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Manifolds which admit $\mathbb{R}^n$ actions


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The purpose of this paper is to determine which $n$-manifolds admit smooth locally free actions of $\mathbb{R}^{n-1}$. We shall restrict ourselves to actions $\varphi$ of $\mathbb{R}^{n-1}$ on $V^n$ which are of class $C^2$ and tangent to $\partial V^n$, i.e., the components of $\partial V^n$ are orbits of $\varphi$. For $n=3$, we know that $V^3$ admits such an $\mathbb{R}^2$ action if and only if $V^3 = S^1 \times \mathbb{T}^2$ or $V^3$ is a bundle over $S^1$ with fibre $\mathbb{T}^2$ [7]. Moreover, the topological type of such $\mathbb{R}^2$ actions has been completely determined [8]. We recall that the rank of $V^n$ is the largest integer $k$ such that $V^n$ admits a smooth locally free action of $\mathbb{R}^k$.

Now suppose that $\varphi$ is a locally free action of $\mathbb{R}^{n-1}$ on $V^n$. We shall prove:

**Theorem 1.** — If $\partial V^n$ is not empty, then $V^n$ is homeomorphic to $S^{n-1} \times \mathbb{T}$ (here $\mathbb{T}$ denotes the torus of dimension $1$).

**Theorem 2.** — If $\partial V^n$ is empty and $\varphi$ has at least one compact orbit, then $V^n$ is a bundle over $S^1$ with fibre $\mathbb{T}^{n-1}$.

**Theorem 3.** — If $\partial V^n$ is empty and $\varphi$ has no compact orbits then $V^n$ is a bundle over a torus $\mathbb{T}^k$ with fibre a torus $\mathbb{T}^{n-k}$.

Theorem 2 follows directly from Theorem 1 by cutting $V^n$ along a compact orbit. Theorem 3 depends upon an observation of Novikov [4], and independently Joubert: suppose $\varphi$ acts on $V^n$ with no compact orbits. By Sacksteder [9], all the orbits of $\varphi$ are $\mathbb{T}^{n-k} \times \mathbb{R}^{k-1}$ for some $k$. Choose linearly independent vector fields $X_1, \ldots, X_{n-1}$ tangent to the orbits of $\varphi$ such that all the integral curves of $X_1, \ldots, X_{n-k}$ are periodic, of period one. Then $X_1, \ldots, X_{n-k}$ define a locally free action of $\mathbb{T}^{n-k}$ on $V^n$ and the orbit space $M$ is a smooth manifold of dimension $k$. Also $M$ admits an action of $\mathbb{R}^{k-1}$ with all the orbits $\mathbb{R}^{k-1}$. It follows that $M$ is homeomorphic to $\mathbb{T}^k$, which proves Theorem 3 ([5] and [3]). Consequently, our main result is Theorem 1. Here is how we proceed to prove Theorem 1: by inductive arguments similar to those used in [7], we restrict ourselves to actions $\varphi$ with no compact orbits in the interior of $V^n$. We then
remark that the foliation defined by the orbits of $\varphi$ is almost without holonomy, i.e. the noncompact leaves have no holonomy. With this, we construct collar neighborhoods $U_i$ of each component $T'_i$ of $\partial V$, such that $\partial U_i = T'_i \cup T'_i$ with $T'_i$ transverse to the foliation. We construct $U_i$ so that some linear field $Y$ (tangent to the orbits of $\varphi$) is transverse to each $T'_i$. We then prove the integral curves of $Y$ go from $T'_i$ to $T'_j$ hence define a homeomorphism of $V^n$ to $T^{n-1} \times I$.

1. Some Preliminaries.

(1.1) Let $\mathcal{F}$ be the foliation of $V$ defined by the orbits of $\varphi$. Then each noncompact leaf of $\mathcal{F}$ has zero holonomy.

Proof. — If $T$ is a compact leaf of $\mathcal{F}$, then the germ of $\mathcal{F}$ in a neighborhood of $T$ is without holonomy outside of $T$, provided $T$ is an isolated compact leaf (page 13 of [8]). This is also true if $T$ is an isolated compact leaf on one side in $V$ and one considers the germ of $\mathcal{F}$ on this side. Now if $\varphi$ has no compact orbits then $\mathcal{F}$ is without holonomy and we are done [9]. So suppose $F$ is a noncompact leaf of $\mathcal{F}$ and $\mathcal{F}$ has compact leaves. Since $\mathcal{F}$ has no exceptional minimal sets [9], there is a compact leaf $T$ of $\mathcal{F}$ such that $T$ is in the closure of $F$. Let $x$ be a point of $F$ and $\alpha(x)$ a non zero element of $\pi_1(F, x)$. Let $X$ be a vector field on $V$ such that the integral curve of $X$ through $x$ is closed and homotopic to $\alpha(x)$, and all the integral curves of $X$ on $F$ are closed. $X$ is easily constructed using the action $\varphi$ (cf. [6]). Since $T$ is in the closure of $F$, we know the integral curves of $X$ on $T$ are also closed. Now $T$ is an isolated compact leaf at least on one side in $V$, the side where $F$ intersects a transverse arc infinitely often. Let $U$ be a neighborhood of $T$, on this side, such that all the leaves of $\mathcal{F}$ in $U$, except $T$, have zero holonomy. Then $U$ contains closed integral curves of $X$ which are on $F$, so such an integral curve $C$ has zero holonomy. Since $C$ is conjugate to $\alpha(x)$, it follows that $\alpha(x)$ has zero holonomy; thus $F$ as well.

(1.2) Suppose $\partial V$ is not empty and $\varphi$ has no compact orbits in the interior of $V$. Let $T$ be a compact orbit of $\varphi$; $T \subset \partial V$. The leaves which contain $T$ in their closure are homeomorphic to $T^k \times \mathbb{R}^{n-k-1}$ where $k =$ the rank of the kernel of the holonomy map on $T$.

Proof. — Let $F$ be an open leaf whose closure contains $T$; $F \approx T^k \times \mathbb{R}^{n-k-1}$. Suppose $Z^k$ is the kernel of the holonomy homomorphism on $T$. Let $T^k$ be a $k$-torus embedded in $T$ which lifts onto nearby leaves by the holonomy. Since $\mathcal{F} \ni T$, we can lift $T^k$ to a $k$-torus $T_i$ in $F$. Also $i_T: \pi_i(T) \to \pi_i(V)$ is injective, where $i: T \hookrightarrow V$ (cf. [4]), hence $\pi_i(T_i)$ embeds in $\pi_i(F)$ and $k \leq j$.

Next we show $j \leq k$. Let $x \in F$ and $\alpha \in \pi_i(F, x)$, $\alpha \neq 0$. Let $X$ be a vector field tangent to the orbits of $\varphi$, such that the integral curves of $X$ on $F$ are closed and the integral curve of $X$ through $x$ is homotopic to $\alpha$. Since $\mathcal{F} \ni T$, all the integral curves
of \( X \) on \( T \) are closed. Let \( C \) be an integral curve of \( X \) on \( T \). We know that \( C \) lifts to a closed curve on \( F \), so by (1.1), the holonomy of \( G \) is trivial; i.e. \( G \) is in the kernel of the holonomy homomorphism. Hence \( j \leq k \).

