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THE INFINITE RANDOM ORDER OF DIMENSION k

by Peter WINKLER

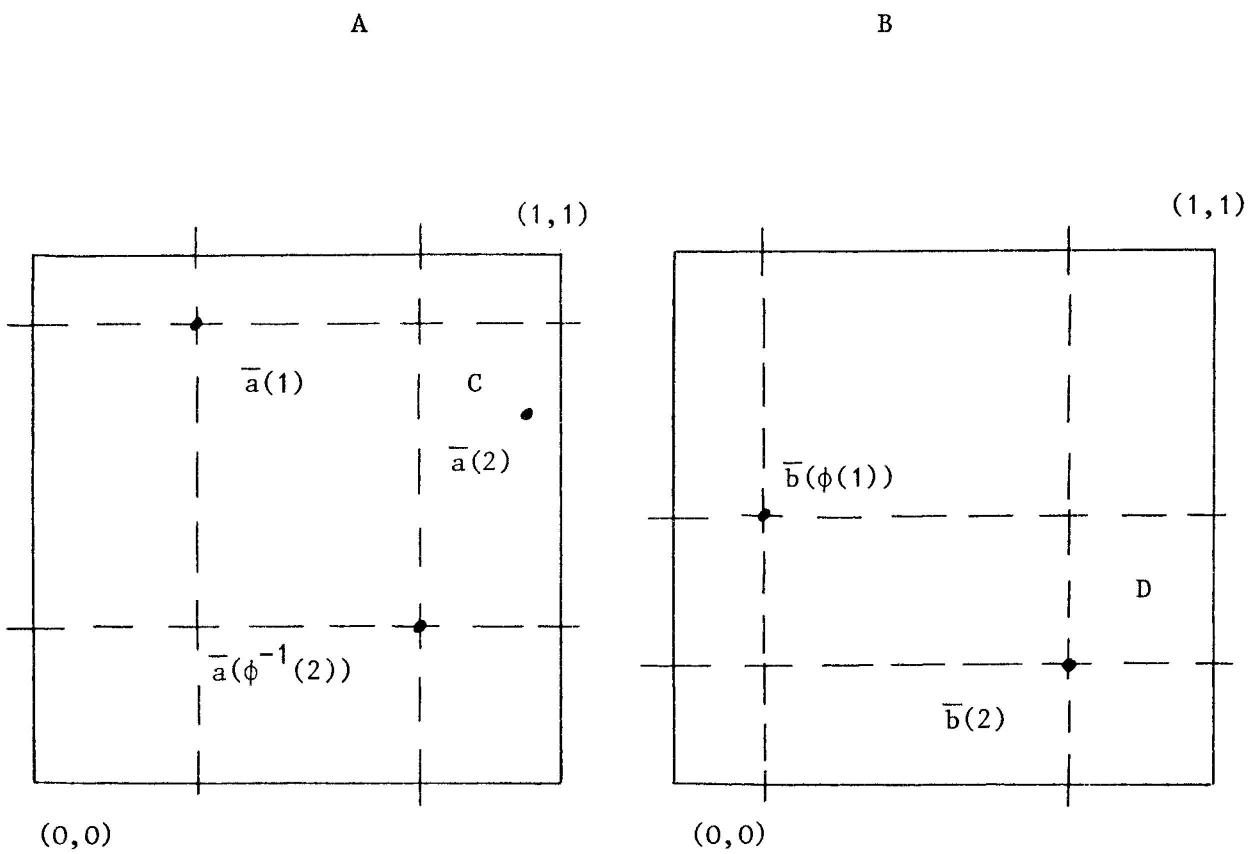
Fix an integer $k > 1$ and let $I^k = \{\bar{x} = (x_1,\ldots,x_k) : 0 < x_i < 1\}$ be the unit hypercube in Euclidean k-space, endowed with the ordinary product order : $\bar{x} < \bar{y}$ iff $x_i < y_i$ for each $i < k$. If n points are chosen randomly and independently from the uniform probability distribution on I^k and given the induced order, the resulting ordered set is what we call the "random order" P^k_n. If instead a countably infinite set of points is chosen, we obtain the infinite random order P^k_ω .

