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Publications mathématiques de l'I.H.É.S., tome 42 (1973), p. 121-127

http://www.numdam.org/item?id=PMIHES_1973__42__121_0

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AN INDEX THEOREM FOR SYSTEMS OF DIFFERENCE OPERATORS ON A HALF SPACE

by David G. SCHAEFFER⁽¹⁾

In our joint paper [3] we presented an index theorem for a certain class of operators analogous to Tœplitz operators. The index which appeared there was real-valued and it involved Breuer's theory [1] of Fredholm operators in a von Neumann algebra of type II — indeed, this result was of interest precisely because it was the first example of an index theorem for the Breuer index. In the present paper we extend the theorem of [3] to the matrix case, and we apply this result to prove the equivalence of existence and uniqueness of solution for elliptic systems of difference equations, generalizing the result of [6] for a single equation.

Our main result is stated in § 1 and proved in § 2. These sections overlap somewhat with [2] and [3] — our slightly different point of view here allows us to give simpler proofs of the results from these papers which are needed. (See the discussion at the end of § 1.) In § 3 we state without proof the application of the index theorem to difference equations.

1. Statement of the main theorem.

For $j \in \mathbf{Z}^n$ a multi-integer let T_j be the translation on $L^2(\mathbf{R}^n)$,

$$(1.1) \quad T_j v(x) = v(x + j),$$

and let M_ν be the $\nu \times \nu$ matrix algebra. If $\varphi \in C(\mathbf{T}^n, M_\nu)$ is a continuous, matrix-valued function on the torus with Fourier series $\sum_j m_j e^{i\langle j, \cdot \rangle}$, then the formula $L_\varphi = \sum_j m_j \otimes T_j$ defines a bounded linear operator on $C^\nu \otimes L^2(\mathbf{R}^n)$. In this paper we study the restriction of such operators to a half space $H = \{x \in \mathbf{R}^n : \langle x, N \rangle \geq 0\}$, where N is a unit vector in \mathbf{R}^n . Let $E = \chi_{[0, \infty)}(\langle x, N \rangle)$ be the projection onto $L^2(H)$, and for $\varphi \in C(\mathbf{T}^n, M_\nu)$ let $W_\varphi = EL_\varphi E$.

Let \mathcal{N}_0 be the von Neumann algebra of operators on $L^2(\mathbf{R}^n)$ generated by multi-

⁽¹⁾ Research supported under NSF grant 22927.

integer translations T_j and multiplication by functions of $\langle x, N \rangle$. Any operator $A \in \mathcal{N}_0$ may be expressed uniquely as a weakly convergent sum :

$$A = \sum_{j \in \mathbf{Z}^n} c_j(\langle x, N \rangle) T_j$$

where $c_j \in L^\infty(\mathbf{R}^1)$. Two such operators multiply according to the extension by linearity of the basic product formula

$$(1.2) \quad c(\langle x, N \rangle) T_j d(\langle x, N \rangle) T_k = c(\langle x, N \rangle) d(\langle x + j, N \rangle) T_{j+k}.$$

In § 1 of [6] we defined a faithful, normal trace $\tau : \mathcal{N}_0^+ \rightarrow [0, \infty]$ such that for non-negative operators

$$(1.3) \quad \tau(\sum_j c_j(\langle x, N \rangle) T_j) = \int_{-\infty}^{\infty} c_0(t) dt,$$

where c_0 is the coefficient of the identity translation in the sum on the left in (1.3). (In particular, for non-negative operators $c_0 \geq 0$ almost everywhere.) Let $\mathcal{N} = M_\nu \otimes \mathcal{N}_0$; we shall also use τ for the natural trace on \mathcal{N} defined by

$$(1.4) \quad \tau(m \otimes A) = \text{tr}(m) \tau(A).$$

Of course $W_\varphi \in \mathcal{N}$ for every $\varphi \in C(\mathbf{T}^n, M_\nu)$.