2. The transverse torus and vector field.

Throughout this section, we suppose \( \varphi \) acts on \( V \) so that there are no compact orbits in the interior of \( V \) and \( T \) is a compact orbit in \( \partial V \). Let \( k \) denote the rank of the kernel of the holonomy map associated to \( T \); \( k \) varies between \( 0 \) and \( n-2 \). Let \( Y_1, \ldots, Y_{n-1} \) be linearly independent commuting vector fields on \( V \) satisfying:

(i) they are tangent to the \( \varphi \)-orbits;
(ii) their integral curves are closed and of period one on \( T \); and
(iii) the integral curves of \( Y_1, \ldots, Y_k \) represent the kernel of the holonomy map on \( T \).

We shall construct an \((n-1)\)-torus \( T' \subset \text{Int} V \) such that \( T \cup T' \) bound a trivial cobordism in \( V \), and \( Y_{n-1} \) is transverse to \( T' \) at each point.

By (1.1), we know the orbits of \( Y_{k+1}, \ldots, Y_{n-1} \) on \( T \) induce germs in \( \text{Diff}(\mathbb{R}^4) \) which are contractions or expansions, via the holonomy. Here \( \text{Diff}(\mathbb{R}^4) \) is the set of \( C^0 \)-germs of diffeomorphisms of \( \mathbb{R}^4 \) to itself, which leave \( 0 \) fixed. After reversing the sign of \( Y_j \) if necessary, we shall assume the germs are all contractions, for \( k+1 \leq j \leq n-1 \).

Choose a metric on \( V \) and let \( U_\varepsilon \) be a geodesic collar neighborhood of \( T \) isometric to \( T \times [0, \varepsilon] \), with the obvious product metric. Clearly, if \( \varepsilon \) is small enough, the geodesics normal to \( T \) in \( U_\varepsilon \) will be transverse to the orbits of \( \varphi \). Let \( f^i_\varepsilon \) be the holonomy diffeomorphism associated to the \( Y_i \) orbit through \( x \); \( f^i_\varepsilon \) is the identity for \( 1 \leq i \leq k \) and a contraction for \( k+1 \leq i \leq n \).

**Proposition (2.1).** — There is an \((n-1)\)-torus \( T' \) contained in \( U_\varepsilon \) such that \( Y_{n-1} \) is transverse to \( T' \) and \( T \cup T' \) bound a trivial cobordism.

In an earlier version of this paper we gave a proof of (2.1) which used calculus. Charles Pugh pointed out to us how one can use a theorem of W. Wilson on the existence of Liapounov functions for uniform stable attractors of vector fields [13]. We present this proof here and in an appendix we give our original proof.

We need some definitions before stating Wilson's theorem. Let \( X \) be a vector field on \( V \) and let \( A \) be a closed invariant subset of \( V \) (here \( V \) is a compact manifold). \( A \) is called a \emph{uniform stable attractor} of \( X \) if the following conditions are satisfied:

a) there exists an increasing function \( \delta \) sending \( \mathbb{R}^+ \) into itself such that
\[
\text{d}(X(p, t), A) < \delta
\]
whenever \( \text{d}(p, A) < \delta(\varepsilon) \) and \( t \geq 0 \).
b) there exists a neighborhood \( U \) of \( A \) such that \( \omega(p) \subset A \) whenever \( p \in U \) (\( \omega(p) \) is the \( \omega \)-limit set of \( p \));

c) let \( D(A) \) be the set of \( p \) such that \( \omega(p) \subset A \); \( D(A) \) is an open set, called the basin of attraction of \( A \).

Wilson has proved [13] that if \( A \) is a uniform stable attractor for \( X \) then there exists a \( C^\infty \) Liapounov function, i.e.

\[ a) \text{ there is a } C^\infty \text{ function } f : D(A) \to \mathbb{R}^+ \text{ with } f^{-1}(0) = A; \text{ and} \]

\[ b) X(f)(p) < 0 \text{ whenever } f(p) \neq 0. \]

Hence \( f \) has no singularities outside \( A \) and all the level surfaces of \( f \) are diffeomorphic. Before proving (2.1) we need three lemmas.

**Lemma (2.2).** (Action box lemma.)

There exists a unique mapping

\[ F : J^{n-1} \times [0, \varepsilon] \to U_c \subset \mathbb{R} \]

(where \( J = [-1, 2] \)) satisfying the following conditions:

a) \( F \) is a \( C^2 \)-immersion;

b) \( F \) sends the horizontal plaques \( J^{n-1} \times \{z\} \) into the leaves of \( \mathcal{F} \);

c) \( F \) sends vertical arcs \( \{A\} \times [0, \varepsilon] \) onto the geodesic arcs normal to \( T \);

d) \( F \), when restricted to \( J^{n-1} \times \{0\} \), is the restriction of the natural covering map: \( \mathbb{R}^{n-1} \to T \) induced by \( \varphi \), which sends the \( i \)-direction line onto the \( Y_i \) circular orbit;

e) let \( x_0 \in T \); then \( F \) sends \( \{0\} \times [0, \varepsilon] \) isometrically onto the geodesic arcs issued from \( X_0 \), normal to \( T \) and pointing inside \( T \).

**Proof.** — Define first \( F \) via e) and d). \( F \) obviously extends to \( J^{n-1} \times [0, \varepsilon] \) using b) and c).

a) is clear, for geodesic arcs are normal to \( \mathcal{F} \) in \( U_c \). Note that each \( Y_i \) orbit on \( T \) is covered three times by \( F \).

**Lemma (2.3).** (Commuting contraction lemma.)

If \( f_1 \) and \( f_2 \) are commuting embeddings \( [0, \varepsilon] \to [0, \infty[ \) and \( f_2 \) is a contraction towards \( 0 \), then there exists a \( K \) so large that \( f_1 f_2^K \) is a contraction to \( 0 \).

**Proof.** — \( f_2 \) commuting with \( f_1 f_2^K \), \( f_1 f_2^K \) is an embedding without fixed point or is the identity (N. Koppel’s Thesis). For sufficiently large \( h \), \( f_1 f_2^K \) is not the identity. Hence \( f_1 f_2^K \) is a contraction or an expansion. For \( f_1 f_2^K [0, \varepsilon] = f_1 f_2^K [0, \varepsilon] \), and \( K \) may be chosen so large that \( f_2^K f_1 [0, \varepsilon] \subset \left[ 0, \frac{\varepsilon}{2} \right] \). \( f_1 f_2^K \) is therefore a contraction.
3. Attraction Lemma.

There exists ε and δ > 0 such that whenever X is a C¹ vector field on $\mathbb{R}^{n-1}$ and $|X|_0 < \delta$, then $Y = \Phi_t \left( \frac{\partial}{\partial \lambda_{n-1}} + X \right)$ generates a flow having $T$ as a uniform and stable attractor, $U_\varepsilon$ being in the basin of attraction of $T$.

