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On K -Boolean Rings

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In this note we define K-Boolean rings which are a generalization of a Boolean ring and obtain conditions for a ring to be K-Boolean in general and in particular for a group ring RG to be K-Boolean, where R denotes a ring and G a group, RG the group ring of G over R.

Definition 1. Let R be a ring with identity we say R is a K-Boolean ring if $x^{2k} = x$ for every $x \in R$ and for some natural number k.

Remark. When k = 1 we trivially get R to be a Boolean ring.

Proposition 2. Every Boolean ring is a K-Boolean ring.

Proof. Given R is a Boolean ring hence $x^2 = x$ for every $x \in R$; so clearly for every $x \in R$, $x^{2k} = x$. Hence R is K-Boolean.

Proposition 3. Every k-Boolean ring need not be a Boolean ring.

Proof. By an example. Take $Z_2 = (0,1)$ to be the field of characteristic two and $G = \langle g \mid g^3 = 1 \rangle$. Clearly $Z_2G = \{0, 1, g, g^2, 1+g, 1+g^2, g+g^2, 1+g+g^2\}$ is 2-Boolean which is clearly not Boolean as 1+g is in Z_2G but $(1+g)^2 \neq 1+g$. Hence the result.

Example. Let $Z_2 = (0,1)$ be the field of characteristic two and $G = \langle g \mid g^2 = 1 \rangle$. Clearly Z_2G is not n - Boolean for any n; as $1+g \in Z_2G$ but $(1+g)^2 = 0$. In view of the above example we have the following.

Proposition 4. Let $Z_2 = (0,1)$ be a field of characteristic two and $G = \langle g | g^{2n}=1 \rangle$ be a cyclic group of even order. Then the group ring Z_2G is not a n-Boolean ring for any n.

Proof. Clearly $1+g^n \in Z_2G$ with $(1+g^n)^2 = 0$; hence Z_2G is not n-Boolean for any n.

The above result can be generalized to any commutative ring of characteristic two as follows.

Theorem 5. Let R be a commutative ring of characteristic two and $G = \langle g | g^{2n}=1 \rangle$ be a cyclic group of even order. Then the group ring RG is not a n-Boolean ring for any n.

Proof. As in the case of proposition 4 we have $1+g^n \in RG$ with $(1+g^n)^2 = 0$, hence the group ring RG is not a n-Boolean ring for any n.

Proposition 6. If R is a n-Boolean ring then R has no non zero nilpotent elements.

Proof. Obvious.

Theorem 7. Let $Z_2 = (0,1)$ be the field of characteristic two and $G = \langle g | g^{2n+1}=1 \rangle$. Then the group ring Z_2G is γ-Boolean with $\gamma = n+1$ and $2(n+1) = 2^s (s>1)$.

Proof. Take any $\alpha \in Z_2G$, since we have for every $g \in G$; $g^{2n+1} = 1$, $\alpha^{2(n+1)} = \alpha$ for every $\alpha \in Z_2G$. Thus Z_2G is (n+1)-Boolean.

Remark. If $2(n+1) \neq 2^s$ for some s we will not have $\alpha^{2(n+1)} = \alpha$. By an example take $G = \langle g | g^5=1 \rangle$, $(1+g)^6 \neq 1+g$.

The above theorem can be true only if we put some conditions on the ring R as follows:

Theorem 8. Let R be a commutative ring of characteristic two with no nontrivial nilpotents and in which every element γ in R is of the form $\gamma^{2(n+1)} = \gamma$ and $G = \langle g \mid g^{2n+1} = 1 \rangle$. Then the group ring RG is γ-Boolean with $\gamma = n+1$ and $2(n+1) = 2^s (s>1)$.

Proof. Obvious.

Theorem 9. Let R be a n-Boolean ring then characteristic of R is two.

Proof. Four possibilities arise (i) characteristic of R is odd (ii) characteristic of R is even not-equal to 2. (iii) characteristic of R is 2^n and (iv) characteristic of R is zero.

Case (i). Let characteristic of R be odd; say 2n+1. Since $1 \in R$ we have an integer $m \in R$ with $m < 2n+1$ with $m^2 = 1$. Thus R is not n-Boolean for any integer n.

Case (ii). Let characteristic of R be even say 2n, since $1 \in R$ clearly $2n-1 \in R$ but $(2n-1)^2 = 1$ hence R is not n-Boolean for any integer n.

Case (iii). Characteristic of R is 2^n $(n>1)$;

Let $m = \frac{2^n}{2} + 1$ then $m^2 = 1$ so R is not n-Boolean.

Case (iv). Let characteristic of R be zero; since $1 \in R$ we have $Z^+ \cup Z^- \cup \{0\} \subseteq R$. Clearly for no natural integer $\pm n \in Z^+ \cup Z^- \cup \{0\}$ we have $n^r = n$. Hence R is n-Boolean. Hence the characteristic of R is two.

Theorem 10. Let RG be the group ring of a group G over the ring R and G a group in which every element is of finite order. RG is n-Boolean if and only if R is a n-Boolean ring of characteristic two and G is a commutative group in which every element is of odd order m with $m+1 = 2^s$ for some $s>1$.

Proof. Since $1 \in G$ we have $R.1 = R \subseteq RG$. Let us asume RG to be n-Boolean this implies by Theorem 9 and definition of a n-Boolean ring characteristic of RG is two and R is commutative since $R \subseteq RG$ and R is n-Boolean ring. G is a commutative group since RG is commutative further every element in G is of odd order for if an element g is of even order we have $g^{2r}=1$ so $(1+g+\ldots+g^{2r-1})^2 = 0$ hence RG is not n-Boolean. Hence the claim.

Conversely if R is n-Boolean ring and G is a commutative group in which every element is of odd order with $m+1 = 2^s$, clearly RG is n-Boolean as RG is commutative with characteristic of RG to be two and every element $\alpha \in RG$ is such that $\alpha^{2k} = \alpha$ for some k, as every element in G is of odd order. Hence the result.

In the above theorem we must have $m+1 = 2^s$ for if $m+1 \neq 2^s$ for any s we will not have $\alpha^{2(m+1)} = \alpha$. By an example take $G = \langle g | g^9=1 \rangle$, $(1+g)^{10} = (1+g)^2(1+g)^2(1+g)^6 = (1+g^2)(1+g^2)(1+g)$

$= (1+g^4)(1+g)^6 = (1+g^4)(1+g)^4(1+g)^2 = (1+g^4)(1+g^4)(1+g^2) =$ $(1+g^8)(1+g^2) \neq 1+g$. Hence we must have $m+1 = 2^s$; then only when we expand we will always have the power of that element to be exactly divided in twos till the end.

Theorem 11. Let R be a finite commutative ring of characteristic two with no non zero nilpotent elements. Then R is a K-Boolean ring if and only if every element in R is of odd order m with $m+1 = 2^s$.

Proof. One way is obvious, for if every element is of odd order m with $(m+1) = 2^s$ then clearly R is a K-Boolean ring.

Conversely if R is a K-Boolean ring then we have $x^{2k} = x$ for every $x \in R$; if K is not a odd number such that $k+1 \neq 2^s$ then we have $(1+x)^{2k} \neq 1+x$.

Hence the theorem.

Reference

[1] Jacobson, N. Structure of Rings, A.M.S. (1956).