If $P \in \mathcal{N}$ is a projection, we shall call $\tau(P)$ the (relative) dimension of the subspace range P . Let \mathcal{K} be the uniformly closed, two sided ideal in \mathcal{N} generated by operators whose range has finite relative dimension. Following Breuer [1] we shall say that $A \in \mathcal{N}$ is (generalized) Fredholm if A is invertible modulo \mathcal{K} . It was shown by Breuer that if A is Fredholm, then $\dim \ker A$ and $\dim \ker A^*$ are finite, so that one may define a real-valued index of A

$$(1.5) \quad i(A) = \dim \ker A - \dim \ker A^*.$$

This index has the same algebraic and invariance properties as the ordinary Fredholm index. In particular, even though the index is a real number, it is unchanged by a continuous deformation.

Theorem 1. — If $\varphi \in C(\mathbf{T}^n, M_\nu)$, W_φ is (generalized) Fredholm if and only if $\det \varphi$ is non-vanishing on \mathbf{T}^n , and in this case the index of W_φ equals the mean winding number of $\det \varphi$ along the line $\{tN\}$,

$$\lim_{T \rightarrow \infty} (2T)^{-1} (\arg \det \varphi(TN) - \arg \det \varphi(-TN)).$$

The above theorem is the main result of this paper. The simplest example to which it applies occurs in the case of a pure shift $S_j = ET_jE$ for $\nu=1$. Note that according to (1.2)

$$(1.6) \quad S_j S_k - S_{j+k} = \begin{cases} \chi_j(\langle x, N \rangle) S_{j+k} & \text{if } \langle j, N \rangle < 0 \text{ and } \langle k, N \rangle > 0 \\ 0 & \text{otherwise} \end{cases}$$

where χ_J is the characteristic function of the interval $J=[0, -\langle j, N \rangle]$. Thus S_j is Fredholm, since by (1.6) S_{-j} is an inverse of S_j modulo \mathcal{K} . Of course $\ker S_j$ and $\ker S_j^*$ may be determined by inspection. For example, if $\langle j, N \rangle \geq 0$ then $\ker S_j^*$ is trivial and $\ker S_j$ equals the range of the projection $\chi_J(\langle x, N \rangle)$ where $J=[0, \langle j, N \rangle]$. Hence for $\langle j, N \rangle \geq 0$:

$$i(S_j) = \tau(\chi_J(\langle x, N \rangle)) - 0 = \langle j, N \rangle,$$

which is equal to the mean winding number of $e^{ik_j \cdot \cdot}$ along $\{tN\}$. The case $\langle j, N \rangle \leq 0$ may be checked similarly.

Let \mathcal{A} be the uniformly closed sub-algebra of \mathcal{N} generated by the operators $\{W_\varphi : \varphi \in C(\mathbf{T}^n, M_\nu)\}$. In § 2 we introduce a symbol calculus for the algebra \mathcal{A} , a homomorphism $\sigma : \mathcal{A} \rightarrow C(\mathbf{T}^n, M_\nu)$ such that $\ker \sigma = \mathcal{A} \cap \mathcal{K}$ and $\sigma(W_\varphi) = \varphi$. Thus an operator $A \in \mathcal{A}$ is Fredholm if and only if $\sigma(A)$ is invertible in $C(\mathbf{T}^n, M_\nu)$; indeed $A - W_{\sigma(A)} \in \mathcal{K}$, so if $\sigma(A)$ is invertible the index of A equals the mean winding number of $\sigma(A)$. Hence it is a trivial matter to extend theorem 1 to an index theorem for any operator in \mathcal{A} .

