GILLES CHATELET
HAROLD ROSENBERG

Manifolds which admit $\mathbb{R}^n$ actions


<http://www.numdam.org/item?id=PMIHES_1974__43__245_0>
MANIFOLDS WHICH ADMIT $\mathbb{R}^n$ ACTIONS
by G. CHATELET and H. ROSENBERG

INTRODUCTION

The purpose of this paper is to determine which $n$-manifolds admit smooth locally free actions of $\mathbb{R}^{n-1}$. We shall restrict ourselves to compact connected orientable manifolds $V^n$ and locally free actions $\varphi$ of $\mathbb{R}^{n-1}$ on $V^n$ which are of class $C^2$ and tangent to $\partial V^n$, i.e. the components of $\partial V^n$ are orbits of $\varphi$. For $n=3$, we know that $V^3$ admits such an $\mathbb{R}^2$ action if and only if $V^3 = \mathbb{T}^2 \times I$ or $V^3$ is a bundle over $S^1$ with fibre $T^2$ [7]. Moreover, the topological type of such $\mathbb{R}^2$ actions has been completely determined [8].

We recall that the rank of $V^n$ is the largest integer $k$ such that $V^n$ admits a smooth locally free action of $\mathbb{R}^k$.

Now suppose that $\varphi$ is a locally free action of $\mathbb{R}^{n-1}$ on $V^n$. We shall prove:

**Theorem 1.** — If $\partial V^n$ is not empty, then $V^n$ is homeomorphic to $\mathbb{T}^{n-1} \times I$ (here $\mathbb{T}^i$ denotes the torus of dimension $i$).

**Theorem 2.** — If $\partial V^n$ is empty and $\varphi$ has at least one compact orbit, then $V^n$ is a bundle over $S^1$ with fibre $\mathbb{T}^{n-1}$.

**Theorem 3.** — If $\partial V^n$ is empty and $\varphi$ has no compact orbits then $V^n$ is a bundle over a torus $\mathbb{T}^k$ with fibre a torus $\mathbb{T}^{n-k}$.

Theorem 2 follows directly from Theorem 1 by cutting $V^n$ along a compact orbit.

Theorem 3 depends upon an observation of Novikov [4], and independently Joubert: suppose $\varphi$ acts on $V^n$ with no compact orbits. By Sacksteder [9], all the orbits of $\varphi$ are $\mathbb{T}^{n-k} \times \mathbb{R}^{k-1}$ for some $k$. Choose linearly independent vector fields $X_1, \ldots, X_{n-k}$ tangent to the orbits of $\varphi$ such that all the integral curves of $X_1, \ldots, X_{n-k}$ are periodic, of period one. Then $X_1, \ldots, X_{n-k}$ define a locally free action of $\mathbb{T}^{n-k}$ on $V^n$ and the orbit space $M$ is a smooth manifold of dimension $k$. Also $M$ admits an action of $\mathbb{R}^{k-1}$ with all the orbits $\mathbb{R}^{k-1}$. It follows that $M$ is homeomorphic to $\mathbb{T}^k$, which proves Theorem 3 ([5] and [3]). Consequently, our main result is Theorem 1. Here is how we proceed to prove Theorem 1: by inductive arguments similar to those used in [7], we restrict ourselves to actions $\varphi$ with no compact orbits in the interior of $V^n$. We then
remark that the foliation defined by the orbits of \( \varphi \) is almost without holonomy, i.e. the noncompact leaves have no holonomy. With this, we construct collar neighborhoods \( U_i \) of each component \( T_i' \) of \( \partial V \), such that \( \partial U_i = T_i' \cup T_i'' \) with \( T_i'' \) transverse to the foliation. We construct \( U_i \) so that some linear field \( Y \) (tangent to the orbits of \( \varphi \)) is transverse to each \( T_i' \). We then prove the integral curves of \( Y \) go from \( T_i' \) to \( T_i'' \) hence define a homeomorphism of \( V^n \) to \( T^{n-1} \times I \).

1. Some Preliminaries.

1.1 Let \( F \) be the foliation of \( V \) defined by the orbits of \( \varphi \). Then each noncompact leaf of \( F \) has zero holonomy.

Proof. If \( T \) is a compact leaf of \( F \), then the germ of \( F \) in a neighborhood of \( T \) is without holonomy outside of \( T \), provided \( T \) is an isolated compact leaf (page 13 of [8]). This is also true if \( T \) is an isolated compact leaf on one side in \( V \) and one considers the germ of \( F \) on this side. Now if \( \varphi \) has no compact orbits then \( F \) is without holonomy and we are done [9]. So suppose \( F \) is a noncompact leaf of \( F \) and \( F \) has compact leaves. Since \( F \) has no exceptional minimal sets [9], there is a compact leaf \( T \) of \( F \) such that \( T \) is in the closure of \( F \). Let \( x \) be a point of \( F \) and \( a(x) \) a non zero element of \( \pi_1(F, x) \). Let \( X \) be a vector field on \( V \) such that the integral curve of \( X \) through \( x \) is closed and homotopic to \( a(x) \), and all the integral curves of \( X \) on \( F \) are closed. \( X \) is easily constructed using the action \( \varphi \) (cf. [6]). Since \( T \) is in the closure of \( F \), we know the integral curves of \( X \) on \( T \) are also closed. Now \( T \) is an isolated compact leaf at least on one side in \( V \), the side where \( F \) intersects a transverse arc infinitely often. Let \( U \) be a neighborhood of \( T \), on this side, such that all the leaves of \( F \) in \( U \), except \( T \), have zero holonomy. Then \( U \) contains closed integral curves of \( X \) which are on \( F \), so such an integral curve \( C \) has zero holonomy. Since \( C \) is conjugate to \( a(x) \), it follows that \( a(x) \) has zero holonomy; thus \( F \) as well.

1.2 Suppose \( \partial V \) is not empty and \( \varphi \) has no compact orbits in the interior of \( V \). Let \( T \) be a compact orbit of \( \varphi \); \( T \subset \partial V \). The leaves which contain \( T \) in their closure are homeomorphic to \( T^k \times R^{n-k-1} \) where \( k = \text{the rank of the kernel of the holonomy map on } T \).

Proof. Let \( F \) be an open leaf whose closure contains \( T \); \( F \approx T^j \times R^{n-j-1} \). Suppose \( Z^k \) is the kernel of the holonomy homomorphism on \( T \). Let \( T^k \) be a \( k \)-torus embedded in \( T \) which lifts onto nearby leaves by the holonomy. Since \( \tilde{F} \supset T \), we can lift \( T^k \) to a \( k \)-torus \( T_i \) in \( F \). Also \( i_k : \pi_k(T) \to \pi_i(V) \) is injective, where \( i : T \hookrightarrow V \) (cf. [4]), hence \( \pi_k(T) \) embeds in \( \pi_i(F) \) and \( k \leq j \).

