DAVID A. KAZHDAN
S. J. PATTERSON

Metaplectic forms


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METAPLECTIC FORMS
by D. A. KAZHDAN and S. J. PATTERSON

INTRODUCTION

The purpose of this work is to study automorphic forms on a class of groups known as metaplectic groups. To give some idea of what these are, let \( k \) be an \( \mathbb{A} \)-field (algebraic number field or field of functions over a finite field) containing the full group of \( n \)-th roots of \( i \), denoted by \( \mu_n(k) \), where \( n \) is coprime to the characteristic of \( k \). Let \( G = \text{GL}_n \) and \( G_\mathbb{A}, G_\mathbb{A} \) be the group of rational points and the associated adelic group respectively. Then there exists a non-trivial extension of groups

\[
(1) \longrightarrow \mu_n(k) \xrightarrow{i} \tilde{G}_\mathbb{A} \xrightarrow{p} G_\mathbb{A} \longrightarrow (1)
\]

which is constructed from analogous local extensions

\[
(1) \longrightarrow \mu_n(k_v) \xrightarrow{i_v} \tilde{G}_v \xrightarrow{p_v} G_v \longrightarrow (1).
\]

In the case \( n = 2 \) these have been known implicitly for a long time but were first made explicit in Weil's memoir [49], although there it is symplectic groups which appear. Theta functions are most naturally regarded in this context. The general case was discovered independently and simultaneously as the result of two different lines of thought. In [25] and [26] T. Kubota constructed quite explicitly the \( n \)-fold extension in the case \( r = 2 \). He had been led to this by the derivation of the law of quadratic reciprocity from the theory of theta functions, or, ultimately from the fact that the 2-fold extension exists and splits over \( G_\mathbb{A} \subseteq G_\mathbb{A} \). He showed instead how the reciprocity law led to the existence of such an extension, and that this splits over \( G_\mathbb{A} \) (this is generally true, not just in the case \( n = 2 \)).

On the other hand R. Steinberg [47] and C. G. Moore [35] investigated the general algebraic problem of determining the central extensions of simply-connected Chevalley groups over arbitrary fields, and they also found metaplectic groups. The relation between the two lines of thought was recognized quite quickly, c.f. the remarks in [2] where the relation to the "congruence subgroup problem" is also discussed. This direction of investigation was completed by Matsumoto [32], who gave the general construction of the metaplectic extension of a Chevalley group; recently a yet more general and intrinsic construction has been given by P. Deligne [5].
Once these groups have been constructed and one has the splitting referred to above, viz,

\[(1) \rightarrow \mu_{\mathbb{A}}(k) \rightarrow \tilde{G}_A \rightarrow G_A \rightarrow (1),\]

one is naturally led to look at automorphic forms (in the sense of [23] § 10, although some further explanation is needed). These we call metaplectic forms. In this respect there are two features of the "classical" theory which can guide us. One is the theory of theta functions, which form a very exceptional class of automorphic forms. We can, and do, seek analogues of these; the construction of such analogues is one of the principal objectives of this work. This goal was first proposed by Kubota (cf. [26]). In a sense it was achieved completely in one very special case \((n = 3, r = 2, k = \mathbb{Q}(\sqrt{-3})\) in [37]. A new and more conceptual treatment of the more general case \((n = 3, r = 2)\) was given by Deligne [6], who followed Gelbart, Piatetski-Shapiro et al. [9], [10], [11] in taking the property of a representation being distinguished (i.e. having unique Whittaker model) as the fundamental one which relates local and global representations. They had used this concept to great effect in studying the classical theta-functions from a representation-theoretic point of view. Deligne showed that in the special case \((n = 3, r = 2)\) there is one factor of a reducible principal series representation (locally) which has a unique Whittaker model. He then constructed a global representation by means of residues of Eisenstein series (as was originally proposed by Kubota [26]) and he showed that the local factors of these were just the representations which he had considered locally. The uniqueness of local Whittaker models then implied that the "Fourier coefficients" of a global form have a product structure; indeed they turn out to be cubic Gauss sums. This type of argument first appears in [23].

This is the problem which we shall study in general, and we shall extend the above argument to as general a case as we can. It turns out that one can construct certain local representations which have several interesting properties, as quotients of reducible principal series representations. One can show that they have a unique Whittaker model only when \(r = n - 1\) or \(n\); it is for this reason that we find ourselves compelled to look at groups of higher rank.

In the case \(n = 3, r = 2, k = \mathbb{Q}(\sqrt{-3})\) considered in [37] the "L-series" of the metaplectic form was equal to the "Fourier coefficient" of an Eisenstein series. The construction in [37] was carried out by making this identification and applying a "converse theorem". The same kind of identity still holds; that is, the "L-series" of the form with \(r = n - 1\) is the "Fourier-coefficient" of an Eisenstein series for \(r = 2\); unfortunately our formalism does not allow us to state this so directly, but it is contained in Theorems II.2.3 and I.4.2. In the case \(r = n\) analogous considerations suggest
that the metaplectic form is associated with a \( \zeta \)-function in the same way that \( \sum_{n \in \mathbb{Z}} \frac{\varepsilon(n)}{n} \) is associated with the Riemann \( \zeta \)-function.

The other "classical" aspect of the theory of forms of half-integral weight is the Shimura correspondence. This is a correspondence (both local and global) between representations of \( \tilde{G}_F \) (\( F \) local) or automorphic representations of \( \tilde{G}_A \) and those of \( G_F \) or \( G_A \) whose characters satisfy certain symmetry conditions. This has been much discussed in the case \( n = 2, \ r = 2 \), and it has been constructed for arbitrary \( n \) and \( r = 2 \) by Flicker [7]. One hopes that in view of the progress on the Selberg trace formula made recently by J. Arthur and others it will be possible in the future to prove this in general. In any case the formulation in [7] makes it quite clear what one should expect and we hope to discuss this in detail in a separate publication. Such a result would be significant for the theory of representations of metaplectic groups on three grounds. Firstly, as described in § 1.5, one can use a knowledge of the characters of the local representations considered here to understand their Whittaker models. Secondly, similar considerations lead to a general and "universal" formula for the "number" of Whittaker models of a general local cuspidal representation of a metaplectic group. Finally, it suggests the existence of certain very special cuspidal representations, both locally and globally, when \( r = n \). In the special case \( r = n = 2 \) these would be the cuspidal \( r^2 \) of \([11]\); compare also the remarks in [7] §§ 2.2, 5.2.

This work is organized into three chapters. Chapter 0 contains essentially known material—namely a discussion of the construction of the metaplectic group, both locally and globally, and of the representation theory of Heisenberg groups. Chapter I is devoted to the local theory, mainly the construction of the representations alluded to above and the investigation of their properties. The techniques are by now fairly standard (cf. [3], [46]) but some of the results are unexpected. In the case \( r = 2 \) similar investigations have been carried out by H. Aritürk ([11], \( n = 3 \)), C. Moen ([34], \( n \) general)—we have not seen this work—and P. Deligne ([6]). We do not presume to undertake a full study of the representation theory of metaplectic groups.

In Chapter II we develop the global theory. The local representations of Chapter I are shown to be the local factors of certain automorphic representations. This is achieved by an application of the theory of Eisenstein series. The final results are to be found in § II. 2. The global considerations also allow us to complete the local theory in some essential points. In § II. 3 we make our results more explicit in order to bring out their arithmetical content.

In [39] the second author promised a work with the same title as this. The original manuscript has now become obsolete. Most of the material referred to in [39] can be found here, especially in § II. 3.

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I. The metaplectic group

For any field $F$ let 

$\mu_n(F) = \{ x \in F^\times : x^n = 1 \}$.

In this paragraph we shall consider a local field $F$ such that 

$\text{Card}(\mu_n(F)) = n$.

We shall summarize here the construction of the metaplectic cover of $GL_n(F)$. 

Let $(\cdot, \cdot)$ be the $n$-th order Hilbert symbol on $F$ (see [2] or [50] XIII-5); it is a map $(\cdot, \cdot) : F^\times \times F^\times \to \mu_n(F)$ satisfying

$$
\begin{align*}
(a,b).(a',b) &= (aa',b), \\
(a,b)(b,a) &= 1, \\
(a,1-a) &= 1
\end{align*}
$$

and 

$$
\{ a : (a,x) = 1 \text{ for all } x \in F^\times \} = F^\times \n
$$

where $F^\times \n = \{ a^n : a \in F^\times \}$.

The group which we would like to construct is a central extension 

$$(1) \longrightarrow \mu_n(F) \longrightarrow \tilde{GL}_n(F) \longrightarrow GL_n(F) \longrightarrow (1)$$

but there are several related but non-isomorphic such extensions which we shall later have to examine. We shall first construct one such extension along with a section $GL_n(F) \rightleftharpoons \tilde{GL}_n(F)$, so that the extension will be described by a 2-cocycle $\sigma$. At this point we should remark that the corresponding extension of $SL_n(F)$ is essentially unique (cf. [33], [35] or [47]) but that this not true of $GL_n(F)$. Once we have described the construction of $GL_n(F)$ we will construct various "twisted" forms of it.

To construct $\tilde{GL}_n(F)$ we shall adapt the construction of $\tilde{SL}_n(F)$ given by Milnor in [33] Ch. 12. Note that $GL_n(F)$ can be regarded as a subgroup of $SL_{n+1}(F)$ by 

$$
g \mapsto \begin{pmatrix} \det(g)^{-1} & 0 \\
0 & g \end{pmatrix},
$$

which shows that, by restriction, one can construct $\tilde{GL}_n(F)$ once one has constructed $\tilde{SL}_{n+1}(F)$. We first set up some notations. Let $G = GL_n(F)$, and let $H$ be the subgroup of diagonal matrices. If $h_i \in F^\times (1 \leq i \leq r)$ let $\text{diag}(h)$
be the diagonal matrix \((\delta_i h_i)\). Let \(W\) be the group of permutation matrices; i.e. those matrices such that each row or column has just one non-zero entry, which is 1. Thus \(W \cong S_r\), the symmetric group on \(r\) symbols. We let \(\Phi\) be the set of roots of \(G\), which we shall often consider as the set of pairs \((i,j), \ i \leqslant i, \ j \leqslant r, \ i \neq j\). If \(h = \text{diag}(h_i)\), \(a = (i,j)\) we write
\[
h^a = h_i/h_j;
\]
and this definition can be extended to the lattice spanned by \(\Phi\) (root lattice). Let now \(N_+\) be the group of unipotent upper-triangular matrices, \(N_-\) the group of unipotent lower-triangular matrices, and \(\Phi^+\) the set of positive roots with respect to \(N_+\). We shall write \(\alpha > 0\) if \(\alpha \in \Phi^+\) and \(\alpha < 0\) if \(-\alpha \in \Phi^+\). We shall write \(\langle , \rangle\) for the Killing form on \(\Phi\).

If \(\alpha \in \Phi\) let \(N_\alpha\) be the corresponding subgroup of unipotent matrices. So, if \(n \in N_\alpha\), \(n = I + \xi e_a\) where \(\xi \in F\) and \(e_a\) is the elementary matrix with 1 at the \(a\)-th position and 0 elsewhere. Thus \(hn^{-1} = I + h^a \xi e_a\).

The group \(W\) acts on \(H\) by \(h \mapsto h^w = w^{-1}hw\), and on \(\Phi\) by \(\alpha \mapsto w\alpha\) where \((h^\alpha)^{\omega} = h^{\omega(\alpha)}\). Clearly \(e_{wx} = we_a w^{-1}\).

We begin by constructing a central extension with a preferred section
\[
0 \longrightarrow \mu_n(F) \longrightarrow \tilde{H} \xrightarrow{\rho} H \longrightarrow (1)
\]
by the 2-cocycle
\[
\sigma(h, h') = \prod_{i < j} (h_i, h'_j) \quad h = \text{diag}(h_i), \quad h' = \text{diag}(h'_i).
\]
Note that, if \(h, h' \in H\) and \(\tilde{h}, \tilde{h}' \in \tilde{H}\) are such that \(\rho(\tilde{h}) = h, \ \rho(\tilde{h}') = h'\), then
\[
\tilde{h} \tilde{h}' \tilde{h}^{-1} \tilde{h}'^{-1} = i(\prod_{h} (h, h')^{-1} (\det(h), \det(h'))).
\]

One can verify that the cohomology class of \(\sigma\) is invariant under \(W\) (although the cocycle \(\sigma\) itself is not). We extend \(\sigma\) now to the group \(M = HW\) by defining
\[
\sigma(w, w') = 1 \quad (w, w' \in W) \\
\sigma(h, w) = 1 \quad (w \in W, \ h \in H) \\
\sigma(w, h) = \prod_{(i,j) \in \Phi(w)} (h_i, h'_j)^{-1} (\det(w), \det(h)) \quad (w \in W, \ h \in H)
\]
where \(\Phi(w) = \{ \alpha \in \Phi^+ : w(\alpha) < 0\}\). Then
\[
\sigma(h_1 w_1, h_2 w_2) = \sigma(h_1, h_2^{\omega(\alpha)}) \sigma(w_1, w_2)
\]
defines a 2-cocycle on \(M\), and a corresponding extension \(\tilde{M}\) and a section \(s_\varphi : M \rightarrow \tilde{M}\).
The construction of $\tilde{G}$ is now achieved by means of the beautiful technique of Matsumoto. Let $R : G \rightarrow M$ be such that $R(m) = m$ ($m \in M$) and $R(ngn') = R(g)$ ($n, n' \in N_+$); this exists and is unique, as one sees from the Bruhat decomposition. Let

$$X = \{(g, \tilde{m}) : g \in G, \tilde{m} \in \tilde{M}, R(g) = \rho(m)\},$$

where $\rho : \tilde{M} \rightarrow M$ is the natural projection.

We shall construct two groups of automorphisms of $X$, $LAut(X)$ and $RAut(X)$, which will be generated by certain elements which we now describe. First $LAut(X)$ will be generated by

$$\lambda(n) (n \in N_+) \text{ where } \lambda(n) (g, \tilde{m}) = (ng, \tilde{m}),$$

$$\lambda(h) (h \in \tilde{H}) \text{ where } \lambda(h) (g, \tilde{m}) = (\rho(h) g, h\tilde{m})$$

and $\lambda(s)$ ($s$ a simple reflection in $W$), which we now specify. (Recall that $s$ is simple if there is just one $\alpha \in \Phi^+$ such that $s\alpha < 0$.) If $m \in M$ let $\tilde{m} = \mathbf{s}_\alpha(m) \in \tilde{M}$. Then $\lambda(s)$ is defined to be

$$\lambda(s). (g, \tilde{m}) = (sg, (R(\rho(g)) R(g)^{-1}) \sim \tilde{m}).$$

It is clear that $LAut(X)$ acts transitively on $X$. Now define $RAut(X)$ which is generated by

$$\lambda^*(n) (n \in N_+) \text{ where } \lambda^*(n) (g, \tilde{m}) = (gn, \tilde{m}),$$

$$\lambda^*(h) (h \in \tilde{H}) \text{ where } \lambda^*(h) (g, \tilde{m}) = (g\rho(h), \tilde{m}h)$$

$\lambda^*(s)$ ($s$ a simple reflection) where $\lambda^*(s) = (gs, \tilde{m}(R(g)R(g)^{-1}) R(g) \sim^{-1})$; $RAut(X)$ is also a transitive group of transformations of $X$. One must next verify that if $g \in LAut(X)$, $g^* \in RAut(X)$, $x \in X$ then

$$(gx) g^* = g(xg^*).$$

Granted that this true, one sees that $LAut(X)$, and $RAut(X)$ act simply transitively on $X$; for if $g_1 x_0 = g_2 x_0$ for some $x_0, g_1, g_2$ then $g_1 x_0 g^* = g_2 x_0 g^*$, and as $RAut(X)$ acts transitively, this means that $g_1 = g_2$. Thus $X$ is a principal homogeneous space of both $LAut(X)$ and $RAut(X)$, which are now seen to be isomorphic. Now as $X \rightarrow G$; $(g, \tilde{m}) \mapsto g$ has fibre isomorphic to $\mu_\alpha(F)$ and as $LAut(X)$ acts transitively on $X$, it follows that $LAut(X)$ is an extension of $G$ by $\mu_\alpha(F)$. Moreover, by construction the subgroup of $LAut(X)$ preserving $\{(\rho(m), \tilde{m}) : \tilde{m} \in \tilde{M}\}$ contains $\tilde{M}$. So $LAut(X)$ is the group we are seeking; and we write $\tilde{G} = LAut(X)$, and

$$(1) \rightarrow \mu_\alpha(F) \rightarrow \tilde{G} \rightarrow G \rightarrow (1).$$

Before going further we shall say a little about the structure of the proof that $g(xg^*) = (gx) g^*$. It suffices to check it for $g$ (resp. $g^*$) a generator of $LAut(X)$ (resp. $RAut(X)$). It is easy to check this in all cases except when $g = s_\alpha'$, $g^* = s_\beta'$ and $s_\alpha'$ (resp. $s_\beta'$) is the simple reflection associated with the simple root $\alpha$ (resp. $\beta$). This
one treats by considering $x = (n, mn_n, \vec{n})$ ($n \in \mathbb{N}_a, n_0 \in \mathbb{N}_b$) and computing; only the cases $\alpha = \pm m\beta$ are difficult, and one merely has to have the patience to check these (cf. [33]). One should note that when $\alpha = -m\beta$ one requires that $\langle x, 1-x \rangle = 1$, and only in this case.

When $r = 2$ one can supplement this construction by describing the cocycle $\sigma$ defining $\tilde{\text{GL}}_2(F)$ explicitly; this was given by Kubota [26], and the following formula is a simpler version of his. It is

$$\sigma(g_1, g_2) = \left( \frac{x(g_1 g_2)}{x(g_1)}, \frac{x(g_1 g_2)}{x(g_2)} \right) \left( \frac{\det(g_1)}{x(g_1)} \right) \left( \frac{\det(g_2)}{x(g_2)} \right)$$

where

$$x \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \begin{array}{l} = c \text{ if } c \neq 0 \\ = d \text{ if } c = 0 \end{array}$$

Now we have described an extension

$$(1) \rightarrow \mu_n(F) \overset{i}{\rightarrow} \tilde{\text{GL}}_r(F) \overset{p}{\rightarrow} \text{GL}_r(F) \rightarrow (1)$$

where the section $s$ is defined by

$$s(nm') = \lambda(n) \vec{m} \lambda(n') \quad (n, n' \in \mathbb{N}_+, m \in M)$$

and $\vec{M}$ is considered as above as a subgroup of $\text{LAut}(X)$. With respect to $s$ this extension is described by a cocycle $\sigma$, which extends the $\sigma$ previously defined on $M$, and which satisfies $\sigma(ng, g'n') = \sigma(g, g')$ ($n, n' \in \mathbb{N}_+$).

The "twisted" extensions, for $e \in \mathbb{Z}/n\mathbb{Z}$,

$$(1) \rightarrow \mu_n(F) \overset{i}{\rightarrow} \tilde{\text{GL}}_r^{(e)}(F) \overset{p}{\rightarrow} \text{GL}_r(F) \rightarrow (1)$$

are defined by a cocycle $\sigma^{(e)}$ where

$$\sigma^{(e)}(g_1, g_2) = \sigma(g_1, g_2) (\det(g_1), \det(g_2))^e.$$  

We shall also write $\tilde{\text{G}}^{(e)}$ for $\tilde{\text{GL}}_r^{(e)}(F)$.

**Remarks.** — It is apposite here to make a simple remark concerning a subtle distinction. Let $\varepsilon : \mu_n(F) \rightarrow \mathbb{C}^\times$ be an embedding. In our applications we can distinguish between two metaplectic coverings

$$(1) \rightarrow \mu_n(F) \rightarrow \tilde{\text{G}}_j \rightarrow G \rightarrow (1) \quad (j = 1, 2)$$

only if the induced extensions

$$(1) \rightarrow \mathbb{C}^\times \rightarrow \tilde{\text{G}}_{j, e} \rightarrow G \rightarrow (1)$$

are inequivalent. This does not depend on the choice of $\varepsilon$. In our case $\tilde{\text{G}}^{(e)}$ and $\tilde{\text{G}}^{(e')}$ are indistinguishable in this sense if and only if

$$2(e - e') \equiv 0 \quad (\text{mod } n).$$

It may or may not happen that the original extensions are equivalent.
The reason for this is the following. Let $\varepsilon_1 : \mu_2(F) \to \mathbb{C}^\times$. Then there exists a function (cf. [49], p. 176)

$$\gamma : F^\times \to \mu_4(\mathbb{C})$$

which factors through $F^\times/F^\times^2$, so that

$$\varepsilon(x, y)_{2, F} = \gamma(xy)/\gamma(x) \gamma(y).$$

Thus $\varepsilon \circ (\cdot, \cdot)_{2, F}$, as a 2-cocycle on $F^\times$ is trivial, whereas $(\cdot, \cdot)_{2, F}$, with values in $\mu_2(F)$ need not be.

In this connection it is worth remarking that J. Klose has shown that $(\cdot, \cdot)_{2, F}$ is trivial in $H^2(F^\times, \mu_2(F))$ if and only if

$$( -1, -1)_{2, F} = 1.$$ 

Now let $B = HN_+$ be the standard Borel subgroup of $G$. $\widetilde{B}(\varepsilon) = (\varepsilon^{(2)})^{-1}(B)$ the group covering $B$ in $\widetilde{G}(\varepsilon)$. Let also

$$Z^{(i)} = \{ \lambda I : \lambda^{-1+2\varepsilon} \in F^\times \}$$

and

$$\widetilde{Z}^{(i)} = (\varepsilon^{(2)})^{-1}(Z^{(i)}).$$

**Proposition 0.1.1.** $\widetilde{Z}^{(i)}$ is the centre of $\widetilde{G}(\varepsilon)$ and of $\widetilde{B}(\varepsilon)$.

**Proof.** Consider first the centre of $\widetilde{B}(\varepsilon)$; this consists of elements of the form $z = s(\lambda I) i(\zeta)$. Then if $h \in H$, $h = \text{diag}(h_k)$ and if $\widetilde{h}$ is such that $\varepsilon^{(2)}(\widetilde{h}) = h$

$$\widetilde{a} h^{-1} \zeta^{-1} = i(\prod(h_k, \lambda)^{-1}. \text{det}(h), \lambda)^{1+2\varepsilon}).$$

By considering the case when all $h_k$ but one are $1$ one sees that the righthand side is identically $1$ if and only if $\lambda^{(1+2\varepsilon)} \in F^\times$. This shows that $\widetilde{Z}^{(i)}$ is the centre of $\widetilde{B}(\varepsilon)$.

If now $z \in \widetilde{Z}^{(i)}$ and if $g \in G$, $\tilde{g} \in \widetilde{G}(\varepsilon)$ with $\varepsilon^{(2)}(\tilde{g}) = g$ then we define a homomorphism $\theta : G \to \mu_2(F)$ by $\tilde{g} z^{-1} = i(\theta(g))$. As it is trivial on $B$, $\theta$ is trivial and hence $\tilde{z}$ is in the centre of $\widetilde{G}(\varepsilon)$.

Now we begin the discussion of the topology on $\widetilde{G}(\varepsilon)$. If $F$ is archimedean then there are two cases. If $F = \mathbb{C}$, $(\cdot, \cdot)$ is trivial and the cover is trivial. We shall not discuss this case further. If $F = \mathbb{R}$ then $n = 2$ and we have constructed the double (spin) cover of $GL_n(\mathbb{R})$; this shall presume known (cf. [33]).

Thus henceforth we shall take $F$ to be non-archimedean. Let $\mathbb{R}_p$ be the ring of integers in $F$, $P_p$ the maximal ideal of $\mathbb{R}_p$. Let $|\cdot|_p$ be the normalized valuation (as in [50] Ch. 1) on $F$. Then one has

**Proposition 0.1.2.** There exists an open subgroup $K \subset GL_n(\mathbb{R}_p)$ on which $\sigma^{(i)}$ splits. If $|n|_p = 1$ we can take $K = GL_n(\mathbb{R}_p)$.

**Proof.** See [35] pp. 54-56.

Let us write $\sigma(g_1, g_2) = \kappa(g_1 g_2)/\kappa(g_1) \kappa(g_2)$ for a splitting with $g_1, g_2 \in K$ (which
we take to be $\text{GL}_r(R_p)$ if $|n|_p = 1$. Note that we can find $K$ as above so that 
$(\det(h_1), \det(h_2)) = 1$ $(h_1, h_2 \in K)$. Hence $\kappa$ also splits $\sigma^{(e)}$.

In the case $r = 2$ Kubota ([26] p. 19) has shown that

$$
k(g) = (\epsilon, d/\det(g)) \quad \left( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0 < |\epsilon|_p < 1 \right)
$$

$$= 1 \quad \left( |\epsilon|_p = 0, 1 \right).$$

Let now $K, K$ be as above and $\kappa : K \to \tilde{\mathbb{G}}^{(e)}$, $g \mapsto i(\kappa(g)) \, s(g)$. Although $\kappa$ is not unique, if $\kappa'_1$ is another such map then $\kappa$ and $\kappa'_1$ are equal on an open sub-
group $K' \subset K$, and $\kappa'_1 = (i, \theta) \cdot \kappa^*$, where $\theta$ is a locally constant homomorphism

$K \to \mu_n(F)$. Thus we can define a topology on $\kappa^*(K) \subset \tilde{\mathbb{G}}^{(e)}$ by giving it the topology

of $K$. Next, if $g \in \tilde{\mathbb{G}}^{(e)}$, the uniqueness of the germ of $\kappa$ shows that there exists a neigh-
bourhood $U$ of $I$ in $K$ so that $g^{-1} \kappa(U) \subset \kappa^*(K)$, and hence we can give $\tilde{\mathbb{G}}^{(e)}$ the struc-
ture of a topological group which agrees with the structure just described on $\kappa^*(K)$. With

described.

This topology one sees in

$$
(1) \to \mu_n(F) \xrightarrow{i} \tilde{\mathbb{G}}^{(e)} \xrightarrow{\rho^{(e)}} G \to (1)
$$

that $\tilde{\mathbb{G}}^{(e)}$ is a Hausdorff group and $\rho^{(e)}$ is a local homeomorphism.

The metaplectic extensions $\tilde{\mathbb{G}}^{(e)}$ can be defined as follows. Let $r' > r$ and
let $t_1, \ldots, t_{r'-r}$ be integers such that $\sum t_j = -1$. Then we have an embedding of $\text{GL}_{r'}$
into $\text{SL}_{r'}$, by

$$
g \mapsto \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & \text{det} \, g & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \text{det} \, g^{r'-r} \end{pmatrix}.
$$

By examining the restriction of $\sigma$ (on $\text{SL}_{r'}$) to the image of $H$, we see that $\sigma$ pulls back
to $\sigma^{(e)}$ with

$$
\epsilon = -1 + \sum_{i < j} t_i t_j
$$

and it is easy to realize every value of $\epsilon$ by a suitable choice of $r', (t_i)$.

We can therefore construct $\kappa$ on $\text{SL}_{r'}$, and pull it back to $\text{GL}_{r'}$. One can verify
that, if $|n|_p = 1$, there is no homomorphism

$$
\text{SL}_{r'}(R_p) \to \mu_n(F)
$$

and hence $\kappa$ on $\text{SL}_{r'}(R_p)$ is uniquely defined by the condition that it should be a lifting. Such a lift we call "canonical".

Before we proceed we observe that

$$
N_+ \to \tilde{\mathbb{G}}^{(e)}; \quad n \mapsto s(n)
$$

is a homomorphism and a continuous section of $\rho^{(e)}$. As such, it is unique. We denote
the image of this map in $\tilde{\mathbb{G}}^{(e)}$ by $N'_+$.
Proposition 0.1.3 — Suppose that $|\pi|_\wp = 1$, and let
\[ \kappa^*: \text{GL}_r(R_\wp) \to \widehat{\mathcal{O}}(\omega) \]
be a canonical lift. Then

i) $\kappa^* | (H \cap K) = s | (H \cap K)$

ii) $\kappa^* | W = s | W$

iii) $\kappa^* | (N_+ \cap K) = s | (N_+ \cap K)$.

Remarks. — 1) i) and ii) (resp. iii)) could have been combined to

$\kappa^* | B = s | B$.

2) Since the three groups generate $\text{GL}_r(R_\wp)$ one sees that $\kappa^*$ is determined by Proposition 0.1.3.

3) It is a consequence of this that the “canonical” lift does not depend on the choice of $\nu', \nu, \ldots, \nu_r$.

Proof. — Consider the restriction of $\kappa^*$ on $\text{SL}_r$, to a copy of $\text{SL}_q(F)$. Since $\kappa^*$ is
uniquely determined on $\text{SL}_q(F)$, this restriction is determined. Moreover if $\text{SL}_q(F)$ is
that copy associated with a positive root $\alpha$ then one sees from an examination of the
construction of $\widehat{\text{SL}}_r(F)$ that $a)$ the restriction of the covering to $\text{SL}_q(F)$ is $\widehat{\text{SL}}_q(F)$ and
$b)$ the restriction of $s$ to $\text{SL}_q(F)$ is the corresponding section for $\text{SL}_q(F)$.

Thus if $h \in H \cap K$ one can write $h$ as a product in $\text{SL}_r(F), \Pi h$, where each $h_j$
lies in such a copy of $\text{SL}_q(F)$. Thus as
\[ \kappa^*(\Pi h_j) = \Pi \kappa^*(h_j) \]
the assertion follows. The same argument is valid on $N_+ \cap K$. This proves (i) and (iii).

The same argument shows that it is only necessary to prove that $\kappa^*(w) = s(w)$
for $w$ a simple reflection as $s | W$ is a homomorphism. We write $w$ as $\delta w_1$ where $\delta \in H$, $\delta^2 = 1$ and $w_1$ belongs to the corresponding copy of $\text{SL}_q(F)$. As $\kappa^*(w) = \kappa^*(\delta) \cdot \kappa^*(w_1)$
and as
\[ \kappa^*(\delta) = s(\delta) \quad \text{(by (i))} \]
\[ \kappa^*(w_1) = s(w_1) \quad \text{(by Kubota's formula)} \]
one has only to verify that
\[ s(w) = s(\delta) \cdot s(w_1) \]
and this follows from the construction of $\sigma$ on $M$ which we have described above. This
proves the proposition.

Observe also that, as $G$ is totally disconnected, there is a continuous section
$G \to \widehat{\mathcal{O}}(\omega)$, but this is not unique. In particular we can choose a continuous 2-cocycle,
$\tau^{(i)}$, cohomologous to $\sigma^{(i)}$, so that $\tau^{(i)} | K \times K = \{1\}$. 


We shall conclude our investigation of the structure of $\tilde{G}^{(e)}$ by considering its conjugacy classes. Let then $C \subseteq G$ be a conjugacy class. Then $(\phi^{(e)})^{-1}(C)$ is a union of conjugacy classes in $\tilde{G}^{(e)}$. If $\tilde{C}_1$ is one of these let $\mu(C) = \{ \zeta \in \mu_n(F) : i(\zeta) \tilde{C}_1 = \tilde{C}_1 \}$; this is a subgroup of $\mu_n(F)$ and

$$(\phi^{(e)})^{-1}(C) = \bigcup_{\zeta \in \mu_n(F) : i(\zeta) \tilde{C}_1 = \tilde{C}_1} i(\zeta) \tilde{C}_1.$$ 

This can be characterized in another way. Let $x \in C$ and let $Z_0(x)$ be the centralizer of $x$ in $G$. Let $\tilde{x} \in (\phi^{(e)})^{-1}(x)$. Define

$$\theta_x : Z_0(x) \to \mu_n(F); \quad z \mapsto i^{-1}(\tilde{x}^x \tilde{x}^{-1})$$

where we use the fact that $G$ acts on $\tilde{G}^{(e)}$ by conjugation. Then $\theta_x$ is a homomorphism and its image is $\mu(C)$.

We shall be interested in those classes $C$ such that $(\phi^{(e)})^{-1}(C)$ consists of $n$ conjugacy classes in $\tilde{G}^{(e)}$, since in any application of the Selberg trace formula these are the only ones that would contribute.

**Proposition 0.1.4.** — Suppose $C \subseteq G$ is a conjugacy class of regular elements. Then $\mu(C) = \{ 1 \}$ if and only if for $g \in G$, $g \in \mu(C)$ if and only if $g \in T_{n}(\gamma \in T_n) \{ \gamma^a : \gamma \in T_n \}$ where $T_n$ is the maximal torus to which $g$ belongs.

**Remarks.** — If $\gamma$ is regular and $g = \gamma^a$ then it does not necessarily follow that $\mu(g^a) = \{ 1 \}$. As an example let $n$ be even, $r = 2$, $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ where $\theta \in F^\times$, $\theta^{n^2} \notin F^{x \times n}$. Then $g = \gamma^a = \begin{pmatrix} \theta^{n^2} & 0 \\ 0 & \theta^{n^2} \end{pmatrix}$ and as $(\theta^{n^2})^{2-1+2 \times 2} \notin F^{x \times n}$ it follows that $\mu(g^a) \neq \{ 1 \}$.

