcal{S}$ (see [15]) and is called the Schwartz topology associated with τ. Often we will write E_o instead of $[E,\tau_o]$.

The topology τ_o was first introduced and studied by Raǐkov [26]. Later contributions can be found e.g. in [1], [20], [14], [33].

The dual of E_o is E' again. Hence we have, besides Λ^m, another convex limit structure Λ^m_o on E' which is defined by $[E',\Lambda^m_o] = \lim\limits_{U \in \mathfrak{U}_o} \text{ind}\, [E'_U, \tau_U]$, where $\mathfrak{U}_o$ is a neighborhood basis of zero in E_o. In every case, Λ^m is coarser than Λ^m_o. Equality holds if and only if $E = E_o$ is true which is equivalent with the statement that E is a Schwartz space. There are, however, some more connections between Λ^m and Λ^m_o. First of all, we have the following characterization of τ_o (see [14], [33]):

(3.4) τ_o is the topology of uniform convergence on the null sequences in $[E',\Lambda^m]$.

This yields that Λ^m and Λ^m_o have always the same null sequences, cf. [33]. A little more can be said:

(3.5) Λ^m_o is the finest bornological limit structure on E' which defines the same null sequences as Λ^m.

Using the result of Schwartz [32] concerning the ultrabornological character of the strong dual of a complete Schwartz space one finds the following characterization of the locally convex topology $\varkappa(\Lambda^m)$ on E' (cf. [1], [33]):

(3.6) For every locally convex space E we have on E'

$$\varkappa(\Lambda^m) = \varkappa(\Lambda^m_o) = \beta(E',\tilde{E}_o).$$

Here $\tilde{E}_o$ denotes the completion of E_o.

We can now give a simple proof of the following result of Raǐkov [26]:

(3.7) Let E be a locally convex space and let F be a Banach space. The continuous linear maps from E_o into F are exactly the compact linear maps from E into F.

Proof. If $T:E_o \longrightarrow F$ is continuous, then it is compact as a map $T:E \longrightarrow F$ since E_o is a Schwartz space. Conversely, if $T:E \longrightarrow F$ is a compact linear map, then there is a continuous semi-norm p on E such that T factors through a compact operator $T_p:E/\overset{-1}{p}(o) \longrightarrow F$, where $E/\overset{-1}{p}(o)$ is normed in the canonical way. Hence $T_p':F' \longrightarrow (E/\overset{-1}{p}(o))'$ is compact which means that $T':F' \longrightarrow [E',\Lambda_o^m]$ is continuous (use (3.4)). Hence $T:E_o \longrightarrow F$ is continuous by [17].

From (3.1) we deduce the following representation theorem for Schwartz spaces:

(3.8) A locally convex space E is a Schwartz space if and only if it is isomorphic with a subspace of some product space $[c_o]^I$, I a suitably chosen set.

Here we denote by $[c_o]$ the space c_o of all scalar null sequences, together with the topology of uniform convergence on the compact subsets of ℓ_1. This space, however, is not complete. The completion is the space ℓ_∞, together with the topology of uniform convergence on the compact subsets of ℓ_1, which is nothing else but the Mackey topology. So we may replace, in the theorem, $[c_o]$ by $[\ell_\infty, \tau(\ell_\infty, \ell_1)]$.

We only scetch the proof of [21]. For another proof see [30]. First of all, it is clear that $[c_o]$ is a Schwartz space and that its topology is the Schwartz topology associated with the sup-norm topology. If now $((u_k^i)_{k\in\mathbb{N}})_{i\in I}$ is the family of all null sequences in $[E',\Lambda^m]$, and if E is a Schwartz space, then we define continuous linear mappings $t_i:E \longrightarrow c_o : x \longmapsto (\langle u_k^i, x\rangle)_{k\in\mathbb{N}}$ whose kernels coincide with the kernels of the gauges of the corresponding neighborhoods $\{u_k^i \mid k\in\mathbb{N}\}^o$ in E. If E_i denotes the corresponding quotient space, normed by the gauge, then t_i admits a unique linear factorization $T_i:E_i \longrightarrow c_o$ which is an isometry. Identifying the completion $\tilde{E}_i$ of E_i with a closed linear subspace of c_o we see that we not only have $\tilde{E} = \lim\limits_{i\in I} \operatorname{proj} \tilde{E}_i$ but even $\tilde{E} = \lim\limits_{i\in I} \operatorname{proj} (\tilde{E}_i)_o$, since the completion $\tilde{E}$ of the Schwartz space E is also a Schwartz space. Since the $(\tilde{E}_i)_o$ are subspaces of $[c_o]$ it follows that $\tilde{E}$ and hence E are subspaces of $[c_o]^I$.