Next we show \( j \leq k \). Let \( x \in F \) and \( a \in \pi_1(F, x) \), \( a \neq 0 \). Let \( X \) be a vector field tangent to the orbits of \( \varphi \), such that the integral curves of \( X \) on \( F \) are closed and the integral curve of \( X \) through \( x \) is homotopic to \( a \). Since \( \tilde{F} \supset T \), all the integral curves
of $X$ on $T$ are closed. Let $C$ be an integral curve of $X$ on $T$. We know that $C$ lifts to a closed curve on $F$, so by (1.1), the holonomy of $G$ is trivial; i.e. $G$ is in the kernel of the holonomy homomorphism. Hence $j \leq k$.

2. The transverse torus and vector field.

Throughout this section, we suppose $\varphi$ acts on $V$ so that there are no compact orbits in the interior of $V$ and $T$ is a compact orbit in $\partial V$. Let $k$ denote the rank of the kernel of the holonomy map associated to $T$; $k$ varies between 0 and $n-2$. Let $Y_1, \ldots, Y_{n-1}$ be linearly independent commuting vector fields on $V$ satisfying:

(i) they are tangent to the $\varphi$-orbits;
(ii) their integral curves are closed and of period one on $T$; and
(iii) the integral curves of $Y_1, \ldots, Y_k$ represent the kernel of the holonomy map on $T$.

We shall construct an $(n-i)$-torus $T' \subset \text{Int} V$ such that $T \cup T'$ bound a trivial cobordism in $V$, and $Y_{n-1}$ is transverse to $T'$ at each point.

By (1.1), we know the orbits of $Y_{k+1}, \ldots, Y_{n-1}$ on $T$ induce germs in $\text{Diff}(\mathbb{R}^+)\setminus \partial \mathbb{R}^+$ which are contractions or expansions, via the holonomy. Here $\text{Diff}(\mathbb{R}^+)$ is the set of $C^2$-germs of diffeomorphisms of $\mathbb{R}^+$ to itself, which leave 0 fixed. After reversing the sign of $Y$, if necessary, we shall assume the germs are all contractions, for $k+i \leq j \leq n-1$.

Choose a metric on $V$ and let $U_\varepsilon$ be a geodesic collar neighborhood of $T$ isometric to $T \times [0, \varepsilon]$, with the obvious product metric. Clearly, if $\varepsilon$ is small enough, the geodesics normal to $T$ in $U_\varepsilon$ will be transverse to the orbits of $\varphi$. Let $f^i_\varepsilon$ be the holonomy diffeomorphism associated to the $Y_i$ orbit through $x$; $f^i_\varepsilon$ is the identity for $1 \leq i \leq k$ and a contraction for $k+1 \leq i \leq n$.

**Proposition (2.1).** — There is an $(n-1)$-torus $T' \subset \text{Int} V$ such that $Y_{n-1}$ is transverse to $T'$ and $T \cup T'$ bound a trivial cobordism.

In an earlier version of this paper we gave a proof of (2.1) which used calculus. Charles Pugh pointed out to us how one can use a theorem of W. Wilson on the existence of Liapounov functions for uniform stable attractors of vector fields [13]. We present this proof here and in an appendix we give our original proof.

We need some definitions before stating Wilson’s theorem. Let $X$ be a vector field on $V$ and let $A$ be a closed invariant subset of $V$ (here $V$ is a compact manifold). $A$ is called a uniform stable attractor of $X$ if the following conditions are satisfied:

a) there exists an increasing function $\delta$ sending $\mathbb{R}^+$ into itself such that
$$d(X(p, t), A) < \delta(t)$$
whenever $d(p, A) < \delta(\varepsilon)$ and $t \geq 0$;
b) there exists a neighborhood U of A such that \( \omega(p) \subseteq A \) whenever \( p \in U \) (\( \omega(p) \) is the \( \omega \)-limit set of \( p \));

c) let \( D(A) \) be the set of \( p \) such that \( \omega(p) \subseteq A \); \( D(A) \) is an open set, called the basin of attraction of \( A \).

Wilson has proved [13] that if \( A \) is a uniform stable attractor for \( X \) then there exists a \( C^\infty \) Liapounov function, i.e.

a) there is a \( C^\infty \) function \( f : D(A) \rightarrow \mathbb{R}^+ \) with \( f^{-1}(0) = A \); and

b) \( X(f)(p) < 0 \) whenever \( f(p) \neq 0 \).

Hence \( f \) has no singularities outside \( A \) and all the level surfaces of \( f \) are diffeomorphic. Before proving (2.1) we need three lemmas.

**Lemma (2.2). (Action box lemma.)**

There exists a unique mapping

\[
F : J^{n-1} \times [0, \varepsilon] \to U \subset V
\]

(where \( J = [-1, 2] \)) satisfying the following conditions:

a) \( F \) is a \( C^2 \)-immersion;

b) \( F \) sends the horizontal plaques \( J^{n-1} \times \{z\} \) into the leaves of \( \mathcal{F} \);

c) \( F \) sends vertical arcs \( \{\Lambda\} \times [0, \varepsilon] \) onto the geodesic arcs normal to \( T \);

d) \( F \), when restricted to \( J^{n-1} \times \{0\} \), is the restriction of the natural covering map: \( \mathbb{R}^{n-1} \to T \) induced by \( \varphi \), which sends the i-direction line onto the \( Y_i \) circular orbit;

e) let \( x_0 \in T \); then \( F \) sends \( \{0\} \times [0, \varepsilon] \) isometrically onto the geodesic arcs issued from \( X_0 \), normal to \( T \) and pointing inside \( T \).

**Proof.** — Define first \( F \) via e) and d). \( F \) obviously extends to \( J^{n-1} \times [0, \varepsilon] \) using b) and c).

a) is clear, for geodesic arcs are normal to \( \mathcal{F} \) in \( U \). Note that each \( Y_i \) orbit on \( T \) is covered three times by \( F \).

**Lemma (2.3). (Commuting contraction lemma.)**

If \( f_1 \) and \( f_2 \) are commuting embeddings \( [0, \varepsilon] \to [0, \infty[ \) and \( f_2 \) is a contraction towards 0, then there exists a \( K \) so large that \( f_1 f_2^K \) is a contraction to 0.

**Proof.** — \( f_2 \) commuting with \( f_1 f_2^K \), \( f_1 f_2^K \) is an embedding without fixed point or is the identity (N. Koppel's Thesis). For sufficiently large \( h \), \( f_1 f_2^K \) is not the identity. Hence \( f_1 f_2^K \) is a contraction or an expansion. For \( f_1 f_2^K [0, \varepsilon] = f_2^K f_1 [0, \varepsilon] \), and \( K \) may be chosen so large that \( f_2^K f_1 [0, \varepsilon] \subset [0, \varepsilon/2] \). \( f_1 f_2^K \) is therefore a contraction.

248
3. Attraction Lemma.