— Look at the application $F$ of Lemma (2.2) (action box lemma). If

$$Y = \Phi_t \left( \frac{\partial}{\partial \lambda_{n-1}} + X \right),$$

$F^*Y$ is a $C^1$ vector field defined on $J^{n-1} \times [0, \varepsilon]$ ($F$ is a $C^2$-immersion); $F^*Y$ has no vertical component and may be chosen arbitrarily close to $\frac{\partial}{\partial \lambda_{n-1}}$ for a suitable choice of $\delta$.

Let $I = [0, 1]$, $A_0 = I^{n-2} \times \{0\} \times [0, \varepsilon]$, $A_1 = I^{n-2} \times \{1\} \times [0, \varepsilon]$ and $x \in A_0$. I being interior to $J$, choose $\delta$ such that the positive orbit of $F^*Y$ through $x$ crosses $A_1$ before reaching the boundary of $J^{n-1} \times [0, \varepsilon]$. Let $x$ be the point of intersection of $A_1$ with the orbit. Via $F$, $x$ is identified with a point $x \in A_0$ and hence may be written in the form $(\lambda', \ldots, \lambda'_{n-1}, 0, z_1)$ where

$$z_1 = f_{k+1}^0 \circ \cdots \circ f_{k+1}^0 \circ \cdots \circ f_{n-1}(z) \quad \text{if} \quad x = (\lambda_1, \ldots, \lambda_{n-1}, 0, z).$$

Recall that for $1 \leq j \leq n-K+1$, $f_{k+j}$ are the contracting holonomy diffeomorphisms associated to the circular $Y_{k+j}$ orbits.

Using the contraction commuting lemma, we choose $N$ such that

$$f_{k+j}^{-1} \circ \cdots \circ f_{n-1}^{-1} \circ f_{n-1}$$

is a contraction. For $\varepsilon$ and $\delta$ small, we may build a sequence $(x, \tilde{x}, x_1, \ldots, x_{n-1}, \tilde{x}_{n-1}, x_N)$ where the $F^*Y$ orbit through $x_i$ crosses $A_i$ at $\tilde{x}_i$ and $\tilde{x}_i$ being identified via $F$ with $x_{i+1}$ in $A_0$. So if $x = (\lambda_1, \ldots, \lambda_{n-1}, 0, z)$, then $x_N = (\lambda''_1, \ldots, \lambda''_{n-1}, 0, h(z))$ where

$$h(z) = \prod_j f_{k+j}^{-1} \circ f_{n-1}(z).$$

Thus we have shown that the vertical coordinate of any $Y$-orbit tends to $0$ in a manner dominated by a fixed contraction $f_{k+1}^{-1} \circ \cdots \circ f_{n-1}^{-1}$ as we proceed along the orbit in forward times, i.e. $T$ is a uniformly stable attractor.

Let us prove now Proposition (2.1).

— The choice of the $Y_i$'s on $T$ allow us to write $T$ as a trivial fibration $\Sigma \times S^1$, where $\Sigma$ is a manifold diffeomorphic to $T^{n-2}$ and transversal to the circular orbits of $Y_{n-1}$ which are the fibers of that fibration. Over these circles, consider the normal geodesic fibers of $U_\varepsilon$. This gives a two dimensional foliation of $U_\varepsilon$ by cylinders. Call it $\mathcal{A}$; $\mathcal{A}$ is clearly transversal to $\mathcal{F}$.
Let \( Y_{n-1} = X + Y \) where \( Y \) is tangent to \( \mathcal{A} \cap \mathcal{F} \) and orthogonal to \( X \); clearly \( Y_{n-1}(x) - Y(x) = X(x) \) tends to 0 when \( d(x, T) \) tends to 0. Due to the attraction lemma, \( Y \) admits \( T \) as a uniform stable attractor. Let \( V_1 = U_{\frac{2}{3}} \), \( V_2 = U_{\frac{1}{4}} \), and let \( \beta \) be a bump function such that \( \beta = 1 \) on \( V_1 \) and \( \beta = 0 \) outside \( V_2 \). Let \( Z = \beta Y + (1 - \beta) Y_{n-1} \). It is easy to check that \( Z \) admits \( T \) as a uniform stable attractor and hence there exists a Liapounov function \( f \) for \( Z \). For \( \epsilon > \varepsilon_0 > \frac{2\pi}{3} \), \( Z = Y_{n-1} \) and \( f^{-1}(\varepsilon) \) is transversal to \( Y_{n-1} \). For \( \frac{2\pi}{3} > \varepsilon_1 > 0 \), \( f^{-1}(\varepsilon_1) \) is transverse to \( Y \); \( f^{-1}(\varepsilon_1) \) is diffeomorphic to \( f^{-1}(\varepsilon_0) \). It remains to prove \( f^{-1}(\varepsilon_1) \) is a \((n-1)\)-dimensional torus for \( f^{-1}(\varepsilon_0) \) will then be a torus satisfying conditions of (2.1).

Let \( Y \) being transverse to \( f^{-1}(\varepsilon_1) \), \( f^{-1}(\varepsilon_1) \) is transverse to \( \mathcal{A} \). Let \( \mathcal{A}_0 \) be the leaf of \( \mathcal{A} \) through \( x \); \( \mathcal{A}_0 \cap f^{-1}(\varepsilon_1) \) is a compact one-dimensional manifold and hence diffeomorphic to a circle. Writing \( T = \Sigma \times S_1 \) and \( x = (\lambda, s) \) here \( \lambda \in \Sigma \) and \( s \in S_1 \), one produces a family of embeddings of \( S_1 \), \( (\pi_0)_\lambda \in \Sigma \) such that \( \pi_0(S_1) = \mathcal{A}_0 \cap f^{-1}(\varepsilon_1) \). We define now an application \( \pi : \Sigma \times S_1 \to f^{-1}(\varepsilon_1) \) by \( \pi(\lambda, s) = \pi_0(s) \) which is clearly an embedding. Proposition (2.1) is thereby proved for \( \Sigma \) is diffeomorphic to \( T^{n-2} \).

Proof of Theorem 1. — We now assume \( \partial V \) is not empty and \( \varphi \) has no compact orbits in the interior of \( V \). Let \( T, T' \), and \( Y_1, \ldots, Y_{n-1} \) be as in section 2; so that \( Y_{n-1} \) is transverse to \( T' \) and pointing into \( V \) along \( T' \), i.e. \( Y_{n-1} \) points out of the tubular neighborhood of \( T \). Let \( F \) be an orbit of \( \varphi \) which intersects \( T' \) and let \( L \) be a connected component of \( F \cap T' \).

Lemma (3.1). — \( \bigcup_{t \in \mathbb{R}} Y_{n-1}(t, L) = F \).