The existence of a Schwartz space, universal in the above sense, has been predicted already by Diestel, Morris and Saxon in their paper [10] on varieties of locally convex spaces.

From (3.7), (3.8) we obtain the following result of Randtke [28] :

(3.9) Let E be a locally convex space and F a Banach space. A linear map $T:E \longrightarrow F$ is compact if and only if there exist a closed linear subspace G_T of c_o and compact linear maps $T_1:E \longrightarrow G_T$, $T_2:G_T \longrightarrow F$ such that $T = T_2 T_1$.

Proof. If T is compact, then it is continuous as a map $T:E_o \longrightarrow F$. Writing $\tilde{E}_o = \lim\limits_{i\in I} \text{proj}\ (E_i)_o = \lim\limits_{i\in I} \text{proj}\ E_i$ as above with closed linear subspaces E_i of c_o, we find, that $\tilde{T}$, the extension of T to $\tilde{E}_o$, factors through a continuous linear map $T_2:(E_i)_o \longrightarrow F$ for some $i\in I$. T_2 is compact as a map $T_2:E_i \longrightarrow F$, the canonical map $T_1:E \longrightarrow E_i$ is compact, and $T = T_2 T_1$ holds.

In his original proof in [28] Randtke used the following result (see [35] and [27]):

(3.10) Let E and F be Banach spaces. A linear map $T:E \longrightarrow F$ is compact if and only if there is null sequence $(u_n)_{n\in\mathbb{N}}$ in the dual Banach space E' of E such that

$$\|Tx\| \le \sup\{|\langle u_n, x\rangle| \mid n\in\mathbb{N}\}, \forall x\in E.$$

This is now also a simple consequence of (3.4) and (3.7).

4. MACKEY CONVERGENCE OF SEQUENCES

Let E be a locally convex space with dual E'. By $\mathfrak{B}, \mathfrak{S}, \ldots$ we will denote coverings of E consisting of bounded subsets of E. We require $\mathfrak{B}, \mathfrak{S}, \ldots$ to be saturated, i.e. closed under the operations of forming subsets, scalar multiples, finite unions and closed absolutely convex covers of their members. Thus $\mathfrak{B}, \mathfrak{S}, \ldots$ are convex bornologies on E. The corresponding topologies of uniform convergence on the sets in $\mathfrak{B}$, $\mathfrak{S}, \ldots$ on E' is denoted by $\tau_{\mathfrak{B}}, \tau_{\mathfrak{S}}, \ldots$. Not every locally convex topology $\mathfrak{a}$ on E' satisfying $\sigma(E',E) \le \mathfrak{a} \le \beta(E',E)$ is expressible in this way since we are working with a fixed duality, namely $\langle E',E\rangle$.

With each of the above $\mathfrak{B}$ we associate the (Hausdorff) bornological limit structure $\lambda_{\mathfrak{B}}$ on E by means of

$$[E,\lambda_{\mathfrak{B}}] = \lim_{\substack{\longrightarrow \\ B\in\mathfrak{B}}} \text{ind}\, [E_B,\tau_B] .$$

In analogy with [14],[23],[33] we call $\lambda_{\mathfrak{B}}$ a Schwartz limit structure and $[E,\lambda_{\mathfrak{B}}]$ a Schwartz limit space if for each $B\in\mathfrak{B}$ there is a $C\in\mathfrak{B}$ containing B such that the inclusion map $[E_B,\tau_B] \longrightarrow [E_C,\tau_C]$ is precompact. If F is a Schwartz space, then $[F',\Lambda^m]$ is a Schwartz limit space. Moreover:

(4.1) $\lambda_{\mathfrak{B}}$ is a Schwartz limit structure on E if and only if $\mathcal{T}_{\mathfrak{B}}$ is a Schwartz topology on E'.

For arbitrary $\mathfrak{B}$, we denote by $\mathfrak{B}_o$ the saturated hull of the system of all null sequences in $[E,\lambda_{\mathfrak{B}}]$. (If τ is the finest locally convex topology on E such that every $B\in\mathfrak{B}$ is τ-bounded, then the null sequences in $[E,\lambda_{\mathfrak{B}}]$ are just the Mackey null sequences [24] in $[E,\tau]$. It is easy to see that $\tau = \varkappa(\lambda_{\mathfrak{B}})$ holds). We have $\mathfrak{B}_o = \mathfrak{B}_{oo}$ (see [14], [23], [33]). Moreover:

(4.2) $\lambda_{\mathfrak{B}_o}$ is a Schwartz limit structure on E. It is the finest bornological limit structure on E which defines the same null sequences than $\lambda_{\mathfrak{B}}$.