Although the infinite random order is of some interest in itself our intention originally was to use it to prove a "0-1 law" for random orders, i.e. a theorem saying that for any first order sentence in the language of ordered sets, the limit as $n \to \infty$ of Pr (the sentence holds in P^k_n) is either 0 or 1. The 0-1 law holds for random graphs, digraphs, tournaments etc. and is proved in these cases by showing that a countably infinite random structure is with probability 1 isomorphic to the unique countable model of a certain $\aleph_0$-categorical theory. (See, for example, [1]).

Infinite random orders seem at first to be equally congenial. Let $A = \{\bar{a}(1),\bar{a}(2),\ldots\}$ and $B = \{\bar{b}(1),\bar{b}(2),\ldots\}$ be two realizations of P^k_ω ; we employ a back-and-forth argument to construct an order-isomorphism from A to B , $\bar{a}(i)$ to $\bar{b}(\phi(i))$.

Assume at stage 2j of the construction that we have a partial isomorphism having 2j pairings which include all the elements $\bar{a}(1),\dots,\bar{a}(j)$ and $\bar{b}(1),\dots,\bar{b}(j)$. Each point $\bar{a}(i)$ on A's side of the partial isomorphism determines k orthogonal planes in R^k and all $2kj$ planes together will (with probability 1) partition A's hypercube into $(2j+1)^k$ small hyperrectangles. If for each coordinate i the partial isomorphism has so far preserved the linear orders of the a_i's , there will be a corresponding partition of B's hypercube.

Now let s be the least integer so that $\bar{a}(s)$ has not yet been included in the partial isomorphism. The point $\bar{a}(s)$ will be found in one of the above hyperrectangles C of A ; let D be the hyperrectangle of B corresponding to C. (See the diagram below for the case k=j=2.)



Since D has (with probability 1) non-zero volume there will, with probability 1, be infinitely many points of B in its interior ; let $\overline{b}(\phi(s))$ be the least-numbered such point which has not yet been used.

At stage 2j+1, of course, we do the same thing but with the roles of A and B reversed ; here the pre-image $a(\phi^{-1}(t))$ of the next point b(t) is found. In the limit all points of A and B will have been included in the isomorphism.

It is not difficult to see that there is a first-order theory in the language of k linear orders whose unique countable model is the one above ; actually only one axiom is needed, saying that for every 2k distinct points there is a point in each of the $(2k+1)^k$ induced hyperrectangles. It is more difficult to see that this theory exists already in the weaker language of partial order ; for details see [2] , last section.

Unfortunately this theory is a finite extension of the theory of ordered sets of dimension $\leq k$. This means that unlike the case of graphs, digraphs etc. a single new axiom suffices to characterize the infinite random structure ; no finite ordered set can satisfy this axiom. Consequently, while it is still the case that every first-order sentence is either implied or contradicted by this axiom, neither tells us anything about its behavior in finite models. This should not really be surprising, in view of the well-know case where k=1 : an infinite random graph may look like a large finite random graph, but a dense linear order without endpoints (our P^1_ω) looks not at all like a finite linear order !

As it happens, the 0-1 law holds anyway for linear orders. In case $k \geq 2$, however, the following sentence in the language of k linear orders can be seen to have limiting probability 1-1/e : "There exist points x and y such that y is the successor of x in the first two linear orders." When k=2 this sentence is expressible also in the weak language, but in higher dimensions we have as yet no similar example. Thus it is conceivable that in the language of partial orders, the 0-1 law fails only for k=2 ; but not very conceivable.

Our method of constructing random orders does not count each n-element k-dimensional order an equal number of times, so even in the two-dimensional case it is not clear whether the 0-1 law holds in the uniform-weight situation. In fact it is unknown at this time whether the 0-1 law holds for general ordered sets. We should tend to conjecture that the answer is yes in general, no for fixed or bounded dimension.

REFERENCES.

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[2] Peter WINKLER, *Random orders,* Order, to appear.

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