**Proof.** — The proof of this proposition depends on another proposition, which we shall now formulate.

Let $A \subseteq M_r(F)$ be a commutative $F$-algebra of dimension $r$. Then we can represent $A$ as $\bigoplus E_j$ where the $E_j$ are fields and

$$\Sigma[E_j : F] = r.$$ 

If $u, v \in A^\times$ then choose $u', v' \in \tilde{G}^{(e)}$ so that

$$\phi^{(e)}(u') = u, \phi^{(e)}(v') = v.$$ 

Then $u', v', u'^{-1}, v'^{-1}$ lies in $i(\mu_n(F))$, and does not depend on the choice of $u', v'$. We can therefore define

$$[u, v] = i^{-1}(u' v' u'^{-1} v'^{-1}).$$

Let $\alpha : \bigoplus E_j \to A$ be the isomorphism between $\bigoplus E_j$ and $A$.

Let $(\ , \ )_B$ be the $n$-th order Hilbert symbol in $E_j$. 

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Proposition 0.1.5. — With the notations above, let \( u = (u_j), v = (v_j) \in \bigoplus E_j \). Then
\[
[a(u), a(v)]_A = \prod_j (u_j, v_j)_{E_j}^{-1} \det(a(u)), \det(a(v))^{1+2r}
\]

Before we proceed to the proof of this proposition let us show how it leads to the proof of Proposition 0.1.4. Let \( C \) be a regular conjugacy class as in the statement of the proposition. Let \( g \in C \). Then the centralizer of \( g \), say, is a maximal torus in \( G \). As such there exists a sub-\( F \)-algebra \( A \) of \( M_r(F) \) such that \( T_g = \Lambda^x \). The algebra \( A \) itself is isomorphic to \( \bigoplus E_j \) for some fields \( E_j \) with \( \Sigma[E_j : F] = r \), as above. Suppose \( g = \alpha(u) \). Then
\[
\mu(C) = \{ [u, v]_A : v \in \bigoplus E_j^x \}
\]
by the second definition. Thus if \( \mu(C) = \{1\} \) we must have
\[
[u, v]_A = 1
\]
for all \( v \in \bigoplus E_j^x \). Let \( i \) be one of the indices and let \( v = (v_j) \) with \( v_j = 1(j \neq i) \). Then we demand that
\[
(u_i, v_i)^{-1} (\det(u), N_{E_i/F}(v_i)^{1+2r}) = 1.
\]
Equivalently, one has, by a functorial property of the Hilbert symbol ([33], p. 177)
\[
(u_i^{-1}, \det(u)^{1+2r}, v_i)_{E_i} = 1,
\]
or, if \( \lambda = \det(u)^{1+2r} \) we demand that
\[
u_i \in \lambda. E_i^{x_n}.
\]
But since \( \det(u) = \prod_j N_{E_j/F}(v_j) \) we have
\[
\lambda^{(1+2r)-1} \in F^{x_n}.
\]
This means that \( u \) is of the form \( u_i^{\nu_i} \lambda \) with \( \lambda^{(1+2r)-1} \in F^{x_n} \). Conversely, if \( u \) is of this form then \([u, v]_A = 1 \) for all \( v \in \bigoplus E_j^x \). These statements are precisely the assertion of the proposition.

Proof of Proposition 0.1.5. — The proof of this proposition is based on the use of the transfer map in K-theory. We shall reduce the proof to the case that \( A \) is a field. To do this we observe that from the definition of \([\cdot, \cdot]_A \) one has that
\[
[y \gamma^{-1}, y' \gamma^{-1}]_{\gamma \gamma^{-1}} = [u, v]_A
\]
for any \( \gamma \in G \). We shall then choose \( \gamma \) so that \( \gamma A_{\gamma^{-1}} \subset M_{[\gamma_1, \ldots, \gamma_l]}(F) \) where
\[
r_j = [E_j : F],
\]
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\( M_{r_1, \ldots, r_l}(F) \) is the subalgebra of \( \text{M}_r(F) \) consisting of blocks of size \( r_1 \times r_1, r_2 \times r_2, \ldots \) astride the diagonal and \( \alpha(E) \) lies in the \( j \)-th block. Note that

\[ M_{r_1, \ldots, r_l}(F)^r = \text{GL}_{r_1}(F) \times \text{GL}_{r_2}(F) \times \cdots \times \text{GL}_{r_l}(F). \]

Denote by \( \rho \) the partition \((r_1, r_2, \ldots, r_l)\) and by \( G_\rho \)

\[ M_{(r_1, \ldots, r_l)}(F)^\times. \]

Fix \( i, i^*; i \leq i^*, i^* \leq t, i \neq i^* \). Suppose \( g, g^* \in G_\rho \) so that \( g = (g_i), g^* = (g_i^*) \) with \( g_i = 1 \) if \( j \neq i \), and \( g_i^* = 1 \) if \( j \neq i^* \). Let \( \tilde{g}, \tilde{g}^* \in \tilde{G}(c) \) with

\[ p^i(\tilde{g}) = g, \quad p^{i^*}(\tilde{g}^*) = g^*. \]

Define \([g, g^*]\) by

\[ i([g, g^*]) = \tilde{g} \tilde{g}^* \tilde{g}^{*-1} \tilde{g}^{*-1}. \]

Again this does not depend on the choice of \( g, g^* \). The map

\[ \text{GL}_{r_1}(F) \times \text{GL}_{r_2}(F) \rightarrow \mu_t(F); \quad (g, g^*) \mapsto [g, g^*] \]

(where we have identified \( \text{GL}_{r_i}(F) \) as its realization as a subgroup of \( G_\rho \)), is a bilinear map. As such it is a function only of \( \det(g) \) and \( \det(g^*) \). Examining this function on the diagonal subgroup we see that

\[ [g, g^*] = (\det(g), \det(g^*))^{1/2}. \]

This permits us to compute the commutant of \( \alpha(E_i) \) and \( \alpha(E_{i^*}) \) (\( i \neq i^* \)), where we have made use of an evident abuse of language. It therefore remains to compute the commutant of two elements of \( \alpha(E_i^\times) \), and this is essentially the same problem as when \( A \) is a field. We shall, however, proceed slightly differently, and take \( A \) to be of the form \( E \otimes E \otimes E \) where \( E \) is a field. We consider the subgroup

\[ S = \{(x, x', x'') \in E^\times \times E^\times \times E^\times \mid xx' x'' = 1\} \]

of \( A^\times \). This maps into \( \text{SL}_{2; \mathbb{F}_p}(F) \). By a simple observation made in [33] p. 95, to know the commutator on \( \alpha(S) \) determines the commutator we need (Hint: consider the commutator of \( (x, x^{-1}, 1) \) and \( (y, 1, y^{-1}) \)). We shall therefore explain how the commutator can be determined. In doing this we shall use the concept of the functor \( K_2 \), and we shall draw freely on the theory of this functor as developed in [33].

Recall that the Hilbert symbol induces

\[ h_{n, p} : K_2(F) \rightarrow \mu_n(F) \]

such that, if

\[ \{ , \}_{p} : F^\times \otimes_{\mathbb{Z}} F^\times \rightarrow K_2(F) \]

is the natural symbol map, then

\[ h_{n, p}(\{x, y\}) = (x, y)_{n, p}. \]
Let $E/F$ be a finite extension of fields. Then in [33] the transfer map
\[ \text{tr} : K_2(E) \to K_2(F) \]
is defined. It has the property (loc. cit.)
\[ \text{tr} \{ x, y \}_E = \{ x, N_{E/F}(y) \}_F \quad (x \in F^\times, y \in E^\times) \]
where $N_{E/F}$ denotes the usual norm. Since
\[ (x, N_{E/F}(y))_{a,E} = (x, y)_{a,F} \quad (x \in F^\times, y \in E^\times) \]
it follows that the following diagram commutes
\[
\begin{array}{ccc}
K_2(E) & \xrightarrow{\text{tr}} & K_2(F) \\
\downarrow h_{n,E} & & \downarrow h_{n,F} \\
\mu_n(E) & \xrightarrow{=} & \mu_n(F)
\end{array}
\]
To see this, note that $h_{n,F} \circ \text{tr}$ must be a power of $h_{n,E}$ by Moore's theorem ([33] p. 165, [35] Theorem (3.1)). Moreover we need only to prove the assertion for $E/F$ totally ramified or unramified. But in these cases there exist $x \in F^\times, y \in E^\times$ so that $(x, N_{E/F}(y))_{a,F}$ is a primitive $n$-th root of unity, as one can easily verify. From these facts the assertion follows.

Now, as earlier in this section, we can construct metaplectic covers

(I) \[ (1) \to K_2(F) \to \tilde{SL}_{r_1}(F) \to SL_{r_1}(F) \to (1) \]

(II) \[ (1) \to K_2(E) \to \tilde{SL}_{r_2}(E) \to SL_{r_2}(E) \to (1) \]

which are the universal covers if $r_1, r_2 \geq 2$. Starting from an identification $F^{n \otimes_p E} \cong F^{n[E:F]}$ we embed
\[ \alpha : SL_{r_2}(E) \to SL_{r_2(E)[E:F]}(F) \]
from which we obtain the map $\text{tr} : K_2(E) \to K_2(F)$ as the map of fundamental groups.

On the other hand, if $h_1, h_2$ lie in the diagonal torus of $SL_{r_2}(E)$ then their commutator in (II) is
\[ \prod_i \{ h_{1,i}, h_{2,i} \}_E^{-1} \quad (\in K_2(E)) \]
and thus the commutator of $\alpha(h_1)$ and $\alpha(h_2)$ in (I) with
\[ r_1 = r_2[E:F] \]
is, by definition of $\text{tr}$, $\prod_i \text{tr} \{ h_{1,i}, h_{2,i} \}_E^{-1}$.  

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If we now apply $h_{n,F}$ we see that the commutator of $\alpha(h_1)$ and $\alpha(h_2)$ in 

$$(1) \rightarrow \mu_n(F) \rightarrow \SL_{\tau_3(E)}(F) \rightarrow \SL_{\tau_3(E)}(F) \rightarrow (1)$$

is

$$\prod_{i<j} (h_{i,,1} h_{j,2})^{-1}.$$ 

This is (in the special case that $r_3 = 3$) the result that we need. This completes the proof of the proposition.

Remarks. — 1) If $g$ is not regular then the situation is more complicated. Suppose that $n \in C$ where $n \in \mathbb{N}_+$. Then $\mu(C) = \{1\}$. But if $\lambda n \in C$, $\lambda \in F^\times$, $n \in \mathbb{N}_+$, $n \neq 1$ then a much more careful analysis is required. Similar questions arise if $g$ is semisimple but not regular.

2) When $r = 2$ one can easily complete the description given above. One then has that if $\lambda n \in C$, $n \in \mathbb{N}_+ - \{1\}$ then $\mu(C) = \{1\}$ if and only if $\lambda^{(1+\omega)} \in F^\times$, but if $\lambda I \in C$ then $\mu(C) = \{1\}$ if and only if $\lambda^{1+\omega} \in F^\times$.

2. The global metaplectic group

In this section we describe the construction of the adelic metaplectic groups associated with $GL_n$ over a global field $k$. This is fairly standard, but it will be useful to set it down here for future reference.

We shall first fix some notation. Let $k$ be a global field and for each place $v$ of $k$ let $k_v$ be the corresponding closure. Let $G = GL_n$, and $G_v = G(k_v)$, $G_k = G(k)$. Let $r_v$ be the ring of integers of $k_v$, $m_v$ its maximal ideal and $q_v = \text{Card}(r_v/m_v)$. Suppose that

$$\text{Card}(\mu_n(k)) = n.$$ 

Let now 

$$\mathcal{G}_v^{(c)}(S) = \left( \prod_{v \in S} \mathcal{G}_v^{(c)} \times \prod_{v' \notin S} \mathcal{K}_v \right)/M$$

be the extension constructed in § 0.1. Here $\mathcal{G}_v$ is a local homeomorphism, a requirement that determines the topology on $\mathcal{G}_v^{(c)}$. For a non-archimedean place $v$ let $K_v \subset G(r_v)$ be a subgroup over which this extension splits, and $K_v = G(r_v)$ for almost all places $v$. We identify $\mu_n(k_v)$ with $\mu_n(k)$. Let us also choose a lift $K_v'$ of $K_v$, which for almost all $v$ should be the canonical lift of Proposition 0.1.3. Let $S$ be a finite set of places containing all the archimedean places. We form

$$\mathcal{G}_k^{(c)}(S) = \left( \prod_{v \in S} \mathcal{G}_v^{(c)} \times \prod_{v' \notin S} \mathcal{K}_v \right)/M$$

where

$$M = \langle \tilde{j}_v \circ i_v(\zeta), j_v' \circ i_v(\zeta)^{-1} | \zeta \in \mu_n(k), v_1, v_2 \in S \rangle,$$

and, if $v' \in S$,

$$\tilde{j}_{v'} : \mathcal{G}_v' \rightarrow \left( \prod_{v \in S} \mathcal{G}_v^{(c)} \times \prod_{v' \notin S} \mathcal{K}_v \right).$$
is the natural injection. Moreover \( \tilde{j}_S \), induces an injection
\[
\tilde{G}_A^{(i)} \to \tilde{G}_A^{(i)}(S)
\]
which we shall also denote by \( \tilde{j}_S \). We give \( \tilde{G}_A^{(i)}(S) \) the obvious topology. Let \( G_A(S) \) be analogous group defined without the metaplectic coverings. Then one has a topological extension of groups
\[
(1) \to \mu_n(k) \to \tilde{G}_A^{(i)}(S) \to G_A(S) \to (1).
\]
Let \( j_S : G_S \to G_A(S) \) be the canonical injection. Then we have the following commutative diagram:
\[
(1) \to \mu_n(k) \to \tilde{G}_A^{(i)}(S) \to G_A(S) \to (1)
\]
Moreover if \( S' \supset S \) then we have an analogous diagram:
\[
(1) \to \mu_n(k) \to \tilde{G}_A^{(i)}(S') \to G_A(S') \to (1)
\]
where \( \tilde{j}_{S',S} \) and \( j_{S',S} \) are the natural injections. We can thus form \( \tilde{G}_A^{(i)} = \lim_{\to} \tilde{G}_A^{(i)}(S) \) and \( G_A = \lim_{\to} G_A(S) \) with the appropriate topologies. Thus one has continuous injections \( j_S, j_S \) and
\[
(1) \to \mu_n(k) \to \tilde{G}_A^{(i)}(S) \to G_A \to (1)
\]
\[
(1) \to \mu_n(k) \to \tilde{G}_A^{(i)}(S') \to G_A \to (1)
\]
Here \( G_A \) is the usual adelic group and \( \tilde{G}_A^{(i)} \) is endowed with the finest topology with respect to which all the \( j_S \) are continuous. Hence \( \tilde{G}_A^{(i)} \to G_A \) is a local homeomorphism, a stipulation which also determines the topology on \( \tilde{G}_A^{(i)} \). There is one further fact which will be crucial to us. Let \( G_k \to G_A \) be the diagonal embedding. We shall show that this lifts to \( G_k \to \tilde{G}_A^{(i)} \); thus
\[
(1) \to \mu_n(k) \to \tilde{G}_A^{(i)} \to G_A \to (1)
\]
Let, for any place \( v \), \( s_v : G_v \to \tilde{G}_v^{(e)} \) be the original section with respect to which \( \sigma^{(e)} \) (over \( k_v \)) was defined. We shall define a section \( s_0 : G_k \to \tilde{G}_k^{(e)} \) such that \( s_0 \mid B(k) \) is an injective homomorphism. From the theory of [33] it will then follow that the restriction of the covering to \( G_k \) is trivial, and this will also imply that \( s_0 \) is an injective homomorphism.

We shall first define \( s_0 \) on \( B(k) \) and \( W(C G_k) \). Let \( \gamma \) belong to either of these two groups. We let \( S \) be a finite set of places containing all the archimedean places (if there are any) and such that if \( w \notin S \) then \( K_w^* \) is the canonical lift, and \( \gamma \in K_w \). Then we define \( s_0(\gamma) \) to be the image of

\[
\prod_{v \in S} s_v(\gamma) \times \prod_{w \notin S} s_w(\gamma) \in \tilde{G}_k^{(e)} \times \prod_{w \notin S} K_w^*
\]

in \( \tilde{G}_k^{(e)}(S) \). This is well-defined by Proposition o.1.3.

By construction

\[ s_0(\gamma_1 \gamma_2) = s_0(\gamma_1) \cdot s_0(\gamma_2) \quad (\gamma_1, \gamma_2 \in W). \]

Let us verify this also for \( \gamma_1, \gamma_2 \in B(k) \). If we write \( \gamma_i \) as \( h_i n_i \) with \( h_i \in H(k) \), \( n_i \in N_+(k) \) then (with \( h_i = \text{diag}(h_i) \))

\[ s_0(\gamma_1 \gamma_2) = s_0(\gamma_1) \cdot s_0(\gamma_2) \cdot i_* ( (h_{1,a}, h_{2,b})_* \langle \det(h_1), \det(h_2) \rangle^*_e ) \]

Thus if \( S \) is chosen to be sufficiently large

\[ s_0(\gamma_1 \gamma_2) = s_0(\gamma_1) \cdot s_0(\gamma_2) \quad (w \notin S) \]

whereas \( \prod_{v \in S} ( (h_{1,a}, h_{2,b})_* \langle \det(h_1), \det(h_2) \rangle^*_e ) = 1 \) by the reciprocity law. Hence in \( \tilde{G}_k^{(e)}(S) \) one has

\[ s_0(\gamma_1 \gamma_2) = s_0(\gamma_1) \cdot s_0(\gamma_2). \]

Now let \( g \in G_k \) be arbitrary. We can write \( g \) as \( b \cdot w \cdot n \) (\( b \in B(k) \), \( w \in W \), \( n \in N_+(k) \)) and can define \( s_0(g) \) by

\[ s_0(g) = s_0(b) \cdot s_0(w) \cdot s_0(n). \]

It is easy to see that this does not depend on the choice of \( b \) or \( n \). This therefore defines a section. Finally, either by the theory of [33], \( \S \) 11, or directly, one verifies that if \( \tau : G_k \times G_k \to \mu_k \) is defined by

\[ s_0(g_1 g_2) = i_* ( \tau(g_1, g_2) ) \cdot s_0(g_1) \cdot s_0(g_2) \]

then \( \tau(g_1, g_2) = 1 \).

Remarks. — 1) One can also use the method described by Kubota in [26] to prove these results. In this approach one constructs an adelic 2-cocycle on \( G_k \). If one does this, then it is again plain that the splitting of the restriction of the cover to \( G_k \) is equivalent to the reciprocity law for the Hilbert symbol.
2) We shall denote $s_0(G_k)$ by $G_0^*$ and $s_0(\gamma)$ by $\gamma^*$ ($\gamma \in G_k$). That $s_0$ is not unique will cause us no problems.

To complete this section we must discuss the notion of a smooth function on $\tilde{G}_A^{(g)}$. Let

$$\varepsilon : \mu_\infty(k) \to \mathbb{C}^\times$$

be an injective homomorphism. Let us assume that we are given, for each $v$, a smooth function $f_v$ (i.e. smooth at the archimedean places and locally constant at the non-archimedean places) such that

$$f_v(\zeta g) = \varepsilon(\zeta) f_v(g) \quad (\zeta \in \mu_\infty(k)),$$

and such that the following condition holds: There exists a finite set of places $S$ containing all the archimedean places and such that if $w \notin S$ then a) $K_*^w$ is the canonical lift, and b)

$$\text{Supp}(f_v) = K_*^w,$$

$$f_v(gh) = f_v(h) \quad (h \in K_*^w),$$

and

$$f_v(I) = 1.$$ Then $\otimes f_v$ is defined to be the function on $\tilde{G}_A^{(g)}(S)$ for any $S \supset S$, such that on the class of

$$(g_v) \times (g_w) \in \prod_{v \in S'} \tilde{G}_v^{(g_v)} \times \prod_{w \notin S'} K_0^w,$$

$\otimes f_v$ has the value $\prod_{v \in S'} f_v(g_v)$. This is well-defined. We regard the tensor product as being with respect to $\mathbb{C}[\mu_\infty(k)]$. The space of smooth functions on $\tilde{G}_A^{(g)}$ will then be the space of finite linear combinations of such functions.

We note here that we shall also have to consider other spaces of functions on $\tilde{G}_A^{(g)}$, but we shall postpone their introduction until they are needed.

3. Heisenberg groups

In this section we summarize some of the basic facts about a certain type of group extension, usually known as Heisenberg groups. Most of the material discussed here is known, but again we have collected together those aspects which will be useful to us. An account of much of the basic theory is given by Weil in [49], Ch. I, but as he has a different type of application in mind, it cannot be used here as it stands.

To define a Heisenberg group, we suppose we are given locally compact abelian groups $G, A$ and a bicharacter $(\ , \ ) : G \times G \to A$, i.e. a continuous function such that $(xx', y) = (x, y)(x', y)$ and $(x, yy') = (x, y)(x, y')$. We shall also assume that $A$ is cyclic in the sense that there exists a continuous faithful unitary representation $\varepsilon : A \to \mathbb{C}^\times$, which we consider fixed. In our case $A$ will generally be a finite group, $G$ will be a torus in GL and $(\ , \ )$ will be derived from some Hilbert symbol. In the case considered
in [49], \( G \) was a vector space (or its adelization), \( A = \{ z \in \mathbb{C} : |z|^2 = 1 \} \) and \((x, y) = e^{2\pi i(x, y)}\) where \( B \) is a bilinear form.

Now, \((\cdot, \cdot)\) is a 2-cocycle on \( G \) and hence there is a locally compact group \( G \) and a central extension

\[
(1) \longrightarrow A \overset{i}{\longrightarrow} \tilde{G} \overset{\rho}{\longrightarrow} G \longrightarrow (1)
\]

where \( \tilde{G} \) is set-theoretically \( G \times A \) and endowed with the multiplication

\[
(g, a) \cdot (g', a') = (gg', (g, g') \cdot a a').
\]

The group \( \tilde{G} \) takes the product topology, and all the maps are continuous, as are multiplication and inversion in \( \tilde{G} \). Then \( \tilde{G} \) is called a Heisenberg group.

Let us note that \( G \) is abelian if and only if \((\cdot, \cdot)\) is symmetric; i.e. if \((g_1, g_2) = (g_2, g_1)\).

The extension is trivial if there is a continuous map \( \varphi : G \to A \) for which

\[
(g_1, g_2) = \varphi(g_1, g_2) \varphi(g_1)^{-1} \varphi(g_2)^{-1},
\]

and which therefore satisfies for \( g_1, g_2, g_3 \in G \)

\[
\varphi(g_1, g_2) \varphi(g_3) \varphi(g_3) = \varphi(g_1, g_2) \varphi(g_2, g_3) \varphi(g_3, g_1) \varphi(1).
\]

Let now, for \( g_1, g_2 \in G \),

\[
\{g_1, g_2\} = i^{-1}(\tilde{g}_1 \tilde{g}_2 \tilde{g}_1^{-1} \tilde{g}_2^{-1})
\]

where \( \rho(\tilde{g}_i) = g_i \). This is well-defined and one has

\[
\{g_1, g_2\} = \{g_1, g_2\} \{g_2, g_1\}^{-1}.
\]

This is a skew-symmetric bicharacter. Let \( G_0 = \{ g \in G : \{g, g'\} = 1 \text{ for all } g' \in G \} \). Now call a subgroup \( H \subseteq G \) isotropic if \( \{h, h'\} = 1 \) for \( h, h' \in H \), and maximal isotropic if it is maximal with respect to this property. Otherwise expressed, \( H \) is such that \( \tilde{H} = \rho^{-1}(H) \) is a maximal abelian subgroup of \( G \). Observe that \( H \) is closed in \( G \), since its closure would have the same properties.

**Examples.** — 1) An interesting example of a group formed in this way is \( \mathbb{Z}/4\mathbb{Z} \) as

\[
o \rightarrow \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to o.
\]

Here \( G = \mathbb{Z}/2\mathbb{Z} \), \( A = \mathbb{Z}/2\mathbb{Z} \) and \((x, y) = 0 \) if \( x \) or \( y \) is \( 0 \) and \((1, 1) = 1 \). In this case \( G = \mathbb{Z}/4\mathbb{Z} \) is abelian, but is not isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Naturally \( (\cdot, \cdot) \) is trivial and \( H = G \).

2) Let \( F \) be a non-archimedean local field in which \(-1 \) is not a square. Let \((\cdot, \cdot)\) be the 2-Hilbert symbol; then \((\cdot, \cdot)\) is a bicharacter as above ([50], XIII, § 5). In [49], p. 176, Weil has shown that there exists a function \( \epsilon_a \) \((a \in F^c)\) with values in the fourth roots of \( 1 \) so that

\[
\epsilon_{ab} \epsilon_a^{-1} \epsilon_b^{-1} = (a, b); \quad \epsilon_a = 1.
\]
From this, \( \epsilon(a, b) = (a, -1) \) and so \( \epsilon \) does not only take the values \( \pm 1 \). In this case \( G_0 = G \). If, moreover, \((-1, -1) = -1\) (which happens when \( F = \mathbb{Q}_p \), \( (\epsilon(-1))^2 = -1 \) so that \( (a, b) \) cannot be written as \( \varphi(ab)/\varphi(a) \varphi(b) \) for any \( \varphi : \Gamma^x \to \mu_q(F) \). Thus \( \widetilde{G}_0 \) is not a trivial covering of \( G_0 \).

Let us now write, for any locally compact abelian group \( \Gamma, \Gamma^x \) for the group of characters, with its usual topology, and let \( \langle \cdot, \cdot \rangle : \Gamma^x \times \Gamma \to \mathbb{C} \times \mathbb{C} \) be the natural pairing between \( \Gamma^x \) and \( \Gamma \). Let now \( H \) be maximal isotropic in \( G \); then set

\[
\alpha : (H/G_0) \to (G/H)^\times; \quad h \mapsto \epsilon([h, \cdot])
\]
\[
\beta : (G/H) \to (H/G_0)^\times; \quad g \mapsto \epsilon([g, \cdot])
\]

and one sees, by the conditions on \( H \), that \( \alpha(H/G_0) \) separates points in \( G/H \), and \( \beta(G/H) \) separates points in \( H/G_0 \). Moreover

\[
\langle \alpha(h), \beta(g) \rangle_{G/H} = \langle \beta(g), \alpha(h) \rangle_{H/G_0},
\]

so that \( \alpha, \beta \) are the transposes of one another, and are injective. Thus \( \alpha, \beta \) are actually isomorphisms, cf. \[48\], § 28.

Likewise one can identify \( G/G_0 \) and \( (G/G_0)^\times \).

Now let \( \widetilde{G}_0 = \rho^{-1}(G_0) \) (this need not be isomorphic to \( G_0 \times A \), although this is frequently the case). Let \( \omega : \widetilde{G}_0 \to \mathbb{C}^\times \) be a quasicharacter, such that \( \omega \circ \epsilon = \epsilon \) on \( A \). Then there exists a quasicharacter \( \omega' : \widetilde{H} \to \mathbb{C}^\times \) extending \( \omega \), (an easy exercise). If \( \omega'' \) is another such extension of \( \omega \) then \( \omega''/\omega' \) can be regarded as a quasicharacter on \( H/G_0 \) and, in view of our assumption that \( A \) be finite, it is a character and hence of the form \( \epsilon([g, \cdot]) \). Thus if \( \omega' \) is as above, every extension can be written in the form \( \omega' \epsilon([g, \cdot]) \). We shall write \( \epsilon_g \) for \( \epsilon([g, \cdot]) \) henceforth.

Now we turn to the representations of \( \widetilde{G} \). We have to bear some topological considerations in mind. We shall assume now that \( A \) is a discrete group. Let us first describe a construction of certain representations and then we shall indicate the relations between them. Let \( H \) be as above and let \( \omega \) be a quasicharacter \( \widetilde{G}_0 \), and \( \omega' \) an extension of \( \omega \) to \( \widetilde{H} \). Then we consider \( \text{ind}^\widetilde{H}_L(\omega') \) which we denote by \( (\pi_{\omega, \omega', H}, V_{\omega, \omega', H}) \)—the process of induction we have to describe in rather more detail. To do this we recall that if we consider all pairs of subgroups \( (L, L') \) with \( L' \subseteq L \subseteq G \), and such that \( L' \) is compact, \( L \) is generated by a compact neighbourhood of \( I \), and \( L/L' \) is elementary then \( G \) can be "exhausted" by all such pairs (cf. \[49\], No. 11). We choose \( L' \) so that \( (L', g) = 1 \) and \( (g, L') = 1 \) for all \( g \in L \) and we assume also that \( \omega' \) is trivial on \( L' \) considered as a subgroup of \( \widetilde{G}_0 \). Under our assumptions such \( L, L' \) exist exhausting \( G \). We suppose also that \( L \cap H \) exhausts \( H \). Then we consider \( V_{\omega, \omega', H}(L, L') \) to be the space of the functions \( f : \widetilde{L} \to \mathbb{C} \) satisfying

\[
f(hg) = \omega'(h)f(g) \quad (h \in \widetilde{H})
\]

and \( f \) is a Schwarz function on \( L/L' \). Then this is naturally endowed with the Schwarz topology and the injective limit over the \( (L, L') \) yields a space \( V_{\omega, \omega', H} \). Compare \[49\].
No. ii. In most cases the construction can be carried out much more easily, but we shall remark on this later. In particular, if $G$ is a $p$-adic group this coincides with the algebraic induction; cf. [49]. For the sake of brevity, we consider $V_{\omega, \omega', H}$ as a space of functions and ignore the topological considerations from now on. As usual, $\pi_{\omega, \omega', H}$ acts on $V_{\omega, \omega', H}$ by right translation.

By an analogue of Mackey's criterion $(\pi_{\omega, \omega', H}, V_{\omega, \omega', H})$ is irreducible. Its class does not depend on the choice of $\omega'$. Indeed, let $\omega''$ be another choice of extension of $\omega$ to $\tilde{H}$; then $\omega'' = e^* \omega'$ and the intertwining map $V_{\omega, \omega', H} \to V_{\omega, \omega'', H}$ is given by $f \mapsto (g \mapsto f(\gamma g))$.

Now we show that in fact it does not depend on the choice of $H$. To this end let $H_1, H_2$ be maximal isotropic subgroups and $\omega_1$ (resp. $\omega_2$) an extension of $\omega$ to $\tilde{H}_1$ (resp. $\tilde{H}_2$). Then let $\tilde{H}_3 = \tilde{H}_1 \cap \tilde{H}_2$. Again every character of $\tilde{H}_3/\mathbb{C}^*$ can be represented as $e_\gamma | \tilde{H}_3$; thus for some $\gamma \in G$

$$\omega_1 | \tilde{H}_1 \cap \tilde{H}_2 = e_\gamma \cdot \omega_2 | \tilde{H}_1 \cap \tilde{H}_2.$$ 

We shall replace $\omega_2$ by $e_\gamma \omega_2$ (making use of the intertwining operator above). Now consider the map

$$F_{21}: V_{\omega, \omega_1, H_2} \to V_{\omega, \omega_2, H_1}, \quad f \mapsto \left( g \mapsto \int_{H_2/\tilde{H}_1 \cap \tilde{H}_2} e_{\omega_1}^{-1}(h) f(hg) \, dh \right)$$

where $dh$ is a Haar measure on $\tilde{H}_2/\tilde{H}_1 \cap \tilde{H}_2$. This integral exists—this was the purpose of the topological considerations above. Analogously one defines $F_{12}: V_{\omega, \omega_1, H_2} \to V_{\omega, \omega_2, H_1}$. These are both $\mathbb{C}$-maps. We need only verify that these two maps are inverses to one another, at least up to scalar multiple. But this is clear as the maps $F_{12}$ and $F_{21}$ are non-trivial and the $V_{\omega, \omega_1, H_j}$ ($j = 1, 2$) are irreducible.

We shall close this section by discussing two minor topics which can be most conveniently accommodated here.

For the first of these we shall show how to construct maximal isotropic subgroups in the cases which will be of interest to us. Let $\Gamma$ be any abelian group and $(\ , \ ) : \Gamma \times \Gamma \to \mathbb{C}^*$ any skew-symmetric bicharacter. Let $G = \Gamma^r$ and, for $c \in \mathbb{Z}$, define $(\ , \ )$ on $G$ by

$$(g, g') = \prod_{i < j} (g_i, g'_i) \cdot (\prod_k g_k, \prod_i g'_i)^c$$

where $g = (g_i), g' = (g'_i)$. Then

$$(g, g') = \prod_i (g_i, g'_i)^{-1} \cdot (\prod_k g_k, \prod_i g'_i)^{1+2c}.$$ 

**Proposition 0.3.1.** Let notations be as above and suppose there are subgroups $\Gamma_1 \subset \Gamma_2 \subset \Gamma$ such that $\Gamma_1 = \{ \gamma \in \Gamma : (\gamma, \Gamma_1) = \{1\} \}$ and $\Gamma_2 = \{ \gamma \in \Gamma : (\gamma, \Gamma_2)^{-1+2c} = \{1\} \} \cap \{ \gamma \in \Gamma : (\gamma, \Gamma_2)^{1-1+2c} = \{1\} \}$. 