There exists $\varepsilon$ and $\delta > 0$ such that whenever $X$ is a $C^1$ vector field on $\mathbb{R}^{n-1}$ and $|X|_0 < \delta$, then $Y = \Phi \left( \frac{\partial}{\partial \lambda_{n-1}} + X \right)$ generates a flow having $T$ as a uniform and stable attractor, $U_e$ being in the basin of attraction of $T$.

— Look at the application $F$ of Lemma (2.2) (action box lemma). If

$$Y = \Phi \left( \frac{\partial}{\partial \lambda_{n-1}} + X \right),$$

$F^*Y$ is a $C^1$ vector field defined on $J^{n-1} \times [0, \varepsilon]$ ($F$ is a $C^2$ immersion); $F^*Y$ has no vertical component and may be chosen arbitrarily close to $\frac{\partial}{\partial \lambda_{n-1}}$ for a suitable choice of $\delta$.

Let $I = [0, 1]$, $A_0 = I^{n-2} \times \{0\} \times [0, \varepsilon]$, $A_1 = I^{n-2} \times \{1\} \times [0, \varepsilon]$ and $x \in A_0$. $I$ being interior to $J$, choose $\delta$ such that the positive orbit of $F^*Y$ through $x$ crosses $A_1$ before reaching the boundary of $J^{n-1} \times [0, \varepsilon]$. Let $x$ be the point of intersection of $A_1$ with the orbit. Via $F$, $x$ is identified with a point $x_1 \in A_0$ and hence may be written in the form $(\lambda_1', \ldots, \lambda_{n-1}', 0, z_1)$ where

$$z_1 = f_{K+1} \circ \cdots \circ f_{n-1}(z) \quad \text{if} \quad x = (\lambda_1', \ldots, \lambda_{n-1}', 0, z).$$

Recall that for $1 \leq j \leq n-K+1$, $f_{K+j}$ are the contracting holonomy diffeomorphisms associated to the circular $Y_{K+j}$ orbits.

Using the contraction commuting lemma, we choose $N$ such that

$$f_{K+1} \circ \cdots \circ f_{n-2} \circ f_{n-1},$$

is a contraction. For $\varepsilon$ and $\delta$ small, we may build a sequence $(x, x_1, \ldots, x_{n-1}, x_N)$ where the $F^*Y$ orbit through $x_i$ crosses $A_i$ at $\bar{x}_i$ and $\bar{x}_i$ being identified via $F$ with $x_{i+1}$ in $A_0$. So if $x = (\lambda_1', \ldots, \lambda_{n-1}', 0, z)$, then $x_N = (\lambda_{n-1}', \lambda_{n-1}', 0, h(z))$ where

$$h(z) = \prod_j f_{K+j} \circ f_{n-1}(z).$$

Thus we have shown that the vertical coordinate of any $Y$-orbit tends to 0 in a manner dominated by a fixed contraction $f_{K+1} \circ \cdots \circ f_{n-1}$ as we proceed along the orbit in forward times, i.e. $T$ is a uniformly stable attractor.

Let us prove now Proposition (2.1).

— The choice of the $Y_j$'s on $T$ allow us to write $T$ as a trivial fibration $\Sigma \times S$, where $\Sigma$ is a manifold diffeomorphic to $T^{n-2}$ and transversal to the circular orbits of $Y_{n-1}$ which are the fibers of that fibration. Over these circles, consider the normal geodesic fibers of $U_e$. This gives a two dimensional foliation of $U_e$ by cylinders. Call it $\mathcal{F}$; $\mathcal{F}$ is clearly transversal to $\mathcal{F}$.
Let \( Y_{n-1} = X + Y \) where \( Y \) is tangent to \( \mathcal{A} \cap \mathcal{F} \) and orthogonal to \( X \); clearly \( Y_{n-1}(x) - Y(x) = X(x) \) tends to 0 when \( d(x, T) \) tends to 0. Due to the attraction lemma, \( Y \) admits \( T \) as a uniform stable attractor. Let \( V_1 = U_{\frac{\varepsilon}{2}} \), \( V_2 = U_{\frac{\varepsilon}{3}} \) and let \( \beta \) be a bump function such that \( \beta = 1 \) on \( V_1 \) and \( \beta = 0 \) outside \( V_2 \). Let \( Z = \beta Y + (1 - \beta) Y_{n-1} \). It is easy to check that \( Z \) admits \( T \) as a uniform stable attractor and hence there exists a Liapounov function \( f \) for \( Z \). For \( \varepsilon > \varepsilon_0 \), \( Z = Y_{n-1} \) and \( f^{-1}(\varepsilon_0) \) is transversal to \( Y_{n-1} \).

For \( \frac{\varepsilon}{3} > \varepsilon_1 > 0 \), \( f^{-1}(\varepsilon_1) \) is transverse to \( Y \); \( f^{-1}(\varepsilon_1) \) is diffeomorphic to \( f^{-1}(\varepsilon_0) \). It remains to prove \( f^{-1}(\varepsilon_1) \) is a \((n-1)\)-dimensional torus for \( f^{-1}(\varepsilon_0) \) will then be a torus satisfying conditions of (2.1).

\( Y \) being transverse to \( f^{-1}(\varepsilon_1) \), \( f^{-1}(\varepsilon_1) \) is transverse to \( \mathcal{A} \). Let \( \mathcal{A}_e \) be the leaf of \( \mathcal{A} \) through \( x \); \( \mathcal{A}_e \cap f^{-1}(\varepsilon_1) \) is a compact one-dimensional manifold and hence diffeomorphic to a circle. Writing \( T = \Sigma \times S_1 \) and \( x = (\lambda, s) \) here \( \lambda \in \Sigma \) and \( s \in S_1 \), one produces a family of embeddings of \( S_1 \), \( (\pi_\lambda)_{\lambda \in \Sigma} \) such that \( \pi_\lambda(S_1) = \mathcal{A}_e \cap f^{-1}(\varepsilon_1) \). We define now an application \( \pi : \Sigma \times S_1 \to f^{-1}(\varepsilon_1) \) by \( \pi(\lambda, s) = \pi_\lambda(s) \) which is clearly an embedding. Proposition (2.1) is thereby proved for \( \Sigma \) is diffeomorphic to \( T^{n-1} \).

**Proof of Theorem 1.** We now assume \( \partial V \) is not empty and \( \varphi \) has no compact orbits in the interior of \( V \). Let \( T, T' \), and \( Y_1, \ldots, Y_{n-1} \) be as in section 2; so that \( Y_{n-1} \) is transverse to \( T' \) and pointing into \( V \) along \( T' \), i.e. \( Y_{n-1} \) points out of the tubular neighborhood of \( T \). Let \( F \) be an orbit of \( \varphi \) which intersects \( T' \) and let \( L \) be a connected component of \( F \cap T' \).