Proof. — We know \( F \) is diffeomorphic to \( T^k \times \mathbb{R}^{n-k-1} \) (in the leaf topology) and we have a covering map \( \pi : \mathbb{R}^{n-1} \to F \) induced by \( \varphi \). Since \( Y_1, \ldots, Y_{n-1} \) define the action \( \varphi \), we can take \( \pi'(Y_{n-1}) = \frac{\partial}{\partial x_{n-1}} \) where \( (x_1, \ldots, x_{n-1}) \) denote the usual coordinates in \( \mathbb{R}^{n-1} \). Let \( X \) denote \( \frac{\partial}{\partial x_{n-1}} \), and let \( W \) be a connected component of \( \pi^{-1}(L) \). It suffices to prove that each orbit of \( X \) starting at a point of the hyperplane \( x_{n-1} = 0 \), intersects \( W \), since this implies \( \bigcup X(t, W) = \mathbb{R}^{n-1} \).

Now \( W \) is a closed submanifold of \( \mathbb{R}^{n-1} \), of codimension one, and \( X \) is transverse to \( W \), and makes an angle with \( W \) that is strictly bounded away from zero, since \( Y_{n-1} \) is transverse to \( T' \). Clearly, the set of points of the hyperplane \( x_{n-1} = 0 \), whose \( X \) orbits intersect \( W \), is an open non empty set \( \Omega \). It suffices to show \( \Omega \) is closed. Let \( z_n \in \partial \Omega \), satisfying \( \lim z_n = z \) and for each \( n \), there exists \( t_n \in \mathbb{R} \), such that \( X(t_n, z_n) \in W \). If some subsequence of \( (t_n) \) converges to a number \( t \) then we have \( X(t, z) \in W \); hence we can suppose no subsequence converges. Let \( (\xi_n) \) be a subsequence of \( (t_n) \) such
that \( |s_n - s_{n+1}| \geq 1 \) and \( |z_n - z_{n+1}| < \frac{1}{n} \). Let \( E(n) \) denote the line segment joining \( z_n \) to \( z_{n+1} \) and consider \( (E(n) \times \mathbb{R}) \cap W \). This is a curve in \( W \) with endpoints \( X(S_n, z_n) \) and \( X(S_{n+1}, z_{n+1}) \). There exists a point \( U_n \) on this curve where the tangent to the curve is parallel to the cord joining the endpoints. The angle this cord makes with \( X \) tends to zero as \( n \to \infty \), which contradicts the fact that the angle between \( X \) and \( W \) is strictly positive.

**Lemma (3.2).** Let \( F, W, L, T \) and \( T' \) be as in (3.1). Then there exists a compact orbit \( T_1 \) of \( \varphi \) such that \( \overline{F \cup T_1} \) and \( T_1 \neq T \).

**Proof.** Let \( W_0 \equiv W \) and \( W_n \equiv X(n, W_0) \) for each positive integer \( n \). By an argument analogous to that of (3.1), one sees that the distances \( d(W_k, W_{k+1}) \) tend to infinity as \( s \to \infty \). Let \( L_0 \equiv L \) and \( L_n \equiv Y_{n-1}(n, L_0) \), so that \( \lim_{s \to \infty} d(L_n, L_{n+1}) = \infty \), where the metric is that induced by \( \pi \). We define \( \Omega = \bigcap_n \overline{E_n} \), where \( E_n \) is the connected component of \( F - L_n \) towards which \( Y_{n-1} \) points on \( L_n \). \( \Omega \) is an intersection of a nested family of compact sets, hence \( \Omega \) is not empty and compact. We claim \( \Omega \) is invariant under the \( \varphi \) action: clearly \( \Omega = \{ y \in V \mid \text{there exists } x_0 \in E_n \text{ and } x_0 \to y \} \). Let \( F(y) \) be the orbit of \( \varphi \) by \( y \in \Omega \) and let \( y' \in F(y) \). Let \( [y, y'] \) denote a path in \( F(y) \) joining \( y \) to \( y' \) and let \( [x_n, x'_n] \) be the holonomy lifting of this path to the leaf of \( x_n \). By construction we have \( d(x_n, x'_n) \) bounded above by some number \( \ell \), independent of \( n \). Since

\[
d(L_n, L_{n+1}) \to \infty
\]
as \( s \to \infty \), we can choose a subsequence of \( (x'_n) \), call it \( (y_n) \), such that \( y_0 \in E_n \). Thus \( y' \in \Omega \) and \( \Omega \) is invariant. Thus \( \Omega \) contains a \( \varphi \)-minimal set, which must be a compact orbit by Sacksteder’s theorem. Since \( Y_{n-1} \) points away from \( T \), this compact leaf \( T_1 \subset \Omega \) is different from \( T \).

**Lemma (3.3)** Let \( V^n \) be of rank \( n-1 \) and let \( \varphi \) be an action of \( \mathbb{R}^{n-1} \) on \( V \) such that the only compact orbits of \( \varphi \) are in \( \partial V \) and \( \partial V \) is not empty. Then \( V \) is homeomorphic to \( T^{n-1} \times I \).

**Proof.** We use the notation of (3.1) and (3.2). From these lemmas, it follows that the open leaves having \( T \) int heir closure are homeomorphic to the open leaves having \( T_1 \) in their closure, i.e. to \( T^k \times \mathbb{R}^{n-k-1} \), where \( k \) is the rank of the kernel of the holonomy map of \( T \). Now since all the integral curves of \( Y_1, \ldots, Y_k \) are closed in \( F \) and \( \overline{F \cup T_1} \), we know they are also closed in \( T_1 \); hence the \( k \)-tori in \( T_1 \) spanned by the orbits of \( Y_1, \ldots, Y_k \) represents the kernel of the holonomy map of \( T_1 \). Now the orbits of \( Y_{k+1}, \ldots, Y_{n-1} \) are not necessarily closed but we can choose vector fields (from lines through the origin in \( \mathbb{R}^{n-1} \)) \( \tilde{Y}_{k+1}, \ldots, \tilde{Y}_{n-1} \), such that \( Y_1, \ldots, Y_k, \tilde{Y}_{k+1}, \ldots, \tilde{Y}_{n-1} \) are linearly independent, commute, are tangent to the \( \varphi \) orbits, and all the integral curves of \( \tilde{Y}_{k+1}, \ldots, \tilde{Y}_{n-1} \) in \( T_1 \) are closed. Clearly this can be done so that \( \tilde{Y}_{n-1} \) is
C^2-close to Y_{n-1}. We choose \( \tilde{Y}_{n-1} \) so close that \( \tilde{Y}_{n-1} \) is also transverse to \( T' \). Now we go through the construction of a torus \( T'_1 \), bounding a collar neighborhood with \( T'_1 \), such that \( \tilde{Y}_{n-1} \) is transverse to \( T'_1 \); this is \((2.1)\). Letting \( Y \) denote \( \tilde{Y}_{n-1} \), we now have a linear vector field \( Y \) transverse to both tori \( T' \) and \( T'_1 \). We know the set of points \( A \) in \( T' \) whose \( Y \)-integral curve intersects \( T'_1 \) is an open non-empty set. By the same reasoning, the complement of \( A \) in \( T' \) is open; hence \( A = T' \). Now using the integral curves of \( Y \), it is easy to construct a homeomorphism between \( V \) and \( T^{n-1} \times I \).