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Then \[ H = \{ (\delta_t) : t \in \Gamma_1, \delta \in \Gamma_2 \} \]
is a maximal isotropic subgroup of \( G \).

**Proof.** — We shall first check that \( H \) is isotropic. As \( \{ , \} \) is bimultiplicative we have to check that \( \{ t, t' \} = 1 \) and \( \{ \delta, \delta' \} = 1 \) where \( t = (t_j) \) \( t' = (t'_j) \), \( t_j, t'_j \in \Gamma_1 \), \( \delta = (\delta, \delta, \ldots) \), \( \delta' = (\delta', \delta', \ldots) \), \( \delta, \delta' \in \Gamma_2 \). That \( \{ t, t' \} = 1 \) is clear as \( (\Gamma_1, \Gamma_1) = \{ 1 \} \). Also

\[ \{ t, \delta \} = (\prod_k t_k, \delta)^{r-1+2re} = 1 \]
as \( (\Gamma_1, \Gamma_2)^{r-1+2re} = 1 \). Likewise

\[ \{ \delta, \delta' \} = (\delta, \delta')^{r(r-1+2re)} = 1 \]
as \( (\Gamma_2, \Gamma_2)^{r(r-1+2re)} = 1 \).

Conversely, suppose \( g \in G \), \( \{ g, h \} = 1 \) for all \( h \in H \). Choose \( h \) to be \( (1, \ldots, t, 1, \ldots, 1) \), \( t \in \Gamma_1 \) in the \( j \)-th entry. Then, if \( g = (g_\delta) \)

\[ t = \{ g, h \} = (g_j^{-1}(\prod_k g_k)^{1+2re}, t) \]

Hence \( g_j^{-1}(\prod_k g_k)^{1+2re} \in \Gamma_1 \). Thus we can write \( g_j \) as \( T_j \Delta \) where \( T_j \in \Gamma_1 \), \( \Delta^{r-1+2re} \in \Gamma_1 \). Also, if \( h = (\delta, \delta, \ldots) \), \( \delta \in \Gamma_2 \)

\[ 1 = \{ g, h \} = (\Delta, \delta)^{r(r-1+2re)} \]
The conditions on \( \Delta \) show that

\[ \Delta \in \{ \gamma \in \Gamma : \{ \gamma, \Gamma_1 \}^{r-1+2re} = \{ 1 \} \} \cap \{ \gamma \in \Gamma_2 : \{ \gamma, \Gamma_2 \}^{r-1+2re} = \{ 1 \} \} = \Gamma_2 \].

Thus \( g \in H \), as required.

Finally we shall prove the following lemma.

**Lemma 0.3.2.** — Let \( F \) be a non-archimedean local field and let \( n' = \text{Card} \mu_n(F) \), where \( n \) is a natural number. If \( R_p \) is the ring of integers of \( F \) then

\[ [R_p^n : R_p^s] = n' |n|_p^{-1} \]

and

\[ [F^s : F^{s+n}] = n'n' |n|_p^{-1} \].

**Proof.** — The second assertion is clearly an immediate consequence of the first. To prove the first let \( d^* x \) be a Haar measure on \( R_p^n \). We shall compute the volume of \( R_p^n \). Let \( a : R_p^n \to R_p^n \) be \( a(x) = x^n \). This map is of degree \( n' \) and its Jacobian is \( |n|_p \). Hence the measure of the image of \( a \) is

\[ n'^{-1} |n|_p \text{ meas}(R_p^n) \]

which is equivalent to the first assertion. This proves the lemma.
I. — LOCAL THEORY

In this chapter we shall construct principal series representations of $\mathbf{GL}_n^\vee(F)$ where $F$ is a non-archimedean local field satisfying $\text{Card}(\mu_n(F)) = n$. These are usually irreducible, but our interest will centre on a certain quotient representation of a reducible principal series representation. This representation we shall call "exceptional", for it is closely related to the "Ausnahmefall" of Hecke in [20], No. 42. In the case $n = 1$ such representations are 1-dimensional, and when $n = 2$ they are amongst those associated with theta functions. They are amongst those investigated by Serre and Stark in [42] and Gelbart, Piatetski-Shapiro et al. [9], [10], [11]. The nature of this condition is most naturally expressed in terms of coinvariants (as in [6], 5.4), and by the "periodicity theorem", Theorem 1.2.9 $\varepsilon$). Note that this does not depend on the existence and uniqueness of a Whittaker model of the representation in question.

As was intimated in the introduction, we shall have to understand the Whittaker models of these exceptional representations in order to be able to draw global consequences. They shall be classified in § I.3.

In § I.4 we give some more detailed information about the Whittaker models of these representations and, in particular, we compute the "class-1 Whittaker functions" associated with them.

In § I.5 we establish a connection between the character of a representation and its Whittaker models. In § I.6 we consider briefly the case when $F$ is an archimedean field.

Our notations will be carried over from Chapter o, except that we shall suppress the dependence of the metaplectic groups on the parameter $\varepsilon$. Thus we shall write $\tilde{G}$ for $\tilde{G}^{(0)}$ but it should not be forgotten that the dependence on $\varepsilon$ is important.

In this chapter $\pi \in F$ will be a uniformizer of $F$ and $q = |\pi|_F^{-1}$.

I.1. Principal Series Representations

Let $G$, $H$, $N_+$, $\tilde{G}$, $\tilde{H}$, $N'_+$ be as before, $B = HN_+$, $\tilde{B} = \tilde{H}N'_+$. We also let $H_n = \{ h^n : h \in H \}$, $\tilde{H}_n = \rho^{-1}(H_n)$. Note that $\tilde{H}_n \tilde{Z}$ is the centre of $\tilde{H}$. We let $\tilde{H}_n \triangleright \tilde{H}_n \tilde{Z}$ be a maximal abelian subgroup of $\tilde{H}$. We fix an injective character $\varepsilon : \mu_n(F) \to \mathbb{C}^\times$.

Let $\Phi$ be the set of roots of $G$ and let $\Phi(Z)$ be the lattice spanned by $\Phi$ in $\text{Hom}_{\mathbb{Z}}(H_\ast, H_\ast)$, where $H_\ast$ is the diagonal sub-algebraic-group of $GL_\ast$ and $H_1 = GL_1$. For any commutative ring $R$ let $\Phi(R) = \Phi(Z) \otimes_{\mathbb{Z}} R$. Let $\langle , \rangle$ be the Killing form on $\Phi(R)$.
If \( u = \sum_{a \in \Phi} u(a) a \in \Phi(C) \) and \( h \in H \) then we define \(|h^a|_p\) to be \( \prod_{a \in \Phi} |h^a|_p^{w(a)} \). We shall also consider the function \( h \mapsto |h^a|_p \) as a function on

i) \( B \) which factors through \( B \to B/N_+ \cong H \),

ii) \( \overline{H} \) which factors through \( \overline{H} \to H \), and

iii) \( \overline{B} \) which factors through \( \overline{B} \to B \to B/N_+ \cong H \).

In particular let

\[
\rho = \frac{1}{2} \sum_{a \in \Phi} a
\]

and set

\[
\mu(h) = |h^a|_p
\]

which we also regard as a function on all of the groups listed above.

Let \( W \) be the group of permutation matrices and let \( s \) be the section of \( \rho \) constructed in \( \S \) 0.1. One then has

\[
s(w_1 w_2) = s(w_1) s(w_2) i((\det(w_1), \det(w_2)))^a.
\]

Thus \( s \) fails to be a homomorphism only in the case that \( c \) is odd and \((-1, -1) = -1\). This last condition implies that \( n = 2 \pmod{4} \) and so if we let \( c' = c + n/2 \) then \( c' \) is even and \( 2c = 2c' \pmod{n} \). We have already remarked in the discussion preceding Proposition 0.1.1 that we cannot distinguish representation-theoretically between \( \overline{G}^{(c)} \) and \( \overline{G}^{(c')} \). We can assume that when \( n = 2 \pmod{4} \) the parameter \( c \) will be taken to be even. With this assumption \( s \) is always a homomorphism and we shall henceforth identify \( s(W) \) and \( W \).

Let \( \omega \) be a quasicharacter of \( \overline{H} \in \mathbb{Z} \) such that \( \omega \circ i = \varepsilon \). Extend \( \omega \) to a quasicharacter \( \omega' \) of \( \overline{H} \). Now let \( \overline{B} = \overline{H} \cap N^+ \) and extend \( \omega' \) to \( \overline{B} \) by

\[
\omega'(hn) = \omega'(h) \quad (h \in \overline{H}, n \in N^+).
\]

Let \( V(\omega') \) be the space of functions \( f: \overline{G} \to \mathbb{C} \) such that

1) \( f(bg) = (\omega' \mu)(b) f(g) \quad (b \in \overline{B}) \);

2) there is an open subgroup \( K_i \subset \overline{G} \) such that \( f(gk) = f(g) \quad (k \in K_i) \), and let \( \overline{G} \) act on this space by right translations. This is an admissible representation of \( \overline{G} \) and we denote it by \( (\pi(\omega'), V(\omega')) \).

Observe that this can be constructed in several steps: first by inducing \( \omega' \) to \( \overline{H} \), yielding \( \pi_0(\omega') \), then by extending this to \( \overline{B} \) by pulling back via \( \overline{B} \to \overline{B}/N^+ \cong \overline{H} \), yielding \( \pi_B(\omega') \), and finally inducing \( \mu \otimes \pi_B(\omega') \) to \( \overline{G} \). This shows that the class \( (\pi(\omega'), V(\omega')) \) depends only on \( \omega \), since the class of \( \pi_B(\omega') \) depends only on \( \omega \), as we saw in \( \S \) 0.3.
Let us observe that \( \omega^{-1} \) is a representation on \( \hat{\mathbb{H}} \) and that \( \omega^{-1} \circ i = \hat{\epsilon} \). The representation \( (\pi(\omega^{-1}), V(\omega^{-1})) \) can be identified with the (algebraic) contragredient of \( (\pi(\omega), V(\omega)) \) by means of the pairing
\[
V(\omega) \times V(\omega^{-1}) \to \mathbb{C}
\]
\[
(f_1, f_2) \mapsto \sum_{n \in \mathbb{Z}} f_1(\eta w_0 n) \overline{f_2(\eta w_0 n)} \, dn.
\]
where \( w_0 = (\delta_{r, r+1-\ell}) \in W \).
The invariance of this pairing follows in the usual way.

Let now \( \omega \) be as above; then there exist real numbers \( \sigma_1, \sigma_2, \ldots, \sigma_{r-1}, \sigma_0 \) so that, if \( h \in \hat{\mathbb{H}}, p(h) = \text{diag}(h) \) then
\[
|\omega(h)| = |h_1/h_2|^\sigma_1 |h_2/h_3|^\sigma_2 \cdots |h_{r-1}/h_r|^\sigma_{r-1} |h_1 \cdots h_r|^\sigma_r.
\]
If we write \( \sigma_\alpha \) for \( \sigma_{\alpha+1} \) if \( \alpha \) is the simple root \((i, i+1)\), then writing
\[
\omega(A) = \prod \sigma_\alpha \alpha
\]
one has in a self-explanatory notation
\[
|\omega(h)| = |p(h)^{\omega(\alpha)}| \prod \det p(h)||\sigma_{\alpha}.
\]
One thinks of \( \omega(\alpha) \) as a kind of "real part" of \( \omega \).

If \( \omega \) is a quasicharacter as above then for \( w \in W, \omega^w \), defined by
\[
(\omega^w)(h) = \omega(h^w)
\]
is also of the same type. Note that
\[
\omega(\alpha) = \omega^w(\alpha).
\]

We say that \( \omega \) is dominant if \( \omega(\alpha) \) lies in the dominant Weyl chamber; that is, if \( \alpha > 0 \) then \( \langle \alpha, \sigma(\alpha) \rangle > 0 \). This means that if we write \( |\omega(h)| = \prod |h_i|^\sigma_i \) then \( t_1 > t_2 > t_3 > \ldots > t_r \). We let \( \omega^w : H \to \mathbb{C}^* \) be the quasicharacter defined by
\[
\omega^w(h) = \omega(h^w).
\]

We call \( \omega \) unramified if \( \omega^w \) is trivial on \( H \cap \text{GL}(R_\mathfrak{p}) \).

We shall end this section with some considerations special to those \( \mathbb{F} \) with \( |n|_p = 1 \).

We first construct a special \( \hat{\mathbb{H}} \), which we will take as standard. In this case we apply Proposition 0.3.1 with \( \Gamma_1 = R^F, F_1 = \mathbb{F} \), \( \Gamma_2 = \mathbb{F}^\times \), \( F_2 = \mathbb{F}^\times \times \mathbb{F}^{n(r-1+2\nu)} \). These satisfy the conditions of Proposition 0.3.1 by [50] XIII-5, Prop. 6. Note that the corresponding \( \hat{\mathbb{H}} \) is normalized by \( W \). When \( |n|_p = 1 \) we shall take \( K^* \) to be the canonical lift of \( \text{GL}(R_\mathfrak{p}) \) characterized by Proposition 0.1.3. We now make an observation which simplifies much of our later work.

**Lemma 1.1.1.** — Let \( \mathbb{F} \) be such that \( |n|_p = 1 \) and let \( \omega \) be an unramified quasicharacter of \( \hat{\mathbb{H}} \). Then there exists a quasicharacter \( \chi \) of \( F^\times \) such that \( \omega \cdot (\chi \circ \det \circ p) \) is trivial on \( \mathbb{Z} \cap K^* \).
Proof. — Let for \( \lambda \in \mathbb{R}_F^\times \), \( \lambda^{-1+2\pi i} \in F^\times \),
\[ \omega_1(\lambda) = \omega(x^*(\lambda I)). \]

This is a character of \( \mathbb{R}_F^\times \). As \( \omega \) is unramified, \( \omega_1 \) is trivial on \( \mathbb{R}_F^\times \). Now we can observe (as \( \chi^{(1+2\pi i)-1} \in F^\times \))
\[ \omega_1(\lambda) = \omega_1(\det(\lambda))^{1+2\pi i} \quad (\lambda \in \mathbb{R}_F^\times). \]

We now let \( \chi \) be an extension of \( \omega^{-(1+2\pi i)} \) to \( F^\times \). Then
\[ \omega(\chi \circ \det \circ \rho) \]
is trivial on \( \mathbb{Z} \cap K^* \).

Corollary 1.1.1. — With the notations above \( \chi \) can be chosen so that \( \omega \circ (\chi \circ \det \circ \rho) \) is unitary on \( \mathbb{Z} \).

Proof. — Clear.

When \( |\pi|_F = 1 \) we shall call \( \omega \) normalized if \( \omega \) is trivial on \( \mathbb{Z} \cap K^* \). Suppose that \( \omega \) is an unramified, normalized quasicharacter of \( \mathbb{H} \cap \mathbb{Z} \). If \( \mathbb{H} \) is as above then there exists a unique extension \( \omega' \) of \( \omega \) to \( \mathbb{H} \), characterized by
\[ \omega' \mid \mathbb{H} \cap \mathbb{Z} = 1. \]

These extensions of \( \mathbb{H} \cap \mathbb{Z} \) and \( \omega \) we shall also term canonical.

Lemma 1.1.3. — Suppose that \( |\pi|_F = 1 \) and that \( \omega \) as above is unramified and normalized. Let \( (\pi, V) = (\pi(\omega'), V(\omega')) \). Then
\[ \dim \{ v \in V : \pi(k) v = v \ (k \in K^*) \} = 1. \]

Proof. — We may choose \( \mathbb{H} \) and \( \omega' \) to be canonical. Then as \( \mathbb{C} = \cup \mathbb{H} \eta K^* \), where \( \eta \) runs through a set of representatives of \( \mathbb{H} \setminus \mathbb{H} \), a function \( f \) in \( V(\omega') \) which satisfies \( \pi(k) f = f \) is determined by its values \( f(\eta) \). If \( h \in \mathbb{H} \cap K^* \) then there is a consistency condition, namely that
\[ f(\eta h) = f(\eta h \eta^{-1}, \eta) \]
so that \( (\omega' \mu) (\eta h \eta^{-1}) = 1 \) if \( f(\eta) \neq 0 \). But this condition is verified if \( \eta \in \mathbb{H} \) (by construction) but not for any \( \eta \notin \mathbb{H} \) (by the maximality of \( \mathbb{H} \)). This proves the lemma.

Notation. — Under the assumptions of the lemma and supposing that \( \mathbb{H}, \omega' \) are canonical we shall let \( v_0(\omega') \in V(\omega') \) be that element which is \( K^* \)-invariant and for which \( v_0(\omega')(I) = 1 \). If no confusion should arise we shall write \( v_0 \) instead of \( v_0(\omega') \).
I.2. Coinvariants and intertwining operators

In this section we shall use the general methods of [3] to investigate the principal series representations introduced in § I.1. In particular, we shall investigate the reducibility of these representations. Similar techniques can also be found in [46] Chapters 1 and 2.

Let $P \subset G$ be a standard parabolic subgroup of $G$. One has $P = M_P U_P$ where $M_P$ is the standard Levi component and $U_P$ is the unipotent radical of $P$. Then we let $U_P \subset N_+$ be the lift of $U_P$ to $G$. Let $\tilde{M}_P = \rho^{-1}(M_P)$. Then if $(\pi, V)$ be an admissible representation, we defined

$$ \varphi_P(V) = V/\langle \pi(n) v - v \mid v \in V, n \in U_P \rangle. $$

This is an admissible $\tilde{M}_P$-module (Jacquet's theorem, cf. [3] i.9 (c)), and $\varphi_P$ is an exact functor from the category of admissible $\tilde{G}$-modules to that of admissible $\tilde{M}_P$-modules (cf. [3] 1.9 (a)). It is often called the Jacquet functor (e.g. in [46]), but in [3] it is called the localization functor, and in [6] it is called the functor of coinvariants.

We shall first make use of this in the case $P = B$, in which case we write $\varphi_0$ for $\varphi_B$.

**Proposition I.2.1.** — The $\tilde{H}$-module $\varphi_0((\pi(\omega'), V(\omega'))) \equiv \langle \omega(\omega^2) \rangle$ has a Jordan-Hölder series whose composition factors are

$$ \text{ind}_{\text{Heis}, w^{-1}}^\tilde{H}(w \omega', \mu) \quad (w \in W). $$

If $w \omega \neq \omega$ for all $w \in W$, $w \neq 1$ then

$$ \varphi_0((\pi(\omega'), V(\omega'))) \cong \bigoplus_{w \in W} \text{ind}_{\text{Heis}, w^{-1}}^\tilde{H}(w \omega', \mu). $$

**Remark.** — The representation $\text{ind}_{\text{Heis}}^\tilde{H}(\omega', \mu)$ is the representation $(\pi_\omega, V_\omega)$ of § 0.4 (by definition).

**Proof.** — This is a variant of the Geometrical Lemma 2.12 of [3]. It follows at once from [3] Theorem 5.2, which is proved in sufficient generality to cover our case (cf. [3] 6.4). The last statement is then immediate.

Let us now call $\omega$ regular if $w \omega = \omega$ for all $w \in W$, $w \neq 1$.

As an immediate application Proposition I.2.1 one has:

**Proposition I.2.2.** — Let $\omega_1$, $\omega_2$ be two quasicharacters of $\tilde{H}$, and let $\omega'_1$, $\omega'_2$ be extensions to $\tilde{H}$. Then, if $\omega_1$ is regular

$$ \dim \text{Hom}(V(\omega'_1), V(\omega'_2)) \leq 1 $$

with equality if and only if $\omega_2 = w \omega_1$ for some $w \in W$. 


Proof. — This follows at once from Proposition I.2.1 and [3] 1.9 (a), (b).

It is now our intention to construct a homomorphism \( V(\omega') \to V(\omega') \) explicitly. We do this by writing, for \( f \in V(\omega') \)

\[
(\mathcal{I}_w f)(g) = \int_{N_w(\omega)} f(w^{-1} ng) \, dn
\]

where \( N_w(\omega) \) is the subgroup of \( N_+ \) corresponding to roots \( \alpha > 0 \) such that \( \omega^{-1} \alpha < 0 \). If this converges for all \( f \in V(\omega') \) and is non-trivial then it is a generator of \( \text{Hom}(V(\omega'), V(\omega')) \). Here \( \omega' \) is a quasicharacter of \( (\tilde{H}_s)^{w-1} \), which is also a maximal abelian subgroup of \( \tilde{H}_s \).

Let \( s \in \Phi(\mathcal{C}) \) (for the notation see § I.1) and let \( \omega_s \) be the quasicharacter \( h \mapsto |h^s|^p \) of \( H, \tilde{H}, B \) and \( \mathcal{B} \). Write \( s \) in the form \( \sum s_a \alpha \), where the sum is over the positive simple roots. Let, for any positive simple root \( \alpha \),

\[
X_{\alpha} = |\pi|^{p^a},
\]

where \( \pi \) is a uniformizer of \( F \).

We let \( \Omega_s : G \to \mathbb{C}^* \) be defined as

\[
\Omega_s(g) = \omega_s(h)
\]

if, in the Iwasawa decomposition \( G = N_+ HK, \, g = nkh \). This we also regard as a function on \( \tilde{G} \). Note that if \( f \in V(\omega') \) then \( \Omega_s f \in V(\omega_s \omega') \); this can be regarded as trivializing the vector bundle whose fibre over the quasicharacter \( \omega \) of \( \tilde{H}_s \mathbb{Z} \) is \( V(\omega') \), when the maps \( \Phi(\mathcal{C}) \to \{ \text{Quasicharacters on } \tilde{H}_s \mathbb{Z} \} : s \mapsto \omega_s \omega' \) are used to define the structure of a (disconnected) complex manifold on the set of quasicharacters of \( \tilde{H}_s \mathbb{Z} \). We shall write \( f_s \) for \( \Omega_s f \).

We now introduce the notion of the length, \( l(*) \) of \( w \in W \); this is the minimal number of simple reflections \( s_1, \ldots, s_l \) needed to write \( w \) as \( s_{l_1} s_{l_2} \ldots s_{l_l} \). There is in \( W \) a unique longest element, which we shall denote by \( w_0 \); it is characterized by \( w_0 \alpha < 0 \) for all \( \alpha > 0 \). It is the same element as was introduced in § I.1.

If \( \omega \) is a quasicharacter of \( \tilde{H}_s \mathbb{Z} \) and \( \alpha \) is a root we let, for \( x \in F^* \),

\[
\omega_\alpha^x(x) = \omega^x(\text{diag}(1, \ldots, 1, x, 1, \ldots, x^{-1}, \ldots, 1)),
\]

where, if \( \alpha = (i, j) \), the \( x \) is in the \( i \)-th position and \( x^{-1} \) in the \( j \)-th position. This is a quasicharacter of \( F^* \). Moreover we recall that given any quasicharacter \( \chi \) of \( F^* \) we can define the associated \( L \)-function \( L(\chi) \) as

\[
L(\chi) = (1 - \chi(\pi))^{-1} \quad (\chi \text{ unramified, } \pi \text{ a uniformizer of } F)
\]

\[= 1 \quad (\chi \text{ ramified}). \]

Proposition I.2.3. — a) Suppose \( w_1, w_2 \in W, \, l(w_1 w_2) = l(w_1) + l(w_2) \). Suppose also that

\[
I_{w_1} : V(\omega_1 \omega') \to V(\omega_1^w \omega'), \quad I_{w_2} : V(\omega') \to V(\omega_2 \omega')
\]
are defined in the sense that the integral above is everywhere absolutely convergent. Then
\[ I_{v_1,v_2} : V(\omega') \to V('m_{v_1,v_2}\omega') \]
is defined and
\[ I_{v_1,v_2} = I_{v_1} I_{v_2}. \]

b) If \( w \in W \) then \( I_w \) is defined for \( \omega \) satisfying
\[ \langle a, \sigma(\omega) \rangle > 0 \quad \text{for} \; a > 0 \quad \text{such that} \; wx < 0. \]
Moreover if \( t \in \Phi(G), \langle a, \sigma(\omega, \omega) \rangle > 0 \) for \( a > 0 \) such that \( wx < 0 \), then, for \( f \in V(\omega') \),
\[ \prod_{\omega > 0} L((\omega, \omega)_{\omega}^{-1})^{-1} I_w(f) \]
is a polynomial in \( \{ X_a(t), X_{-a}(t) : a > 0 \} \).

Proposition I.2.4. — Let \( F \) be such that \( |\pi|_F = 1 \) and suppose that \( \omega \) is unramified and normalized. Let \( \nu_0(\omega') \) be defined as in § I.1. Let \( m = \text{meas}(\mathcal{R}_F) \) be the measure of \( \mathcal{R}_F \) with respect to the additive measure. Then if \( \omega \) satisfies the conditions of Proposition I.2.3
\[ I_w \nu_0(\omega') = \{ \prod_{\omega > 0} mL((\omega)_{\omega}(-1)) \nu_0(\omega') \}_{wx < 0}. \]

Proofs. — We have grouped these two propositions together since their proofs involve the same techniques.

First of all, Proposition I.2.3 is a simple application of Fubini's theorem.

Now let us assume we have proved Propositions I.2.3 and I.2.4 when \( w = s \), a simple reflection. Then one deduces the general case by induction on \( l(w) \). More specifically, assume that for some \( l \) that these have been proved for all \( w' \in W \) with \( l(w') < l \). Suppose that \( l(w) = l \). Then we can find a simple reflection \( s \) such that \( l(w) = l(ws) + 1 \). By Proposition I.2.3
\[ I_w = I_{ws} I_s. \]

Moreover, if \( \alpha \) is the simple root associated with \( s \) then
\[ \{ \beta : \beta > 0, w\beta < 0 \} = \{ \alpha \} \cup \{ \beta' > 0, w\beta' < 0 \}, \]
the union being disjoint. From these two facts it is easy to derive the validity of the two assertions for \( w \) from the corresponding ones for \( ws \) and \( s \).

Thus we have reduced the general case to that when \( w = s \), a simple reflection. Recall that, if \( f \in V(\omega') \) then
\[ I_s f_s(g) = \int_{N_{\mathcal{F}}(s)} f_s(s^{-1} ng) \, dn. \]
We have to verify that the integral is absolutely convergent. Consider now a fixed function \( f \in V(\omega') \) and let \( L \) be an open neighbourhood of \( I \) in \( \mathcal{G} \) such that \( f(gl) = f(g) \) \( (l \in L) \). Let \( \eta \) run through a fixed set of representatives \( R \) of \( \hat{\mathcal{G}} \). Then we can find
a compact neighbourhood $L_{\Lambda}$ of $I$ in $N^{\ast}_{\Lambda}(s)$ such that if $n \in N^{\ast}_{\Lambda}(s) - L_{\Lambda}$, $s^{-1}ng$ can be written as

$$s^{-1}ng = b(n) \eta(n) gl(n) \quad (b(n) \in \tilde{B}_{\ast}, \eta(n) \in \mathbb{R}, l(n) \in L).$$

Thus

$$I_{\ast}f_{i}(g) = \int_{L_{\Lambda}} f_{i}(s^{-1}ng) \, dn + \sum_{s \in \mathbb{R}} \left( \int_{(n \in N^{\ast}_{\Lambda}(s) - L_{\Lambda} : \eta(n) = s)} \omega_{i} \omega'(b(n)) \, dn \right) f_{i}(\gamma_{g}). \quad (1)$$

Let $\alpha$ be the root associated with $s$. Then it is clear that, as $L_{\Lambda}$ is compact, the first term is a polynomial in $X_{\alpha}(t), X_{\alpha}(t)^{-1}$. Thus we have to verify that the integral in the second term is absolutely convergent and we must compute it. The computation involved is very similar to the one involved in computing $I_{\ast}v_{0}(\omega')$, so we shall discuss this and then treat the two computations together. The integral above we shall call the "first case".

Since $v_{0}$ is unique (Lemma I.1.3) $I_{\ast}v_{0}(\omega')$ is a multiple of $v_{0}(\omega')$. However, as $v_{0}(\omega')$ can be regarded as a function on $\tilde{G}$, and its value at $I$ is $1$, we see that the multiple is $I_{\ast}v_{0}(\omega')(I)$. This is

$$\int_{N^{\ast}_{\Lambda}(s)} v_{0}(sn) \, dn.$$  

This integral is the sum of $m \cdot v_{0}(s) = m$ and

$$\int_{N^{\ast}_{\Lambda}(s) - N^{\ast}_{\Lambda}(s) \cap K^{\ast}} v_{0}(sn) \, dn.$$

If $n \in N^{\ast}_{\Lambda}(s) - N^{\ast}_{\Lambda}(s) \cap K^{\ast}$ we can write

$$s^{-1}n = b(n) \eta(n) k_{0}(n) \quad (b(n) \in \tilde{B}_{\ast}, \eta(n) \in \mathbb{R}, k_{0}(n) \in K^{\ast}).$$

Then this second integral is

$$\int_{(n \in N^{\ast}_{\Lambda}(s) - N^{\ast}_{\Lambda}(s) \cap K^{\ast} : \eta(n) = 1)} \omega'(b(n)) \, dn$$

which is the same type of integral as above. This we call the "second case".

If we write $n = s(I + \xi e_{\alpha})$ then

$$s^{-1}n = h_{\alpha}(\xi) \eta_{\alpha} s(I - \xi e_{\alpha}) s(I + \xi^{-1} e_{-\alpha})$$

where $h_{\alpha}(\xi) = s(\text{diag}(i, 1, \ldots, 1, \xi^{-1}, i, 1, \ldots, 1))$ (where, if $\alpha = (i, i + 1)$, $\xi^{-1}$ is in the $i$-th place, $\xi$ in the $(i + 1)$-th, and $\eta_{\alpha} = s(\text{diag}(1, i, \ldots, 1, i, 1, \ldots, 1))$, where the $-$ is in the $i$-th position. The first case, if $L_{\Lambda}$ is large enough $s(I + \xi^{-1} e_{-\alpha}) \in gLg^{-1}$, whereas in the second case $s(I + \xi^{-1} e_{-\alpha}) \in K^{\ast}$. Thus in the first case we can consider the integral split into classes modulo $n$-th powers, so that it is a sum of integrals of the form

$$\int_{(\xi : |\xi| > e)} (\omega_{1} \omega' \mu)(h_{\alpha}(\xi^{n})) |n|_{\mathbb{R}} |\xi|_{\mathbb{R}} d^{\times} \xi$$

$$= n^{-1} \int_{(\xi : |\xi| > e)} (\omega_{1} \omega)_{n}^{\alpha}(\xi)^{-1} |n|_{\mathbb{R}} d^{\times} \xi.$$
This is a familiar integral, convergent whenever

\[ |(\omega, \omega)^{n}_{\mathfrak{a}}(\pi)|^{-1} > 1, \]

which is precisely the stated condition. Moreover this integral is 0 if \( \omega^{n}_{\mathfrak{a}} \) is ramified, and is of the form \( mX_{\mathfrak{a}}(t)^{k}/(1 - X_{\mathfrak{a}}(t)^{n} \omega^{n}_{\mathfrak{a}}(\pi)) \) if \( \omega_{\mathfrak{a}} \) is unramified. This proves Proposition 1.2.3 when \( \omega = s \).

In the second case we write \( \xi = \pi^{-1}x \) where \( j \equiv 0 \pmod{n} \), \( j > 0 \) and \( x \) ranges over \( R_{\overline{\mathfrak{a}}}^{\times} \). The integral is then

\[
\sum_{j \geq 0} \int_{R_{\overline{\mathfrak{a}}}^{\times}} f(h_{\mathfrak{a}}(\pi^{-j}x) \eta_{\mathfrak{a}}) \, dx \cdot |\pi|^{-j} = \sum_{j = 1}^{m} \omega_{\mathfrak{a}}^{n}(\pi)^{j} m(1 - |\pi|_{\mathfrak{a}}) \\
= m(1 - |\pi|_{\mathfrak{a}}) \omega_{\mathfrak{a}}^{n}(\pi)/(1 - \omega_{\mathfrak{a}}^{n}(\pi)) \tag{2}
\]

Thus we have

\[
\int_{N_{\mathfrak{a}}(0)} \rho_{\mathfrak{a}}(s^{-1} n) \, dn = m(1 + (1 - |\pi|_{\mathfrak{a}}) \omega_{\mathfrak{a}}^{n}(\pi)/(1 - \omega_{\mathfrak{a}}^{n}(\pi))) \\
= m(1 - q^{-1} \omega_{\mathfrak{a}}^{n}(\pi))/(1 - \omega_{\mathfrak{a}}^{n}(\pi)),
\]

as required. This completes the proofs of the propositions.

As a further application of the computations carried out above we can deduce a rather technical result which will be needed later. Let \( \alpha \) be a simple root, \( \iota \) the associated reflection. We shall investigate the case that \( \omega_{\mathfrak{a}} = 1 \), or, equivalently, \( 4\omega = \omega \).