**Lemma (3.1).** \( \bigcup_{t \in \mathbb{R}} Y_{n-1}(t, L) = F \).

**Proof.** We know \( F \) is diffeomorphic to \( T^k \times \mathbb{R}^{n-k-1} \) (in the leaf topology) and we have a covering map \( \pi : \mathbb{R}^{n-1} \to F \) induced by \( \varphi \). Since \( Y_1, \ldots, Y_{n-1} \) define the action \( \varphi \), we can take \( \pi(Y_{n-1}) = \frac{\partial}{\partial x_{n-1}} \) where \( (x_1, \ldots, x_{n-1}) \) denote the usual coordinates in \( \mathbb{R}^{n-1} \). Let \( X \) denote \( \frac{\partial}{\partial x_{n-1}} \), and let \( W \) be a connected component of \( \pi^{-1}(L) \). It suffices to prove that each orbit of \( X \) starting at a point of the hyperplane \( x_{n-1} = 0 \), intersects \( W \), since this implies \( \bigcup X(t, W) = \mathbb{R}^{n-1} \).

Now \( W \) is a closed submanifold of \( \mathbb{R}^{n-1} \), of codimension one, and \( X \) is transverse to \( W \), and makes an angle with \( W \) that is strictly bounded away from zero, since \( Y_{n-1} \) is transverse to \( T' \). Clearly, the set of points of the hyperplane \( x_{n-1} = 0 \), whose \( X \) orbits intersect \( W \), is an open non empty set \( \Omega \). It suffices to show \( \Omega \) is closed. Let \( z_n \in \Omega \), and \( z_n \in \Omega \), satisfying: \( \lim_{n \to \infty} z_n = z \) and for each \( n \), there exists \( t_n \in \mathbb{R} \), such that \( X(t_n, z_n) \in W \). If some subsequence of \( (t_n) \) converges to a number \( t \) then we have \( X(t, z) \in W \); hence we can suppose no subsequence converges. Let \( (\xi_n) \) be a subsequence of \( (t_n) \) such
that $|s_n - s_{n+1}| \geq 1$ and $|z_n - z_{n+1}| < \frac{1}{n}$. Let $E(n)$ denote the line segment joining $z_n$ to $z_{n+1}$ and consider $(E(n) \times \mathbb{R}) \cap W$. This is a curve in $W$ with endpoints $X(S_n, z_n)$ and $X(S_{n+1}, z_{n+1})$. There exists a point $U_n$ on this curve where the tangent to the curve is parallel to the cord joining the endpoints. The angle this cord makes with $X$ tends to zero as $n \to \infty$, which contradicts the fact that the angle between $X$ and $W$ is strictly positive.

**Lemma (3.2).** — Let $F$, $W$, $L$, $T$ and $T'$ be as in (3.1). Then there exists a compact orbit $T_0$ of $\varphi$ such that $\mathbb{F} \supset T_0$ and $T_0 + T$.

**Proof.** — Let $W_0 = W$ and $W_n = X(n, W_0)$ for each positive integer $n$. By an argument analogous to that of (3.1), one sees that the distances $d(W_k, W_{k+s})$ tend to infinity as $s \to \infty$. Let $L_0 = L$ and $L_n = Y_{n-1}(n, L_0)$, so that $\lim_{s \to \infty} d(L_n, L_{k+s}) = \infty$, where the metric is that induced by $\pi$. We define $\Omega = \bigcap_n E_n$, where $E_n$ is the connected component of $F - L_n$ towards which $Y_{n-1}$ points on $L_n$. $\Omega$ is an intersection of a nested family of compact sets, hence $\Omega$ is not empty and compact. We claim $\Omega$ is invariant under the $\varphi$ action: clearly $\Omega = \{ y \in \mathbb{V} \mid \text{there exists } x_n \in E_n \text{ and } x_n \to y \}$. Let $F(y)$ be the orbit of $\varphi$ by $y \in \Omega$ and let $y' \in F(y)$. Let $[y, y']$ denote a path in $F(y)$ joining $y$ to $y'$ and let $[x_n, x'_n]$ be the holonomy lifting of this path to the leaf of $x_n$. By construction we have $d(x_n, x'_n)$ bounded above by some number $\ell$, independent of $n$. Since $d(I_n, L_{n+s}) \to \infty$ as $s \to \infty$, we can choose a subsequence of $(x'_n)$, call it $(y_n)$, such that $y_n \in E_n$. Thus $y' \in \Omega$ and $\Omega$ is invariant. Thus $\Omega$ contains a $\varphi$-minimal set, which must be a compact orbit by Sacksteder's theorem. Since $Y_{n-1}$ points away from $T$, this compact leaf $T_0 \subset \Omega$, is different from $T$.

**Lemma (3.3)** Let $V^n$ be of rank $n-1$ and let $\varphi$ be an action of $\mathbb{R}^{n-1}$ on $V$ such that the only compact orbits of $\varphi$ are in $\partial V$ and $\partial V$ is not empty. Then $V$ is homeomorphic to $T^{n-1} \times I$. 

**Proof.** — We use the notation of (3.1) and (3.2). From these lemmas, it follows that the open leaves having $T$ in their closure are homeomorphic to the open leaves having $T_0$ in their closure, i.e. to $T^k \times \mathbb{R}^{n-k-1}$, where $k$ is the rank of the kernel of the holonomy map of $T$. Now since all the integral curves of $Y_1, \ldots, Y_k$ are closed in $F$ and $\mathbb{F} \supset T_0$, we know they are also closed in $T_0$; hence the $k$-tori in $T_0$ spanned by the orbits of $Y_1, \ldots, Y_k$ represents the kernel of the holonomy map of $T_0$. Now the orbits of $Y_{k+1}, \ldots, Y_{n-1}$ are not necessarily closed but we can choose vector fields (from lines through the origin in $\mathbb{R}^{n-1}$) $\tilde{Y}_{k+1}, \ldots, \tilde{Y}_{n-1}$, such that $Y_1, \ldots, Y_k, \tilde{Y}_{k+1}, \ldots, \tilde{Y}_{n-1}$ are linearly independent, commute, are tangent to the $\varphi$ orbits, and all the integral curves of $\tilde{Y}_{k+1}, \ldots, \tilde{Y}_{n-1}$ in $T_0$ are closed. Clearly this can be done so that $\tilde{Y}_{n-1}$ is
We go through the construction of a torus $T'_1$, bounding a collar neighborhood with $T_1$, such that $\tilde{Y}_{n-1}$ is transverse to $T'_1$; this is (2.1). Letting $Y$ denote $\tilde{Y}_{n-1}$, we now have a linear vector field $Y$ transverse to both tori $T'$ and $T'_1$. We know the set of points $A$ in $T'$ whose $Y$-integral curve intersects $T'_1$ is an open non empty set. By the same reasoning, the complement of $A$ in $T'$ is open; hence $A = T'$. Now using the integral curves of $Y$, it is easy to construct a homeomorphism between $V$ and $T^{n-1} \times I$.