**Proof of Theorem 1.** — The proof follows from \((3.3)\), and a reasoning identical to that on page 462 of [7].

**Remarks 1.** — A basic question remains unanswered: suppose \( \varphi \) is a locally free action of \( R^+ \) on a closed manifold \( V^n \), with no compact orbits. Then we know \( V^n \) fibres over a torus with fibre a torus, hence \( V^n \) fibres over \( S^1 \) with fibre \( F \) (this also follows from \([10]\)). Is \( F \) homeomorphic to \( T^{n-1} \)?

2. Suppose \( V^n \) is a closed, orientable, bundle over \( S^1 \) with fibre \( M \). Then there exists a diffeomorphism \( f: M \to M \) such that \( V \) is obtained from \( M \times I \) by identifying points \( (f(x), 1) \) with \( (x, 0) \) for \( x \in M \). We claim that if \( f^* : H^1(M, \mathbb{R}) \to H^1(M, \mathbb{R}) \) does not have one as an eigenvalue, then every locally free action of \( R^+ \) on \( V \) has a compact orbit. To see this, first observe that \( f^* \) does not have \( 1 \) as an eigenvalue if and only if rank \( H^1(V, \mathbb{R}) = 1 \) \([11]\). Now suppose \( \mathcal{F} \) is any foliation of \( V \) of codimension one, class \( C^2 \) and with no compact leaves. By \([12]\), we can suppose \( L \) is a covering space of \( M \) for \( L \) a leaf of \( \mathcal{F} \). We have an exact sequence of free abelian groups:

\[
o \to \pi_1(F)/\pi_1(L) \to \pi_1(V)/\pi_1(L) \to \pi_1(V)/\pi_1(F) \to 0.
\]

Since \( H^1(V, \mathbb{R}) \approx \mathbb{R} \), the last two groups are of rank one. Hence \( \pi_1(L) = \pi_1(F) \) and \( L \) must be compact.

**APPENDIX**

**Proof of (2.1)**

**Notation.** — If \( X \) is a vector field on \( V \), \( t \to X(t, x) \) will denote the integral curve of \( X \) passing through \( x \) at \( t = 0 \). For \( A \subset V \), \( X(t, A) = \{X(t, x) \mid x \in A\} \), and

\[
X([a, b], A) = \bigcup_{a \leq s \leq b} X(t, A).
\]

If \( x \in T \), we define \( q_i(x) = Y_i([0, 1], x) \) for \( i = 1, \ldots, n-1 \), and \( T_i(x) \) is the \( i \)-torus in \( T \) which is the orbit through \( x \) of the \( R^i \)-action determined by \( Y_1, \ldots, Y_i \). If \( \bar{x} \) is on the normal arc through \( x \in T \) and if the holonomy germs are defined on \( \bar{x} \),
then we denote by \( \overline{T}_{x}(x) \) the lifting of \( T_{x}(x) \) into the leaf of \( x \), given by the holonomy.

Let \( N \) be a unit vector field on \( V \), normal to the orbits of \( \varphi \) and pointing into \( V \) along \( T \) (with respect to some metric on \( V \)). Let \( U = N(I,T) \) where \( I = [0,1] \). We may suppose \( U \) is a tubular neighborhood of \( T \) in which the holonomy liftings of \( \alpha_{i}(x) \), \( \ldots, \alpha_{n-1}(x) \) are defined, for \( x \in T \). Let \( f^{i} \) be the holonomy diffeomorphism of \( \alpha_{k+i}(x) \); \( 1 \leq i \leq n-k-1 \).

Let \( \pi : U \to T \) be the projection along \( N \) orbits. If \( x \in T \) and \( \bar{x} \in \pi^{-1}(x) \), let \( \bar{\alpha}_{i}(\bar{x}) \) denote the holonomy lifting of \( \alpha_{i}(x) \) starting at \( \bar{x} \); for \( 1 \leq i \leq k \), \( \bar{\alpha}_{i}(\bar{x}) \) is an embedded circle, and for \( i > k \), \( \bar{\alpha}_{i}(\bar{x}) \) is diffeomorphic to \( I \). For \( x \in T \) and for all \( \bar{x} \in \pi^{-1}(x) \), the \( \bar{\alpha}_{i}(\bar{x}) \) form a one dimensional foliation in \( U \). Let \( C_{i} \) be a vector field in \( T \), tangent to this foliation, and coinciding with \( Y_{i} \) on \( T \).

We fix a base point \( x_{0} \in T \) and we let \( \alpha_{i} = \alpha_{i}(x_{0}) \), \( T_{i} = T_{i}(x_{0}) \), etc., and define \( A_{i} = N(I,T_{i}) \).

Let \( E_{\ell}(A_{j}) \) be the vector bundle of exterior products of order \( \ell \) of vectors tangent to \( A_{j} \). We identify \( E_{\ell}(A_{j}) \) with \( A_{j} \times \wedge^{\ell} \mathbb{R} \); so sections of \( E_{\ell}(A_{j}) \) are functions from \( A_{j} \) to \( \wedge^{\ell} \mathbb{R} \). We give these sections the canonical norm.

Let \( f \) be a function defined in a neighborhood of \( 0 \) such that \( \lim_{x \to 0} f(x) = 0 \). We write \( f = \sigma_{i}(x) \) if

\[
    f(x) = ax + x\sigma(x),
\]

with \( a \neq 0 \) and \( \sigma(x) \to 0 \) when \( x \to 0 \). Finally, we let \( \beta_{k+j} = Y_{k+j} \wedge \ldots \wedge Y_{n-1} \).

**Proposition (2.1).** — For each \( j \), \( 1 \leq j \leq n-k-1 \), there is a family of tori \( G(k+j) \), satisfying:

1. There is a neighborhood \( U_{j} \) of \( T_{k+j} \) and the \( G(k+j)'s \) are a foliation of \( U_{j} \) by tori of dimension \( k+j \);
2. There is a section \( g_{k+j} \) of \( E_{k+j}(A_{k+j}) \) such that \( g_{k+j}(x) \) represents the tangent space at \( x \) to \( G(k+j)(x) \) and

\[
    (g_{k+j} \wedge \beta_{k+j})_{x} \neq 0
\]

for all \( p \in U_{j} - T_{k+j} \);
3. On \( T \), \( G_{k+j}(x) = T_{k+j} \).

**Remark.** — In particular \( e_{k+j} \) implies

\[
    g_{n-1} \wedge Y_{n-1} = 0 \quad \text{in} \quad U_{n-1} - T.
\]

Hence there exist \((n-1)\)-tori, transverse to \( Y_{n-1} \), as close to \( T \) as we wish.

**Proof of (2.1).** — We proceed by induction on \( j \); first "cylinders" are constructed in \( T^{j} \times I \), \( k+1 \leq j \leq n-2 \), and then these cylinders are closed, to give tori, by the map \( F_{j} \), defined by the holonomy of \( \alpha_{j} \).