In this case \( \omega' \) is an extension of \( \omega \) to \( \tilde{\mathbb{H}}_{\mathfrak{a}}^{\times} \). Thus \( V(\omega') \) and \( V(\omega') \) are equivalent and we wish to make the relationship between them explicit, following the principles set out in § 0.3.

First of all note that on \( \tilde{\mathbb{H}}_{\mathfrak{a}} \cap \tilde{\mathbb{H}}_{\mathfrak{a}}^{\times} \), \( \omega' \) and \( \omega' \) are extensions of \( \omega \). Thus \( \omega'(\omega')^{-1} \) is a character of \( \tilde{\mathbb{H}}_{\mathfrak{a}} \cap \tilde{\mathbb{H}}_{\mathfrak{a}}^{\times} \) trivial on \( \tilde{\mathbb{H}}_{\mathfrak{a}} \tilde{\mathbb{Z}} \). Since the quotient group is of exponent \( n \) it follows that \( \omega'(\omega')^{-1} \) is of order dividing \( n \). Indeed,

\[
\omega'(\omega')^{-1}(h) = \omega'(h_{\mathfrak{a}}^{-1} h^{-1} \xi) \\
= \xi(-1, h, \det(h)) \epsilon(h/\xi)
\]

where \( \alpha = (i,j) \) and \( \epsilon(\xi) = \omega'(\text{diag}(1, \ldots, 1, \xi, \xi^{-1}, 1, \ldots, 1)) \), when this is defined \( \xi \) being in the \( i \)-th position.

One knows, as \( \omega'(\omega')^{-1} \) is a character trivial on \( \tilde{\mathbb{H}}_{\mathfrak{a}} \tilde{\mathbb{Z}} \), that \( \epsilon \) is a character on a subgroup of \( F^{\times} \) containing \( F^{\times n} \) and \( \epsilon \) is trivial on \( F^{\times n} \). As \( \epsilon \) can be extended to a character of \( F^{\times} \) we see that \( \epsilon(\xi) = \epsilon(x, \xi) \) for some \( x \in F^{\times} \). Thus, with the notations above,

\[
\omega'(\omega')^{-1}(h) = \epsilon \circ \iota^{-1}(h_{\mathfrak{a}}(x) \eta_{\mathfrak{a}})^{-1} h(h_{\mathfrak{a}}(x) \eta_{\mathfrak{a}}) h^{-1}.
\]

Hence we construct a map \( \alpha_{\omega'} : V(\omega') \to V(\omega') \) such that

\[
\alpha_{\omega'}(f)(g) = \int_{\tilde{\mathbb{H}}_{\mathfrak{a}} \cap \tilde{\mathbb{H}}_{\mathfrak{a}}^{\times}} (\omega'(\mu)(x)^{-1} |x|_{\mathfrak{a}}^{-1} f(h_{\mathfrak{a}}(x) \eta_{\mathfrak{a}})^{-1} yg) \, dy.
\]

Here the integral is simply a finite sum; let \( A_{\mathfrak{a}} \) be the measure of \( \tilde{\mathbb{H}}_{\mathfrak{a}} \cap \tilde{\mathbb{H}}_{\mathfrak{a}}^{\times} \) with respect to the chosen measure. Then the lemma referred to above is the following:
Lemma 1.2.5. — Let notations be as above. Then, for \( f \in V(\omega') \) one has
\[
\alpha_s(\lim_{t \to 0} L(| |_F (\omega_1 \omega_2)^t) L((\omega_1 \omega_2)^{-1})^{-1} I_s f) = Af
\]
where
\[ A = mn^{-1} |n| F \alpha_0. \]

Proof. — From the formulae (1), (2) encountered in the proof of Proposition 1.2.3 one has that
\[
\lim_{t \to 0} L(| |_F (\omega_1 \omega_2)^t) L((\omega_1 \omega_2)^{-1})^{-1} I_s f(g)
= \sum_{\xi \in R_s} |\xi|_F f(h_\alpha(\xi) \eta_0 g) \lim_{t \to 0} L(| |_F (\omega_1 \omega_2)^{-1})^{-1} \int_{x \in F^{x, n}: |x|_F > 1} (\omega_2 \omega_1)(h_\alpha(\xi)) d\xi
\]
since \( L(| |_F) = (1 - |\pi|_F)^{-1} \).

The limit appearing here has in essence already been evaluated and the right-hand side of the above equation becomes
\[
mn^{-1} |n|_F \sum_{\xi \in R_s} |\xi|_F f(h_\alpha(\xi) \eta_0 g)
\]
since \([F^x : F^{x, n}] = n^|n|_F^{-1} \) (Lemma 0.3.2). The result of applying \( \alpha_s \) to this is
\[
mn^{-1} |n|_F \sum_{\xi \in R_s} \int_{\bar{H}_n \cap H_i \cap H_n} |\xi|_F f(h_\alpha(\xi) \eta_0(h_\alpha(\xi))) dy.
\]

After carrying out the integration over \( y \) the only non-zero summand is that arising from \( \xi = x \). This yields the assertion.

Now we shall discuss the question of the “regularization” of the \( I_\alpha \), that is, the definition of the intertwining operators \( V(\omega') \to V(\omega) \) for arbitrary \( \omega \). This is done by noting that for \( t \) such that \( \omega_1 \omega_2 \) is dominant \( I_\alpha : V(\omega_1 \omega_2) \to V(\omega_2 \omega_1) \) is defined and
\[
\prod_{\alpha > 0, w \alpha < 0} L((\omega_1 \omega_2)^{n})^{-1} I_\alpha(f_i)
\]
is a polynomial in \( \{X_\alpha(t), X_\alpha(t)^{-1} : \alpha > 0, w \alpha < 0\} \). This means that we can define a homomorphism of \( G \)-modules
\[
\begin{align*}
\prod_{\alpha > 0, w \alpha < 0} L((\omega_1 \omega_2)^{n})^{-1} I_\alpha'' : V(\omega') \to V(\omega')
\end{align*}
\]
by
\[
\begin{align*}
\prod_{\alpha > 0, w \alpha < 0} L((\omega_1 \omega_2)^{n})^{-1} I_\alpha''(f) = \prod_{\alpha > 0, w \alpha < 0} L((\omega_1 \omega_2)^{n})^{-1} I_\alpha(f_i)\big|_{t = 0}.
\end{align*}
\]

This we shall use as a definition henceforth. Moreover, if \( L(\omega_1 \omega_2) \) is finite for all \( \alpha > 0 \) we can speak of \( I_\alpha \) as
\[
\prod_{\alpha > 0, w \alpha < 0} L((\omega_1 \omega_2)^{n})'' \prod_{\alpha > 0, w \alpha < 0} L((\omega_1 \omega_2)^{n})^{-1} I_\alpha''.
\]
There is another approach, due to I. N. Bernstein, which can be used to define the $I_\alpha$ in general, and which, since we find it very illuminating, we discuss here. The underlying idea is to remark that if we regard the $X_\alpha(t)$ as indeterminates which we shall now write as $X_\alpha$, the space $V(\omega_1 \omega')$, which can be regarded as a vector space over $\mathbb{C}(X_{(1)}, X_{(2)}, \ldots, X_{(r-1), r})$ and which we write as $\mathbb{C}(X_\alpha)$, becomes a representation space of $\tilde{G}$ over this field.

Let us ask whether there exists a homomorphism of $\tilde{G}$-modules

$$V((\omega_1 \omega')) \to V((\omega_1 \omega'))$$

where $\omega_i$ is regarded as a function of the $X_\alpha$. Clearly, for such a homomorphism to exist is equivalent to the solubility of a countable set of equations. These equations are polynomial in the $X_\alpha (\alpha > 0, \text{ simple})$, since only such functions arise. Thus either this set of equations is identically satisfied, or there exists a countable collection of algebraic sets in $(\mathbb{C}^*)^{-1}$ such that $(X_\alpha)$ must lie in one of these so that the equations may have a solution. But we have already seen in Proposition 1.2.3 that for $X_\alpha$ lying in the open set such that $\omega_1 \omega$ is dominant there is a solution. Hence this second possibility is untenable, and so the equations have a solution for generic $X_\alpha$.

This means that there is a map $V((\omega_1 \omega')) \to V((\omega_1 \omega'))$ in this generic sense. However, by Proposition 1.2.2, this map is unique up to a scalar multiple. Moreover, it coincides up to a scalar function of the $X_\alpha (\text{a priori not necessarily rational})$ with our $I_\alpha$. But we have shown in Proposition 1.2.3 b) that our $I_\alpha$ do in fact induce a homomorphism $V(\omega_1 \omega') \to V((\omega_1 \omega'))$ of the above type.

Now, for any quasicharacter $\chi$ of $\mathbb{F}^\times$ let $P^{(\chi)}_\chi$ be its conductor. We set

$$I'_\alpha = \prod_{\alpha > 0} \left( \prod_{\omega < 0} \frac{1}{|F| \frac{1}{2} \frac{1}{2} l(\omega \omega') m^{-1} L(1, \omega_0^\alpha \omega_0^\alpha, \omega_0^{-\alpha^\alpha})} \right) \cdot I_\alpha$$

which is defined, as a $\tilde{G}$-homomorphism $V(\omega') \to V((\omega')$, for all $\omega'$.

**Theorem 1.2.6.** — $I'_{\omega', \omega} = I'_{\omega, \omega'}$.

**Proof.** — The proof, following [12], Ch. 3, §7.6, depends on a preliminary reduction to the case $\omega_1 = \omega_2 = s$, a simple reflection. To see this, notice that $W$ is generated by the simple reflections subject to the relations

a) $s^2 = 1$

b) $ss^1 s = s^1 ss^1$

where, in b), $s$ and $s^1$ are "neighbouring" reflections. To verify the relations of the type b) is elementary as

$$l(ss^1 s) = l(ss^1) + 1 = l(s) + 2 = 3$$

$$l(s^1 ss^1) = l(s^1 s) + 1 = l(s^1) + 2 = 3$$
and so b) follows by Proposition 1.2.3 (a), since
\[ I_{n', n} = I_{n', n} I_n, \quad I_{n', n} = I_{n'} I_n I_n, \]
where these are defined, and this is equivalent to the statement.

Thus we have to prove that \( I_r \) is the identity. Let us show that this can be reduced to the corresponding question in \( G \). Let \( V_1(\omega) \) be the space of locally constant functions on \( G \) which satisfy
\[ f(bg) = (\omega(b))(b)f(g) \quad (b \in \hat{G}). \]
Notice that \( I_n \) can be defined on \( V_1(\omega) \) just as before. As \( \hat{H} \) acts on the left on \( V_1(\omega) \), we obtain the different \( V_1'(\omega') \) by decomposing \( V_1(\omega) \) with respect to characters of \( \hat{H} \).

Thus all statements about the \( I_n \) on \( V_1'(\omega') \) are valid for \( V_1(\omega) \) and conversely. We shall study the \( I_n' \) on \( V_1'(\omega) \). Let us consider a copy of \( \hat{GL}_n \) in \( G \), containing the roots \( \pm \alpha \), where \( \alpha \) is associated with \( s \). We can restrict to this (which would not have been valid if we had used \( V_1'(\omega') \)), and thus it suffices to prove the equality in the case \( r = 2 \). Note also that, by Proposition 1.2.2, \( I_r' \) is a multiple of the identity. Our problem is to determine which multiple. This we do by writing \( I_r' \) as an integral transform and then using techniques from the theory of such transforms.

Let then \( r = 2 \) and let \( f \in V_1(\omega) \). To this we ascribe a family \([f](h, \cdot) \) \((h \in \hat{H})\) of functions on \( F \) which are defined by
\[ [f](h, \xi) = f\left(hs\left(\begin{array}{c}
1 \\
1
\end{array}\right)\left(\begin{array}{c}
1 \\
0
\end{array}\right)\right). \]
Thus \([f](h)\) is locally constant and as \( y \to \infty\)
\[ [f](h, \xi, y^n) \omega_\gamma(y) |y|^p \]
is ultimately constant (as \( f \) is constant near \( h' \), for \( h' \in \hat{H} \)). Note also that
\[ [f](\eta h, \xi) = \omega_\mu(\eta) [f](h, \xi) \quad (\eta \in \hat{H}). \]
For this reason we can regard \([f](h, \cdot)\), as \( h \) runs through a set of representatives of \( \hat{H} \setminus \hat{H} \), as a finite family of functions satisfying the continuity and asymptotic properties above. Conversely, given such a family we can reconstruct a corresponding \( f \).

After a brief computation one verifies that
\[ [I_n, f](h, \xi) = \sum_{x \in F^n \setminus F_0^n} n^{-1} |n||\mu(h)| |x|^{-1} \int_{F^n} [f](h' \Delta(x) \eta_0, \xi + xy^n) \omega_\gamma(y) d^n y \]
where \( \Delta(x) = s(\text{diag}(x, x^{-1})) \) and \( \eta_0 = s(\text{diag}(1, 1)) \). Iterating this yields
\[ [I_n, I_n, f](h, \xi) = \sum_{x_1, x_2} n^{-2} |n|^2 |x_2/x_1|^{\varepsilon(x_2, x_1)} \int_{F^n} [f](h \Delta(-x_1/x_2), \xi + x_1 y_1^n + x_2 y_2^n) \omega_\gamma(y_1) d^n y_1 \omega_\gamma(y_2)^{-1} d^n y_2 \]
where the integrals are to be understood as their regularizations, and \( x_1, x_2 \) run through a set of representatives of \( F^\times \setminus F^\times \).

Since this is a multiple of \([ f ] (h, \xi)\) it follows that we need only consider those terms with \(-x/y, x \in F^\times n\). Thus we can assume that \( x = -x_1\); the integral is then, on replacing \( y_2 \) by \( y_1 y \) and writing \( F^\times = \bigcup x_1 F^\times n\),

\[
n^{-1} |n|_F \int_{F^\times} \left( \int_{F^\times} [f] (h, \xi + (y^n - 1) x) \omega_n^\prime(y) \, d^\times x \right) \, d^\times x.
\]

This can be reformulated as saying that, if \( \varphi \) is a Schwarz function on \( F \), then

\[
n^{-1} |n|_F \int_{F^\times} \left( \int_{F^\times} \varphi(x(y^n - 1)) \omega_n^\prime(y) \, d^\times x \right) \, d^\times x
\]
is a multiple of \( \varphi(0) \). Our problem is to determine which multiple.

By definition of the regularization, the integral above is equal to

\[
n^{-1} |n|_F \lim_{\lambda \to 0} | \int_{F^\times} \left( \int_{F^\times} \varphi(x(y^n - 1)) \omega_n^\prime(y) \, d^\times x \right) \, d^\times x.
\]

The double integral is convergent for \( \Re(\lambda) \) large enough, and so we can write it as

\[
n^{-1} |n|_F \lim_{\lambda \to 0} \left( \int_{F^\times} \omega_n^\prime(y) \, d^\times x \right) \, d^\times x.
\]

However

\[
\lim_{\lambda \to 0} \int_{F^\times} \varphi(x) \, d^\times x = m(1 - q^{-1}) (\log q)^{-1} \varphi(0)
\]

and thus the multiple which we are trying to find is

\[
nm^{-1} |n|_F (1 - q^{-1}) (\log q)^{-1} \lim_{\lambda \to 0} \lambda^{-1} \int_{F^\times} \omega_n^\prime(y) \, d^\times x.
\]

We shall next express the final integral in more familiar terms. By means of the exact sequence

\[(1) \to \mu_n(F) \to F^\times \to F^\times n \to (1)\]

we can regard \( \omega_n^\prime \), which is a quasicharacter of \( F^\times \) trivial on \( \mu_n(F) \), as a quasicharacter on \( F^\times n \). Let \( \omega \) be an extension of \( \omega_n^\prime \) to \( F^\times \). Let \( X_n \) be the group of characters of \( F^\times \) of order \( n \), or, equivalently, of characters trivial on \( F^\times n \). Then as \( [F^\times : F^\times n] = n^2 |n|_{\mathbb{F}}^{-1} \) (cf. Lemma 0.3.2) we have

\[
n^{-1} |n|_F \int_{F^\times} \omega_n^\prime(y) \, d^\times x = \sum_{\chi \in X_n} n^{-2} |n|_F \int_{F^\times} (\omega \chi)(y) \, d^\times y.
\]

The integrals appearing on the right-hand side of this formula are beta-functions, in the sense of [12] p. 145. By the analogue of Euler's formula (loc. cit.) this is

\[
n^{-2} |n|_F \sum_{\chi \in X_n} \Gamma(\omega \chi) \Gamma(|1 - y^{-1}\lambda|_{\mathbb{F}}) / \Gamma(\omega \chi |1 - y^{-1}\lambda|_{\mathbb{F}}).
\]
Since \( \Gamma(| \frac{1}{2} |) \) has a zero at \( t = 1 \) and, as \( t \to 1 \),
\[
\Gamma(| \frac{1}{2} |) \sim - (\log q) m. (1 - q^{-1})^{-1}(t - 1)
\]

it follows that the multiple which we are seeking is

\[
n^{-1} |n|_p m^2 \sum_{x \in X_n} \Gamma(\omega x) / \Gamma(\omega x | \frac{1}{p})
\]

It therefore remains to evaluate this sum.

By [12] p. 150 (18) and (18')
\[
\Gamma(\omega) / \Gamma(\omega | \frac{1}{p}) = \frac{(1 - q^{-1})(1 - q^{-1 - 1})}{(1 - q^{-1})(1 - q^{-1})} \quad (\omega \text{ unramified, } \omega(\pi) = q^{-1})
\]

Thus we can compute the multiple above explicitly.

First of all suppose that \( \omega^n \) is unramified, \( \omega^n(x) = |x|_p^n, \) say. Then we can take \( \omega(x) = |x|_p \). There exist \( n \) characters in \( X_n \) which are unramified. Let \( X_n' \) be the group of characters on \( R_x' \), of order \( n \); this group is of order \( n |n|_p^{-1} \). The multiple is
\[
n^{-1} |n|_p m^2 \sum_{x \in X_n'} (1 - \zeta q^{-1})(1 - \zeta^{-1} q^{-1 - 1}) + n^{-1} |n|_p m^2 \sum_{x \in X_n' - \{1\}} q^{-f(x)}
\]

If, however, \( \omega^n \) is ramified then the multiple is
\[
n^{-1} |n|_p m^2 \sum_{x \in X_n} q^{-f(\omega x)}
\]

Now let us evaluate these when \( |n|_p = 1 \). When \( \omega^n \) is unramified then \( q^{-f(x)} = q^{-1} \) \( (x \in X_n' - \{1\}) \) and so we obtain
\[
|n|_p m^2 (1 - q^{-1 - 1})(1 - q^{-1 - 1} - 1)(1 - q^{-1} - 1),
\]
as required. If \( \omega^n \) is ramified then the multiplier is, as required, \( m^2 q^{-f(\omega^2)} \).

To deal with the cases when \( |n|_p < 1 \) it seems to be most convenient to resort to global methods. The point is that a global analogue of the functional equations in question can be proved by means of the theory of Eisenstein series (Theorem II.1.4 below).

This is independent of any local theorems and therefore can be exploited to complete the proof here. We shall merely sketch the latter; it runs in tandem with the proof of Theorem I.6.3.

Let \( k \) be a global field with \( \text{Card}(\mu_n(k)) = n \), and which has a place \( v \) such that \( k_v \cong F \). Then we can make use of the considerations of § II.1. There we have shown that if \( I_u \) is \( \bigotimes \mu_n(k) \), the tensor product being taken over all places of \( k \), and with respect to \( \mu_n(k) \), then
\[
(I_u)^n = \text{Id}.
\]
Let now the global \( \omega \) be such that \( \omega^\bullet_u \) is ramified at all places \( u \) of \( k \) with \( |n|_u < 1 \) except \( u = v \). \textit{A priori} we know that at these places one has \( (I_u, \omega) = (\text{const}) \text{Id} \). By Theorem I.6.3 we can assume that the archimedean analogue is true. It thus follows from the global functional equation that when \( (\omega^\bullet_u) \) is unramified, \( (I_u, \omega) \) is of the form

\[
(\text{const}) (1 - q^{-n-1}) (1 - q^{n-1}) (1 - q^{-m-1}) (1 - q^{m-1}) \text{Id}.
\]

However from the expression which we have already derived it is clear that the constant here has to be \( |n|_v m^2 \). This proves the theorem in this case.

Finally we observe that \( \{ (\omega^\bullet_u) : \omega | H^* = 1, \omega^\bullet_u \text{ unramified at } u \text{ with } |n|_u < 1, u \neq v \} \) is dense in the set of all local \( \omega^\bullet_u \) at \( k_v \). Therefore, to prove the local assertion, it suffices to prove it for these \( (\omega^\bullet_u) \). But since it is true at all other places of \( k \), and is true globally, it follows that it is true at \( v \). This proves the theorem.

\begin{remark}
These methods have shown that

\[
\sum_{x \in X_u} q^{-f(\omega x)} = 1 + (n - 1) q^{-1} \quad (\omega^\bullet \text{ unramified})
\]

\[
= nq^{-f(\omega)} \quad (\omega^\bullet \text{ ramified}).
\]

It seems possible to verify these assertions by purely local considerations; at least Bob Coleman has shown us how to do this in certain cases.

After these rather lengthy preliminaries we can return to our principal objective, which is the understanding of the \( V(\omega') \). We base our analysis on the following lemma.

\begin{lemma}
Let \( (\pi, W) \) be an admissible representation of \( \tilde{G} \) and let \( \varphi_0 \) be as in Proposition I.2.1. Then \( \text{ind}_{\tilde{H}}(\omega' \mu) \) is a sub-\( \tilde{H} \)-representation of \( \varphi_0(W) \) if and only if there exists a non-zero morphism of \( \tilde{G} \)-representations \( \alpha : W \rightarrow V(\omega') \).

\begin{proof}
This follows from [3] 1.9.
\end{proof}

\begin{corollary}
Suppose that \( \omega^\bullet_u + |x|^{-1} (x \in \Phi^+) \). Then \( V(\omega') \) is irreducible.

\begin{proof}
By Theorem I.2.6, under the assumptions of the corollary,

\[
\prod_{\alpha > 0, \alpha < 0} L(\omega^{-1})^{-1} \text{I}_\alpha : V(\omega') \rightarrow V(\omega')
\]

is an isomorphism. Thus if \( W \subset V(\omega') \) is a non-zero subrepresentation \( W \) can also be realized as a subrepresentation of \( V(\omega') \). Hence, by the lemma, \( \varphi_0(W) \) contains \( \text{ind}_{\tilde{H}}(\omega' \mu) \), for each \( \omega \in W \). So, by Proposition I.2.1, \( \varphi_0(W) = \varphi_0(V(\omega')) \).

As \( W \subset V(\omega') \) and as \( \varphi_0 \) is exact, \( \varphi_0(V(\omega'))/W = \{ 0 \} \); hence, by [3] 2.4 \( V(\omega')/W = \{ 0 \} \). Thus \( W = V(\omega') \), and so \( V(\omega') \) is irreducible.

We now call \( \omega \) \textit{exceptional} if \( \omega^\bullet_u = |x|_1 \) for all positive simple roots \( \alpha \). Such an \( \omega \) is clearly dominant. Then we can construct the representations which we shall study.
Theorem 1.2.9. — Let \( \omega \) be exceptional. Let
\[
V_\omega(\omega') = \text{Im}(I_\omega : V(\omega') \to V(\omega')),
\]
where \( \omega_0 \) is the longest element of \( W \). Then

a) \( V_\omega(\omega') \) is the unique irreducible subrepresentation of \( V(\omega') \),
b) \( V_\omega(\omega') \) is the unique irreducible quotient representation of \( V(\omega') \),
c) \( V_\omega(\omega') = \bigcap_s \text{Ker}(I_s : V(\omega') \to V(\omega s)) \), where \( s \) runs through the simple reflections,
d) \( \text{Ker}(I_\omega : V(\omega') \to V(\omega')) \) is generated by the set of \( \text{Im}(I_s : V(\omega') \to V(\omega')) \) as \( s \) runs through the set of simple reflections in \( W \),
e) \( \varphi_\omega(V_\omega(\omega')) \cong \text{ind}_{\mu, \omega}\omega(\omega') \mu \),
f) if \( |\pi|^p = 1 \) and \( \omega \) is normalized then \( V_\omega(\omega') \) contains a \( K \)-invariant vector.

A representation \( V_\omega(\omega') \) will be called exceptional.

Proof. — We begin with a general remark concerning the \( I_\omega \) in those cases when \( \omega_\alpha \neq 1 \) for any \( \alpha \). Then \( I_\omega : V(\omega') \to V(\omega') \) is defined as the regularized value of
\[
I_\omega f(g) = \int_{N(\omega)} f(w^{-1}g) \, dn.
\]
Let \( L \) be a sufficiently small compact open subgroup of \( \tilde{G} \), and let \( f \) be the right \( L \)-invariant function in \( V(\omega') \) with support \( \tilde{B} \setminus L \) and such that \( f(1) = 1 \). Let \( g = w \); then the integral is convergent and so can be understood in the usual sense. By choosing \( L \) small enough one has \( I_\omega f(w) = 0 \), and hence \( I_\omega = 0 \). Notice that \( \omega' \) is exceptional if and only if \( L(|\pi|^p \omega_\alpha) \omega_\alpha^{-1} = 0 \) for every positive simple root \( \alpha \).

With \( \omega' \) fixed as in the theorem let
\[
M(\omega) = \text{ind}_{\mu, \omega}\omega(\omega').
\]
Then we shall first prove e), i.e. that \( \varphi_\omega(V_\omega(\omega')) \cong M(\omega_0) \). Clearly by Lemma I.2.7
\( M(\omega_0) \) is a component of \( \varphi_\omega(V_\omega(\omega')) \). Conversely, if \( M(\omega) \) is another component then, by Lemma I.2.7, there would be a morphism \( \alpha : V_\omega(\omega') \to V(\omega') \). Then the composite map \( \alpha 
\alpha \circ I_\omega : V(\omega') \to V(\omega') \) is, by Proposition I.2.2, a multiple of \( I_\omega \). It cannot be \( O \) by construction. Thus for some \( \epsilon \neq O \) one has
\[
\alpha \circ I_\omega = \epsilon I_\omega.
\]
Suppose now \( s \) is a simple reflection such that \( l(\omega s) > l(\omega) \). Then by Theorem I.2.6
\( I_{\omega s} I_\omega = O \) but \( I_{\omega s} \neq O \). This is a contradiction and hence we have proved e).

Since \( M(\omega_0) \) is an irreducible \( \tilde{H} \)-module it now follows from [3] 2.4 that \( V_\omega(\omega') \) is irreducible. Part a) of the theorem follows at once from this and Lemma I.2.7.

Next we shall prove b). Let \( V \) be another irreducible quotient of \( V(\omega') \) and let \( M(\omega) \) be a component of \( \varphi_\omega(V) \). Then \( V \) can be realized as a subspace of \( V(\omega') \), let \( \alpha : V(\omega') \to V(\omega') \) be the map whose image is \( V \). By Proposition I.2.2 this is a
multiple of $I_w$, and as $I_w = I_{w_i w^{-1}} I_w$ it follows that $V_0(\omega')$ is a quotient of $V$. As $V$ is irreducible $V \cong V_0(\omega')$ (and $w = w_0$). This proves b).

We next prove c). Let $V_0$ be the space defined by the right-hand side of the statement. Then as $I_w I_w = 0$, $V_0 \supset V_0(\omega')$. Next note that the argument above shows that $M(\omega w')$ is a component of $\varphi_0(I_w : V(\omega' \omega) \to V(\omega' \omega'))$, and hence $M(\omega w')$ is not a component of $\varphi_0(Ker(I_w : V(\omega' \omega) \to V(\omega' \omega'))$. Taking $w' = w_0$, we see that $M(\omega w_0)$ is not a component of $\varphi_0(Ker(I_w))$. But this shows, as $\varphi_0$ is exact, that

$$\varphi_0(\bigcap_{w \in W} Ker(I_w V(\omega' \omega')))$$

is either 0 or $M(\omega_0)$. Since by Theorem 1.2.6

$$V_0(\omega') \subseteq Ker(I_w V(\omega' \omega'))$$

we see that

$$V_0(\omega') = \bigcap_{w \in W} Ker(I_w V(\omega' \omega')).$$

But now if we choose $s$ such that $l(w) = l(ws) + 1$ then

$$Ker(I_w V(\omega' \omega')) \subseteq Ker(I_w V(\omega' \omega'))$$

and c) follows at once. Part d) follows by a "dual" argument.

Finally, f) is a corollary of Proposition 1.2.4.

**Remarks.** — The class of the representation $V_0(\omega')$ depends only on $\omega$. Our notation will imply a realization of the representation in the form given by either a) or b) of the theorem.

We shall refer to Theorem 1.2.9 e), which plays an especially important role in our investigation, as the "Periodicity Theorem". The reason for the name will become clearer in § 1.4.

### I.3. Whittaker models

In this section we shall discuss the Whittaker models of $V(\omega')$ and of $V_0(\omega')$. To explain the relevant concepts, let $\epsilon_0$ be a non-trivial additive character on $F$ and define the character $\epsilon : N^+ \to G^\times$ by

$$\epsilon(n) = \epsilon_0\left(\sum_{1 \leq i \leq r} n_i (i+1)\right),$$

which we also regard as a character of $N^+_r$. Fix a quasicharacter $\omega_0$ of $\widehat{Z}$, and form the character $\omega_0 \times \epsilon$ of $\widehat{Z} N^+_r$ by

$$(\omega_0 \times \epsilon)(zn) = \omega_0(z) \epsilon(n) \quad (z \in \widehat{Z}, \ n \in N^+_r).$$
We shall suppose that \( \omega_0 \circ i = e \) on \( \mu_\circ(F) \). Then we form the representation \( (\rho, L_\circ) \) of \( \hat{G} \) induced from \( \omega_0 \times \circ ; \) that is, we regard \( L_\circ \) as the space of functions \( f: \hat{G} \to G \) such that

\[
(a) \quad f(\gamma g) = (\omega_0 \times \circ)(\gamma) f(g) \quad (\gamma \in \mathbb{Z}^N_+) ,
\]

\[
(b) \quad \text{for each } f \in L_\circ \text{ there exists an open subgroup } K_\gamma \subset \hat{G} \text{ such that } f(kg) = f(g) \quad (k \in K_\gamma)
\]
on which \( G \) acts by right translation. The representation \( (\rho, L_\circ) \) is algebraic.

We say that a Whittaker model of an admissible representation \( (\pi, V) \) is a injection \( (\pi, V) \to (\rho, L_\circ) \). Let us recall how these are in general constructed. Let \( (\pi', V') \) be the dual representation to \( (\pi, V) \), i.e. \( V' \) is the dual space of \( V \), the action of \( v' \in V' \) on \( v \in V \) being denoted by \( \langle v', v \rangle \). Then \( \pi' \) is defined by

\[
\langle \pi'(g) v', v \rangle = \langle v', \pi(g)^{-1} v \rangle.
\]

Now suppose that \( \lambda \in V' \) and satisfies

\[
\pi'(n) \lambda = e(n) \lambda
\]
and that \( \pi | \hat{Z} = \omega_0 \). Then if \( v \in V \), \( (g \mapsto \langle \lambda, \pi(g) v \rangle) \in L_\circ \). If \( (\pi, V) \) is irreducible and \( \lambda \neq 0 \) this is clearly injective. Conversely if \( t: (\pi, V) \to (\rho, L_\circ) \) then \( \lambda(v) = t(v)(1) \) is an element of \( V' \) as above. We denote the vector space of such \( \lambda \) by \( \text{Wh}(V) \).

Another characterization can be given. Let \( (\pi, V) \) be as above and let

\[
\varphi'(\pi, V) = V/\langle \pi(n) v - e(n) v \mid v \in V, n \in N_+ \rangle ;
\]
then \( \varphi'(\pi, V) \) is the \( e \)-localization of \( \pi \) in the terminology of [3]. Clearly \( \varphi'(\pi, V) \) is a \( \mathbb{Z}^N_+ \)-space, and it is dual to \( \text{Wh}(V) \).

Remark. — Let \( Z_0 \) be the centre of \( G \) and \( \hat{Z}_0 = \rho^{-1}(Z_0) \). Then \( \text{Wh}(V) \) is naturally a \( \hat{Z}_0 \)-module.

Let us now consider the case of \( (\pi, V) = (\pi(\omega'), V(\omega')) \) so that \( \omega_0 = \omega | \hat{Z} \).

For \( \eta \in \hat{H} \) we form

\[
\langle \lambda_\eta, f \rangle = \int_{N_+} \overline{e(n)} f(\eta^0 \eta^{-1} n) \, dn.
\]

As in Lemma 1.2.3 (b) this exists if \( \omega \) is dominant (as we have shown there that the integral is absolutely convergent). Clearly \( \lambda_\eta \) satisfies the above condition, so that \( \lambda_\eta \in \text{Wh}(V) \).