Let \mathcal{G} be the group of invertible elements in $C(\mathbf{T}^n, M_\nu)$. Both the analytic and topological indices depend only on the symbol, and they are constant on any homotopy class of \mathcal{G} . Therefore to prove theorem 1 it would be sufficient to check the index formula on one representative from each homotopy class. In the scalar case $\nu=1$, for any $\varphi \in \mathcal{G}$ we may write :

$$\varphi(\xi) = e^{ik \cdot \xi} \psi(\xi),$$

where $k \in \mathbf{Z}^n$ is a vector whose l^{th} component is the winding number of φ around the l^{th} factor of \mathbf{T}^n and ψ is homotopic to a constant. Thus the components of \mathcal{G} are classified by the winding numbers, and since $\sigma(S_k) = e^{ik \cdot \cdot}$, the computation above of the index of a pure shift suffices to prove theorem 1 when $\nu=1$. In the matrix case, however, more invariants are required to classify the components of \mathbf{T} . For example note that SU_2 is homeomorphic with S^3 ; thus any symbol $\varphi \in C(\mathbf{T}^3, SU_2)$ of non-zero Brouwer degree cannot be homotopic to a constant, although the winding numbers of $\det \varphi$ certainly vanish, since $\det \varphi(\xi) \equiv 1$. In this paper we make no attempt to classify the components of \mathcal{G} ; instead we prove theorem 1 by computing the analytic index for a wider class of operators.

The original result in this area, theorem (2.2) of [3], was an index theorem for a certain algebra of operators on $L^2(0, \infty)$. Our representation on a half space in \mathbf{R}^n rather than a half line is suggested by the application to difference equations in § 3. Moreover on a half space the analytic index may be computed directly in the von Neumann algebra on $L^2(H)$ generated by \mathcal{A} , avoiding the slightly unnatural passage to a representation on $L^2(0, \infty) \otimes \ell^2(\mathbf{R}_d)$ that was required in [3]. In any event, the formulas of § 2 have obvious analogues which apply to the matrix generalization of the algebra considered in [3].

2. Proof of the main theorem.

In this paragraph we summarize certain facts about the trace class that will be needed below. (See chap. I, § 6 of Dixmier [4] for proofs.) Let

$$\mathcal{K}_1^+ = \{K \in \mathcal{N}^+ : \tau(K) < \infty\}$$

and let \mathcal{K}_1 be the linear span of \mathcal{K}_1^+ . Then \mathcal{K}_1 is a two-sided ideal of \mathcal{N} , contained in \mathcal{K} , whose closure equals \mathcal{K} . The trace τ extends uniquely to a linear functional on \mathcal{K}_1 with the property that $\tau(AK) = \tau(KA)$ for $A \in \mathcal{N}$, $K \in \mathcal{K}_1$. If $K \in \mathcal{K}_1$ let $\|K\|_1 = \tau(|K|)$ where $|K| = (KK^*)^{1/2}$; then $\|\cdot\|_1$ is a norm on \mathcal{K}_1 such that $|\tau(K)| \leq \|K\|_1$ and moreover

$$(2.1) \quad \|AK\|_1 \leq \|A\| \|K\|_1 \quad \text{and} \quad \|KA\|_1 \leq \|A\| \|K\|_1$$

for any $A \in \mathcal{N}$. Finally if $K = \sum_l K_l$ is a series that converges absolutely with respect to both the operator norm and the trace norm, then $K \in \mathcal{K}_1$.

Lemma (2.1). — If $\varphi, \psi \in C(\mathbf{T}^n, \mathbf{M}_v)$ then $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}$. Indeed if

$$\varphi = \sum_j m_j e^{i\langle j, \cdot \rangle} \quad \text{and} \quad \psi = \sum_j n_j e^{i\langle j, \cdot \rangle}$$

are smooth, then $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}_1$ and

$$(2.2) \quad \tau(W_\varphi W_\psi - W_{\varphi\psi}) = \sum_{\langle j, \mathbf{N} \rangle < 0} \langle j, \mathbf{N} \rangle \text{tr}(m_j n_{-j})$$