**Proof of Theorem 1.** — The proof follows from (3.3), and a reasoning identical to that on page 462 of [7].

**Remarks 1.** — A basic question remains unanswered: suppose $\phi$ is a locally free action of $R^a_{-1}$ on a closed manifold $V^n$, with no compact orbits. Then we know $V^n$ fibres over a torus with fibre a torus, hence $V^n$ fibres over $S^1$ with fibre $F$ (this also follows from [10]). Is $F$ homeomorphic to $T^{n-1}$?

2. Suppose $V^n$ is a closed, orientable, bundle over $S^1$ with fibre $M$. Then there exists a diffeomorphism $f : M \to M$ such that $V$ is obtained from $M \times I$ by identifying points $(f(x), 1)$ with $(x, 0)$ for $x \in M$. We claim that if $f^* : H^0(M, R) \to H^0(M, R)$ does not have one as an eigenvalue, then every locally free action of $R^a_{-1}$ on $V$ has a compact orbit. To see this, first observe that $f^*$ does not have 1 as an eigenvalue if and only if rank $H^0(V, R) = 1$ [11]. Now suppose $\mathcal{F}$ is any foliation of $V$ of codimension one, class $C^2$ and with no compact leaves. By [12], we can suppose $L$ is a covering space of $M$ for $L$ a leaf of $\mathcal{F}$. We have an exact sequence of free abelian groups:

$$0 \to \pi_1(F)/\pi_1(L) \to \pi_1(V)/\pi_1(L) \to \pi_1(V)/\pi_1(F) \to 0.$$ 

Since $H^1(V, R) \approx R$, the last two groups are of rank one. Hence $\pi_1(L) = \pi_1(F)$ and $L$ must be compact.

**APPENDIX**

**Proof of (2.1)**

*Notation.* — If $X$ is a vector field on $V$, $t \mapsto X(t, x)$ will denote the integral curve of $X$ passing through $x$ at $t = 0$. For $A \subset V$, $X(t, A) = \{X(t, x) \mid x \in A\}$, and 

$$X([a, b], A) = \bigcup_{s \leq t \leq b} X(t, A).$$

If $x \in T$, we define $q_i(x) = Y_i([0, 1), x)$ for $i = 1, \ldots, n-1$, and $T_i(x)$ is the $i$-torus in $T$ which is the orbit through $x$ of the $R^i$-action determined by $Y_1, \ldots, Y_i$. If $\bar{x}$ is on the normal arc through $x \in T$ and if the holonomy germs are defined on $\bar{x}$,
then we denote by $\bar{T}(\bar{x})$ the lifting of $T(x)$ into the leaf of $\bar{x}$, given by the holonomy.

Let $N$ be a unit vector field on $V$, normal to the orbits of $\varphi$ and pointing into $V$ along $T$ (with respect to some metric on $V$). Let $U = N(I, T)$ where $I = [0, 1]$. We may suppose $U$ is a tubular neighborhood of $T$ in which the holonomy liftings of $\alpha_i(x), \ldots, \alpha_{n-1}(x)$ are defined, for $x \in T$. Let $f_\omega^t$ be the holonomy diffeomorphism of $\alpha_{k+i}(x); 1 \leq i \leq n-k-1$.

Let $\pi : U \to T$ be the projection along $N$ orbits. If $x \in T$ and $\bar{x} \in \pi^{-1}(x)$, let $\bar{\alpha}(x)$ denote the holonomy lifting of $\alpha_i(x)$ starting at $\bar{x}$; for $1 \leq i \leq k$, $\bar{\alpha}(x)$ is an embedded circle, and for $i > k$, $\bar{\alpha}(x)$ is diffeomorphic to $I$. For $x \in T$ and for all $\bar{x} \in \pi^{-1}(x)$, the $\bar{\alpha}(\bar{x})$ form a one dimensional foliation of $U$. Let $G_i$ be a vector field in $T$, tangent to this foliation, and coinciding with $Y_i$ on $T$.

We fix a base point $x_0 \in T$ and we let $\alpha_i = \alpha_i(x_0), T_i = T_i(x_0)$, etc., and define $A_i = N(I, T_i)$.

Let $E_i(A_i)$ be the vector bundle of exterior products of order $\ell$ of vectors tangent to $A_i$. We identify $E_i(A_i)$ with $A_i \times \bigwedge A_i^\ell$; so sections of $E_i(A_i)$ are functions from $A_i$ to $\bigwedge A_i^\ell$. We give these sections the canonical norm.

Let $f$ be a function defined in a neighborhood of $0$ such that $\lim_{x \to 0} f(x) = 0$. We write $f = \sigma(x)$ if $f(x) = ax + x\sigma'(x)$, with $a \neq 0$ and $\sigma'(x) \to 0$ when $x \to 0$. Finally, we let $\beta_{k+j} = Y_{k+j} \wedge \ldots \wedge Y_{n-1}$.

Proposition (2.1). — For each $j, 1 \leq j \leq n-k-1$, there is a family of tori $G(k+j)$, satisfying:

1) there is a neighborhood $U_j$ of $T_{k+j}$ and the $G(k+j)$'s are a foliation of $U_j$ by tori of dimension $k+j$;
2) there is a section $g_{k+j}$ of $E_{k+j}(A_{k+j})$ such that $g_{k+j}(x)$ represents the tangent space at $x$ to $G(k+j)(x)$ and $(g_{k+j} \wedge \beta_{k+j})_p \neq 0$

for all $p \in U_j - T_{k+j};$
3) on $T$, $G_{k+j}(x) = T_{k+j}$.

Remark. — In particular $c_2$ implies

$g_{n-1} \wedge Y_{n-1} \neq 0$ in $U_{n-1} - T$.

Hence there exist $(n-1)$-tori, transverse to $Y_{n-1}$, as close to $T$ as we wish.

Proof of (2.1). — We proceed by induction on $j$; first "cylinders" are constructed in $T^j \times I, k+1 \leq j \leq n-2$, and then these cylinders are closed, to give tori, by the map $F_j$ defined by the holonomy of $z_j$.

We start by constructing the foliation $G(k+1)$. Let $U_i = \pi^{-1}(T_{k+1}),$ and
let \((\theta, z, \lambda)\) be coordinates for \(T^k \times I \times J\) where \(\theta = (\theta_1, \ldots, \theta_k) \in T^k\), and \(I = J = [0, 1]\). Let \(F_i : T^k \times I \times J \rightarrow U_i\) be defined by:

\[
F_i(o, o, o) = x_o \\
F_i(o, z, o) = N(z, x_o)
\]

\(F_i(\theta, z, \lambda)\) is the endpoint of the holonomy lifting of the arc in \(T\) given by:

\((t, Y(\lambda t, F_i(\theta, o, o)))\),

\(0 \leq t \leq 1\), starting at \(F_i(\theta, z, o) = N(z, F_i(\theta, o, o))\).