Clearly if \( h \in \hat{H}_\circ \) and \( \eta \in \hat{H} \)

\[
\lambda_{h \eta} = (\omega' \mu)(h) \lambda_\eta.
\]

Lemma 1.3.1. — There exists \( f_\eta \in V(\omega') \) so that

\[
\lambda_\eta(f_\eta) = \begin{cases} 0 & \text{if } \eta^{-1} \eta \notin \hat{H}_\circ, \\ \eta & \text{if } \eta^{-1} \eta \in \hat{H}_\circ. \end{cases}
\]
Proof. — Choose a fixed set of representatives \( R \) of \( \tilde{G} \mod \tilde{H} \). Then let \( L \) be an open compact subgroup of \( \tilde{G} \), which we shall take to be sufficiently small. Let \( f_\eta (\eta \in R) \) be defined by

\[
f_\eta (b \eta \omega_0) = (\omega' \mu)(b) \quad (b \in \tilde{B}, \ell \in L),
\]

\[
f_\eta (g) = 0 \quad (g \not\in \tilde{B}, \gamma \ell \omega_0).
\]

Then we shall check that this has the required properties. One has

\[
\langle \lambda_{\eta'}, f_\eta \rangle = \int_{N_{\tilde{B}}} f_\eta (\eta' \omega_0 \eta) \overline{\epsilon(n)} \ dn.
\]

But there is only a contribution when

\[
\gamma' \omega_0 \eta \in \tilde{B}, \gamma \ell \omega_0,
\]

or

\[
(\gamma^{-1} \gamma')(\omega_0 \eta \gamma^{-1} \omega^{-1} \eta') \in \tilde{B}, L.
\]

This however requires that \( \omega_0 \eta \gamma^{-1} \omega^{-1} \eta' \in \tilde{H}, \) if \( L \) is sufficiently small. Thus the integral is zero if \( \gamma' \not\in \tilde{H} \). If \( \gamma^{-1} \gamma' \in \tilde{H} \), we can assume that \( \gamma' = \gamma \). Then

\[
\langle \lambda_{\eta'}, f_\eta \rangle = \int_{N_{\tilde{B}} \cap \gamma \ell \omega_0} 1 \ dn \neq 0.
\]

This proves the lemma.

Lemma 1.3.a. — One has

\[
\dim(\text{Wh}(V(\omega'))) = \text{Card}(\tilde{H} \backslash \tilde{H}).
\]

Here \( \text{Card}(S) \) means the cardinality of \( S \); the notation will only be used for finite sets \( S \).

Proof. — The equivalent statement that

\[
\dim(\varphi(V(\omega'))) = \text{Card}(\tilde{H} \backslash \tilde{H})
\]

is contained in [3], Theorem 5.2.

Thus if \( \omega \) is dominant the \( \lambda_\eta (\eta \in \tilde{H} \backslash \tilde{H}) \) span the vector space \( \text{Wh}(V(\omega')) \). Since one can easily verify that the \( \lambda_\eta (f_\eta) \) are rational functions of the \( X(t) \), the method of Bernstein explained after Lemma 1.2.4 applies and shows that the \( \lambda_\eta \) are defined generically.

The objective of this section is to describe \( \varphi(V_\omega(\omega')) \) or equivalently \( \text{Wh}(V_\omega(\omega')) \) from the description of \( V_\omega(\omega') \) given by Theorem 1.2.9. We can unfortunately only obtain complete results when \( |n|_p = 1 \). In § 1.5 we shall discuss a different approach which permits us to avoid some of the problems occurring when \( |n|_p < 1 \).

First of all we shall explain the basic idea of the approach of this section. If \( w \in W \) the linear form on \( V(\omega') \)

\[
f \mapsto \langle \lambda_\eta, I_w f \rangle,
\]

where \( \lambda_\eta \in \text{Wh}(V(\omega')) \), is defined in Bernstein's generic sense. Since it lies in \( \text{Wh}(V(\omega')) \) it can be written as

\[
\sum_{\eta'} \tau(w, \omega', \eta, \eta') \langle \lambda_{\eta'}, f \rangle,
\]
where \( \lambda_{\gamma'} \in \text{Wh}(V(\omega')) \). We shall write \( \tau_{\omega}(\gamma, \gamma') \) for \( \tau(\omega', \omega, \gamma, \gamma') \) if no confusion should arise. The sum in \( \gamma' \) is taken over \( \widehat{H} \setminus \widehat{H} \). These equations define the transpose action of the \( I_w \) on \( \text{Wh}(V(\omega')) \).

Now let \( e \) be a function on \( \widehat{H} \) which satisfies
\[
\mathbf{e}(hk) = (\omega' \mu)(k)^{-1} \mathbf{e}(\eta) \quad (h \in \widehat{H}, \eta \in \widehat{H}).
\]
Define
\[
\lambda(e) = \sum_{\eta \in \widehat{H}} \mathbf{e}(\eta) \lambda_{\eta}.
\]

Suppose next that \( \omega' \) is exceptional. Then \( A \in \text{Wh}(V_0(\omega')) \) yields an element of \( \text{Wh}(V(\omega')) \) via
\[
V(\omega') \xrightarrow{I_w} V_0(\omega'),
\]
and this is then a \( \lambda(e) \). Since \( \lambda(e) \) is trivial on the kernel of \( I_w \), we must have
\[
\langle \lambda(e), I_s f \rangle = 0 \quad (f \in \text{V}(\omega'))
\]
where \( s \) runs through the simple reflections of \( W \). Conversely if \( \lambda(e) \) satisfies these conditions it follows from Theorem 1.2.9 d) that \( \lambda(e) \) is trivial on \( \text{Ker}(I_w) \) and hence defines an element of \( \text{Wh}(V_0(\omega')) \).

Moreover \( \langle \lambda(e), I_s f \rangle = 0 \) if and only if
\[
\sum \mathbf{e}(\eta) \tau(\omega', s, \eta, \gamma') = 0.
\]
As \( s \) runs through the set of simple reflections in \( W \) we obtain a system of linear equations in \( e \). If \( e \) is a solution of this system we can define an element \( \Lambda(e) \) of \( \text{Wh}(V_0(\omega')) \) by
\[
\langle \Lambda(e), I_s f \rangle = \langle \lambda(e), f \rangle
\]
and every element of \( \text{Wh}(V_0(\omega')) \) arises uniquely in this way.

We shall now explain how the \( \tau_s(\gamma, \gamma') \) can be computed. Let
\[
K_m = \{ k \in \text{GL}_r(\mathbb{R}) : k \equiv I \pmod{P^m} \},
\]
and suppose that \( m \geq 1 \) is large enough so that the covering \( p \) splits over \( K_m \). Let \( k \mapsto k' \) be such a splitting, and let \( K_m^+ \) be the image of \( K_m \) in \( \widehat{G} \). Let \( K_m^+ = N_+ \cap K_m \).

Then, as in the proof of Lemma 1.3.1 we define \( f^{(m)}_n \) to be that element of \( V(\omega') \) with
\[
\text{a)} \quad \text{Supp}(f^{(m)}_n) = \widehat{B}_+ \eta\omega \quad K_m^* = \widehat{B}_+ \eta\omega((K_m^*)^*),
\]
\[
\text{b)} \quad f^{(m)}_n \text{ is right-}K_m^* \text{-invariant, and}
\]
\[
\text{c)} \quad f^{(m)}_n(\omega_0) = 1.
\]
If \( m \) is sufficiently large, such an \( f^{(m)}_n \) exists. As we have seen in the proof of Lemma 1.3.1 :
\[
\langle \lambda_{\eta}, f^{(m)}_n \rangle \neq 0 \quad (\eta = \eta')
\]
\[
= 0 \quad (\eta \neq \eta').
\]
Then one has
\[
\langle \lambda_{\eta}, I_w f^{(m)}_n \rangle = \tau_{\omega}(\eta, \eta') \langle \lambda_{\eta'}, f^{(m)}_n \rangle,
\]
so that
\[ \tau_m(\eta, \eta') = \langle \lambda_n, I_{m} f_{\eta'}^{(m)} \rangle / \langle \lambda_n, f_{\eta'}^{(m)} \rangle. \]

Moreover in the proof of Lemma 1.3.1 we saw that \[ \langle \lambda_n, f_{\eta'}^{(m)} \rangle = \text{meas}((K^+_m)^*) \]
if \( m \) is large enough; consequently one has
\[ \tau(\omega, \omega', \eta, \eta') = \lim_{m \to \infty} \langle \lambda_n, I_{m} f_{\eta'}^{(m)} \rangle / \text{meas}((K^+_m)^*). \]

Now that we have explained the approach we shall carry it out when \( |n|_p = 1 \).
We shall take \( \tilde{H} \) to be the standard group constructed at the end of \$1.1. \) Note that \( \rho(\tilde{H}) \) is then
\[ \{ \text{diag}(h_j) : \text{ord}_p(h_j) = (2c + 1) \sum \text{ord}_p(h_k) \pmod{n} \}. \]

Let \( \pi \) be a uniformizer of \( F \), let \( f = (f_1, \ldots, f_s) \in \mathbb{Z}^s \) and let \( \pi' = \text{s}((\text{diag}(\pi(f))) \).
If we let
\[ \Lambda = \{ f \in \mathbb{Z}^s : f_j = (2c + 1) \sum f_k \pmod{n} \} \]
and construct a set of representatives \( A \) for \( \mathbb{Z}^s / \Lambda \) then \( \{ \pi' : f \in A \} \) is a set of representatives for \( \tilde{H} \setminus \tilde{H} \).

Suppose now that the conductor of \( \epsilon_0 \) is \( R_p \) and let \( dx \) be the self-dual Haar measure on \( F \). For \( J \in \mathbb{Z}/n\mathbb{Z} \) define the Gauss sum
\[ g^{(J)} = q \int_{\mathbb{Z}^s} \epsilon(\pi, x)^J \epsilon_0(x/\pi) \, dx. \]

Suppose next that \( \omega \) is a quasicharacter of \( \tilde{H} \) \( \mathbb{Z} \) such that \( \omega_{\epsilon_0} \) is unramified for a given simple positive root \( \alpha \). Since \( \tilde{H} \) is the standard maximal isotropic subgroup we can choose \( \omega' \) on \( \tilde{H} \) extending \( \omega \) such that \( \omega' \) is trivial on
\[ \kappa'\{ \text{diag}(1, \ldots, 1, y, y^{-1}, 1, \ldots, 1) : y \in R_{\mathbb{F}_p} \} \]
where \( \kappa' \) is the canonical lift of \( K \) and, if \( \alpha = (i, i + 1) \), the \( y \) (resp. \( y^{-1} \)) is in the \( i \)-th (resp. \( (i + 1) \)-th) position. Let
\[ u_j = \kappa'(\text{diag}(1, 1, \ldots, 1, - 1, 1, \ldots, 1)) \]
where the \(-1\) is in the \( j \)-th position. If \( \omega, \omega' \) are as above and if \( |n|_p = 1 \), let
\[ \theta = \omega'(u_j), \]
it is not difficult to see that this is well-defined. Then one has:

**Lemma 1.3.3.** Suppose that \( F \) is such that \( |n|_p = 1 \). Let \( \alpha = (i, i + 1) \) be a simple root and \( s \in W \) the associated reflection. Suppose that \( \omega_{\epsilon_0} \) is unramified and that \( \omega' \) is chosen as above. Then
\[ \tau(s, \omega', h, h^*) = \theta(\tau_s^*(h, h^*) + \tau_s^*(h, h^*)), \]

where \( \tau_s^*(\cdot, \cdot) \) denotes the twisted metaplectic form and \( \theta \) is the Gauss sum associated to \( \omega'(u_j) \).
where

a) for \( j = 1, 2, \eta, \eta' \in \mathfrak{H}, h, h' \in \mathfrak{H} \), one has
\[
\tau^1_{\ast}(h\eta, h\eta') = \omega' \mu(h) \cdot \omega' \mu(h')^{-1} \tau^1_{\ast}(\eta, \eta'),
\]

b) if \( f, f^* \in \mathbb{Z}^n \) then
\[
\tau^1_{\ast}(\pi, \pi^\pi) = 0 \quad \text{unless} \quad f^* \equiv f \pmod{\Lambda},
\]
and, with
\[
f_1 = (f_1, f_2, \ldots, f_{i-1}, f_{i+1} - 1, f_i + 1, f_{i+2}, \ldots, f_r)
\]
\[
\tau^1_{\ast}(\pi, \pi^\pi) = 0 \quad \text{unless} \quad f^* \equiv f_1 \pmod{\Lambda},
\]

c) one has
\[
\tau^1_{\ast}(\pi, \pi^\pi) = (1 - \omega^n(\pi))^{-1}(1 - q^{-1}) \omega^\pi(h_{i+1}, q^h q^h_{i+1}, \omega^\pi(\pi(h_{i+1}, i))) - 1
\]
and
\[
\tau^2_{\ast}(\pi, \pi^\pi) = \varepsilon(-1, \pi) q_{i+1} q^2 q^{h_{i+1} - h - 1}.
\]

Remark. — This lemma clearly determines \( \tau(s, \omega', h, h^\pi) \) completely.

The proof is an application of the formula derived above, and we shall postpone it to the end of this section.

Let \( \rho = (1, 2, \ldots, r) \in \mathbb{Z}^r \). We define an action of \( W \) on \( \mathbb{Z}^r \) by
\[
w[\mathbf{m}] = w(\mathbf{m} - \rho) + \rho \quad (\mathbf{m} \in \mathbb{Z}^r, w \in W)
\]
where if \( \mathbf{m} = (m_1, m_2, \ldots, m_r) \)
we set
\[
w(\mathbf{m}) = (m_{r-1(1)}, m_{r-1(2)}, \ldots, m_{r-1(r)}).
\]
Note that if \( |\mathbf{m}| = \sum m_j \) then
\[
|w[\mathbf{m}]| = |\mathbf{m}|.
\]
The action of \( W \) on \( \mathbb{Z}^r \) induces one on \( \mathbb{Z}^r/n\mathbb{Z}^r \).

Corollary 1.3.4. — Suppose that \( F \) is such that \( |\pi|_F = 1 \) and that \( \omega \) is exceptional. Let \( c \) be a function on \( \mathfrak{H} \) satisfying
\[
c(\eta h) = (\omega' \mu(\eta)^{-1} c(h) \quad (\eta \in \mathfrak{H}, h \in \mathfrak{H})
\]
for which \( \lambda(c) \) induces an element of \( Wh(V_0(\omega')) \). Then
\[
c(\pi^\pi) = q^h q^h_{i+1} - 1 - q^h q^h_{i+1} - 1 \cdot \varepsilon(-1, \pi) q^h q^h_{i+1} q^{h_{i+1} - h - 1} c(\pi^\pi),
\]
if \( s \) is the simple reflection associated with the root \( (i, i + 1) \). Conversely, given a non-zero function \( c \) satisfying these conditions \( \lambda(c) \) induces a non-zero element of \( Wh(V_0(\omega')) \).

Proof. — We have seen that we must have
\[
\sum_t \tau(s, \omega', \pi^\pi, \pi^\pi) c(\pi^\pi) = 0,
\]
where $f$ is summed over a set of representatives for $\mathbb{Z}/\Lambda$, in order that $\lambda(c)$ should induce an element of $\text{Wh}(V_0(\omega))$. We show now that this is equivalent to the stated condition for each $s$. Suppose $s$ is associated with the root $(i, i + 1)$. By Lemma 1.3.3 the condition above is

$$
\tau(s, \omega', \pi', \pi') c(\pi') + \tau(s, \omega', \pi', \pi') c(\pi') = 0.
$$

This is, as $\omega'$ is exceptional,

$$
-\theta_{\beta_{i+1}} - h_{i+1} - 1 + [h_{i+1}^{-1}] \cdot c(\pi')
+ q^{-2} e(-1, \pi)(h_{i+1}^{-1}) g^{(h_{i+1}^{-1})} c(\pi') = 0.
$$

In this we replace $f_i$ by $f$ and note that $f_i = s[f]$; the quoted condition follows immediately.

This leads at once the following theorem, which is the objective on this section.

**Theorem 1.3.5.** Suppose that $F$ is such that $|n| = 1$ and that $\omega$ is exceptional. Let $N$ be the number of images of free orbits of $W$ acting on $\mathbb{Z}/n\mathbb{Z}$ (orbits of size $r$) in $\mathbb{Z}/\Lambda$. Then

$$
\dim(\text{Wh}(V_0(\omega))) = N.
$$

As an immediate corollary one has:

**Corollary 1.3.6.** With the above notations one has that

$$
\dim(\text{Wh}(V_0(\omega))) = 1
$$

if and only if either

$$
r = n - 1, \quad 2(c + 1) \equiv 0 \pmod{n},
$$
or

$$
r = n.
$$

**Remarks.**

1) Although we shall not dwell on it here it is interesting to observe that $\text{Wh}(V_0(\omega))$ is a $\hat{\mathbb{Z}}$-module on which $\hat{\mathbb{Z}}$ acts by $(w\lambda)^{-1}$. For example, if $r = n$ then $\hat{\mathbb{Z}}$ is abelian but $\hat{\mathbb{Z}} \neq \mathbb{Z}$. Thus $\text{Wh}(V_0(\omega))$ determines a distinguished extension of $(\omega\lambda)^{-1}$ to $\hat{\mathbb{Z}}$, which could be computed. The significance of this extension is unclear. When $r = n$ or $n - 1$ Theorem 1.3.5 implies that $\text{Wh}(V_0(\omega))$ is an irreducible $\hat{\mathbb{Z}}$-module.

2) Although Theorem 1.3.5 does not hold if $|n| = 1$ is not assumed, a weaker version does. Moreover this assumption is not needed for Corollary 1.3.6. Such results will be derived by global methods—see Corollary II.2.6.

**Proofs.** The corollary is a simple combinatorial exercise given the theorem and we shall leave it as such for the reader. We shall therefore just prove the theorem.

As $\hat{H}$ is standard $c$ is determined by the values $c(\pi')$ ($f \in \Lambda$). Moreover from Corollary I.3.4 one sees that if $c(\pi')$ is non-zero then it is of the form $t(w, f) c(\pi')$ where $t(w, f)$ is computable. Hence $c(\pi')$ determines the $c(\pi')$. Suppose that $f$ lies in a non-free orbit under $W$. Then, as one can easily see, there exists a $w \in W$ and a simple reflection $s$ so that

$$
s[w[f]] = w[f] \pmod{n}.
$$
Replace $f$ by $w[t]$. Then if $s$ corresponds to the simple reflection $(i, i + 1)$ one has that

$$f_i = f_{i+1} - 1 \pmod{n}$$

and, by Corollary I.3.4,

$$c(\pi^{w_0}) = q^{i-h+1-[h+1-h-2]/n} c(\pi^t).$$

But also

$$c(\pi^{w_0}) = (\omega' \mu)(\pi^{w_0}, (\pi^t)^{-1} - 1. c(\pi^t)$$

and so, if $c(\pi^t) = 0$ we must have

$$\omega' \mu(\pi^{w_0}, (\pi^{w_0})^{-1} = -q^{i-h+1-[h+1-h-2]/n}.$$ 

But as $\omega$ is exceptional the absolute value of the left hand side is $q^{i+1/h}$, with $F = f_i - f_{i+1} + 1$, and that of the right-hand side is $q^{r/h+1}$. This is a contradiction; thus $c(\pi^t) = 0$ if $f$ lies in a non-free orbit.

Suppose now that $f$ lies in a free orbit. With the notations above one can easily verify that

a) if $s$ is a simple reflection then $t(s, f).t(s, s[f]) = 1$, and

b) if $s, s'$ are "neighbouring" simple reflections then

$$t(s, s's[s[f]]).t(s', s[f]).t(s, f) = t(s', s's'[f]).t(s, s'[f]).t(s', f).$$

Thus given $c(\pi^t)$ we can inductively define $c(\pi^{w_0})$ for all $w \in W$. If for some $w, w' \in W$ we have

$$\pi^{w_0} = h \cdot \pi^{w_0}$$

with $h \in \hat{H}$ then, as $|w[f]| = |w'[f]|$, we would have $w[f] = w'[f] \pmod{n}$ and, by assumption, this would imply that $w = w'$. Hence $h = 1$. Thus the prescription

$$c(\pi^{w_0}) = (\omega' \mu(h))^{-1} c(\pi^{w_0}) \quad (h \in \hat{H}),$$

$$c(\gamma) = 0 \quad (\gamma \notin \bigcup_{w \in W} \hat{H} \cdot \pi^{w_0}),$$

defines a solution to the system of linear equations which $c$ had to satisfy. If

$$\pi^r \in \bigcup_{w \in W} \hat{H} \cdot \pi^{w_0}$$

then, for some $w \in W$, $f' - w[f]$ is of the form $(a, a, \ldots, a)$ where

$$a(r - 1 + 2a) = 0 \pmod{n}.$$ 

From this the theorem follows at once.

It now remains to prove Lemma I.3.3.

**Proof of Lemma I.3.3.** — We have

$$\tau_s(\gamma, \gamma') = \langle \lambda_{\gamma}, I_s f_{\eta}^{(\eta)} \rangle / \text{meas}((K_+^\eta)^*)$$
for sufficiently large \( m \). This then takes the form
\[
\frac{1}{\text{meas}((K_m^+)^*)} \int_{\mathbb{R}^t} \left( \int_{\mathbb{R}^{t(s)}} f_{\eta}(sn_1 \eta w_0^{-1} n_2) \, dn_1 \right) \tilde{e}(n_2) \, dn_2
\]
where the integrals are to be understood as their regularized values. We shall first evaluate the inner integral. To do so we first determine when \( sn_1 \eta w_0^{-1} n_2 \) can lie in \( \tilde{B}, \eta', K_m^* w_0 = \tilde{B}, \eta' w_0^{-1}(K_m^*)^* \). Suppose that
\[
\begin{align*}
    sn_1 \eta w_0^{-1} n_2 &= b \eta' w_0^{-1} k \quad (b \in \tilde{B}, k \in (K_m^*)^*),
\end{align*}
\]
or
\[
\begin{align*}
    \rho(s) \rho(n_1) &= \rho(b) \rho(\eta') \rho(w_0^{-1} k n_2^{-1} w_0) \rho(\eta)^{-1}.
\end{align*}
\]
We shall express matrices in \( G \) in \( (r_1 + 2 + r_2) \times (r_1 + 2 + r_2) \) block form so that \( \rho(s) \) is
\[
\begin{bmatrix}
    I & 0 & 0 \\
    0 & (0 & 1) & 0 \\
    0 & 0 & I
\end{bmatrix}
\]
We can then write \( \rho(n_1) \) as
\[
\begin{bmatrix}
    I & 0 & 0 \\
    0 & (1 & \xi) & 0 \\
    0 & 0 & I
\end{bmatrix}
\]
with \( \xi \in \mathbb{F}, \rho(b) \) as
\[
\begin{bmatrix}
    b_1 & x_1 & x_2 \\
    o & b_2 & x_3 \\
    o & o & b_3
\end{bmatrix}
\]
with \( b_1, b_2, b_3 \) upper-triangular, \( \rho(\eta) \) (resp. \( \rho(\eta') \)) as
\[
\begin{bmatrix}
    \eta_1 & 0 & 0 \\
    0 & \eta_2 & 0 \\
    0 & 0 & \eta_3
\end{bmatrix} \quad \text{(resp.} \begin{bmatrix}
    \eta_1' & 0 & 0 \\
    0 & \eta_2' & 0 \\
    0 & 0 & \eta_3'
\end{bmatrix}\text{)}
\]
with \( \eta_j \) (resp. \( \eta_j' \)) diagonal, and \( \rho(w_0^{-1} k n_2^{-1} w_0) \) as
\[
\begin{bmatrix}
    \bar{n}_1 & 0 & 0 \\
    u_1 & \bar{n}_2 & 0 \\
    u_2 & u_3 & \bar{n}_3
\end{bmatrix}
\]
where $\tilde{n}_1, \tilde{n}_2, \tilde{n}_3$ are lower-triangular matrices. From
$$\varphi(\omega_1 \varphi(\omega_2 \varphi(w_0^{-1} k n_2^{-1} w_0) \varphi(\omega))^{-1}$$
one obtains after a short computation that
$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0,$$
$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0,$$
$$n_1 = I, \quad n_2 = I$$
and
$$b_1 = \eta_1 \eta_1^{-1}, \quad b_2 = \eta_2 \eta_2^{-1}$$
and
$$\begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix} = b_2 \eta_2^{-1} \tilde{n}_2 \tilde{n}_2^{-1}.$$

One also demands that
$$\begin{bmatrix} \eta_1 \eta_1^{-1} & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & \eta_3 \eta_3^{-1} \end{bmatrix}$$
should belong to $\tilde{B}$. From the last of the equations above
$$b_2 = \begin{pmatrix} -\xi^{-1} & 1 \\ 0 & \xi \end{pmatrix} \eta_2^{-1} \eta_2^{-1}$$
and
$$\tilde{n}_2 = \begin{pmatrix} 1 & 0 \\ \nu \xi & 1 \end{pmatrix}$$
where
$$\nu = \varphi(\omega)^{\omega}.$$

Let now
$$b(\xi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\xi^{-1} & 1 \\ 0 & 0 & \xi \end{pmatrix} \varphi(\omega) \varphi(\omega)^{-1}$$
and
$$\tilde{\mathcal{A}}(\xi) = s \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\xi^{-1} & 0 \\ 0 & 0 & \xi \end{pmatrix}.$$

The integral is then
$$\sum_{\mu \in \mathcal{P}_r} \left( \int_{\Xi'(\omega, \eta, \eta')} (\omega' \mu) (\tilde{\mathcal{A}}(\xi) \eta \eta'^{-1}) d\xi \right) \tilde{e}_0(x)$$
where
$$\Xi'(x, \eta, \eta') = \{ \xi \in F : \nu \xi^{-1} + z \in \mathcal{P}_r, \tilde{\mathcal{A}}(\xi) \eta \eta'^{-1} \in \tilde{B} \}. $$

It must still be regarded as a regularized value, first of the integral inside the braces and then of the sum.
If \( z \) does not represent the zero class of \( \mathbb{F}/\mathbb{P}_q \) then the inner integral is simply

\[
(\omega' \mu)(\widehat{\kappa}_u(- vz^{-1}) \eta \eta^{-1}) \bar{e}_0(z) q^{m-2},
\]

if \( \widehat{\kappa}_u(- vz^{-1}) \eta \eta^{-1} \in \widehat{\mathbb{H}} \) and is zero otherwise.

The integral over \( \Xi'(o, \eta, \eta') \) is

\[
A(m) = \int \omega' \mu(\widehat{\kappa}_u(\xi) \eta \eta^{-1}) d\xi
\]

integrated over those \( \xi \in \nu \mathbb{P}_q^m \) such that \( \widehat{\kappa}_u(\xi) \eta \eta^{-1} \in \widehat{\mathbb{H}} \). This integral "is" a geometric series if we regard it as a sum over subintegrals of the form \( |\xi| = R \). The regularization process simply means that we take the formal sum. It proves however more convenient to compute the integral by an indirect method.

Consider now the sum over \( z \); this sum we also arrange as the sum of the integral over \( \Xi'(o, \eta, \eta') \) and the sum of the subsums over \( |z| = q^j (j > -m) \). In these subsums we replace \( z \) by \( z\theta \) with \( \theta \in \mathbb{R}_q \), \( \theta \equiv 1 \) (mod \( \mathbb{P}_q \)), and sum over all such \( \theta \) modulo \( \mathbb{P}_q \).

Since the conductor of \( \epsilon_0 \) is \( \mathbb{R}_q \) this sum vanishes unless \( |z| \leq q \).

Since the sum in question converges in an open set of \( \omega \) this argument is valid there and the consequence remains valid after regularization. The terms remaining are those with \( |z| = q \), which yield

\[
\sum_{\xi \in \mathbb{R}_q / \mathbb{R}_q} (\omega' \mu)(\widehat{\kappa}_u(- vz^{-1}) \eta \eta^{-1}) \bar{e}_0(z) q^{m-2}
\]

and those with \( |z| \leq 1 \), namely

\[
B(m) = \sum_{\xi \in \nu \mathbb{P}_q^m \setminus 0} (\omega' \mu)(\widehat{\kappa}_u(- vz^{-1}) \eta \eta^{-1}) q^{m-1} |z|^{-2}.
\]

In these last two sums there is an additional restriction on the \( z \), viz. that the argument of \( (\omega' \mu) \) should lie in \( \widehat{\mathbb{H}} \).

Hence \( \tau (s, \omega', \eta, \eta') \) is the sum of

\[
\sum_{\xi \in \mathbb{R}_q / \mathbb{R}_q} (\omega' \mu)(\widehat{\kappa}_u(- vz^{-1}) \eta \eta^{-1}) \bar{e}_0(z) q^{m-2}
\]

and \( A(m) + B(m) \) for \( m \) sufficiently large. Thus \( A(m) + B(m) \) is constant, as a function of \( m \), for \( m \) sufficiently large.

If in the integral defining \( A(m + n) \) we replace \( \xi \) by \( \pi^{-n} \xi_1 \), we see that for all \( m \)

\[
A(m + n) = \omega^n(\pi) A(m).
\]

Moreover, if \( m \geq 0, m \equiv 0 \) (mod \( n \)) one has

\[
B(m + n) = B(m) + \sum_{\xi \in \nu \mathbb{P}_q^m \setminus 0} (\omega' \mu)(\widehat{\kappa}_u(- vz^{-1}) \eta \eta^{-1}) q^{m-n} |z|^{-2}.
\]
In the second term here we replace $z$ by $\pi^m z_1$ and obtain

$$B(m + n) = B(m) + \omega^m(\pi^{m(n)}).B(n).$$

From these it follows firstly that for large $m$, $m \equiv 0 \pmod{n}$, one has

$$\omega^m(\pi) A(m) + B(m) + \omega^m(\pi^{m(n)}) B(n) = A(m) + B(m),$$

and secondly that this equation then holds for all $m \geq o$, $m \equiv o \pmod{n}$. Thus one has, for $m \equiv 0 \pmod{n}$,

$$A(m) = \omega^m(\pi^{m(n)}) (1 - \omega^m(\pi))^{-1}.B(n)$$

and so

$$A(m) + B(m) = A(o) + B(o) = B(n) (1 - \omega^m(\pi))^{-1}.$$

Therefore $\tau(z, \omega', \eta, \eta')$ is the sum of

$$\sum_{s \in \mathbb{F}^d_{/R_p}} (\omega', \mu) \tilde{h}_a(-\nu z^{-1} \eta \eta'^{-1}) \tilde{e}_w(z) \cdot q^{-2}$$

and

$$(1 - \omega^m(\pi))^{-1} \sum_{s \in \mathbb{F}^d_{/R_p}} (\omega', \mu) \tilde{h}_a(-\nu z^{-1} \eta \eta'^{-1}) q^{-n \mid z \mid^{-2}}.$$

We are now in a position to deliver the coup de grâce. In the first term set $z = \gamma/\pi$. Take $\eta$ to be of the form $\eta^i$. Under the assumption made on $\omega'$ the sum becomes

$$\sum_{y \in \mathbb{F}_{/R_p}} (\omega', \mu) \tilde{h}_a(-\nu \pi \eta \eta'^{-1}) \tilde{e}(\nu \pi, y, \gamma^{-1}) q^{-n \mid \gamma \mid^{-2}}.$$

Clearly the class of $\eta'$ in $\tilde{H} \less \tilde{H}$ is determined and we see that it is the class of $\pi^h$. If now $J = f_{i+1} - f_i - 1$

then the sum is

$$q^{-2}(\omega', \mu) \tilde{h}_a(-\nu \pi \eta \eta'^{-1}) g^{(j)} \cdot$$

which after a short computation can be seen to be

$$\theta q^{-2} g^{(j)} \tilde{e}(-1, \pi \eta^i f_{i+1}).$$

Now consider the second term. It is invariant under replacing $z$ by $z\theta$ ($\theta \in \mathbb{R}_x^d$) and from this one sees that the only terms which contribute are those with $\text{ord}(\nu z^{-1}) = o \pmod{n}$. But the class of $\eta'$ in $\tilde{H} \less \tilde{H}$ has to be that of $\eta$ if the sum is not to vanish. Thus we can restrict our attention to the case $\eta' = \eta$, and here we obtain

$$(1 - \omega^m(\pi))^{-1}(1 - q^{-1}) (\omega', \mu) \tilde{h}_a(-\nu z^{-1}) \mid z \mid^{-1}$$

where $z$ is chosen with $0 \leq \text{ord}(z) < n$. Since $\nu = \pi^{i-f_{i+1}}$ we choose

$$z = \pi^f_{i-f_{i+1}-n(f_{i-f_{i+1}}/n)}$$
which yields
\[
\theta(1 - \omega_n^{\sigma}(\pi))^{-1}(1 - q^{-1}) q^{h - h_{s1}} \omega_n^{\sigma}(\tau(h - h_{s1}m));
\]
with this the proof of the lemma is complete.

\section*{1.4. Hecke Theory}

In the last section we constructed the space $\text{Wh}(\Lambda_0(\omega'))$; in this section we shall study the function
\[
\tilde{\Lambda} : \gamma \mapsto \langle \gamma, \pi_0(h) v \rangle
\]
for $\gamma \in \text{Wh}(\Lambda_0(\omega'))$ and $v \in V_0(\omega')$.