Proof. — Suppose that $\varphi, \psi \in C^\infty(\mathbf{T}^n, \mathbf{M}_v)$ have the indicated Fourier series; then

$$(2.3) \quad W_\varphi W_\psi - W_{\varphi\psi} = \sum_{j,k} m_j n_k \otimes (S_j S_k - S_{j+k}).$$

Note from (1.6) that for any $m \in \mathbf{M}_v$:

$$\|m \otimes (S_j S_k - S_{j+k})\|_1 \leq \langle j, \mathbf{N} \rangle |\text{tr}(m)|$$

Since φ and ψ are smooth, the Fourier coefficients of these functions are rapidly decreasing. Hence the series in (2.3) is absolutely convergent in both the operator and trace norms, so $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}_1$. Moreover the trace of the sum in (2.3) may be evaluated term by term to yield (2.2). Finally it follows by limits that $W_\varphi W_\psi - W_{\varphi\psi} \in \mathcal{K}$ for general $\varphi, \psi \in C(\mathbf{T}^n, \mathbf{M}_v)$. This completes the proof.

Lemma (2.2). — If $W_\varphi \in \mathcal{K}$ then $\varphi = 0$.

Proof. — We may assume without loss of generality in the proof of this lemma that $v = 1$, for the matrix operator W_φ is generalized compact if and only if each of its entries is generalized compact. If $a \geq 0$ let E_a be the projection $\chi_{[0,a]}(\langle x, \mathbf{N} \rangle)$. Suppose that $\varphi \in C(\mathbf{T}^n)$ has Fourier series $\sum_j c_j e^{i\langle j, \cdot \rangle}$. We claim that

$$(2.4) \quad \lim_{a \rightarrow \infty} a^{-1} \tau(S_{-j} W_\varphi E_a) = c_j.$$

It follows from (1.6) that $\tau(S_{-j}S_k E_a) = 0$ if $j \neq k$ and that

$$\tau(S_{-j}S_j E_a) = \begin{cases} a & \text{if } \langle j, N \rangle \leq 0 \\ a - \langle j, N \rangle & \text{if } 0 < \langle j, N \rangle < a \\ 0 & \text{if } \langle j, N \rangle \geq a. \end{cases}$$

Thus for large a

$$a^{-1}\tau(S_{-j}S_k E_a) = \delta_{jk} + O(a^{-1}).$$

Therefore (2.4) holds if φ is an exponential polynomial, and by limits it holds in the general case.

Suppose now that $W_\varphi \in \mathcal{K}$ for some $\varphi \in C(\mathbf{T}^n)$. Since \mathcal{K}_1 is dense in \mathcal{K} , for every $\varepsilon > 0$ there exists $K \in \mathcal{K}_1$ such that $\|W_\varphi - K\| \leq \varepsilon$. Thus

$$\begin{aligned} |\tau(S_{-j}W_\varphi E_a)| &\leq |\tau(S_{-j}(W_\varphi - K) E_a)| + |\tau(S_{-j}K E_a)| \\ &\leq \varepsilon a + \|K\|_1 \end{aligned}$$

If $\varphi = \sum_j c_j e^{i\langle j, \cdot \rangle}$, then

$$|c_j| \leq \limsup_{a \rightarrow \infty} a^{-1} |\tau(S_{-j}W_\varphi E_a)| \leq \varepsilon.$$

Therefore every Fourier coefficient of φ vanishes, so φ itself vanishes. This completes the proof.