Here we have identified \(T^k = R^k / Z^k\) with \(T^k\) by the linear diffeomorphism \((o, \ldots, o, \ldots, o) \mapsto Y_i(\theta)(o)\).

By definition of \(F_i\) we have:

- \(DF_i\left(\frac{\partial}{\partial \theta_j}\right)\) is colinear with \(C_i\) for \(1 \leq j \leq k\),
- \(F_i\) sends the tori \(T^k \times \{z\} \times \{\lambda\}\) to the holonomy liftings of the tori \(T^k(F_i(o, o, \lambda))\) to the point \(F_i(o, z, \lambda)\),
- \(DF_i\left(\frac{\partial}{\partial \lambda}\right) = C_{k+1} = Y_{k+1}\) on \(T\),
- \(F_i\) sends the segments \(\{\theta\} \times I \times \{\lambda\}\) to the orbits of \(N\) starting at \(F_i(\theta, o, \lambda)\),
- the segments \(\{\theta\} \times \{z\} \times J\) are sent to \(\alpha_{k+1}(F_i(\theta, z, o))\),
- \(F_i\) is a local diffeomorphism to \(U_i\).

From these remarks, it is easy to see that the map \(z \mapsto F_i(\theta, z, \lambda)\) (respectively \(\lambda \mapsto F_i(\theta, z, \lambda)\)) is a reparametrization of the \(N\)-orbits (orbits of \(C_{k+1}\)). Hence there exist functions \(\varphi_i\) and \(\psi_i\), invertible in \(z\) and \(\lambda\) such that

\[
DF_i\left(\frac{\partial}{\partial z}\right) = \frac{\partial \psi_i}{\partial z} N, \\
DF_i\left(\frac{\partial}{\partial \lambda}\right) = \frac{\partial \varphi_i}{\partial \lambda} C_{k+1}
\]

(both \(\varphi_i\) and \(\psi_i\) have strictly positive derivatives on \(T^k \times I \times J\).

Now we construct a family of curves, \(\gamma_i(\theta, z)\), in \(T^k \times I \times J\)

\[
\gamma_i(\theta, z) : \lambda \mapsto (Z_i(\theta, Z, \lambda), \lambda)
\]

satisfying conditions \(A\) and \(B\):

\(A\) For fixed \(\theta, z, F_i(\gamma_i(\theta, z))\) is a closed curve in \(U_i\), of class \(C^1\). For \(\theta\) and \(z\) in a neighborhood of \(T^k \times \{0\}\), the \(F_i(\gamma_i(\theta, Z))\) form a one dimensional foliation of a neighborhood of \(T_{k+1}\) in \(A_{k+1}\),

\[254\]
B) Let $\Delta_{1}(\theta, z) = z - f_{x}^{1}(z)$, where $x = F_{1}(\theta, 0, 0)$, and let $\Delta_{1}(z) = \Delta_{1}(0, z)$. Then we require that:

$$\frac{\partial Z_{1}}{\partial \lambda} = a_{1}(\Delta_{1})$$

(here $\Delta_{1}$ is the function $z \mapsto \Delta_{1}(z)$). The condition $B)$ is not necessary to construct $G(k + j)$; however, it is necessary to insure the transversality relation $c_{j}$ when we construct $G(k + j)$, $j > 1$.

**Lemma (2.2).** — There exists in $T^{8} \times I \times J$, a family of curves $\gamma_{l}(\theta, z)$, satisfying conditions $A)$ and $B)$.

**Proof of (2.2).** — Let $\gamma'$ be the tangent vector field to the $\gamma_{l}$ curves, with the $\lambda$-parametrization. Then condition $A)$ can be written:

$$(1) \quad (DF_{1})_{a} \gamma'_{a} \wedge (DF_{1})_{b} \gamma'_{b} = 0$$

where $b = (\theta, f_{x}^{1}(z), 1)$, $x = F_{1}(\theta, 0, 0)$, and $a = (\theta, z, 0)$ (cf. figure 1).

An easy calculation shows that $(1)$ can be written:

$$\frac{\partial Z_{1}}{\partial \lambda} \bigg|_{b} = K(\theta, z) \frac{\partial Z_{1}}{\partial \lambda} \bigg|_{a}$$

where $K$ is a strictly positive function. Therefore, we can rewrite $A)$ and $B)$ as:

$$\frac{\partial Z_{1}}{\partial \lambda} \bigg|_{b} = K \frac{\partial Z_{1}}{\partial \lambda} \bigg|_{a}$$

$$\frac{\partial Z_{1}}{\partial \lambda} = a_{1}(\Delta_{1}).$$

A tedious, but simple calculation, shows that the cubics (see fig. 1):

$$\lambda \rightarrow (\lambda, Z_{4} = \Delta_{1}(\theta, z) \left[ \frac{1 + K}{1 + K_{0} - 2} \lambda^{3} + \frac{K + 2}{K_{0} + 1} \lambda^{2} + \frac{\lambda}{1 + K_{0}} \right] + f_{x}^{1}(z)$$

255
satisfy these equations, where
\[ K_0 = \sup K(\theta, z), \quad (\theta, z) \in T^k \times I \times \{1\}. \]

Now, one can write:
\[ \frac{\partial Z_1}{\partial \lambda} = \Delta_1(\theta, z) g(\theta, z, \lambda), \]
where \( g > 0 \) on \( T^k \times I \times J \). Also \( \frac{\partial Z_1}{\partial z} > 0 \) on \( T^k \times [0, h] \times J \) for a suitable \( h, \ o < h < 1 \).

Hence the curves \( \gamma_1 \) form a foliation of \( T^k \times [0, h] \times J \). Their image by \( F_1 \) is a foliation (of class \( C^1 \)) of a neighborhood \( V_1 \subset U_1 \) of \( T_{k+1} \) in \( A_{k+1} \). This completes the proof of (2.2) (see fig. 2).

![Fig. 2](image-url)

We can now define \( G(k+1) \). The submanifolds:
\[ H(\pi) = \bigcup_{\theta \in T^k} \gamma_1(\theta, z) \times \{\theta\} \]
are diffeomorphic to \( T^k \times [0, 1] \) and form a foliation of \( T^k \times [0, h] \times J \). Hence their image by \( F_1 \) is a foliation of \( V_1 \) by tori \( G(k+1) \) (figure 2). We now check conditions \( c_2 \) and \( c_3 \).