To describe the results we regard $V_0(\omega')$ as a subrepresentation of $V(\omega')$. As such it is the image of $\Lambda_0$. Let $c : \tilde{H} \to \mathbb{C}$ be a function satisfying
\[
c(\eta h) = \omega'(\mu(\eta)^{-1}, c(h) \quad (\eta \in \tilde{H}, h \in \tilde{H})
\]
and which satisfies the conditions discussed in § 1.3, namely
\[
\Sigma \tau(s, \omega', \eta, \eta') c(\eta) = 0
\]
for all simple reflections $s$ and $\eta' \in \tilde{H}$. If $v \in V_0(\omega')$ and $\bar{v} \in V(\omega')$ is chosen so that $I_{\omega}(\bar{v}) = v$
then $\Lambda(c) : V_0(\omega') \to \mathbb{C}$ can be defined by
\[
\langle \Lambda(c), v \rangle = \langle \lambda(c), \bar{v} \rangle
\]
and yields a Whittaker functional on $V_0(\omega')$.

With these notations we can state our first main result.

\textbf{Theorem 1.4.1.} — With the notations above, let $c$ be fixed and $v \in V_0(\omega')$. Then there exist numbers $A, A' > 0$ so that
\begin{enumerate}
  \item[a)] if $h \in \tilde{H}$ and for some positive simple root $\alpha$ we have
  \[|\rho(h)| > A\]
  then
  \[\langle \Lambda(c), \pi_0(h) v \rangle = 0\]
  \item[b)] if $h \in \tilde{H}$ and for all positive simple roots $\alpha$ we have
  \[|\rho(h)| \leq A'\]
  then
  \[\langle \Lambda(c), \pi_0(h) v \rangle = \sum c(\eta) \mu(\eta)^{\alpha} v(\eta w_0 h)\]
end{enumerate}
the sum being taken over $\tilde{H} \setminus \tilde{H}$.

\textit{Proof.} — Part a) of the theorem is a triviality as $V_0(\omega')$ is admissible so that we need only give a proof for part b).
To do this first let \( h_1 \in \hat{H}_n \hat{Z} \) and form

\[
u = \pi_0(h_1) v - \omega \omega_\mu(h_1) v;
\]

from Theorem I.2.9 e) it follows that \( \varphi_0(u) = 0 \), where \( \varphi_0 \) is the coinvariants functor of § I.2. Hence there exists an open compact subgroup \( N_0 \) of \( N^+ \) so that

\[
\int_{N_+} \pi_0(n) u \, dn = 0
\]

(cf. [46], Lemma 2.2.1). Thus for \( h \in \hat{H}, \lambda \in \text{Wh}(V_0(\omega')) \) we have

\[
\int_{N_+} \langle \lambda, \pi_0(h) \pi_0(n) u \rangle \, dn = 0.
\]

But for a suitable \( A_1 > 0 \)

\[
|\rho(h)|^\alpha \leq A_1
\]

for all positive simple \( \alpha \) implies that

\[
e \mid hN_0 h^{-1} = 1.
\]

Thus, under this condition we have that the integrand is constant and hence

\[
\langle \lambda, \pi_0(h) u \rangle = 0.
\]

Thus we obtain

\[
\langle \lambda, \pi_0(hh_1) v \rangle = (\omega_\mu(h_1)) \langle \lambda, \pi_0(h) v \rangle.
\]

Since \( V_0(\omega') \) is admissible

\[
\hat{H}_n \hat{Z}/\{ h \in \hat{H}_n \hat{Z} \mid \pi_0(h) v = v \}
\]

is finitely generated and from this one deduces that given \( v \in V_0(\omega') \) \( h \in \hat{H} \) there exists \( A_2 > 0 \) so that the function

\[
h_1 \mapsto \omega_\mu(h_1)^{-1} \langle \lambda, \pi_0(h_1 h) v \rangle
\]

declared on

\[
\{ h_1 \in \hat{H}_n \hat{Z} : |\rho(h_1)|^\alpha < A_2 \text{ for all positive simple } \alpha \}
\]

is constant. We next compute the right-hand side if \( \lambda = \Lambda(e) \).

Replace \( n \) by \( h_1 nh_1^{-1} \). Then we have that this expression is equal to

\[
\sum_{e(\eta)} \bar{e}(h_1 nh_1^{-1}) \int_{N^+} \bar{e}(h_1 nh_1^{-1}) i(\omega_\mu^{-1} nh) \, dn.
\]

As \( h_1 \rightarrow 0 \)

\[
\bar{e}(h_1 nh_1^{-1}) \rightarrow 1
\]
and as the integral is absolutely convergent we see, by Lebesgue's dominated convergence theorem, that
\[
\lim_{h_0 \to 0} \omega \mu(h_0)^{-1} \langle \Lambda(e), \pi_0(h_0 h) v \rangle = \sum \mathbf{c}(\eta) \int_{N^*_0} \bar{\nu}(\eta h_0^{-1} nh) d\eta
\]
\[
= \sum \mathbf{c}(\eta) \mu(\eta)^2 \int_{N^*_0} \bar{\nu}(h_0^{-1} \eta^\omega h) d\eta
\]
The integral is however simply \( I_w(\bar{\nu}) (\eta_w^\omega h) \), or \( \nu(\eta_w^\omega h) \). This proves the theorem.

Let us now assume that \( \| n \|_p = 1 \) and that \( e_0 \) has conductor \( R_p \). If \( \omega \) is unramified and \( \omega' \) normalized we let \( v_0 \in V_\sigma(\omega') \) be that \( K^- \)-invariant element for which one has \( \nu_0(I) = 1 \). Call \( h \in \hat{H} \) weakly dominant if \( |h(h)| \leq 1 \) for all positive roots \( \alpha \).

**Theorem I.4.2.** — With the notations above
\[
\langle \Lambda(e), \pi_0(h) v_0 \rangle = \mathbf{c}(h^\omega)^{-1} \mathbf{c}(h) \mathbf{c}(\eta)^2
\]
if \( h \in \hat{H} \) is weakly dominant and \( 0 \) otherwise.

Note that Theorem I.4.1 is the "asymptotic" form of this theorem.

It will be a consequence of the following theorem, of interest in itself:

**Theorem I.4.3.** — With the notations above, if \( h \in \hat{H} \hat{Z} \) is weakly dominant we have
\[
\int_{N^*_0 \cap K^+} \pi_0(nh) v_0 d\eta = \omega \mu(h) \cdot v_0
\]

**Reminder.** — The assumption that \( \omega' \) was normalized carried with it the assumption that \( K^+ \) is canonical. Hence \( N^*_+ \cap K^+ \) is a lift of \( N^*_+ \cap K \).

**Deduction of Theorem I.4.2 from Theorem I.4.3.** — Let \( h \in \hat{H}, h_1 \in \hat{H} \hat{Z} \) be weakly dominant. Then by Theorem I.4.3 one has
\[
\int_{N^*_0 \cap K^+} \pi_0(hnh_1) v_0 d\eta = \omega \mu(h) \pi_0(h) v_0.
\]

Apply \( \Lambda(e) \) to both sides of this; as
\[
h(N^*_+ \cap K^+) h^{-1} \subset N^*_+ \cap K^+
\]
we obtain
\[
\langle \Lambda(e), \pi_0(hh_1) v_0 \rangle = \omega \mu(h) \langle \Lambda(e), \pi_0(h) v_0 \rangle.
\]
We choose now \( h_1 \) so that \( hh_1 \) lies in the region where Theorem I.4.1 \( b) \) is valid. This theorem yields for the left-hand side the expression
\[
\sum \mathbf{c}(\eta) \mu(\eta)^2 \nu_0(\eta^\omega(hh_1))
\]
and, from the construction of \( \nu_0 \) this is simply
\[
\mathbf{c}(h_1^\omega)^{-1} \cdot \mathbf{c}(h_1)^2.
\]
Thus if \( h \) is weakly dominant we see that
\[
\langle \Lambda(e), \pi_0(h) v_0 \rangle = (\mu(h_1))^{-1} . e((h h_1)^{w}) \mu(h_1)^2
\]
\[
= e((h u_{1})^{-1}) \mu(h)^2.
\]

This proves the required formula when \( h \) is weakly dominant. If \( h \) is not weakly dominant then it is easy to show that \( \langle \Lambda(e), \pi_0(h) v_0 \rangle = 0 \).

We now turn to the proof of Theorem 1.4.3. For this we make use of some auxiliary considerations. Let \( \kappa^* : K \to K^* \) be the canonical lift. Let \( K_1 \subset K \) be the Iwahori subgroup of \( K \), i.e. that subgroup of matrices which are, modulo \( \pi \), upper-triangular; let \( K_1^* = \kappa^*(K_1) \).

**Proposition 1.4.4.** — One has, with the notations above,
\[
\{ v \in V_0(\omega) : \pi_0(h) v = v, \ h \in K_1^* \} = C_v.
\]

We shall first assume this and deduce Theorem 1.4.3. Let \( N_- \) be the group of lower-triangular unipotent matrices and choose \( n' \in \kappa^* (N_- \cap K_1) \). If now \( n \in N_+ \cap K^* \) it follows from the Bruhat decomposition that we can write \( n' n \) in the form
\[
n' n = u(n, n').a(n, n').u'(n, n')
\]
where
\[
u(n, n') \in N_+ \cap K^*
\]
\[
a(n, n') \in \tilde{H} \cap K^*
\]
\[
u'(n, n') \in \kappa^*(N_- \cap K^*).
\]
Moreover one sees that \( n \mapsto u(n, n') \) is a measure-preserving map, by first proving this statement for \( SL_n(F) \) and then applying induction. We shall now verify that, for \( h \in \tilde{H}_n \),
\[
\int_{N_1 \cap K^*} \pi_0(nh) v_0 \ dn
\]
is invariant under \( K_1^* \). To prove this it is sufficient to prove that it is invariant under \( \kappa^*(N_- \cap K_1) \), since it is clearly invariant under \( N_+ \cap K^* \) and \( K^* \cap H \) and these groups together generate \( K_1^* \). Thus, we let \( n' \in \kappa^*(N_- \cap K_1) \) and we shall verify that the vector above is left fixed by \( n' \). One has
\[
\pi_0(n') \int_{N_1 \cap K^*} \pi_0(nh) v_0 \ dn = \int_{N_1 \cap K^*} \pi_0(u(n, n') a(n, n') u'(n, n') h) v_0 \ dn.
\]
Since \( v_0 \) is \( K^* \)-invariant and since \( h \) and \( a(n, n') \) commute the right-hand side is
\[
\int_{N_1 \cap K^*} \pi_0(u(n, n') h) v_0 \ dn.
\]
As \( n \mapsto u(n, n') \) is measure-preserving and is clearly injective it is also surjective. Thus this last integral can be written as
\[
\int_{N_1 \cap K^*} \pi_0(uh) v_0 \ dn,
\]
which demonstrates the invariance claimed. Hence, by Proposition 1.4.4 there exists a scalar \( \theta(h) \) so that

\[
\int_{N^\ast \cap K^*} \pi_0(nh) v_0 \, dn = \theta(h) v_0.
\]

It is easy to see that \( \theta(h) = \omega_\mu(h) \) by applying the functor \( \varphi_\theta \) to this equation. This completes the proof of Theorem 1.4.3, once Proposition 1.4.4 has been proved.

**Proof of Proposition 1.4.4.** — We continue to regard \( V_\varphi(\omega') \) as the image of \( V(\omega') \) by \( L^\ast \) in \( V^\ast /K'_1 \). Then if \( \varphi \in V_\varphi(\omega') \) is \( K'_1 \)-invariant there exists \( \tilde{\varphi} \in V(\omega') \) which is also \( K'_1 \)-invariant and such that \( L^\ast (\tilde{\varphi}) = \varphi \); this follows by the familiar averaging argument since \( K'_1 \) is compact. We shall define a basis \( f_w (w \in W) \) of

\[
Y = \{ v \in V(\omega') : \pi_0(h) v = v \ (h \in K'_1) \},
\]

and it will suffice to show that all the \( I_w (f_w) \) are equal. To that effect it will suffice to show that

\[
I_s (f_w - f_{sw}) = 0
\]

for all simple reflections \( s \). The element \( f_w \) of \( V(\omega') \) is defined to be that \( K'_1 \)-invariant element of \( V(\omega') \) with

\[
\text{Supp}(f_w) = \overline{B}_w K'_1
\]

and

\[
f_w (w) = 1.
\]

It is easy to verify that this exists. To see that these form a basis of such functions we observe that \( \{ \eta w : \eta \in T, w \in W \} \), where \( T \) is set of representatives in \( \widehat{H} \) for \( \widehat{H} \setminus \widehat{H}_1 \), is a set of representatives for \( \widehat{B}_1 \setminus \widehat{G} / K'_1 \). But it is easy to verify that if \( \varphi \) is \( K'_1 \)-invariant then \( \varphi(\eta w) = 0 \) when \( \eta \notin \widehat{H}_1 \). Hence the \( f_w \) \((w \in W)\) do form a base. Moreover we see that if \( \varphi \) is \( K'_1 \)-invariant then \( \varphi(w) = 0 \) \((w \in W)\) implies that \( \varphi = 0 \).

Now let us verify that

\[
I_s (f_w - f_{sw}) = 0
\]

for a simple reflection \( s \). By the principle which we have just explained, it suffices to show that

\[
I_s (f_w - f_{sw}) (w') = 0
\]

for all \( w' \in W \). However, if \( w' \neq w, sw \), then \( I_s (f_w) = 0, I_s (f_{sw}) = 0 \) as the integrand of the integral defining \( I_s \) is zero. It will therefore suffice to show that

\[
I_s (f_w - f_{sw}) (w) = 0,
\]

since the other condition, that with \( w' = sw \), arises from replacing \( w \) by \( sw \).

Thus we have to show that

\[
\int_{N^\ast (a)} f_w (sw) \, dn = \int_{N^\ast (a)} f_{sw} (sw) \, dn.
\]
Let $a$ be the root corresponding to $s$. Then one finds that in order that $s(1 + \xi e_a)^* w$ should lie in $\tilde{B}_s w K_1$ one requires that $|\xi| > 1$ and then

$$s(1 + \xi e_a)^* w = \tilde{h}_a(\xi) (1 + \xi^{-1} e_{-a})^* w$$

where $\tilde{h}_a(\xi)$ has the same meaning as in § 1.3. Since $\omega'$ is exceptional it follows that

$$f_a(s(1 + \xi e_a)^* w) = |\xi|^{-\frac{n+1}{n}}$$

if $\text{ord}(\xi) \equiv 0 \pmod{n}$ and $|\xi| > 1$,

$$= 0$$

otherwise.

Assume now that $\text{meas}(R_p) = 1$.

Then the left-hand side in the equality which we have to prove is

$$(1 - g^{-1}) \sum_{k > 0} q^{\frac{n}{2}k^2} g^{-k(n+1)} = q^{-1}.$$ 

Consider next the right-hand side of the proposed equality. The condition that $(1 + \xi e_a)^*$ should lie in $\tilde{B}_s w K_1$ is $|\xi| < 1$. But the integrand is then $1$ and the integral is again $q^{-1}$. This proves the equality and the proposition.


Our objective in this section is to establish a connection between a numerical invariant derived from the character $\chi_\nu$ of an irreducible, admissible representation $V$ of $\tilde{G}$, and $\dim(\text{Wh}(V))$. This result could be used with a knowledge of the character of $V_0(\omega)$, if this were available, to compute $\dim(\text{Wh}(V_0(\omega)))$ when $|\pi|_p < 1$. We shall give a simple example of this (Theorem I.5.7 below), based on Flicker's formulation of the Shimura Correspondence in [7].

Before we can formulate the main result of this section we have to recall some standard facts about characters. Thus let $(\pi, V)$ be an admissible representation of $\tilde{G}$, and let $f$ be a smooth function of compact support on $\tilde{G}$. Then the map

$$\pi(f) : V \to V; \quad v \mapsto \int f(x) \cdot \pi(x) v \, dx$$

has finite rank, and has therefore a trace, which we denote by $\text{Tr}(\pi(f))$. Since

$$f \mapsto \text{Tr}(\pi(f))$$

is a distribution, we can write it as

$$\int \chi_\nu(x) f(x) \, dx$$

where $\chi_\nu$ is a generalized function (or distribution) on $\tilde{G}$. It is invariant in the sense that $\chi_\nu(x^y) = \chi_\nu(x)$ $(y \in G)$.

Let $G_{\text{reg}}$ be the set of regular elements of $G$, and let $\tilde{G}_{\text{reg}} = p^{-1}(G_{\text{reg}})$. 

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Let now \( \alpha = (r_1, r_2, \ldots, r_n) \) be a partition of \( r \) and let \( \Pi_\alpha \) be a standard parabolic subgroup of \( G \) with Levi component isomorphic to \( \text{GL}_{r_1}(F) \times \cdots \times \text{GL}_{r_n}(F) \). Let \( \Omega_\alpha \) be the character of the representation of \( G \) induced from the identity representation of \( \Pi_\alpha \). Then one can compute the \( \Omega_\alpha \) explicitly (cf. [14] § 20.4, § 21.5, [12] Ch. 2, § 5). We shall not give the explicit formulae here, but we note the following consequences:

1. \( \Omega_\alpha \) does not depend on the choice of \( \Pi_\alpha \);
2. let \( U \) be a neighbourhood of \( I \) in \( G \); then the \( \Omega_\alpha \mid U \) are linearly independent, and
3. let \( \alpha_0 = (1, 1, \ldots, 1) \); then
   \[
   \lim_{h \to 1} \frac{\Omega_\alpha(h)}{\Omega_{\alpha_0}(h)} = 0 \quad (\alpha \neq \alpha_0).
   \]

Let \( \widetilde{\Omega}_\alpha = \Omega_\alpha \circ p \).

**Theorem 1.5.1.** — Suppose that \( \text{Char}(F) = 0 \), and that \( \mathcal{V} \) is an irreducible admissible representation of \( \widetilde{G} \). Then one has:

(i) \( \chi_\mathcal{V} \) can be represented by a locally constant function on \( \widetilde{G}_{\text{reg}} \); (ii) there exists a neighbourhood \( U \) of \( I \) so that \( \chi_\mathcal{V} \) is represented on \( U \) by a locally integrable function; more precisely there exist \( c_\mathcal{V} \in \mathcal{C} \) (\( \alpha \) a partition of \( r \)) such that

\[
\chi_\mathcal{V} \mid U = \sum c_\mathcal{V} \cdot \Omega_\alpha \mid U.
\]

**Proof.** — Part (i) follows by writing the proof of [19], Cor. to Theorem 4 in the context of metaplectic groups. Part (ii) will follow from [18], Theorem 20 once we have verified:

a) there exist \( G \)-invariant open neighbourhoods \( \tilde{Y} \) of \( I \) in \( \widetilde{G} \), and \( Y \) of \( I \) in \( G \) such that \( p \mid \tilde{Y} : \tilde{Y} \to Y \) is a homeomorphism, and

b) the \( \Omega_\alpha \mid U \) have the same span as the "\( \nu_\mathcal{V} \)" of [18].

Since the \( \Omega_\alpha \mid U \) are linearly independent, and since the \( \Omega_\alpha \) actually appear as the germs of characters, b) follows easily.

We shall now verify a). Let \( K_1 \subset \text{GL}_n(R_p) \) be an open normal subgroup such that

(i) there exists a \( \text{GL}_n(R_p) \)-invariant section \( \kappa_1 \) of \( p \) over \( K_1 \) (i.e. \( \kappa_1(x^\gamma) = \kappa_1(x)^\gamma \) if \( x \in K_1, \gamma \in \text{GL}_n(R_p) \)), and
(ii) if \( K_2 = \{ x^\gamma : x \in K_1 \} \) then \( K_2 \) is a subgroup of \( K_1 \),

\[
K_1 \to K_2; \quad x \mapsto x^\gamma
\]

is a bijection and the inverse map is given by the binomial series for \( (I + (y - I))^{1/n} \).

It is easy to see that such a \( K_1 \) exists.

We now let

\[
\tilde{Y} = \kappa_1(K_2)^0, \quad Y = K_2^0;
\]
we have to verify that \( p \) is injective. To do this suppose that \( x, x' \in K \), \( g, g' \in G \) are such that \( (x)^p = (x')^p \); we have to verify that
\[
\kappa_1(x)^p = \kappa_1(x')^p.
\]

To do this we can clearly assume that \( g' = I \). Let \( x = \xi^n \), \( x' = \eta^n \). Since the binomial series for \( (1 + (\xi^n - I)^{1/n}) \) converges and is equal to that for \( ((I + (\xi^n - I))^{1/n}) \), we obtain \( \xi = \eta \). But then \( \kappa_1(x)^p = \kappa_1(\xi^n)^p = (\kappa_1(\xi)^p)^n \); as \( \kappa_1(\xi)^p = \kappa_1(\eta)^p \) for some \( \zeta \in \mu_n(F) \), this yields
\[
\kappa_1(x)^p = \kappa_1(\eta)^n = \kappa_1(\eta^n) = \kappa_1(x').
\]

This completes the proof of the theorem.

We now choose a non-degenerate character \( \epsilon : N_+ \to \mathbb{C} \) as in § I.3 and define \( \text{Wh}(V) \) as there for any admissible representation \( V \) of \( G \).

**Theorem 1.5.2.** — (i). Let \( V \) be an irreducible, admissible representation of \( \tilde{G} \). Then \( \text{Wh}(V) \) is finite-dimensional.

(ii) Suppose that \( V \) is cuspidal (i.e. \( \varphi_p(V) = \{0\} \) for all \( P \neq G \)). Then \( \text{Wh}(V) \neq \{0\} \).

Here \( \varphi_p \) is the functor of coinvariants associated with the parabolic subgroup \( P \) of \( G \) as introduced in § I.2.

**Proofs.** — (i) This seems to be well-known but we could not find a reference; therefore we sketch a proof here. It suffices to prove the statement for \( V \) cuspidal, since, by the obvious analogue to [3] 4.5 for the metaplectic group, the general case can be deduced from this one. Under the assumption that \( V \) is admissible and irreducible there exists a finite subset \( V_0 \) of \( V \) such that \( V = \langle \mathcal{B}, V_0 \rangle \).

Thus if \( \lambda \in \text{Wh}(V) \), \( \lambda \) is determined by the functions
\[
\tilde{H} \to \mathbb{C}; \quad h \mapsto \lambda(\pi(h) v) \quad (v \in V_0).
\]

Since \( V \) is admissible there exists \( \epsilon_1 > 0 \) so that, if for some \( \alpha \in \Phi^+ \) one has
\[
|\rho(h)^\alpha|_\varpi > \epsilon_1,
\]
then
\[
\lambda(\pi(h) v) = 0 \quad (v \in V_0).
\]

We shall show that likewise there exists \( \epsilon_2 > 0 \) so that, if for some \( \alpha \in \Phi^+ \) one has
\[
|\rho(h)^\alpha|_\varpi < \epsilon_2,
\]
then
\[
\lambda(\pi(h) v) = 0 \quad (v \in V_0).
\]

Granted this, it follows that the functions above are determined by a finite collection of their values.

Let now \( \alpha \) be a simple positive root and let \( P_a \supset B \) be that maximal parabolic subgroup of \( G \) whose unipotent radical \( N_a \) contains the unipotent subgroup of \( G \) corresponding to \( \alpha \). Then as
\[
\varphi_{P_a}(V) = 0
\]
there exists a compact open subgroup $Y$ of $N$ such that
$$\int_Y \pi(n) \nu \, dn = 0 \quad (\nu \in V_0).$$
Let $c_2(a)$ be such that, if $|p(h)^a|_F < c_2(a)$, then
$$hYh^{-1} \subset \text{Ker}(\epsilon);$$
this being so we have
$$\lambda(\pi(h) v) = 0,$$
for all $h$ with $|p(h)^a|_F < c_2(a)$. This clearly implies the assertion we needed.

We now come to the central theorem of this section. To formulate it we must first introduce some notations. If $g \in G$ let $P(g)$ be the discriminant of the characteristic polynomial of $g$ and let
$$\Delta(g) = \frac{|P(g)|_{F^2} \det(g)}{|F^{(r-1)^2}|.}$$
If $g$ is split with eigenvalues $\lambda_1, \ldots, \lambda_r$, then one has
$$\Delta(g) = \prod_{i<j} |\lambda_i - \lambda_j|_F \det(g) \frac{|F^{(r-1)^2}|}{F}. $$
Naturally, $\Delta$ is a class-function on $G$; we call it the Weyl factor.

Theorem 1.5.3. — Suppose that $\text{Char}(F) = 0$. Let $V$ be an irreducible, admissible representation of $\tilde{G}$ with character $\chi_V$. Then
$$\dim(\text{Wh}(V)) = \frac{1}{r} \lim_{t \to 1} \Delta(h) \chi_V(s(h)),$$
where $h \in H$ with $h$ regular.

Remark. — It is not difficult to show using Kubota’s formula (following Proposition 0.1.2) that $s \mid H$ is a continuous section of $p \mid \tilde{H}$ (although it is not a homomorphism).

This theorem is a variant of one of Rodier’s [41].

Proof. — The proof of this theorem is based on the relationship between $\tilde{G}$ and the group $\tilde{P}_r = p^{-1}(P_r)$ where $P_r = \{ g \in G : (0, \ldots, 0, 1) g = (0, 0, \ldots, 0, 1) \}$. We shall make use of the structure theory of representations of $\tilde{P}_r$, and, in particular, the notion of derived representations as discussed in [3] § 3. These are considered there only on $P$ but the results apply also to $\tilde{P}$.

We shall paraphrase those results of [3] § 3 which we need. Let $W$ be an algebraic representation of $\tilde{P}_r$. Then there exist representations (derived representations) $W^{(k)}$ of $\tilde{G}_{r-k}$ and a filtration $0 \subset W_r \subset W_{r-1} \subset \ldots \subset W_1 = W$ of $W$ such that
$$W_k/W_{k+1} \cong \text{ind}_{G_{r-k}}^{\tilde{P}_r}(W^{(k)} \otimes \epsilon_{k-1}).$$
where

\[ N_k^* = \{ n \in N^r : p(n)_{ij} = 0 \text{ if } i < j \leq r - k \}, \]

and

\[ e_{k-1}(n) = e_0 \left( \sum_{r-k < i < r} p(n)_{i,i+1} \right) \quad (n \in N_k^*), \]

here the subscripts denote the relevant matrix entries. Note also that \( W^{(k)} \otimes e_{k-1} \) is well-defined on \( \tilde{G}_{r-k}.N_k^* \). We shall here consider the \( W^{(k)} \) as the coinvariants defined without the intervention of the modulus function which is used in [3] 1.8 (b); as we have used the usual induction functor we have also used the corresponding modification in 1.8 (a) (1). Thus this leads merely to a slight difference in definition.

Let \( \varphi \) be a locally constant function of compact support on \( \tilde{G} \). We shall call \( \varphi \) \textit{cuspidal} if for every parabolic subgroup \( \Pi \) of \( G \), such that \( \Pi \subseteq \tilde{P} \) one has

\[ \int_{U \Pi} \varphi(ug) \, du = 0 \]

where \( U \Pi \) is the unipotent radical of \( \Pi \) and \( U \Pi^* \) is the canonical lift of \( U \Pi \) to \( \tilde{P} \). Examples of such functions are given by the matrix coefficients of cuspidal representations of \( \tilde{G} \) restricted to \( \tilde{P} \). Denote the space of cuspidal functions on \( \tilde{P} \) by \( C_0(\tilde{P}) \).

**Proposition 1.5.4.** — Let \( \varphi \in C_0(\tilde{P}) \) and let \( W \) be an algebraic representation of \( \tilde{P} \). Suppose moreover that \( W^{(k)} \) is finite dimensional. Let \( \tau \) (resp. \( \tau_j \)) be the action of \( \tilde{P} \) on \( W \) (resp. \( W_j \)). Then \( \tau(\varphi) \) has finite rank and

\[ \text{Tr}(\tau(\varphi)) = \text{Tr}(\tau(\varphi)), \]

**Proof.** — Since \( \varphi \) is continuous and of compact support one has that \( \tau(\varphi) \) exists and that it respects the filtration \( 0 \subset W_r \subset W_{r-1} \subset \ldots \subset W_1 = W \). We shall show that

\[ \tau(\varphi) \cdot W_1 = \tau(\varphi) \cdot W_r. \]

This will be proved by an inductive argument, of which the \( k \)-th step consists of the statements

\begin{enumerate}
  \item [a)] \( \tau(\varphi) \cdot W_j \subset W_{j+k} \quad (1 \leq j + k < r) \), and,
  \item [b)] if \( \tau(\varphi) : W_j/W_{j+1} \rightarrow W_{j+k}/W_{j+k+1} \quad (1 \leq j + k < r) \)
  is the induced map, then \( \tau(\varphi) = 0. \)
\end{enumerate}

Clearly \( b_k \) implies \( a_{k+1} \)). On the other hand \( a_k \) implies that \( \tau(\varphi) \) is well-defined. If \( k = 0 \) \( b_0 \) also is meaningful and \( a_k \) is a triviality. Now note that \( b_0 \) is true since \( \varphi \in C_0(\tilde{P}) \), and so is \( b_k \) \( (k \geq 1) \) by [3] 3.2. Thus \( a_k \) is true for all \( k \). In particular \( \tau(\varphi) \cdot W_1 \subset W_r \). The filtration gives \( 0 \subset \tau(\varphi) \cdot W_{r-1} \subset \ldots \subset \tau(\varphi) \cdot W_1 \subset W_r \). Observe that the natural induced maps

\[ W_j/W_{j+1} \rightarrow \tau(\varphi) \cdot W_j/\tau(\varphi) \cdot W_{j+1} \quad (1 \leq j < r) \]

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are surjective, and \( o \) (by \( b^o \)); hence it follows that
\[
\tau(\varphi) W_1 = \tau(\varphi) W_2 = \ldots = \tau(\varphi) W_r
\]
as required.

Thus it remains to prove that \( \tau(\varphi) \) has finite rank in the case that \( (\tau, W) \) is
\[
(\tau_0, W_0) = \text{ind}_{\mu(P) \times H}^{\bar{P}}(\varepsilon \times \varepsilon).
\]
The representation space for this latter representation consists of locally constant functions \( f \) on \( \bar{P} \) such that
\[
\begin{align*}
a) & \quad f(g, i(\zeta)) = \varepsilon(\zeta) f(g) \quad (\zeta \in \mu(P), g \in \bar{P}) \\
b) & \quad f(n g) = \varepsilon(n) f(g) \quad (n \in N, g \in \bar{P}) \\
c) & \quad |f|, \text{ which is defined on } N \backslash P, \text{ has compact support.}
\end{align*}
\]
Then
\[
(\tau_0(\varphi)f)(g) = \int_{\bar{P}} f(\gamma) \varphi(g^{-1} \gamma) \, d\gamma,
\]
and \( d\gamma \) is a left-invariant Haar measure on \( \bar{P} \).

Since there is a compact subgroup \( K_1(\varphi) \), which depends only on \( \varphi \), such that
\( \tau(\varphi)f \) is \( K_1(\varphi) \)-invariant, it will suffice to show that there exists a compact subset \( K(\varphi) \),
which again depends only on \( \varphi \), such that
\[
\text{Supp}(\tau_0(\varphi)f) \subset N. K(\varphi).
\]
To demonstrate this we shall use induction on \( r \).

Let us write matrices in \( P \) in \( ((r - 1) + 1) \times ((r - 1) + 1) \)-block form; let
\( N_r \) be the subgroup of matrices of the form
\[
\begin{pmatrix}
I & x \\
0 & 1
\end{pmatrix}
\]
(\( x \in F^{r-1} \)), and \( G_{r-1} \) the subgroup
\[
\begin{pmatrix}
g & 0 \\
o & 1
\end{pmatrix}
\]
(\( g \in \text{GL}_{r-1}(F) \)). Let \( n \mapsto n^* \) be the canonical lift of \( N_r \) to \( \bar{P} \) and
let \( \bar{G}_{r-1} = P^{-1}(G_{r-1}) \). We have a bijection
\[
F^{r-1} \times \bar{G}_{r-1} \rightarrow \bar{P}; \quad (x, g) \mapsto \begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}^* \cdot g.
\]
We can therefore parameterize \( \bar{P} \) by this map, the multiplication becomes
\[
(x, g) \cdot (x', g') = (x + gx', gg'),
\]
where \( gx' \) is, by definition, \( \psi(g) x' \).