Let \mathcal{A} be the uniformly closed sub-algebra of \mathcal{N} generated by the operators $\{W_\varphi : \varphi \in C(\mathbf{T}^n, M_\nu)\}$, and let π be the canonical projection $\pi: \mathcal{A} \rightarrow \mathcal{A}/(\mathcal{A} \cap \mathcal{K})$. By lemma (2.1) the formula $\rho(\varphi) = \pi(W_\varphi)$ defines a norm decreasing *-homomorphism of $C(\mathbf{T}^n, M_\nu)$ into $\mathcal{A}/(\mathcal{A} \cap \mathcal{K})$. (It is easily verified that $\rho(\varphi)^* = \rho(\varphi^*)$.) According to lemma (2.2) the kernel of ρ is trivial. The range of ρ contains a generating set for $\mathcal{A}/(\mathcal{A} \cap \mathcal{K})$; since the range of a homomorphism of two C^* -algebras is closed (corollary 1.8.3 of [5]), ρ defines an isomorphism of $C(\mathbf{T}^n, M_\nu)$ onto $\mathcal{A}/(\mathcal{A} \cap \mathcal{K})$. Therefore if $\varphi \in C(\mathbf{T}^n, M_\nu)$, W_φ is invertible mod \mathcal{K} if and only if φ is invertible in $C(\mathbf{T}^n, M_\nu)$; in other words, W_φ is Fredholm if and only if $\det \varphi$ is non-vanishing on \mathbf{T}^n .

If φ is an invertible element of $C(\mathbf{T}^n, M_\nu)$, then there exists $\psi \in C(\mathbf{T}^n, M_\nu)$ homotopic to φ , such that $\psi(\xi)$ belongs to the unitary group U_ν for each ξ . Indeed, for $0 \leq t \leq 1$, let :

$$\Phi(t, \xi) = (\varphi(\xi) \varphi^*(\xi))^{-t/2} \varphi(\xi);$$

then $\Phi(0, \xi) = \varphi(\xi)$ and $\Phi(1, \xi) \in U_\nu$. Of course both the analytic and topological indices are unchanged by this homotopy. Moreover any function in $C(\mathbf{T}^n, U_\nu)$ may be uniformly approximated by a smooth function. Therefore, to prove theorem 1 it suffices to check the index formula for $\{W_\varphi : \varphi \in C^\infty(\mathbf{T}^n, U_\nu)\}$.

Suppose that W_φ is a Fredholm operator. Let $W_\varphi = HV$ be the polar decomposition of W_φ , where $H = (W_\varphi W_\varphi^*)^{1/2}$ and V is a partial isometry. Then $V^*V = I - P$

and $VV^* = I - P'$, where P and P' are projections onto $\ker W_\varphi$ and $\ker W_\varphi^*$ respectively. Hence $[V, V^*] = P - P'$ belongs to \mathcal{K}_1 and $\tau([V, V^*]) = i(W_\varphi)$. However if

$$\varphi \in C^\infty(\mathbf{T}^n, U_\nu),$$

then by lemma (2.1) :

$$H^2 - I = W_\varphi W_\varphi^* - W_{\varphi\varphi^*} \in \mathcal{K}_1,$$

so that

$$W_\varphi - V = (H - I)V = (H + I)^{-1}(H^2 - I)V \in \mathcal{K}_1.$$

Hence $\tau([W_\varphi, W_\varphi^*]) = \tau([V, V^*])$, and we may use (2.2) to evaluate this trace. If $\varphi = \sum_j m_j e^{i\langle j, \cdot \rangle}$ one finds that :

$$\tau([W_\varphi, W_\varphi^*]) = \tau(W_\varphi W_\varphi^* - I) - \tau(W_\varphi^* W_\varphi - I) = \sum_j \langle j, N \rangle \operatorname{tr}(m_j^* m_j).$$

On the other hand, by a trivial computation with Fourier series one sees that if $\varphi = \sum_j m_j e^{i\langle j, \cdot \rangle}$ belongs to $C^\infty(\mathbf{T}^n, M_\nu)$, then

$$(2\pi)^{-n} \int_{\mathbf{T}^n} \operatorname{tr}(\varphi^*(\xi) \nabla_N \varphi(\xi)) d\xi = \sum_j \langle j, N \rangle \operatorname{tr}(m_j^* m_j),$$

where $\nabla_N = \frac{1}{i} \sum_{\ell=1}^n N_\ell \frac{\partial}{\partial \xi_\ell}$. Of course if $\varphi \in C^\infty(\mathbf{T}^n, U_\nu)$ then $\varphi^*(\xi) = \varphi^{-1}(\xi)$. Observe that

$$\operatorname{tr}(\varphi^{-1}(\xi) \nabla_N \varphi(\xi)) = \nabla_N (\log \det \varphi(\xi)).$$