We have a \( C^1 \) vector field \( \gamma' \) in \( T^k \times [0, h] \times J \); \( \gamma' \) is the tangent field to the \( \gamma_1 \) curves with the \( \lambda \)-parametrization. This field induces a natural action of \( R \) on the exterior products of vector fields, which we note by \( \Gamma(\lambda) : \Gamma(\lambda) \) is the differential of the map induced by \( \gamma' \) of \( T^k \times [0, h] \times \{0\} \) to \( T^k \times [0, h] \times \{\lambda\} \). Then the tangent space to \( H(\pi) \) at the point \((\theta, z, \lambda)\) is given by:
\[ \Gamma(\lambda) \left( \frac{\partial}{\partial \theta_1} \wedge \ldots \wedge \frac{\partial}{\partial \theta_k} / (\theta, z, 0) \right) \wedge \gamma'_{(\theta, z, \lambda)}. \]
Now $F_1$ sends the tori $T^k \times \{z\} \times \{o\}$ to the trivial holonomy liftings of the $T_k$; hence the tangent space to $G(k+1)$ at $p = F_1(\theta, z, \lambda)$ is given by:

$$\langle g_{k+1} \rangle_p = \langle DF_1 \circ \Gamma(\lambda) \left( \frac{\partial}{\partial \theta_1} \wedge \ldots \wedge \frac{\partial}{\partial \theta_k} \right) \wedge DF_1 \circ \gamma' \rangle(\theta, z, \lambda)$$

$$\gamma' = \frac{\partial}{\partial \lambda} + \sigma_1(\Delta_1) \frac{\partial}{\partial Z}.$$  

We recall that:

$$DF_1 \left( \frac{\partial}{\partial \lambda} \right) = \frac{\partial}{\partial \lambda} N, \quad DF_1 \left( \frac{\partial}{\partial Z} \right) = \frac{\partial}{\partial \lambda} C_{k+1}.$$

Let $\bar{\sigma}(\Delta_1)$ denote a function such that $\frac{\bar{\sigma}(\Delta_1)}{\Delta_1}$ tends towards a limit $\bar{a}$; then

$$\bar{\sigma}(\Delta_1) = a \Delta_1 + \sigma(\Delta_1),$$

(with $a$ not necessarily different from $o$ and $\sigma(\Delta_1) = \Delta_1 \varepsilon(\Delta_1)$, $\varepsilon(\Delta_1) \rightarrow o$ whenever $\Delta_1 \rightarrow o$). We take the tangent spaces to the trivial holonomy liftings of the $T_k$, to be given by a section of $E_k(A_k)$, equal to $Y_1 \wedge \ldots \wedge Y_k$ on $T_k$.

Let $C_{k+1}(\lambda)$ denote the action induced by $C_{k+1}$ on the vectors tangent to $A_k$ (if $Y$ is tangent to the $\varphi$-orbits, then so is $C_{k+1}(\lambda)(Y)$). Then we obtain for $g_{k+1}$:

$$\langle g_{k+1} \rangle_p = \langle C_{k+1}(\lambda) \circ (g_k)_{F_1(\theta, z, o)} + \bar{\sigma}(\Delta_1) \Omega_p \rangle \wedge (C_{k+1} + \sigma_1(A_k) N)_p,$$

where $p = F_1(\theta, z, \lambda)$ and $\Omega$ is a section of $E_k(A_k+1)$ defined on $V_1$. We can rewrite this as:

$$C_{k+1} \wedge C_{k+1}(\lambda) \circ g_k + \sigma_1(\Delta_1) C_{k+1}(\lambda) \circ g_k \wedge N + \sigma(\Delta_1) C_{k+1} \wedge \Omega + \sigma(\Delta_1) N \wedge \Omega.$$

Now $\beta_{k+1} \wedge C_{k+1}(\lambda) \circ g_k$ is zero, since it is a linear combination of exterior products of $n$ vectors tangent to the $\varphi$-orbits. Hence:

$$g_{k+1} \wedge \beta_{k+1} = \sigma_1(\Delta_1) C_{k+1}(\lambda) \circ g_k \wedge N \wedge \beta_{k+1} + \sigma(\Delta_1) C_{k+1} \wedge \Omega \wedge \beta_{k+1} + \sigma(\Delta_1) N \wedge \Omega \wedge \beta_{k+1}.$$

We have:

$$C_{k+1}(\lambda) \circ g_k \wedge N \wedge \beta_{k+1} = Y_1 \wedge \ldots \wedge Y_k \wedge N \wedge Y_{k+1} \wedge \ldots \wedge Y_n$$

on $T_{k+1}$. Hence for all points $p$ in a neighborhood $V_2$ of $T_{k+1}$, we have:

$$|C_{k+1}(\lambda) \circ g_k \wedge N \wedge \beta_{k+1}|_p > a > o.$$

We want to show $\langle g_{k+1} \wedge \beta_{k+1} \rangle_p > 0$, for $p$ in a suitable neighborhood of $T_{k+1}$ in $A_{k+1}$. Dividing by $\sigma_1(\Delta_1) (\neq 0$ if $z = 0$):

$$\frac{g_{k+1} \wedge \beta_{k+1}}{\sigma_1(\Delta_1)} = \rho_k + C_{k+1} \wedge \Omega' \wedge \beta_{k+1} + \bar{\sigma}(\Delta_1) N \wedge \Omega \wedge \beta_{k+1},$$

where $|\rho_k|_p > a > o$ in $V_3$, and $\Omega'$ is a bounded section on $V_2 - T_{k+1}$, $\bar{\sigma}(\Delta_1) \rightarrow o$ as $\Delta_1 \rightarrow o$. Since $C_{k+1} = Y_{k+1}$ on $T_{k+1}$, the second term is less than $a/3$ if $p$ is in some
neighborhood $V$ of $T$. Also $|\sigma(A)| |N\wedge \Sigma_{k+1}| < \pi/3$ if $p$ is in some neighborhood $V$ of $T$. Hence for $p \in V \cap V \cap V$, 
\[
|\Sigma_{k+1} \wedge \Sigma_{k+1}| > \pi/3 > 0,
\]
which proves the theorem for $j=1$.

It is useful for the induction to write $\Sigma_{k+1}$ in the form:
\[
\Sigma_{k+1} = \alpha_{k+1} + \sigma(A) \Omega'
\]
where $\alpha_{k+1}$ is a section of $E_{k+1}(A_{k+1})$, defined in a neighborhood of $T_{k+1}$ and equal to $Y_1 \wedge \cdots \wedge Y_{k+1}$ on $T_{k+1}$.

Construction of $G(k+j+1)$. — Let $\Delta_{k+j}(s) = s - f^k(s)$, where $s \geq 0$ denotes the normal $N$-coordinate.