As another consequence we get:

(4.3) $\mathcal{T}_{\mathfrak{B}_o}$ is the finest Schwartz topology on E' which is coarser than $\mathcal{T}_{\mathfrak{B}}$ and which is simultaneously an $\mathfrak{S}$-topology in the sense defined above.

If $\mathcal{T}_{\mathfrak{B}} \leq \tau(E',E)$, then we have of course $\mathcal{T}_{\mathfrak{B}_o} = (\mathcal{T}_{\mathfrak{B}})_o$. But in general only $\mathcal{T}_{\mathfrak{B}_o} \leq (\mathcal{T}_{\mathfrak{B}})_o$ is true, and both topologies may very well be different, see [33].

Of course, $\lambda_{\mathfrak{B}}$ is a Schwartz limit structure if and only if it coincides with $\lambda_{\mathfrak{B}_o}$.

Let now $\check{E}$ denote the dual of the locally convex space $[E',\mathcal{T}_{\mathfrak{B}}]$. Since $\sigma(E',E) \leq \mathcal{T}_{\mathfrak{B}}$, we may identify E with a subspace of $\check{E}$. It is not hard to see (cf. [17]) that $[E,\lambda_{\mathfrak{B}}]$ is a subspace of E under its Λ^m-structure, and even under its Λ^c-structure provided $[E,\lambda_{\mathfrak{B}}]$ is a

Schwartz limit space.

We are mainly interested in two special cases of the above situation. Namely, we take for $\mathfrak{B}$ the system $\mathcal{B}$ of all bounded subsets of E (the von Neumann bornology in [13]), or we take for $\mathfrak{B}$ the saturated hull $\mathcal{L}$ of all absolutely compact subsets of E. The null sequences of $[E,\lambda_{\mathcal{B}}]$ are the <u>Mackey null sequences</u> on E, cf. [24]. The null sequences of $[E,\lambda_{\mathcal{L}}]$ are the <u>fast null sequences</u> introduced by de Wilde [8],[9].

Following Köthe [24] we denote the topology $\mathcal{T}_{\mathcal{B}_o}$ on E' by $\tau_{c_o}(E',E)$. Similarly, we write $\tau_{c_f}(E',E)$ instead of $\mathcal{T}_{\mathcal{L}_o}$. These two topologies are Schwartz topologies, a fact which seems to have been completely overlooked during the development of the theory of bornological and ultrabornological spaces.

It is quite clear that $\tau_{c_o}(E',E)$ is independent from the special choice of the topology on E which is only required to be consistent with the duality $\langle E,E'\rangle$. The same result is true for $\tau_{c_f}(E',E)$, but this is less obvious. One proof of this uses the fact that in a Banach space or even in a Fréchet space every null sequence converges fast. But this result is proved by making a more or less complete copy of the corresponding proof of the fact that in a metrizable locally convex space every null sequence is a Mackey null sequence. There is, however, another way to get the desired result and even a little more (see [33]):

(4.4) $\tau_{c_o}(E',E)$ <u>is the finest Schwartz topology on</u> E' <u>which is an</u> σ-<u>topology in our sense</u>.

$\tau_{c_f}(E',E)$ <u>is the finest Schwartz topology on</u> E' <u>which is consistent with</u> $\langle E',E\rangle$, <u>i.e.</u>

$$\tau_{c_f}(E',E) = (\tau(E',E))_o .$$

<u>Proof.</u> The first statement follows from (4.3). To prove the second one, we observe first that $\tau_{c_f}(E',E) \leq (\tau(E',E))_o$ is a triviality.

Let us denote by $\mathfrak{S}$ the saturated hull of the system of all absolutely convex, compact sets in $[E,\sigma(E,E')]$. Then we have $(\tau(E',E))_o = \mathcal{T}_{\mathfrak{S}_o}$. If $(x_n)_{n\in\mathbb{N}}$ is a null sequence in $[E,\lambda_{\mathfrak{S}}]$ we find some absolutely convex $\sigma(E,E')$-compact set $K\subset E$ such that $(x_n)_{n\in\mathbb{N}}$ is a null sequence in the Banach space $[E_K,\tau_K]$. It follows that the closed absolutely convex cover of $\{x_n \mid n \in \mathbb{N}\}$ is compact in E. So we get $(\tau(E',E))_o \leq \mathfrak{T}_{\mathfrak{L}} \leq \tau(E',E)$, hence $\tau_{c_f}(E',E) = \mathfrak{T}_{\mathfrak{L}_0} = (\mathfrak{T}_{\mathfrak{L}})_o = (\tau(E',E))_o$.