We start by constructing the foliation \( G(k+1) \). Let \( U_{1} = \pi^{-1}(T_{k+1}) \), and
let \((\theta, z, \lambda)\) be coordinates for \(T^k \times I \times J\) where \(\theta = (\theta_1, \ldots, \theta_k) \in T^k\), and \(I = J = [0, 1]\). Let \(F_i : T^k \times I \times J \to U_i\) be defined by:

\[
F_i(0, 0, 0) = \eta_0 \\
F_i(0, z, 0) = N(z, \eta_0)
\]

\(F_i(\theta, z, \lambda)\) is the endpoint of the holonomy lifting of the arc in \(T\) given by:

\[
(t, Y(t, F_i(\theta, 0, 0)))
\]

\(0 \leq t \leq 1\), starting at \(F_i(\theta, z, 0) = N(z, F_i(\theta, 0, 0))\).

Here we have identified \(T^k = R^k / Z^k\) with \(T_k\) by the linear diffeomorphism

\((o, \ldots, \theta_j, \ldots, o) \mapsto Y_i(\theta_j)(\eta_0)\).

By definition of \(F_i\) we have:

- \(DF_i \left(\frac{\partial}{\partial \theta_j}\right)\) is colinear with \(C_i\) for \(1 \leq j \leq k\),
- \(F_i\) sends the tori \(T^k \times \{z\} \times \{\lambda\}\) to the holonomy liftings of the tori \(T_i(F_i(0, 0, \lambda))\) to the point \(F_i(0, z, \lambda)\),
- \(DF_i \left(\frac{\partial}{\partial \lambda}\right) = \eta_{k+1} = Y_{k+1}\) on \(T\),
- \(F_i\) sends the segments \(\{r\} \times I \times \{\lambda\}\) to the orbits of \(N\) starting at \(F_i(\theta, 0, \lambda)\),
- the segments \(\{r\} \times \{z\} \times J\) are sent to \(\eta_{k+1}(F_i(\theta, z, 0))\),
- \(F_i\) is a local diffeomorphism to \(U_i\).

From these remarks, it is easy to see that the map \(z \mapsto F_i(\theta, z, \lambda)\) (respectively \(\lambda \mapsto F_i(\theta, z, \lambda)\)) is a reparametrization of the \(N\)-orbits (orbits of \(C_{k+1}\)). Hence there exist functions \(\varphi_i\) and \(\psi_i\), invertible in \(z\) and \(\lambda\) such that

\[
DF_i \left(\frac{\partial}{\partial z}\right) = \frac{\partial \varphi_i}{\partial z} N \\
DF_i \left(\frac{\partial}{\partial \lambda}\right) = \frac{\partial \varphi_i}{\partial \lambda} C_{k+1}
\]

(both \(\varphi_i\) and \(\psi_i\) have strictly positive derivatives on \(T^k \times I \times J\)).

Now we construct a family of curves, \(\gamma_i(\theta, z)\), in \(T^k \times I \times J\)

\[
\gamma_i(\theta, z) : \lambda \mapsto (Z_i(\theta, \lambda), \lambda)
\]

satisfying conditions \(A\) and \(B\):

\(A\) For fixed \(\theta, z, F_i(\gamma_i(\theta, z))\) is a closed curve in \(U_i\), of class \(C^1\). For \(\theta\) and \(z\) in a neighborhood of \(T^k \times \{0\}\), the \(F_i(\gamma_i(\theta, Z))\) form a one dimensional foliation of a neighborhood of \(T_{k+1}\) in \(A_{k+1}\).
B) Let $\Delta_1(\theta, z) = z - f^1_z(z)$, where $x = F_1(\theta, 0, 0)$, and let $\Delta_1(z) = \Delta_1(0, z)$. Then we require that:

$$\frac{\partial Z_1}{\partial \lambda} = \sigma_1(\Delta_1)$$

(here $\Delta_1$ is the function $z \mapsto \Delta_1(z)$). The condition B) is not necessary to construct $G(k+1)$; however, it is necessary to insure the transversality relation $c_a)$ when we construct $G(k+j), j > 1$.

**Lemma (2.2).** There exists in $T^k \times I \times J$, a family of curves $\gamma_1(\theta, z)$, satisfying conditions A) and B).

**Proof of (2.2).** — Let $\gamma'$ be the tangent vector field to the $\gamma_1$ curves, with the $\lambda$-parametrization. Then condition A) can be written:

$$\left(DF_1\right)_b \gamma'_a \wedge \left(DF_1\right)_b \gamma_a = 0$$

where $b = (\theta, f^1_z(z), z), x = F_1(\theta, 0, 0)$, and $a = (\theta, z, 0)$ (cf. figure 1).

An easy calculation shows that (1) can be written:

$$\frac{\partial Z_1}{\partial \lambda} \bigg|_b = K(\theta, z) \frac{\partial Z_1}{\partial \lambda} \bigg|_a$$

where $K$ is a strictly positive function. Therefore, we can rewrite A) and B) as:

$$\frac{\partial Z_1}{\partial \lambda} \bigg|_b = K \frac{\partial Z_1}{\partial \lambda} \bigg|_a$$

$$\frac{\partial Z_1}{\partial \lambda} = \sigma_1(\Delta_1).$$

A tedious, but simple calculation, shows that the cubics (see fig. 1):

$$\lambda \rightarrow \lambda, Z_4 = \Delta_i(\theta, z) \left[ \frac{1 + K}{1 + K_0 - 2} \lambda^2 + \left( \frac{K + 2}{K_0 + 1} \right) \lambda^2 + \frac{\lambda}{1 + K_0} + f^2_z(z) \right].$$
satisfy these equations, where
\[ K_0 = \sup K(\theta, z), \quad (\theta, z) \in T^k \times I \times \{1\}. \]

Now, one can write:
\[ \frac{\partial Z_1}{\partial \lambda} = \Delta_1(\theta, z) \ g(\theta, z, \lambda), \]
where \( g > 0 \) on \( T^k \times I \times J \). Also \( \frac{\partial Z_1}{\partial z} > 0 \) on \( T^k \times [0, h] \times J \) for a suitable \( h \), \( 0 < h < 1 \).

Hence the curves \( \gamma_1 \) form a foliation of \( T^k \times [0, h] \times J \). Their image by \( F_1 \) is a foliation (of class \( C^1 \)) of a neighborhood \( V_1 \subset U_1 \), of \( T_{k+1}^k \) in \( A_{k+1} \). This completes the proof of (2.2) (see fig. 2).

We can now define \( G(k+1) \). The submanifolds:
\[ H(z) = \bigcup_{\theta \in T^k} \gamma_1(\theta, z) \times \{ \theta \} \]
are diffeomorphic to \( T^k \times [0, 1] \) and form a foliation of \( T^k \times [0, h] \times J \). Hence their image by \( F_1 \) is a foliation of \( V_1 \) by tori \( G(k+1) \) (figure 2). We now check conditions \( c_2 \) and \( c_3 \).