Fondamental little lemma:
\[
\lim_{s \to 0} \frac{\Delta_{k+j}}{\Delta_{k+j+1}} \text{ exists.}
\]

Following (1), one may find an homeomorphism $H : [0, \epsilon] \to [0, \epsilon']$ such that $H^{-1}f_{k+j}H = \lambda_{k+j}$ and $H^{-1}f_{k+j+1}H = \lambda_{k+j+1}$ where $\lambda_{k+j}$ and $\lambda_{k+j+1}$ are the homotheties the ratio of which are $\lambda_{k+j}$ and $\lambda_{k+j+1}$ (recall $f_{k+j}$ and $f_{k+j+1}$ are contractions). Define on $[0, \epsilon]$ a metric $\delta$ such that $\delta(x, x') = \left| H^{-1}(x) - H^{-1}(x') \right|$ — this metric is topologically equivalent to the classical one — and $\delta(x, 0) \to 0$ whenever $x \to 0$. We prove $\frac{\delta(x, f_{k+j}(x))}{\delta(x, f_{k+j+1}(x))}$ has a limit when $x \to 0$ (with respect to $\delta$); we shall then be able to define $\delta(x, 0) \to 0$ whenever $x \to 0$. Then let $f = f_{k+j}$, $g = f_{k+j+1}$.

\[
\frac{\delta(x, f(x))}{\delta(x, g(x))} = \frac{|H^{-1}(x) - H^{-1}(f(x))|}{|H^{-1}(x) - H^{-1}(g(x))|} = \frac{|H^{-1}(x) - \lambda_{k+j} f^{-1}(x)|}{|H^{-1}(x) - \lambda_{k+j+1} f^{-1}(x)|}
\]

\[
\rho = \frac{\delta(x, f(x))}{\delta(x, g(x))} = \frac{1 - \lambda_{k+j}}{1 - \lambda_{k+j+1}} \times H^{-1}(x)
\]

when $x \to 0$, $\delta(x, 0) \to 0$ and $\rho \to \frac{1 - \lambda_{k+j}}{1 - \lambda_{k+j+1}}$.

Our inductive hypothesis asserts the existence of the foliation $G(k+j)$ and a section $\Sigma_{k+j}$ satisfying:
\[
\Sigma_{k+j} = \alpha_{k+j} + \sigma(\Delta_{k+j}) \Omega,
\]
where $\alpha_{k+j}$ is a section of $E_{k+j}(A_{k+j})$, defined in a neighborhood $U$ of $T_{k+j}$ in $A_{k+j}$, which is a linear combination of vectors tangent to the $\phi$-orbits, and equal to $Y_1 \wedge \cdots \wedge Y_{k+j}$ on $T_{k+j}$. $\Omega$ is a section of $E_{k+j}(A_{k+j})$ defined in $U$. Henceforth, we work in $U$. 

258
Since the $G(k+j)$ form a foliation of $U$ transverse to the normals, we can construct, by the holonomy, a map $F_{j+1}$ satisfying:

- $F_{j+1}$ sends $T^{k+j} \times I \times J$ to $U$ and is of maximal rank;
- $F_{j+1}(o, z, o) = N(z, o)$;
- $F_{j+1}$ sends the tori $T^{k+j} \times \{z\} \times \{o\}$ to the tori $G(k+j)$ passing by $F_{j+1}(o, z, o)$;

restricted to $T^{k+j} \times \{o\} \times J$, we have:

\[
DF_{j+1} \left( \frac{\partial}{\partial \theta_1} \right) = Y_{\ell}, \quad 1 \leq \ell \leq k+j,
\]

\[
DF_{j+1} \left( \frac{\partial}{\partial \theta_{k+j+1}} \right) = g_{k+j} \quad \text{in } A_{k+j};
\]

$F_{j+1}$ is the holonomy lifting, restricted to the plaques $\{\theta\} \times I \times J$, i.e. $(F = F_{j+1})$ $F(\theta, z, \lambda)$ is the endpoint of the holonomy lifting of the path $\gamma_{j+1}([o, \lambda], F(\theta, o, o))$ to the point $F(\theta, z, o)$.

Exactly as the case $j=1$, we have a family of curves $\gamma_{j+1}(\theta, z)$ satisfying the conditions $\Lambda$ and $\Lambda'$, with $F_1$ replaced by $F = F_{j+1}$. This gives us a foliation of $T^{k+j} \times I \times J$ by submanifolds diffeomorphic to $T^{k+j} \times I$, and closing the cylinders by $F$ we obtain a foliation by tori $G(k+j+1)$. We must verify $\omega_{j+1}$.

Let $g = g_{k+j+1}$, $G = C_{k+j+1}$, $\Delta = \Delta_{k+j+1}$ and $G^* = C_{k+j+1}^*$. Then we have:

\[
g = (G + \sigma_1(\Delta)N) \wedge (G^* + \sigma(\Delta)\Omega) \sigma_{k+j} + \sigma(\Delta)\Omega'
\]

\[
= C \wedge C^*(\lambda) \sigma (k+j) \wedge \sigma(\Delta)N \wedge \sigma(\Delta)\Omega + \sigma(\Delta)\Omega' + \sigma(\Delta)N \wedge \sigma(\Delta)\Omega''.
\]

Now we write $g_{k+j} = \alpha_{k+j} + \sigma(\Delta)\Omega$ (a section defined in $A_{k+j+1}$), to obtain (defined in $A_{k+j+1}$):

\[
g = C \wedge C^*(\lambda) \alpha_{k+j} + (C \wedge C^*(\lambda) \Omega) \sigma(\Delta)\Omega + \sigma(\Delta)\Omega' + \sigma(\Delta)N \wedge \sigma(\Delta)\Omega''.
\]

Notice that $C \wedge C^*(\lambda) \alpha_{k+j}$ is a linear combination of products of order $k+j+1$ of vectors tangent to the leaves, and on $T_{k+j+1}$, it equals $Y_{k+j+1} \wedge Y_{k+j+1} \wedge \ldots \wedge Y_{k+j}$.

Composing $g$ with $\beta_{k+j+1}$:

- the term $\sigma_1(\Delta)\Omega \alpha_{k+j} \beta_{k+j+1} = 0$ since it is a multiple of $n$ vectors tangent to the $\varphi$ orbits;
- $N \wedge C^*(\lambda) \alpha_{k+j} = N \wedge Y_{k+j}$ on $T_{k+j+1}$.

Dividing by $\sigma_1(\Delta)$ we obtain:

\[
\frac{g \wedge \beta_{k+j+1}}{\sigma_1(\Delta)} = N \wedge C^*(\lambda) \alpha_{k+j} + \frac{\sigma(\Delta)\Omega}{\sigma_1(\Delta)} C \wedge C^*(\lambda) \Omega + \sigma(\Delta)\Omega''.
\]
Now, as $s \to 0$, $\Delta_{k+j}/\Delta$ is bounded hence $\sigma(\Delta_{k+j})/\sigma(\Delta)$ is bounded. $\Omega''$ is a bounded section of $E_{k+j+1}(\Delta_{k+j+1})$ and $\delta(\Delta) \to 0$ as $\Delta \to 0$. Hence, in a small enough neighborhood of $T_{k+j+1}$, we have $g \wedge \beta_{k+j+1} \neq 0$, which completes the proof of (2.1).

BIBLIOGRAPHY


*Manuscrit reçu le 14 juin 1973.*