In the above integral we now replace \( g \) and \( \gamma \) by \( (o, g_1) \) and \( (\xi, \gamma_1) \). The left-invariant measure can be taken to be
\[
d\gamma = |\det(\gamma_1)| |x|^{-1} |d\xi| \cdot d\gamma_1
\]
where \( d\gamma_1 \) is a bi-invariant Haar measure on \( \bar{G}_{r-1} \). Thus
\[
(\tau_0(\varphi)f)(o, g_1)
= \int_{\bar{G}_{r-1}} f(o, \gamma_1) |\det(\gamma_1)| |x|^{-1} \int_{F^{r-1}} \varepsilon_0(\bar{G}_{r-1}) \cdot \varphi(\bar{G}_{r-1} \xi, g_1^{-1} \gamma_1) \, d\xi \, d\gamma_1.
\]
It is now convenient to replace $\xi$ by $g_l \xi$; then the right-hand side of the above expression becomes

$$\int_{\mathcal{O}_{r-1}} f(o, \gamma_1) \left| \det(g_l^{-1} \gamma_1) \right| p^{-1} \int_{\mathcal{O}_{r-1}} e_0(\sum_{j=1}^{r-1} g_{r-1,j} \xi_j) \varphi(\xi, g_l^{-1} \gamma_1) |d\xi| \, dq.$$ 

In this expression $\xi = (\xi_1, \ldots, \xi_{r-1})$ and $g_0$ is the $ij$-th matrix coefficient of $p(g_l)$.

Note that there exist compact sets $K_1 \subset F^{-1}$, $K_2 \subset G_{r-1}$ such that

$$\text{Supp } \varphi \subset K_1 \times K_2$$

and also that $\varphi$ is locally constant on $N \times G_{r-1}$.

It follows from the last fact that there exist $c_1 > 0$ such that the integral over $\xi_j$ is zero unless $|g_{r-1,j}|_F \leq c_1$. Secondly, as $\varphi$ has compact support in the above sense, there exists a constant $c_2 > 0$ such that, if $|g_{r-1,j}|_F \leq c_2$ for all $j$, then

$$\int_{\mathcal{O}_{r-1}} e_0(\sum_{j=1}^{r-1} g_{r-1,j} \xi_j) \varphi(\xi, g_l^{-1} \gamma_1) |d\xi| = \int_{\mathcal{O}_{r-1}} \varphi(\xi, g_l^{-1} \gamma_1) \, d\xi = 0$$

since $\varphi$ is cuspidal.

It follows now that in order that $(\tau(\varphi), f)(o, g_l)$ should be non-zero we need that $|g_{r-1,j}|_F \leq c_1$ for all $j$, and that there should exist at least one $j$ such that $|g_{r-1,j}|_F > c_2$. It follows easily from this that there is a compact subset $K_3$ of $G_{r-1}$ such that $\tau(\varphi), f(o, g_l) = 0$ unless $g_l \in \overline{P}_{r-1} K_3$, where $\overline{P}_{r-1}$ is the subgroup of $G_{r-1}$, defined as $\overline{P}$ was in $G$. Moreover, in the integral representation of $\tau(\varphi), f(o, g_l)$ above $g_l^{-1} \gamma_1 \in K_2$; thus the integral need only be extended over $\overline{P}_{r-1} K_3 K_2$. Observe here that $K_3, K_2$ is also compact.

We have now to show that there exists a compact set $K_4 \subset G_{r-1}$ such that, if $g_l \in K_2$, then the map

$$\overline{P}_{r-1} \to \mathbb{C}; \quad p \mapsto \tau(\varphi), f(o, pg_l)$$

has support contained in $(N_r \cap \overline{P}_{r-1}) K_4$. However, since $\tau(\varphi), f(o, g)$ is, as a function of $g$, right-invariant under an open subgroup which depends only on $\varphi$, we see that it suffices to prove this latter statement for any fixed $g_l \in K_3$. This we shall do by reducing the question to $\overline{P}_{r-1}$, when it will be justified by the induction hypothesis.

Suppose that $\varphi(\xi, g)$ is, as a function of $g$, right-invariant under a compact open subgroup $K_5 \subset G_{r-1}$. Then we can assume that $f(o, g)$ is too, since we can average it over $K_5$. Thus there exists a finite set $\gamma_{1,j} \in G_{r-1}$ ($1 \leq j \leq N$), and positive constants $c_j$ ($1 \leq j \leq N$), so that $\tau_\varphi(o, pg_l)$ (where $p \in \overline{P}_{r-1}$) is given by

$$\sum_{1 \leq j \leq n} c_j \int_{\overline{P}_{r-1}} f(o, q\gamma_{1,j}) \left| \det(p^{-1} \gamma_{1,j}) \right| p^{-1} \left| \det(g_l^{-1} \gamma_{1,j}) \right| p^{-1} \int_{\overline{P}_{r-1}} e_0(\sum_{k=1}^{r-1} g_{r-1,k} \xi_k) \varphi(\xi, g_l^{-1} \gamma_{1,j}) |d\xi| \, dq.$$
where \( dq \) is a left-invariant Haar measure on \( \mathbb{P}_{r-1} \). Thus our contention will be proved, by virtue of the induction hypothesis, if we verify that the function on \( \mathbb{P}_{r-1} \) given by

\[
\rho \mapsto |\det(\rho)|^{-1} \int_{\mathbb{P}_{r-1}} \sum_{1 \leq k \leq r-1} \epsilon_{\theta}(a_{g_{r-1}} k \xi_{k}) \varphi(\xi, g_{r-1}^{-1} \rho g_{r-1}, i) \, d\xi
\]

satisfies the same conditions on \( \mathbb{P}_{r-1} \) as \( \varphi \) did on \( \mathbb{P}_r \); for then the integral above can be interpreted as a "\( \tau_0(\varphi), f(\rho) \)" and the conclusion follows.

This function is clearly of compact support and locally constant. That it is cuspidal is almost immediate from the corresponding condition on \( \varphi \). Hence the induction is completed, and Proposition 1.5.4 is proved.

We next need the existence of sufficiently many \( \varphi \). We let \((\tau_0, W_0)\) be the representation of \( \bar{\mathbb{P}} \):

\[
(\tau_0, W_0) = \text{ind}_{\mathbb{N} \times u \mathbb{F}}(e \times e).
\]

**Lemma 1.5.5.** — Let \( L \subset \bar{\mathbb{P}} \) be an open compact subgroup of \( \bar{\mathbb{P}} \). Then there exists \( \varphi \in C_0(\bar{\mathbb{P}}) \) such that

\[
\text{Supp}(\varphi) \subset L
\]

and

\[
\text{Tr}(\tau_0(\varphi)) \neq 0.
\]

**Proof.** — We shall first show that there exists such a \( \varphi \neq 0 \) but without the restriction that \( \text{Tr}(\tau_0(\varphi)) \neq 0 \). We can also assume that \( L \) is of the form \( K_\infty \cap \bar{\mathbb{P}} \) where \( K_\infty \) is the lift of

\[
K_m = \{ k \in GL_n(\mathbb{R}_p) : k = I \ (\text{mod } \pi^m) \},
\]

the lift exists for \( m \) large enough.

It will now suffice to show that there exists a non-zero locally constant function \( \varphi_1 \) on \( \bar{\mathbb{G}} \) with \( \text{Supp}(\varphi_1) \subset K_\infty \) such that for every parabolic subgroup \( \Pi \subset G \) one has

\[
\int_{u \Pi} \varphi_1(u \gamma) \, du = 0.
\]

This is so since, for some \( k \in K (= GL_n(\mathbb{R}_p)) \), one will have that the function \( \rho \mapsto \varphi_1(\rho^k) \) on \( \bar{\mathbb{P}} \) is non-zero.

Let now \( k \in K \) be such that its image under the reduction map

\[
GL_n(\mathbb{R}_p) \to GL_n(\mathbb{R}_p/\pi^m)
\]

is elliptic. We let \( \alpha \in F^\times \), to be chosen later. Then define

\[
\varphi_1(x) = \epsilon_{\theta}(\alpha \text{Tr}(xh)) \quad (x \in K_\infty)
\]

\[
= 0 \quad \text{(otherwise)}.
\]

Here we are identifying \( K_\infty \) with \( K_m \) in writing \( \text{Tr}(xh) \). We have clearly only to show that \( \varphi_1 \) is cuspidal. To do this let \( \Pi \) be parabolic in \( G \). Then \( \Pi \) is conjugate to a standard parabolic subgroup \( \Pi_1 \), and \( U_\Pi \) is then also conjugate, by the same element, to \( U_{\Pi_1} \).
However, $\Pi_1$ normalizes $U_{\Pi_1}$ and $G = \Pi_1 K$ so that there exists $k \in K$ such that $U_\Pi = (U_{\Pi_1})^k$. Thus it suffices to show that

$$\int_{(U_{\Pi_1})^k} \varphi_L(u^k g) \, du = 0$$

for all $k \in K$ and all standard parabolic subgroups $\Pi \subseteq G$. Now since $K^*_m$ is (for $m$ large enough) normalized by $K$ it follows that one can assume that $k = 1$. It then follows from the Iwasawa decomposition that if $(U_{\Pi})^* g \cap K^*_m \neq 0$ then $g \in U_{\Pi}^* K^*_m$. Thus we need only show that

$$\int_{U_{\Pi}^* \cap K^*_m} \varphi_L(uk) \, du = 0$$

where $k \in K^*_m$. However, written out explicitly this is

$$\int_{U_{\Pi}^* \cap K^*_m} \varphi_L(x \cdot \text{Tr}(uhk)) \, du = 0.$$

Let $\Pi'$ be the opposite parabolic subgroup to $\Pi$; then since $kh$ has elliptic reduction (mod $\pi$)

$$\rho(kh) \notin K_1 (\Pi' \cap K).$$

Thus

$$\text{Tr}((kh) \rho(U_{\Pi}^* \cap K^*_m)) = \pi^n R_{\mathbb{F}}$$

and so, for a suitable fixed $\alpha$ (depending only on $m$) the above integral is $0$. Thus $\varphi_L$ is cuspidal.

Now let $\varphi$ be as constructed. We observe that there exists an algebraic representation $(\rho, Y)$ of $\overline{P}$ such that $\rho(\varphi) \neq 0$ (for example the regular representation). Thus from the discussion in the proof of Proposition 1.5.4 one has $\rho_r(\varphi) \neq 0$. Since $\rho_r$ is of the form $\tau_0 \otimes \chi$, where the action of $\overline{P}_r$ on $X$ is trivial, it follows also that $(\tau_0)_r(\varphi) \neq 0$. Note that

$$(\tau_0, W_0) = \text{ind}_{\overline{P}_r \times U_{\Pi}/P}(\epsilon \times \varepsilon)$$

is unitarizable. Hence if $\varphi' = \varphi \ast \tilde{\varphi}$ one has

a) $\tau_0(\varphi') \neq 0$ and $\text{Tr}(\tau_0(\varphi')) \neq 0$

b) $\text{Supp}(\varphi') \subseteq I$, and

c) $\varphi' \in C_0(\overline{P})$,

as one sees at once. Thus the lemma is proved.

We are now in a position to prove Theorem 1.5.3. We shall consider the restriction of $V$ to $\overline{P}$. This is algebraic and the multiplicity of $(\tau_0, W_0)_r$ in $V_r$, or, for that matter $V$, is $\dim(\text{Wh}(V))$; this we write as $N(V)$, which is finite by Theorem 1.5.2. Hence by Proposition 1.5.4 one has, if $\pi$ denotes the action of $\overline{G}$ on $V$ and if $\varphi \in C_0(\overline{P})$,

$$\text{Tr}(\pi(\varphi)) = N(V) \cdot \text{Tr}(\tau_0(\varphi)).$$
However, let $L \subset \tilde{G}$ be a sufficiently small compact open subgroup, and let $\chi_L$ be the normalized characteristic function of $L$. Here "sufficiently small" means that the image of $\pi(\varphi)$ is to be $L$-invariant. Then $\chi_L \ast \varphi$ is locally constant and of compact support on $\tilde{G}$, and $\pi(\varphi) = \pi(\chi_L \ast \varphi)$. Thus we see that

$$\langle \chi_V, \gamma_L \ast \varphi \rangle = N(V) \text{Tr}(\tau_0(\varphi)).$$

From Theorem 1.5.1 we have that there exists an open neighbourhood $\mathcal{O}$ of $I$ in $\tilde{G}$ and an expansion of the following form:

$$\chi_V \mid \mathcal{O} = \sum_a c_a(V) \tilde{\Omega}_a \mid \mathcal{O}.$$ 

We now choose $L \subset \mathcal{O}$ and $\varphi$ such that $\text{Supp}(\varphi) \subset \mathcal{O}$. Then we verify at once that

$$\langle \tilde{\Omega}_a, \chi_L \ast \varphi \rangle = 0$$

for all $a \neq a_0$, where $\Pi_{a_0} = B$, since $\chi_L \ast \varphi$ is cuspidal (or since $\text{dim}(\text{Wh}(\text{ind}^{G}_{H}(\tau))) = 0$). Thus

$$\langle \chi_V, \chi_L \ast \varphi \rangle = c_{a_0}(V) \langle \tilde{\Omega}_{a_0}, \chi_L \ast \varphi \rangle;$$

hence there exists a constant $\gamma_1$ such that

$$N(V) = \gamma_1 c_{a_0}(V).$$

This involves $\chi_V$ only in a neighbourhood of $I$; since the representation of $G$ of which $\Omega_{a_0}$ is the character has a unique Whittaker model it follows that $\gamma_1 = 1$. Thus one has

$$c_{a_0}(V) = \lim_{h \to 1} \Omega_{a_0}(h)^{-1} \chi_V(s(h))$$

where $h$ is taken to be regular. It then follows that

$$N(V) = (r!)^{-1} \lim_{h \to 1} \Delta(h) \chi_V(s(h))$$

since, in a neighbourhood of $I$ on $\tilde{H}$

$$\Omega_{a_0}(h) = r!/\Delta(h),$$

as one can see from Proposition 1.5.6 below. This proves the theorem.

Let $V(\omega)$ be the representation defined in § I.1. Then one has

**Proposition 1.5.6.** — The character of $V(\omega)$ is represented by a locally integrable function $\chi_{V(\omega)}$ which is supported on $(\tilde{H}^n \tilde{Z})^0$ and whose value at a regular element $h$ of $\tilde{H}^n \tilde{Z}$ is given by

$$[\tilde{H} : \tilde{H}] \Delta(h)^{-1} \sum_{w \in W} w_\omega(h).$$
Remarks. — 1. This proposition is a generalization of the computation of characters of principal series representations, which have been known for a long time (cf. [14] § 20.4, [12] Ch. 2, § 5).

2. If \( n = 1, \omega = 1 \) we obtain the formula for \( \Omega_n \), used above.

3. From Theorem I.5.3 it would follow that
\[
\dim \text{Wh}(V(\omega)) = [\tilde{H}: H];
\]
this agrees with Lemma I.3.2.

4. In the case \( r = 2 \) this formula was given by Flicker [7] p. 141, but the factor \( [\tilde{H}: H] \) has been inadvertently omitted.

Proof. — This is a generalization of [7] Lemma 2.1, and the proof is essentially the same. We shall merely note the modifications which have to be made. Firstly, if \( h \in H \) then the Jacobian of \( N_+ \to N_+ \); \( n \mapsto nhn^{-1}h^{-1} \) is \( \Delta(h)/\mu(h) \); thus the computation at the bottom of p. 141 of [7] can easily be generalized. The other point to note is that in [7] p. 142 the author has omitted the integration (in his terminology) over \( A_0 \backslash A \) and this accounts for the missing factor \( [\tilde{H}: H] \).

Theorem I.5.7. — Let \( n = 3, r = 2 \) and \( e = 1 \pmod{3} \). Then
\[
\dim \text{Wh}(V_0(\omega)) = 1.
\]

Proof. — In the sense of [7], Definition 1, p. 172 and Theorem 5.2 the representation \( V_0(\omega) \) corresponds to a one-dimensional representation of \( G \), which, by computing its \( N_+^* \)-coinvariants by [7] Theorem 5.2 is seen to be \( \chi \) where
\[
\chi(h) = \omega(s(h)) \mu(h)^{-1} \quad (h \in H).
\]
It is then easy to compute \( \dim(\text{Wh}(V_0(\omega))) \) by Theorem I.5.3. This yields the result quoted.

Remarks. — Theorem I.5.7 is also a special case of Corollary II.2.6. The proof sketched here is quite different and appears to us to give different insights. We plan to return to the discussion of the generalization of the Shimura Correspondence as formulated in [7] and its consequences in a later publication.

I.6. Archimedean fields

In this section we shall discuss the analogues of the results obtained in the previous sections of this chapter for archimedean fields. We have three distinct cases to consider. These are
1. \( n = 1, F = \mathbb{R}, \mathbb{C} \)
2. \( n = 2, F = \mathbb{R} \)
3. \( n > 1, F = \mathbb{C} \).
In cases 1 and 3 $G$ is isomorphic to $G \times \mu_n(F)$, and we need only review some well-known results. In case 2 the situation is a little more involved, but since $G$ is a real reductive Lie group, locally isomorphic to $G$, it is also covered by the general theory.

We shall begin by defining the analogues of the $V(\omega')$ and discussing the intertwining operators. Then we shall consider the irreducibility of the representations, as in Theorem 1.2.9. In cases 1 and 2 the results are analogous to those of §I.2; in case 3 the "relevant" $V(\omega)$ is actually irreducible. Finally we shall derive those results concerning Whittaker models which we shall later need.

We define $H$, $N_+$ as before, likewise $H$, $H_0$, $N^+$. In cases 1, 3 $H_n = H \cong H \times \mu_n(F)$, whereas in case 2 the index of $H_n$ in $H$ is $2^{|\rho|}$. We let $H_n = \rho^{-1}(b^\alpha : h \in H)$; this is $H$ in cases 1 and 3, and of index $2^i$ in case 2. Let $Z$ be the centre of $G$ (and of $B$); it is $\rho^{-1}\{\lambda : \lambda \in F^\times\}$. Thus in cases 1 and 3, $Z = \rho^{-1}\{\lambda : \lambda \in F^\times\}$, and in case 2

$$Z = \rho^{-1}\{\lambda : \lambda \in R^\times\} \quad (r \text{ odd})$$

$$= \rho^{-1}\{\lambda : \lambda \in R^\times\} \quad (r \text{ even}).$$

The function $\mu$ can be defined as before; note, however, that we shall always take $|x|_0 = \tilde{x}^\alpha$.

Now let $\omega$ be a quasicharacter of $\hat{H}_n \hat{Z}$, extending $\varepsilon : \mu_n(F) \to C^\times$, and let $\omega'$ be an extension of $\omega$ as a quasicharacter to $\hat{H}_n$. We let $V(\omega')$ be the space of functions $f : G \to C$ which satisfy

(i) $f(bg) = (\omega' \mu)(b) f(g) \quad (b \in \hat{B}_n(= \hat{H}, N^+))$

where we have extended $\omega'$ to $\hat{B}_n$ by requiring that $\omega' | N^+ = 1$,

(ii) $g \mapsto f(g)$ is a smooth function.

Now we define, as before, $L(\omega) : V(\omega) \to V(\omega')$ by

$$(L(\omega) f)(g) = \int_{N^+} f(w^{-1} ng) \, d\alpha$$

which, as we shall see below, converges when $\omega$ is dominant.

Let us also define $\omega_n$ and $\sigma(\omega)$ as before. We shall define, as usual, the $L$-function associated with a quasicharacter $\chi$ of $F^\times$ as follows:

$$L(\chi) = \pi^\frac{\ell}{2} \Gamma(s) \quad F = R, \quad \chi(x) = |x|_b^\alpha, \text{ or } |x|_b^{-\frac{\ell}{2}} x^{-1}$$

$$= (2\pi)^{1-s} \Gamma(s) \quad F = C, \quad \chi(x) = |x|^\alpha, \text{ or } |x|^\alpha x^{-m} \quad (m \geq 0).$$

We can also define $\omega_n, f$, exactly as in § I.2.

The analogue to Proposition 1.2.3 (a) is clearly valid. Consequently, as in § I.2, most of the problems involving $L_\omega$ can be reduced to the corresponding problems in the case of $r = 2$. In particular, one has the following analogue to Proposition 1.2.3 (b).
Proposition 1.6.1. — If \( w \in W \) then the integral defining \( I_w \) converges if \( \langle \alpha, \sigma(\omega) \rangle > 0 \) for all \( \alpha > 0 \) such that \( wx < 0 \) and the function \( t \mapsto I_w(f_t) \) is analytic in the corresponding region. Moreover the function
\[
 t \mapsto \prod_{\alpha > 0, \omega < 0} L((\omega, \omega)_{\alpha}^{-1}) I_w(f_t)
\]
can be extended from its original domain of definition, viz.
\[
 \{ t : \text{Re}(\langle \alpha, t \rangle) > - \langle \alpha, \sigma(\omega) \rangle, \quad \text{for} \quad \alpha > 0, \; wx < 0 \},
\]
to a holomorphic function on \( \Phi(C) \).

We shall not give the proof in detail here; we remark merely that it reduces to the case where \( w = s \), a simple reflection, in which case we can treat the integral directly in order to determine its poles. They are amongst those of \( L((\omega, \omega)_{\alpha}) \) and the holomorphicity follows from this. One should note that the holomorphic function above has zeros.

We shall now formulate a version of Lemma 1.2.5. As in § 1.2 we can define \( \alpha : V(\omega') \to V(\omega) \), in the case that \( \omega'' = 1 \) where \( \alpha \) is the positive simple root corresponding to \( s \).

Lemma 1.6.2. — Suppose that \( s \) is a simple reflection associated with the root \( \alpha \). Then, for \( f \in V(\omega') \), one has
\[
 \alpha_n \lim_{t \to 0} L(| f(\omega, \omega)| (\omega, \omega)_{\alpha}^{-1}) I_{s^t}(f) = A f
\]
where
\[
 A = n^{-1} \int \nu A_0
\]
and \( A_0 \) is the measure of \( \overline{H}_1 \cap \overline{H}_s \) used in the definition of \( \alpha_s \).

Remark. — As usual here, we take the measure associated with \( dx \) on \( \mathbb{R} \), and with \( dz \wedge d\bar{z} \) on \( \mathbb{C} \).

Proof. — We write \( \xi \) for the complex conjugate of \( \xi \) when \( F = \mathbb{C} \), and \( \bar{\xi} = \xi \) when \( F = \mathbb{R} \). Then one readily verifies that
\[
 \lim_{t \to 0} L(| f(\omega, \omega)| (\omega, \omega)_{\alpha}^{-1}) I_{s^t}(g)
\]
is equal to
\[
 \lim_{t \to 0} L(| f(\omega, \omega)| (\omega, \omega)_{\alpha}^{-1}) \int_{|\xi| > 1} (\omega, \mu)_{\alpha}(1 + \xi \xi) - 1 f(\tilde{h}_s(\xi) \rho_0) d\xi
\]
where \( \rho_0 \) is as in § 1.2 and \( h_s(\xi) = \tilde{h}_s(\xi/|\xi|^2) \), with \( \tilde{h}_s \) as before.

When \( F = \mathbb{R} \), we set \( R_o = \{ 1, \; -1 \} \) and the limit becomes
\[
 n \sum_{z \in \mathbb{R}_s} f(\tilde{h}_s(z) \rho_0 g)
\]
from which it follows that
\[ \alpha_i \lim_{i \to 0} L\left( \left| \zeta \omega^{\alpha} \right| L\left( \left| \omega \right| \zeta \omega^{\alpha} \right) \right) = \frac{1}{n} f^{(\alpha)} \]
is equal to \( \Lambda f(\delta) \) both when \( n = 1 \) and \( n = 2 \). This proves the lemma in this case.

When \( F = \mathbb{C} \), then \( f(h^\ast(\delta) \eta_0 \delta) = f(\gamma_0 \delta) \) (as \( \omega = 1 \)) and then the limit becomes \( n f(\gamma_0 \delta) \). This yields, for \( \alpha_i \lim_{i \to 0} L\left( \left| \zeta \omega^{\alpha} \right| L\left( \left| \omega \right| \zeta \omega^{\alpha} \right) \right) = \frac{1}{n} f^{(\alpha)} \), the value \( n f(\gamma_0 \delta) \), as required.

As a consequence of Proposition I.6.1, \( \prod_{\alpha > 0} L(\omega^{\alpha})^{-1} I_{\omega} \) can be given a meaning for all \( \omega \); this we understand to be the regularization of \( I_{\omega} \). Note that it involves a process of analytic continuation.

Since \( \prod_{\alpha > 0} L(\omega^{\alpha})^{-1} I_{\omega} \) is sometimes zero we need some finer information, which will be given by the analogue of Theorem 1.2.6. We define
\[ I_{\omega} = \prod_{\alpha > 0} \left( \prod_{\omega < 0} \left| \zeta \omega^{\alpha} \right| L\left( \left| \zeta \omega^{\alpha} \right| \zeta \omega^{\alpha} \right) \right) I_{\omega} \]
which is defined at generic \( \omega \) but which does have singularities.

**Theorem I.6.3.** — \( I_{\omega_1 \omega_2} = I_{\omega_1} I_{\omega_2} \).

**Proof.** — We begin by making some simplifications. First of all, as in the proof of Theorem I.2.6, we can restrict ourselves to proving that \( I_{\zeta}^2 = I \), where \( \zeta \) is a simple reflection. Next, we note that when \( F = \mathbb{C} \),
\[ \left| \zeta \omega^{\alpha} \right| L\left( \left| \zeta \omega^{\alpha} \right| \zeta \omega^{\alpha} \right) = L\left( \left| \zeta \omega^{\alpha} \right| \zeta \omega^{\alpha} \right) \]
where \( \omega(x) = \omega(h_0(x)) \); thus in this case we can assume \( n = 1 \).

It would be possible to mimic the proof of Theorem I.2.6, but it is easier here to exploit the theory of Eisenstein series as we did at the end of the proof of Theorem I.2.6.

Let us first deal with the case \( F = \mathbb{C} \). Since we can take \( n = 1 \) we choose an imaginary quadratic field \( k \) (eg. \( \mathbb{Q}(\sqrt{-1}) \)). Then, since we did not need the theory of Eisenstein series to prove the functional equation at those places \( \omega \) with \( \left| \zeta \right|_\omega = 1 \), we can assume that Theorem I.2.6 is valid for all finite places of \( k \). Since the global functional equation is also known, its validity at the one archimedean place of \( k \) follows immediately.

The same proof, with \( n = 1, k = \mathbb{Q} \), shows that the assertion is also valid for \( F = \mathbb{R}, n = 1 \).

Now let \( k' \) be an imaginary quadratic field with \( k' \otimes \mathbb{Q} \mathbb{Q}_2 \cong \mathbb{Q}_2 \otimes \mathbb{Q}_2 \). The proof of Theorem I.2.6 applies to this case and so we derive the validity of Theorem I.2.6 for \( F = \mathbb{Q}_2 \), with \( n = 2 \). Now finally we can apply the argument above to \( \mathbb{Q} \) to derive
the validity of Theorem 1.6.3 in the case $F = R, n = 2$. The remaining cases of Theorem 1.2.6 now follow.

We next call an $\omega$ as above exceptional if for all positive simple $\alpha$ one has $\omega^\alpha = |\gamma|_F$. We also let, as before, $V_0(\omega')$ be the image in $V(^{w_0}\omega')$ of $V(\omega)$ under $I_{w_0}$. We suppose here that $\omega'$ is exceptional. Then one has

Theorem 1.6.4. — The representation $V_0(\omega')$ is the unique irreducible subrepresentation of $V(^{w_0}\omega')$ and the unique irreducible quotient representation of $V(\omega')$. Moreover a) if $n = 1$ then $V_0(\omega')$ is the one-dimensional representation the restriction of which to $H$ is $w_0\omega_\mu$, b) if $n = 2, F = R$ then $V_0(\omega')$ is a proper subrepresentation of $V(^{w_0}\omega')$ and a proper quotient representation of $V(\omega')$, c) if $r \leq n, F = C$, $V_0(\omega') \cong V(\omega') \cong V(^{w_0}\omega')$.

Proof. — The initial statement follows from the general principles of [31]. Statement a) is well-known. To prove c) regard $G$ as $G \times \mu_n(k)$. Then $V(\omega) \cong (W \otimes \chi) \times \varepsilon$ where $\chi$ is a one-dimensional representation and $W$ is induced from the representation $1^{\mu_n}. \mu$ of $H$. Since, if $r \leq n$ there is no root $\alpha$ for which $1^{\mu_n_\alpha} = |\gamma|_F$ it follows that $W$ is a non-degenerate representation in a complementary series.

To prove b) we check that if $s$ is a simple reflection associated with the simple root $\alpha$, then the composite map

$V(^{w_0}\omega') \xrightarrow{1^{\omega_\mu_\alpha}^{-1} I_s} V(\omega') \xrightarrow{I_s} V(\omega')$

is 0. Thus, by Theorem 1.6.3, the image of $V(\omega')$ under $I_s$ lies in the kernel of $I_{w_0}$. (One observes that, by Proposition 1.6.1, both maps are defined, that the composite is 0 follows from Theorem 1.6.3.) On the other hand on examining the proof of Proposition 1.6.1 one sees that “$I(\omega_\mu)^{-1} I_s$” is non-zero. Hence the kernel of $I_{w_0}$ is non-zero. From [31] it follows also that the image of $V(\omega')$ under $I'_{w_0}$ is non-zero. Hence $V(\omega')$ is reducible.

Now we can discuss the Whittaker models of the $V_0(\omega')$. Let $\varepsilon$ be a non-degenerate character of $N^\mu$. Then if $\omega$ is dominant, $\eta \in \tilde{H}$, we can define the linear functional $\lambda_\eta$ on $V(\omega')$ by

$\langle \lambda_\eta, f \rangle = \int_{N^\mu} \varepsilon(n) f(\eta w_0^{-1} n) \, dn.$

If $h \in \tilde{H}$, then one has

$\lambda_{\eta h} = (\omega' \mu)(h) \lambda_\eta.$

These define Whittaker functionals. Let, as before $\text{Wh}(V)$ be the space of all Whittaker functionals on $V$.

Theorem 1.6.5. — a) When $n = 1$, $\text{dim}(\text{Wh}(V_0(\omega'))) = 0.$

b) When $n = 2$, $r = 2, F = R$

$\text{dim}(\text{Wh}(V_0(\omega')))) = 1.$
c) When $r \leq n$, $F = \mathbb{C}$, then $V_\psi(\omega')$ is isomorphic to $V(\omega')$ and
\[ \text{Wh}(V(\omega')) = \langle \lambda_1 \rangle. \]

**Proof.** — *a)* is obvious in view of Theorem I.6.4.

*b)* is proved in [8] § 4.

*c)* is well-known (see, e.g. [44], Corollary to Theorem 2.2 which is much stronger).

**Remark.** — It would be possible to show, as in [44], that the $\lambda_n$ can be regularized, and that these give rise to all of $\text{Wh}(V(\omega'))$. Then, by studying the effect to the intertwining operators $I'_\psi$ on the $\lambda_n$ (as in § 1.3, in the $p$-adic case) one could show that, when $n = 2$, $r > 2$, $F = \mathbb{R}$, $\text{Wh}(V_\psi(\omega')) = 0$. We shall not need this result. More generally, if $r > n$ then $\text{Wh}(V_\psi(\omega')) = 0$.

We shall need however one particular result of this type when $F = \mathbb{C}$ and $\omega$ is arbitrary. This result is a special case of the functional equation proved by Jacquet in [22] (see also [43], [44]), so that we shall merely reformulate these known results in our terminology.

We let $\lambda = \lambda_1$. Then the function $t \mapsto \langle \lambda, f \rangle$ has a meromorphic extension to $\Phi(\mathbb{C})$; hence $\lambda$ can be defined on a generic $V(\omega')$. If $w \in W$ then $f \mapsto \langle \lambda, I'_\psi f \rangle$ is also a Whittaker functional on $V(\omega')$. Since we know that $\dim(\text{Wh}(V(\omega'))) = 1$, this must be proportional to $\lambda$. We shall determine the factor of proportionality.

If $\chi$ is a quasicharacter of $\mathbb{C}^*$, then $\chi(x)$ can be written either as $x^{-m}|x|^i$ or $x^{-m}|x|^i$ where $m$ is a positive integer. We define then
\[ \varepsilon(\chi) = i^{-m}. \]

We fix the additive character $\varepsilon$ of $\mathbb{C}$ to be
\[ \varepsilon(x) = e^{2\pi i x}. \]

If $\omega$ is a quasicharacter of $H$ we let, for $x \in \Phi$,
\[ \omega_x(x) = \omega(h(x)). \]

Moreover we define, for $x \in \Phi$, $\eta_x \in H$ to be diag$(1, 1, \ldots, 1, -1, 1, \ldots, 1)$ where, if $x = (y)$, the $-1$ is in the $i$-th position. Then the functional equation is given by:

**Theorem I.6.6.** — With the notations above, $f \in V(\omega')$
\[ \langle \chi, I'_\psi f \rangle = \prod_{\alpha > 0} \omega(\eta_x) \varepsilon(\omega_x)^{-1} L(\omega_x) L(| \omega_x^{-1} )^{-1} \langle \lambda, f \rangle. \]

**Remark.** — We could also deduce this from the "global" functional equation of Theorem II.1.3, the procedure being the inverse of that which we shall use in the proof of Corollary II.2.4. This suffices to determine the factor of proportionality up to a constant. (To complete the proof one reduces the evaluation of this constant to the case $r = 2$ and then makes a special choice of $f$.)
In this chapter we shall apply the local theory of Chapter I and the theory of Eisenstein series to construct an automorphic representation of the global metaplectic group $\tilde{G}_k$ over a global field $k$ with $\text{Card}(\mu_n(k)) = n$. We shall show that this representation is, in an appropriate sense, a tensor product of the "exceptional" representations of Theorem I.2.9 and § I.6.