We have a \( C^2 \) vector field \( \gamma' \) in \( T^k \times [0, h] \times J \); \( \gamma' \) is the tangent field to the \( \gamma_1 \) curves with the \( \lambda \)-parametrization. This field induces a natural action of \( R \) on the exterior products of vector fields, which we note by \( \Gamma(\lambda) : \Gamma(\lambda) \) is the differential of the map induced by \( \gamma' \) of \( T^k \times [0, h] \times \{0\} \) to \( T^k \times [0, h] \times \{\lambda\} \). Then the tangent space to \( H(z) \) at the point \( (\theta, z, \lambda) \) is given by:
\[ \Gamma(\lambda) \left( \frac{\partial}{\partial \theta_1} \wedge \ldots \wedge \frac{\partial}{\partial \theta_k/(\theta, z, \lambda)}, \wedge \gamma'_1(\theta, z, \lambda) \right). \]
Now $F_1$ sends the tori $T^k \times \{z\} \times \{o\}$ to the trivial holonomy liftings of the $T_k$; hence the tangent space to $G(k+1)$ at $p = F_1(\theta, z, \lambda)$ is given by:

$$(g_{k+1})_p = (DF_1 \circ \Gamma(\lambda) \left( \frac{\partial}{\partial \theta_1} \wedge \ldots \wedge \frac{\partial}{\partial \theta_k} \right) \wedge DF_1 \circ \gamma')(\theta, z, \lambda)$$

$$\gamma' = \frac{\partial}{\partial \lambda} + \sigma(\Delta) \frac{\partial}{\partial Z}.$$ 

We recall that:

$$DF_1 \left( \frac{\partial}{\partial Z} \right) = \frac{\partial \theta_1}{\partial Z} N,$$

$$DF_1 \left( \frac{\partial}{\partial \lambda} \right) = \frac{\partial \psi_1}{\partial \lambda} C_{k+1}.$$

Let $\mathcal{D}(\Delta)$ denote a function such that $\frac{\mathcal{D}(\Delta)}{\Delta}$ tends towards a limit $a$; then

$$\mathcal{D}(\Delta) = a \Delta + \sigma(\Delta),$$

(with $a$ not necessarily different from $o$ and $\sigma(\Delta) = \Delta \epsilon(\Delta)$, $\epsilon(\Delta) \to o$ whenever $\Delta \to o$). We take the tangent spaces to the trivial holonomy liftings of the $T_k$, to be given by a section of $E_k(A_k)$, equal to $Y_1 \wedge \ldots \wedge Y_k$ on $T_k$.

Let $C_{k+1}(\lambda)$ denote the action induced by $C_{k+1}$ on the vectors tangent to $A_k$ (if $Y$ is tangent to the $\varphi$-orbits, then so is $C_{k+1}(\lambda)(Y)$). Then we obtain for $g_{k+1}$:

$$(g_{k+1})_p = (C_{k+1}(\lambda) \circ (g_k)|_{(\theta, z, o)} + \mathcal{D}(\Delta) \Omega_p) \wedge (C_{k+1} + \sigma(\Delta) N)_p,$$

where $p = F_1(\theta, z, \lambda)$ and $\Omega$ is a section of $E_{k+1}(A_{k+1})$ defined on $V_1$. We can rewrite this as:

$$C_{k+1} \wedge C_{k+1}(\lambda) \circ g_k + \sigma(\Delta) C_{k+1}(\lambda) \circ g_k \wedge N + \sigma(\Delta) C_{k+1}(\lambda) \wedge \Omega + \sigma(\Delta) N \wedge \Omega.$$

Now $\beta_{k+1} \wedge C_{k+1}(\lambda) \circ g_k$ is zero, since it is a linear combination of exterior products of $n$ vectors tangent to the $\varphi$-orbits. Hence:

$$g_{k+1} \wedge \beta_{k+1} = \sigma(\Delta) C_{k+1}(\lambda) \circ g_k \wedge N \wedge \beta_{k+1} + \sigma(\Delta) C_{k+1} \wedge \Omega \wedge \beta_{k+1} + \sigma(\Delta) N \wedge \Omega \wedge \beta_{k+1}.$$

We have:

$$C_{k+1}(\lambda) \circ g_k \wedge N \wedge \beta_{k+1} = Y_1 \wedge \ldots \wedge Y_k \wedge N \wedge Y_{k+1} \wedge \ldots \wedge Y_{n-1}$$

on $T_{k+1}$. Hence for all points $p$ in a neighborhood $V_2$ of $T_{k+1}$, we have:

$$|C_{k+1}(\lambda) g_k \wedge N \wedge \beta_{k+1}|_p > a > o.$$

We want to show $(g_{k+1} \wedge \beta_{k+1})_p > o$, for $p$ in a suitable neighborhood of $T_{k+1}$ in $A_{k+1}$. Dividing by $\sigma(\Delta)$ ($\pm o$ if $z \pm o$):

$$\frac{g_{k+1} \wedge \beta_{k+1}}{\sigma(\Delta)} = \rho_k + C_{k+1} \wedge \Omega' \wedge \beta_{k+1} + \mathcal{D}(\Delta) N \wedge \Omega \wedge \beta_{k+1},$$

where $|\rho_k|_p > a > o$ in $V_2$, and $\Omega'$ is a bounded section on $V_2 - T_{k+1}$, $\mathcal{D}(\Delta) \to o$ as $\Delta \to o$. Since $C_{k+1} = Y_{k+1}$ on $T_{k+1}$, the second term is less than $a/3$ if $p$ is in some
neighborhood \( V_3 \) of \( T_{k+1} \). Also \( |\sigma(\Delta_{k+1})| |N^\Delta \wedge \beta_{k+1}| < \alpha/3 \) if \( p \) is in some neighborhood \( V_4 \) of \( T_{k+1} \). Hence for \( p \in V_2 \cap V_3 \cap V_4 \),

\[
|\sigma(\Delta_{k+1})| |N^\Delta \wedge \beta_{k+1}| > \alpha/3 > 0,
\]

which proves the theorem for \( j = 1 \).

It is useful for the induction to write \( \delta_{k+1} \) in the form:

\[
\delta_{k+1} = \alpha_{k+1} + \sigma(\Delta_1) \Omega''
\]

where \( \alpha_{k+1} \) is a section of \( E_{k+1}(\Lambda_{k+1}) \), defined in a neighborhood of \( T_{k+1} \) and equal to \( Y_{1+k} \wedge \cdots \wedge Y_{k+1} \) on \( T_{k+1} \).

Construction of \( G(k+j+1) \). — Let \( \Delta_{k+j}(s) = s - f^{k+j}(s) \), where \( s \geq 0 \) denotes the normal \( N \)-coordinate.