Now we may write :

$$(2.5) \quad \log \det \varphi(\xi) = i \langle k, \xi \rangle + \psi(\xi),$$

where $\psi \in C^\infty(\mathbf{T}^n)$ and the ℓ^{th} component of $k \in \mathbf{Z}^n$ is the winding number of $\det \varphi$ around the ℓ^{th} factor of \mathbf{T}^n . Note that by (2.5), the mean winding number of $\det \varphi$ along $\{tN\}$ equals $\langle k, N \rangle$. But

$$i(W_\varphi) = \tau([W_\varphi, W_\varphi^*]) = (2\pi)^{-n} \int_{\mathbf{T}^n} \nabla_N (\log \det \varphi(\xi)) d\xi = \langle k, N \rangle,$$

since $\int \nabla_N \psi d\xi$ vanishes by an integration by parts. This completes the proof of theorem 1.

3. Application to elliptic difference equations.

In this section we assume the reader is familiar with the result of [6]. Suppose

$$Q_{\alpha\beta}(D) = \sum_{j \in \mathbf{Z}^n} c_j^{\alpha\beta} T_j, \quad \alpha, \beta = 1, \dots, \nu$$

is a system of difference operators on \mathbf{R}^n . Let m_j be the $\nu \times \nu$ matrix with entries $c_j^{\alpha\beta}$, and let

$$Q(\xi) = \sum_j m_j e^{i\langle j, \xi \rangle}.$$

We shall call $Q_{\alpha\beta}(D)$ properly elliptic if $\det Q(\xi) \neq 0$ for $\xi \in \mathbf{R}^n$ and $\arg \det Q$ is periodic. Consider a boundary value problem for an elliptic system:

$$(3.1) \quad \sum_{\beta=1}^{\nu} Q_{\alpha\beta}(D)v_{\beta}(x) = 0 \quad \text{for } \langle x, N \rangle \geq a_{\alpha}, \alpha = 1, \dots, \nu$$

$$(3.2) \quad \sum_{\beta=1}^{\nu} q_{\gamma\beta}(x, D)v_{\beta}(x) = g_{\gamma}(x) \quad \text{for } 0 \leq \langle x, N \rangle < b_{\gamma}, \gamma = 1, \dots, M.$$

Here the unknown functions v_{β} belong to $L^2(H)$; we shall require that $c_j^{\alpha\beta} = 0$ for $\langle j, N \rangle < -a_{\alpha}$ so that in (3.1) no attempt to evaluate v_{β} in $\mathbf{R}^n - H$ is made. As in [6] we assume the boundary operators $q_{\gamma\beta}(x, D)$ depend on x only through $\langle x, N \rangle$.

We regard the boundary data g as an element of $\bigoplus_{\gamma=1}^M L^2([0, b_{\gamma}] \times \partial H)$. Finally we suppose that

$$\sum_{\alpha=1}^{\nu} a_{\alpha} = \sum_{\gamma=1}^M b_{\gamma}.$$

Theorem 2. — *Under the above hypotheses, the following two statements are equivalent.*

(i) Equation (3.1) with homogeneous boundary conditions has the unique solution zero in $\bigoplus_1^{\nu} L^2(H)$.

(ii) For a dense set of boundary data g there is at least one solution of (3.1), (3.2) in $\bigoplus_1^{\nu} L^2(H)$.

We omit the proof of this theorem, as it is completely analogous to that of [6].

ACKNOWLEDGEMENT

I am indebted to Lewis Coburn and Isadore Singer for discussions about the topology of the general linear group which influenced my thinking about this problem.

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Manuscrit reçu le 1^{er} mars 1972.