As a corollary we obtain:

(4.5) $[E,\sigma(E,E')]$ <u>and</u> $[E,\tau(E,E')]$ <u>have the same fast null sequences</u>.

5. APPLICATIONS TO BORNOLOGICAL AND ULTRABORNOLOGICAL SPACES

A locally convex space E is said to satisfy the <u>Mackey convergence condition</u> (<u>MCC</u>) if every null sequence in E is a Mackey null sequence. E satisfies the <u>fast convergence condition</u> (<u>FCC</u>) if every null sequence in E is even fast convergent. Every metrizable locally convex space satisfies MCC and is easily seen to satisfy FCC if and only if it is a Fréchet space. Moreover, there are even complete (DFM)-spaces which do not satisfy MCC. For details we refer to [33].

The problem of giving a precise internal characterization of those locally convex spaces which satisfy MCC (or FCC) is still open. But something can be said in the case of bornological or ultrabornological spaces, as we shall see below.

First of all, the above results yield a simple dual characterization of the spaces in question. If $\tau_n(E',E)$ is the topology of uniform convergence on all null sequences in the locally convex space E, then one has (cf. [33]):

(5.1) E <u>satisfies</u> MCC <u>if and only if</u> $\tau_n(E',E)$ <u>is a Schwartz topology</u>.
E <u>satisfies</u> FCC <u>if and only if</u> $\tau_n(E',E)$ <u>is a Schwartz topology and consistent with the duality</u> $\langle E',E\rangle$.

As an easy consequence it follows that every (DS)-space (see [34]) satisfies MCC, and that in these spaces every bounded (=precompact) subset is contained in the closed absolutely convex hull of some null-sequence.

We now turn to bornological and ultrabornological spaces and refer on some of the results recently proved in [22].

A locally convex space E is bornological (ultrabornological) if and only if it has its Mackey topology and $\tau_{c_o}(E',E)$ $(\tau_{c_f}(E',E))$

is complete ([24],[9]).

To begin with, let E be bornological. Then $[E',\tau_{c_0}(E',E)]$ is a complete Schwartz space. Hence it is semi-reflexive, and its dual $\breve{E}$ is ultrabornological with respect to $\tau(\breve{E},E') = \beta(\breve{E},E')$ ([15],[32]). Using the Grothendieck construction of the completion $\tilde{E}$ of E we see that $\breve{E}$ is isomorphic with a subspace of $\tilde{E}$. Thus $\tau(\breve{E},E')$ induces $\tau(E,E')$ on E which is the original topology since E is bornological.

By (3.8) we may identify $[E',\tau_{c_0}(E',E)]$ with a closed subspace of the product $[\ell_\infty, \tau(\ell_\infty,\ell_1)]^I$ where I is a suitably chosen set. From duality theory (see e.g. [24]) it follows that we may regard $[\breve{E},\tau(\breve{E},E')]$ as a certain quotient space $\bigoplus_I \ell_1/Q$. Hence E occurs as a dense subspace of this quotient. Writing the direct sum $\bigoplus_I \ell_1$ as the locally convex inductive limit $\varinjlim_\alpha \ell_1^\alpha$, $\ell_1^\alpha := \bigoplus_\alpha \ell_1$, where α ranges over all finite subsets of I, we get a representation of the ultrabornological space $\breve{E}$ of the form $\breve{E} = \varinjlim \breve{E}_\alpha$ where $\breve{E}_\alpha$ is the separable Banach space $\ell_1^\alpha/(Q\cap\ell_1^\alpha)$. So we have:

(5.2) Every bornological space E is isomorphic to a dense subspace of a locally convex space $\breve{E}$ with the following properties:

(a) $\breve{E}$ is ultrabornological and can be written in the form $\breve{E} = \varinjlim_\alpha \breve{E}_\alpha$ with separable Banach spaces $\breve{E}_\alpha$.

(b) Every Mackey null sequence in E can be considered as a null sequence in some $\breve{E}_\alpha$.

Working with $\tau_{c_f}(E',E)$ instead of $\tau_{c_0}(E',E)$ and using that this topology is consistent with $\langle E',E\rangle$ we obtain in the same manner:

(5.3) Every ultrabornological space E has a representation as the inductive limit $E = \varinjlim E_\alpha$ of separable Banach spaces E_α such that every fast null sequence in E converges in some E_α.

And making use of the construction of the E_α we find:

(5.4) An ultrabornological space E satisfies FCC if and only if it is isomorphic to some quotient $\bigoplus_I \ell_1/Q$ such that the corresponding quotient map is sequentially invertible.

Of course, every space of the above type is sequentially complete.

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(Mathematisches Institut
Universität Zürich
Freie Strasse 36
CH 8032 Zürich / Switzerland)