Lefondamental little lemma:

\[
\lim_{s \to 0} \frac{\Delta_{k+j}}{\Delta_{k+j+1}} \text{ exists.}
\]

Following (1), one may find an homeomorphism \( H : [0, \varepsilon] \to [0, \varepsilon'] \) such that \( H^{-1} f_{k+j} H = \lambda_{k+j} \) and \( H^{-1} f_{k+j+1} H = \lambda_{k+j+1} \) where \( \lambda_{k+j} \) and \( \lambda_{k+j+1} \) are the homotheties the ratio of which are \( \lambda_{k+j} \) and \( \lambda_{k+j+1} \) (recall \( f_{k+j} \) and \( f_{k+j+1} \) are contractions). Define on \([0, \varepsilon]\) a metric \( \delta \) such that \( \delta(x, x') = |H^{-1}(x) - H^{-1}(x')| \) — this metric is topologically equivalent to the classical one — and \( \delta(x, 0) \to 0 \) whenever \( x \to 0 \). We prove \( \frac{\delta(x, f_{k+j}(x))}{\delta(x, f_{k+j+1}(x))} \) has a limit when \( x \to 0 \) (with respect to \( \delta \)); we shall then be over. Then let \( f = f_{k+j} \), \( g = f_{k+j+1} \).

\[
\rho = \frac{\delta(x, f(x))}{\delta(x, g(x))} = \frac{1 - \lambda_{k+j}}{1 - \lambda_{k+j+1}} \times H^{-1}(x)
\]

when \( x \to 0 \), \( \delta(x, 0) \to 0 \) and \( \rho \to \frac{1 - \lambda_{k+j}}{1 - \lambda_{k+j+1}} \).

Our inductive hypothesis asserts the existence of the foliation \( G(k+j) \) and a section \( \psi_{k+j} \) satisfying:

\[
\psi_{k+j} = \alpha_{k+j} + \sigma(\Delta_{k+j}) \Omega,
\]

where \( \alpha_{k+j} \) is a section of \( E_{k+j}(\Lambda_{k+j}) \), defined in a neighborhood \( U \) of \( T_{k+j} \) in \( A_{k+j} \), which is a linear combination of vectors tangent to the \( \varphi \)-orbits, and equal to \( Y_{1+k} \wedge \cdots \wedge Y_{k+j} \) on \( T_{k+j} \). \( \Omega \) is a section of \( E_{k+j}(\Lambda_{k+j}) \) defined in \( U \). Henceforth, we work in \( U \).
Since the $G(k+j)$ form a foliation of $U$ transverse to the normals, we can construct, by the holonomy, a map $F_{j+1}$ satisfying:

- $F_{j+1}$ sends $T^{k+j} \times I \times J$ to $U$ and is of maximal rank;
- $F_{j+1}(0, z, o) = N(z, x_0)$;
- $F_{j+1}$ sends the tori $T^{k+j} \times \{z\} \times \{o\}$ to the tori $G(k+j)$ passing by $F_{j+1}(0, z, o)$;
- restricted to $T^{k+j} \times \{o\} \times J$, we have:
  \[
  DF_{j+1} \left( \frac{\partial}{\partial x_i} \right) = Y_i, \quad 1 \leq i \leq k+j,
  \]
  \[
  DF_{j+1} \left( \frac{\partial}{\partial z} \right) = Y_{k+j+1};
  \]
  \[
  DF_{j+1} \left( \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_{k+j}} \right) = g_{k+j} \text{ in } A_{k+j};
  \]

$F_{j+1}$ is the holonomy lifting, restricted to the plaques $\{0\} \times I \times J$, i.e. $(F = F_{j+1}) F(\theta, z, \lambda)$ is the endpoint of the holonomy lifting of the path $Y_{k+j+1}([0, \lambda], F(0, 0, 0))$ to the point $F(\theta, z, o)$.

Exactly as the case $j'=i$, we have a family of curves $\gamma_{j+1}(\theta, z)$ satisfying the conditions $A)$ and $B)$, with $F_i$ replaced by $F = F_{j+1}$. This gives us a foliation of $T^{k+j} \times I \times J$ by submanifolds diffeomorphic to $T^{k+j} \times I$, and closing the cylinders by $F$ we obtain a foliation by tori $G(k+j+1)$. We must verify $c)$.

Let $g = g_{k+j+1}$, $C = C_{k+j+1}$, $\Delta = \Delta_{k+j+1}$ and $C' = C'_{k+j+1}$. Then we have:

\[
\begin{align*}
g &= (C + \sigma_1(\Delta)N) \wedge (C'(\lambda) \circ g_{k+j} + \sigma(\Delta)\Omega') \\
&= C \wedge C'(\lambda) \circ g_{k+j} + \sigma(\Delta)N \wedge \Omega' + \sigma_1(\Delta)N \wedge C'(\lambda) \circ g_{k+j} + \sigma(\Delta)C \wedge \Omega'.
\end{align*}
\]

Now we write $g_{k+j} = \alpha_{k+j} + \sigma(\Delta_{k+j})\Omega$ (a section defined in $A_{k+j}$), to obtain (defined in $A_{k+j+1}$):

\[
g = C \wedge C'(\lambda) \circ \alpha_{k+j} + (C \wedge C'(\lambda) \Omega) \sigma(\Delta_{k+j}) + \sigma_1(\Delta)N \wedge C'(\lambda) \circ \alpha_{k+j} + \sigma(\Delta)\sigma(\Delta_{k+j})\Omega + \alpha(\Delta)\Omega + \sigma(\Delta)N \wedge \Omega'.
\]

Notice that $C \wedge C'(\lambda) \circ \alpha_{k+j}$ is a linear combination of products of order $k+j+1$ of vectors tangent to the leaves, and on $T_{k+j+1}$, it equals $Y_{k+j+1} \wedge Y_{1} \wedge \ldots \wedge Y_{k+j}$.

Composing $g$ with $\beta_{k+j+1}$:

- the term $C \wedge C'(\lambda) \circ \alpha_{k+j} \wedge \beta_{k+j+1} = 0$ since it is a multiple of $n$ vectors tangent to the $\varphi$ orbits;
- $N \wedge C'(\lambda) \circ \alpha_{k+j} = N \wedge Y_{1} \wedge \ldots \wedge Y_{k+j}$ on $T_{k+j+1}$.

Dividing by $\sigma_1(\Delta)$ we obtain:

\[
\frac{g \wedge \beta_{k+j+1}}{\sigma_1(\Delta)} = N \wedge C'(\lambda) \circ \alpha_{k+j} + \frac{\sigma(\Delta_{k+j})}{\sigma_1(\Delta)} C \wedge C'(\lambda) \Omega + \delta(\Delta) \Omega'.
\]
Now, as $s \to 0$, $\Delta_{k+j}/\Delta$ is bounded hence $\sigma(\Delta_{k+j})/\sigma(\Delta)$ is bounded. $\Omega''$ is a bounded section of $E_{k+j+1}(T_{k+j+1})$ and $\mathcal{F}(\Delta) \to 0$ as $\Delta \to 0$. Hence, in a small enough neighborhood of $T_{k+j+1}$, we have $g \wedge \beta_{k+j+1} \neq 0$, which completes the proof of (2.1).

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