In § II.1 we shall recall those aspects of the theory of Eisenstein series which we need. The automorphic representation referred to above is then described in Theorem II.2.1. Although this is of interest in itself one can only derive consequences of arithmetic interest after some further work. The automorphic forms have "Fourier coefficients" which may be described in two different ways.

Firstly, as we have represented the representation as a restricted tensor product, these Fourier coefficients are described by the local Whittaker models as in Theorem II.2.2. One consequence of this is that one can use global methods to complete the local theory in certain points. Thus, for example, one can show that if $r = n$ or if $r = n - 1$ and $2(\varepsilon - 1) \equiv 0 \pmod{n}$ in our customary notations then all the local factors of the representation have a unique Whittaker model whereas we could only deduce the corresponding statement for almost all local factors from the local theory at our disposal. For this see Theorem II.2.5 and Corollary II.2.6.

The other approach is to observe that the Fourier coefficients can be expressed as residues of the corresponding Fourier coefficients of Eisenstein series. These, in their turn, are Dirichlet series, here denoted by $\Psi$ or $\Psi_r$ whose coefficients are typically Gauss sums.

Although we shall not discuss these functions in general here we shall investigate the special case $r = 2$ in § II.3. Thus our treatment of the arithmetic implications of the theory of metaplectic forms will be rather cursory, but our intention has been to stress the representation theoretic aspects here. Moreover we shall restrict our attention to the case of a number field $k$, although the case of a function field is also of interest, and is in some senses simpler since $|n_v| = 1$ for all places $v$ of $k$.

The arithmetically interesting consequences follow from a comparison of these two approaches and are given in Theorem II.2.3, Corollary II.2.4 and in the consequences of Theorem II.3.3.

We conclude this introduction with a notational remark; we shall write $\tilde{G}_k$ in place of $\tilde{G}_k^{(c)}$, but the dependence on $c$ should not be forgotten.
II.i. Eisenstein series on the metaplectic group

Let \( k \) be a global field satisfying \( \text{Card}(\mu_n(k)) = n \), and let \( \tilde{G}_A \) be constructed as in § 0.2. Let \( \tilde{H}_{n,A} \) (resp. \( \tilde{Z}_A \)) be the subgroup each of whose local components lies in \( \tilde{H}_{n,A_v} \) (resp. \( \tilde{Z}_v \)). Let \( G^*_\psi, H^*_\psi \) be the images of \( G^*_\psi, H^*_\psi \) under the lifting described in § 0.2.

**Lemma II.i.1.** — \( H^*_\psi \tilde{H}_{n,A} \tilde{Z}_A \) is a maximal abelian subgroup of \( \tilde{H}_A \).

**Proof.** — Recall that, relative to the \( n \)-th order Hilbert symbol on \( k^\times \) the subgroup \( k^\times k^\times \) is maximal isotropic; cf. [50] XIII-5, Prop. 8. The assertion then follows from Proposition 0.3.1.

Next for each place \( v \) of \( k \) we choose a maximal abelian subgroup \( \tilde{H}_{n,v} \) of \( \tilde{H}_v \) which, at all but a finite number of places, is the one appearing in the discussion preceding Lemma 1.1.1. Then let \( \tilde{H}_{n,A} \) be the subgroup of \( \tilde{H}_A \) all of whose components lie in the corresponding \( \tilde{H}_{n,v} \). This \( \tilde{H}_{n,A} \) is a second maximal abelian subgroup of \( \tilde{H}_A \); the material in § 0.3 was introduced chiefly in order to compare it with \( H^*_\psi \tilde{H}_{n,A} \tilde{Z}_A \).

Let \( \omega \) be a quasicharacter of \( \tilde{H}_{n,A} \tilde{Z}_A \) which is trivial on \( H^*_\psi \cap (\tilde{H}_{n,A} \tilde{Z}_A) \), and let \( \omega_0 \) be the extension of \( \omega \) to \( H^*_\psi \tilde{H}_{n,A} \tilde{Z}_A \) which is trivial on \( H^*_\psi \). Let \( \omega \) be any extension of \( \omega \) to \( \tilde{H}_{n,A} \). As in the local case we shall fix an injective homomorphism \( \varepsilon : \mu_n(k) \to \mathbb{C}^\times \), and we shall only consider those \( \omega \) for which \( \omega \circ \varepsilon = \varepsilon \).

We remark also that there exist \( \sigma(\omega) \) in \( \Phi(\mathbb{R}) \), \( \tau \in \mathbb{R} \) so that, if \( \sigma(\omega) = \Sigma \sigma_x(\omega) \cdot x \) then

\[
| \omega(h) | = \Pi_x \||\rho_A(h)^x||_{A^x}^x \cdot || \det(\rho_A(h)) ||_{A^x}^x
\]

where \( || \cdot ||_{A^x} : k^\times \to \mathbb{R}^+ \) is the idele norm.

We observe next that, as the map \( \rho_A \) in

\[
(t) \xrightarrow{i} \mu_n(k) \xrightarrow{i} \tilde{G}_A \xrightarrow{\rho_A} G_A \xrightarrow{} (t)
\]

is a finite covering map, then the usual consequences of reduction theory ([4], [15]) hold also for the discrete subgroup \( G^*_A \) of \( \tilde{G}_A \); see for example the remark in the proof of Theorem II.1.3. In particular the assumptions necessary for the development of the theory of Eisenstein series in [16], [29], [36] are fulfilled.

Let \( \mu_A(h) = ||\rho_A(h)^x||_A \) where \( \varphi = \frac{1}{2} \Sigma \xi \in \Phi \cdot x \).

The theory of Eisenstein series begins by constructing (analogously to the \( (\pi(\omega), V(\omega)) \) of Chapter I) representations \( (\pi_\psi, V_\psi) \) of \( \tilde{G}_A \) which are realized in a space of slowly increasing functions on \( G^*_A \). The Eisenstein series form the intertwining operator
from the standard representation into this space. They are first defined when \( \sigma(\omega) - \rho \) lies in the dominant Weyl chamber, this definition is then extended to almost all \( \omega \) by analytic continuation (due to Roelcke, Selberg and Langlands).

To be able to express these results we must first describe the analytic structure on the set of all \( \omega \). To do this we define for \( s \in \Phi(\mathbb{C}) \) the quasicharacter \( \omega_s^* \) of \( \tilde{H} \) by

\[
\omega_s(h) = \prod_{a \in \Phi} \frac{||p_a(h)||^s}{||a||^s} \quad \text{if} \quad s = \sum s(a) \alpha.
\]

This is trivial on \( i(\mu_n(k)) \) and \( H_n^0 \). We let \( \Omega_\omega = \{ \omega, \omega: s \in \Phi(\mathbb{C}) \} \) where \( \omega \) is as above, then the map \( \Phi(\mathbb{C}) \to \Omega_\omega; \ s \mapsto \omega_s \) defines an analytic structure on \( \Omega_\omega \). We regard \( \Omega_\omega \) as the component of \( \omega \) in the set \( \Omega \) of all quasicharacters \( \omega: \tilde{H}_{n,A} \tilde{Z}_A \to \mathbb{C}^* \), with \( \omega \circ i = \varepsilon, \ \omega \) trivial on \( H_n^0 \cap (\tilde{H}_{n,A} \tilde{Z}_A) \).

Let \( \tilde{K}_A = p_n^{-1}(K_A) \) where \( K_A \) is the standard maximal compact subgroup of \( G_A \).

Let \( K_{A,\omega} \) (resp. \( K_{A,f} \)) be the product of the archimedean (resp. non-archimedean) components of \( K_A \). Then there is a natural product decomposition

\[
K_A = K_{A,\omega} \times K_{A,f}
\]

Corresponding to this one obtains a product decomposition of \( \tilde{K}_A \),

\[
\tilde{K}_A = \tilde{K}_{A,\omega} \times \tilde{K}_{A,f}
\]

where \( \tilde{K}_{A,\omega} \) (resp. \( \tilde{K}_{A,f} \)) is a \( \mu_n(k) \)-covering of \( K_{A,\omega} \) (resp. \( K_{A,f} \)). We therefore have a map from the usual product to \( \tilde{G}_A \),

\[
\tilde{K}_{A,\omega} \times \tilde{K}_{A,f} \to \tilde{G}_A; \quad (k_\omega, k_f) \mapsto (k_\omega \times k_f)
\]

(this defines the notation to be used).

We call a function \( f: \tilde{G}_A \to \mathbb{C} \) right \( \tilde{K}_A \)-smooth if

a) for all \( g \in \tilde{G}_A \) the function

\[
\tilde{K}_{A,\omega} \to \mathbb{C}; \quad k \mapsto f(g(k_\omega \times k_f))
\]

is smooth, and

b) there exists an open subgroup \( L \) of \( \tilde{K}_{A,f} \) so that for all \( g \in \tilde{G}_A \) one has

\[
f(g(I \times k_f)) = f(g) \quad (k_f \in L).
\]

Naturally this definition does not depend on the choice of \( K_A \).

For each \( \omega \in \Omega \) let \( \mathcal{F}_0(\omega) \) be the space of right \( \tilde{K}_A \)-smooth functions \( f_0 \) on \( \tilde{G}_A \) which satisfy

\[
f_0(bg) = (\omega_0. \mu_\Delta)(b).f_0(g) \quad (b \in (H_n^0. \tilde{H}_{n,A} \tilde{Z}_A) \tilde{N}_{n,A})
\]

(\( \omega_0. \mu_\Delta \) is taken to be trivial on \( \tilde{N}_{n,A} \)). There is a trivial holomorphic vector bundle \( \mathcal{F}_0(\omega) \) over \( \Omega \) with fibre \( \mathcal{F}_0(\omega) \) at \( \omega \). We shall also have occasion to speak of the space \( \mathcal{F}_0(U) \) of holomorphic sections over an open subset \( U \subseteq \Omega \).

Let now \( \omega_0 \) be an extension of \( \omega \) to \( \tilde{H}_{n,A} \); as

\[
((H_n^0. \tilde{H}_{n,A} \tilde{Z}_A) \cap \tilde{H}_{n,A})/\tilde{H}_{n,A} \tilde{Z}_A
\]
is finite we can assume that \( \omega_0 = \omega_0 \) on \( (\mathbb{H}_a^* \tilde{\mathbb{H}}_{n,a} \tilde{Z}_A) \cap \tilde{\mathbb{H}}_{n,A} \). Let \( \mathcal{F}_*(\omega_0) \) be the space of right \( \tilde{\mathbb{K}}_A \)-smooth functions \( f_\omega \) on \( \tilde{\mathbb{G}}_A \) which satisfy
\[
 f_\omega(bg) = (\omega_0 \nu_A)(b) f_\omega(g) \quad (b \in \tilde{\mathbb{H}}_{n,A} \cdot N_{n,A}^*),
\]
just as above one can form a vector bundle \( \mathcal{F}_* \) and so on if we choose \( \omega_0 \) so that \( (\omega_0 \omega_0)_* = \omega_0 \omega_0 \).

From § 0.3 one sees that there are inverse isomorphisms \( \mathcal{F}_0(\omega_0) \rightarrow \mathcal{F}_*(\omega_0) \), \( \mathcal{F}_*(\omega_0) \rightarrow \mathcal{F}_0(\omega_0) \). To simplify notations we let \( f_0, f_\omega \) be functions in \( \mathcal{F}_0(\omega_0) \) and \( \mathcal{F}_*(\omega_0) \) respectively which correspond under these isomorphisms. We recall that
\[
 f_\omega = f_\omega (g), \quad g \in G_{\mathbb{K}}^* \tilde{G}_A.
\]
Notice that from its definition, \( f_\omega \) can be written as a finite sum of functions of the form \( \otimes f_{\omega,v}(\gamma_v) \) where \( g = (\gamma_v) \in \tilde{G}_A \) in the sense discussed in § 0.3, and \( f_{\omega,v}(\gamma_v) \) satisfies
\[
 f_{\omega,v}(h, \gamma_v) = (\omega_{\gamma_v} \nu_{\gamma_v})(h) f_{\omega,v}(\gamma_v)
\]
\( (i.e. f_{\omega,v} \in \mathcal{V}(\omega_{\gamma_v})) \). Note that at almost all \( v \), \( f_{\omega,v} \) is the \( \mathbb{K}_v^* \)-invariant vector with \( f_{\omega,v}(1) = 1 \). Also, \( \omega_{\gamma_v} \) is the \( v \)-th component of \( \omega_0 \), and \( \nu_{\gamma_v} \) is \( \mu \) as in Chapter I with \( F = \mathbb{K}_v \).

Now we can define the Eisenstein series themselves; if \( f_0 \in \mathcal{F}_0(\omega) \) let
\[
 E(g, \omega_0, f) = \sum_{\gamma \in \mathcal{G}(\gamma)} f_0(\gamma g).
\]
This converges absolutely if \( \sigma(\omega) - \rho \) lies in the dominant Weyl chamber ([16], [29]). The convergence is locally uniform and if \( U \subset \omega \in \Omega : \sigma(\omega) - \rho \) lies in the dominant Weyl chamber}, \( f_0 \in \mathcal{F}_0(U) \), then
\[
 E(\cdot, \cdot, f_0) : U \rightarrow \text{Slowly increasing functions on } G_{\mathbb{K}}^* \tilde{G}_A
\]
is analytic.

One of the major achievements of the theory of Eisenstein series is the proof that \( E \) has an analytic continuation as a meromorphic function, in the sense that \( E \) is defined as a meromorphic function on any \( U \times \mathcal{F}_0(U) \) (with values in the space of slowly increasing functions on \( G_{\mathbb{K}}^* \tilde{G}_A \)) with the obvious compatibility under restriction from \( U \) to \( U \cap \mathcal{C} \) and which agrees with the original class of \( U \). The analytic properties of \( E \) can be described quite accurately. As we shall need a few of the intermediate results we shall sketch some of the important features of the theory. The first concept which we need is that of the constant term.

Normalize the additive Haar measure on \( \mathbb{K}_A \) so that it is selfdual with respect to a character \( \psi \) trivial on \( \mathbb{K} \), then \( \text{meas}(\mathbb{K} \mathbb{H}_a^* \tilde{\mathbb{H}}_{n,a} \tilde{Z}_A) = 1 \) (loc. cit. Cor. 2 to Thm. 1). We can assume that this measure is a product measure formed from the local Haar measure.
self-dual with respect to the \( v \)-th component \( \epsilon_{0,v} \) of \( \epsilon_0 \). Other measures on adelic groups will be those given be the Tamagawa construction.

**Proposition 2.1.2.** — Let notations be as above. Then

\[
\int_{\mathbf{G}^*_A \backslash \mathbf{G}^*_A} E(n g, \omega, f_0) \ dn = \sum_{\omega \in \mathcal{W}} I^*_\omega(f_0)(g)
\]

where

\[
I^*_\omega : \mathcal{F}_0(\omega) \to \mathcal{F}_0(\omega^*)
\]

is defined by

\[
I^*_\omega(f_0)(g) = \int_{\mathbf{N}[w]_A} f_0(w^{-1}n g) \ dn.
\]

Moreover if \( I^*_\omega : \mathcal{F}_0(\omega) \to \mathcal{F}_0(\omega^*) \) is the corresponding map, and if \( f = \bigotimes f_{v,v} \), then

\[
I^*_w f = \bigotimes (I_{w,v} f_{v,v}),
\]

where \( I_{w,v} \) is, with \( F = k_v \), the operator introduced in \( \S 1.2 \) or \( \S 1.6 \).

**Proof.** — The proof is based on the Bruhat decomposition of \( \mathbf{G}^*_A \), viz.

\[
\mathbf{G}^*_A = \bigcup_{w \in \mathcal{W}} \mathbf{B}^*_A w^{-1} \mathbf{N}^*_A(w)_A.
\]

Since

\[
E(g, \omega, f_0) = \sum_{\gamma \in \mathbf{N}[w]_A} f_0(\gamma g)
\]

we have immediately that

\[
\int_{\mathbf{G}^*_A \backslash \mathbf{G}^*_A} E(n g, \omega, f_0) \ dn = \sum_{w} \int_{\mathbf{N}[w]_A \backslash \mathbf{N}^*_A(w)_A} \sum_{n \in \mathbf{N}[w]_A} f_0(w^{-1}n g) \ dn.
\]

On factoring the integral into two parts we see that the right-hand side is equal to

\[
\sum_{w} \int_{\mathbf{N}[w]_A \backslash \mathbf{N}^*_A(w)_A} f_0(w^{-1}n g) \ dn.
\]

where \( \mathbf{N}^*_A[w] \) corresponds to those \( \alpha \in \Phi^+ \) such that \( w^{-1} \alpha > 0 \) (as \( \mathbf{N}_+(w) \) corresponds to those \( \alpha \in \Phi^+ \) such that \( w^{-1} \alpha < 0 \)). Since \( f_0 \) is left \( \mathbf{N}^*_A \)-invariant, and as \( \dn \) is a product of self-dual measures, this is

\[
\sum_{w} \int_{\mathbf{N}[w]_A} f_0(w^{-1}n g) \ dn.
\]

This proves the first assertion of the proposition. The second statement, the translation in terms of \( \mathcal{F}_0 \), is formal. This proves the proposition.

Now let \( \alpha \in \Phi, \ \alpha = (ij) \); we let \( H^*_A : k^*_A \to \overline{H}_A \) be the homomorphism such that

\[
H^*_A(\xi) = s(\text{diag}(1, \ldots, 1, \xi^n, \ldots, \xi^{-n}, \ldots, 1)) \quad \text{where} \quad \xi^n \text{ sits at the } i \text{-th place, } \xi^{-n} \text{ at the } j \text{-th.}
\]

Then we let \( \omega^v = \omega \circ H^*_A; \) this is a Grössencharacter, which is to say, a quasi-character of \( k^*_A \) trivial on \( k^* \). If \( v \) is any place of \( k \) we let \( L_v(\omega^v) \) be the corresponding \( L \)-function, and

\[
L(\omega^v) = \Pi L_v(\omega^v).
\]

We now redefine \( I^*_w \) by setting, when \( f_v = \bigotimes f_{v,v}, \)

\[
I^*_w(f_v) = \{ \prod_{\alpha > 0} L(\omega^v_{\alpha}) L(|| \cdot ||_A \omega^v_{\alpha})^{-1} \} \bigotimes v [L(\omega^v_{\alpha}) L(\omega^v_{\alpha})^{-1} \cdot I_{w,v}(f_{v,v})].
\]

"
The expression in $[\ ]$ is, by Proposition I.2.4, the canonical $K^*$-invariant vector at almost all places. Thus this operator has an analytic continuation as a meromorphic operator since the $L$-functions have analytic continuations as meromorphic functions.

Let
\[ \Omega_+ = \{ \omega \in \Omega : \langle \sigma(\omega), \alpha \rangle \geq 0 \ (\alpha \in \Phi^+) \} \]
and
\[ \Omega_+^\circ = \{ \omega \in \Omega : \langle \sigma(\omega), \alpha \rangle > 0 \ (\alpha \in \Phi^+) \}. \]
Note that the only singularities of $I^\circ_\omega(f_\omega)$ in $\Omega_+^\circ$ lie on the "hyperplanes"
\[ T_\alpha = \{ \omega : \omega_\alpha^\circ = \| \|_\lambda \} \ (\alpha \in \Phi^+). \]

**Theorem II.1.3.** — The operator $E^\circ$ can be analytically continued to a meromorphic operator on $\Omega$ in the sense explained above, and satisfies the following functional equations
\[ E(g, \omega, f_\omega) = E(g, \omega, I^\circ_\omega(f_\omega)) \ (w \in W), \]
and
\[ I^\circ_{w'w} = I^\circ_w T^\circ_{w'}. \]

Let $U$ be an open subset of $\Omega$, $U \supset \Omega_+$ and let $f_\omega \in \mathcal{F}_\omega(U)$. Then the function
\[ \omega \mapsto \prod_{\alpha > 0} L(|| \|_\lambda \omega^\alpha_\omega ||) E(g, \omega, f_\omega) \]
in analytic on $\Omega_+ - \bigcup_{\alpha > 0} T_\alpha$.

**Proof.** — This follows from the general results of [16], [29], [36]; we shall first discuss the analytic continuation and functional equation.

In the case that $k$ is an algebraic number field we need only remark that if $K$ is a compact open subgroup of $\prod_{\omega \in \omega} \mathbb{G}_\omega$ then $G_\omega / G_\mathbb{A} K$ can be covered by a finite number of spaces of the form $\Gamma \setminus \prod_{\omega \in \omega} (\mathbb{G}_\omega / \mathbb{Z}_\omega)$, where $\Gamma$ is arithmetic: thus the assumptions of [29] are satisfied. When $k$ is a function field the arguments of [36] can be modified, as L. Morris indicated to us. We do observe, however, that since in this case the analytic continuation of the $I^\circ_\omega$ is known, the analytic continuation follows from the "principle of the constant term" ([29], Lemma 6.2). This leads to substantially simpler proofs in our case. The functional equations follow from the Maass-Selberg relations, which will be discussed in the course of the proof of Theorem II.1.4. They are essentially equivalent to the local functional equations (Theorems I.2.6 and I.6.3), along with the functional equation of Hecke-Tate.

It will be useful to make this a little more explicit. Let us recall the functional equation of Hecke-Tate; it is
\[ L(\omega) = \varepsilon(\omega) L(|| \|_\lambda \omega^{-1} ||) \]
where, with a choice of $\varepsilon_\omega$ as above, $\varepsilon(\omega)$ has the form $\prod_\omega \varepsilon_\omega(\omega_\alpha, \varepsilon_\alpha, n)$ where the $\varepsilon$-functions are monomials and are given in [23] pp. 104-5, 194-5, or, essentially, [50] VII-7.
We consider instead of the $I^w$, the $I^e$. Then if $f = \bigotimes_{v} f_{v,e} \in \mathcal{F}_e(\omega)$, we can write

$$I^e f = \prod_{a>0} \{ L(\omega_a^n) L(1 \mid \mid A \omega_a^n)^{-1} \} \otimes [L_a(1 \mid \mid A \omega_a^n) L_a(\omega_a^n)^{-1} I_{w,e}(f_{v,e})]$$

$$= \prod_{a>0} \{ L(1 \mid \mid A \omega_a^n)^{-1} \} \otimes \prod_{v} \{ \left[ \prod_{a>0} \frac{1}{\omega_a^n} e_u(\omega_a^n, e_{0,v}) L_a(1 \mid \mid A \omega_a^n)^{-1} \right] \} I_{w,e}(f_{v,e}).$$

If we write $I''_{w,e}(f_{v,e})$ for

$$\prod_{a>0} \left[ \prod_{u < 0} \frac{1}{\omega_a^n} e_u(\omega_a^n, e_{0,v}) L_a(1 \mid \mid A \omega_a^n)^{-1} \right] I_{w,e}(f_{v,e})$$

then one verifies easily that, equivalently to Theorems 1.2.6 and 1.6.3, one has

$$I''_{w,0,e} = I''_{w,0,e} I''_{w,0}$$

and the global functional equation follows from these local ones. This, however, is not a proof since we used the global functional equations in the proofs of the local ones, but it serves to demonstrate the equivalence.

Now that we have made these remarks we can return to the study of the singularities of

$$L(1 \mid \mid A \omega_a^n) E(g, \omega, f_0).$$

By the principle of the constant term these are determined by the singularities of the constant term. We shall write the constant term of the above function in terms of $f_r$, which we assume to have the form $\bigotimes_{v} f_{v,e};$ it is

$$\sum_{w \in W} \prod_{a>0} \prod_{u < 0} L_a(1 \mid \mid A \omega_a^n)^{-1} \otimes \left( I''_{w,e}(f_{v,e}) \right).$$

Since $L(\chi)$ is holomorphic except where $\chi = 1, \mid \mid A$, we see that the singularities in $\Omega_+$ lie in the sets $\omega_a^n = \mid \mid A, 1 \mid \mid$, $\chi \in \Phi^+$. The content of the assertion of the theorem is that there is (essentially) no singularity along $\omega_a^n = 1$.

We shall first show that it suffices to prove that the constant term is analytic at those $\omega \in \Omega_+ - \bigcup T_a$ for which there exists exactly one $\beta \in \Phi^+$ with $\omega_\beta^n = 1$; such a $\beta$ is necessarily simple. Consider a singular point

$$\omega_\beta \in \Omega_+ - (\Omega_\beta^n \cup \bigcup T_a).$$

It follows from the functional equation and Proposition 1.2.3 b) that the divisor of the singularity is $\text{Stab}_w(\omega_\beta)$-invariant. Since the divisor is disjoint from $\Omega_\beta^n - \bigcup T_a$ it follows that it must be a union of some of the hyperplanes

$$\{ \omega : \omega_\beta^n = 1 \} \quad (\chi \in \Phi^+)$$
which pass through $\omega_1$. It therefore suffices to show that the constant term is analytic at those points of $\Omega_+ - \Omega_+$ described above.

We shall therefore consider a positive simple root $\alpha_1$ with associated reflection $s_1$ and show that at $\omega \in \Omega_+ - \bigcup \mathbb{T}_\alpha$ with $\omega_1^\prime = 1$ but $\omega_2^\prime \neq 1 \ (\alpha \in \Phi_+ - \{\alpha_1\})$ the constant term is analytic. The constant term consists of a sum over $W$; we shall show that the sum of the terms corresponding to $\omega$ and $\omega s_1$ is analytic at $\omega$. To see this observe that $\{ \alpha > 0; \omega s_1 \alpha < 0 \}$ differs from $\{ \alpha > 0; \omega \alpha < 0 \}$ by exactly $\alpha_1$. Thus, as $\omega_1^\prime \to 1$ the products of $L$-functions differ by one factor, which has a pole, and in the two terms the residue of one is just the negation of the residue of the other. Thus we have to show that

$$\otimes_\omega I_{\omega_1^\prime, \omega} f_{s_1, \omega} = \otimes_\omega I_{\omega_1^\prime, \omega} f_{s_1, \omega} = 0,$$

or, equivalently,

$$\otimes_\omega I_{\omega_1^\prime, \omega} f_{s_1, \omega} = \otimes_\omega f_{s_1, \omega}.$$

Now recall that $\omega_2$ was an extension of $\omega$; the assumption that $\omega_2^\prime = 1$ is equivalent to $s_1 \omega = \omega$. Thus $s_1 \omega$ is a second extension of $\omega$. As we have seen there is locally an isomorphism $\alpha_{n, \omega}: V(\omega_2, \omega) \to V(\omega_2, \omega)$ (see the discussion leading up to Lemma 1.2.5). These yield an isomorphism $\alpha_{n, \omega}: \mathcal{F}(\omega_2, \omega) \to \mathcal{F}(\omega_2, \omega)$, and one verifies from the construction of the maps involved that the diagram

\[
\begin{array}{ccc}
\mathcal{F}_0(\omega) & \xrightarrow{\text{Id}} & \mathcal{F}_0(\omega_2) = \mathcal{F}_0(\omega) \\
\downarrow & & \downarrow \\
\mathcal{F}_n(\omega_2) & \xleftarrow{\alpha_{n, \omega}} & \mathcal{F}_n(\omega)
\end{array}
\]

commutes. Thus we have to show that

$$\alpha_{n}(\otimes_\omega (I_{\omega_1^\prime, \omega} f_{s_1, \omega})) = \otimes_\omega f_{s_1, \omega}$$

where now the equation takes place inside $\mathcal{F}_n(\omega)$. But by Lemmas 1.2.5 and 1.6.2 the left-hand side is a positive multiple of the right-hand side. Moreover, as $I_{s_1, \omega} = 1$ it follows that this multiple can only be 1. Hence we have proved the equality, and with it, the theorem.

The final result of this section is a simple, but important consequence of the Maass-Selberg relations. We call $\omega$ exceptional if $\omega_2^\prime = 1$ for all positive, simple roots $\alpha$. At such points the Eisenstein series have their greatest singularity. Let $\omega$ be exceptional and let

$$\theta(g, f_0) = \lim_{\omega \to \omega_1} (\prod \mathcal{L}_n((\omega')_2^{-1} \omega) E(g, \omega', f_0).$$
This depends on the choice of $\omega$, and we shall write $\theta(g, \omega, f_0)$ if it is necessary to make the choice of $\omega$ explicit.

By Proposition II.1.2
\[
\int_{N^*_A \backslash G} \theta(g, f_0) \, dn = \mathcal{V}_\omega(f_0, \omega)(g)
\]
where
\[
\mathcal{V}_\omega(f_0, \omega) = \lim_{\omega \to \omega} \left( \prod_{\gamma > 0} L((\omega')_\gamma)^{-1} \right) \mathcal{V}_\omega(f_0, \omega').
\]
If we express this in terms of $f_\bullet$, as usual taken to be in the form $\otimes f_{\bullet, \nu}$, then
\[
\mathcal{V}_\omega(f, \omega') = Z_A(2)^{-(r-1)} Z_A(3)^{-(r-2)} \cdots Z_A(r)^{-1} \otimes \mathcal{V}_\omega'(f_{\bullet, \nu}),
\]
where
\[
Z_A(s) = L(||||^s_A).
\]
Note that if $\omega^{(1)}$ and $\omega^{(2)}$ are exceptional then $\omega^{(1)} (\omega^{(2)})^{-1}$ is of the form $\chi \circ \det \circ \rho_A$ where $\chi$ is a Grössencharakter. Thus we can alter a given $\omega$ by such a character so that $\omega | \tilde{Z}_A$ is unitary. This we shall do, and fix such an exceptional $\omega^1$ such that $\omega^1 | \tilde{Z}_A$ is unitary. Then the result which we referred to above is the inner product formula in the following theorem.

\begin{theorem}

With the notations above, if $\omega^1 | \tilde{Z}_A$ is unitary
\[
\int_{0 \backslash \tilde{H}_A} \theta(g, f_{\bullet, 0}). \overline{\theta(g, f_{\bullet, 0})} \, dg
\]
is equal to
\[
c \int_{H^*_A \backslash H_A} \int_{N^*_A \backslash G} f_{\bullet, 0}(\gamma \omega_0 \eta) \mathcal{V}_\omega(f_{\bullet, 0})(\gamma \omega_0 \eta) \, dn \, d\eta,
\]
where $c$ is a non-zero constant which depends on the choices of Haar measures.

\end{theorem}

\begin{remark}

We observe, before beginning the proof, that under the assumptions of the theorem,
\[
\omega \omega(h)^{-1} = \overline{\omega(h)} \quad (h \in H^*_A \tilde{H}_{n, A} \tilde{Z}_A)
\]
as one easily verifies. Thus the integral is well-defined.

\end{remark}

\begin{proof}

The proof is an adaption of the technique used by Langlands in [30]. We must begin by developing some general principles. Let $\omega_s (s \in \Phi(C))$ be as above. If $\alpha_1, \ldots, \alpha_{r-1}$ are the positive simple roots in the usual order we can write $s = \sum \alpha_j$. We then let $ds$ be the differential form
\[
d s_1 \wedge ds_2 \wedge \ldots \wedge ds_{r-1}
\]
on $\Phi(C)$. From now on, we take $k$ to be a number field, leaving the modifications for the function field case to the reader.

\end{proof}
Fix now a quasicharacter \( \chi \) on \( \hat{\mathbb{Z}}_A \), trivial on \( \hat{\mathbb{Z}}_A \cap H^*_A \), and consider the space \( \Psi \) of functions on \( \hat{\mathcal{G}}_A \) which satisfy the following conditions:

if \( \psi \in \Psi \) then

(i) \( \psi(hng, z) = \chi(z) \psi(g) \quad (h \in H^*_A, n \in N^*_A, z \in \hat{\mathbb{Z}}_A) \)

(ii) \( \psi \) is right \( \hat{K}_A \)-finite.

(iii) \( \psi \) has a representation of the form

\[
\psi(g) = \sum_{\omega} (2\pi)^{-(r-1)} \int_{(0)} \hat{\psi}(\xi, \omega) \, ds \quad (\sigma \in \Phi(\mathbb{R}))
\]

where \( (a) \) \( \omega \) is a quasicharacter of \( H^*_A \hat{H}^*_n, \hat{A} \), trivial on \( H^*_A \) and

\[
\hat{\psi}(hng, \omega) = \omega_H(k) \hat{\psi}(g, \omega) \quad (h \in H^*_A \hat{H}^*_n, \hat{A}, n \in N^*_A, \hat{A}),
\]

b) if \( \Omega \) is the set of quasicharacters satisfying \( a) \) and \( \Omega_0 = \{ \omega_n : s \in \Phi(\mathbb{C}) \} \) then (noting that \( \Omega \) is a \( \Omega \)-homogenous space), the \( \omega \)-sum is over a set of representatives of \( \Omega_0 \backslash \Omega \),

c) the function \( s \mapsto \psi(g, \omega) \) is analytic and there exists \( B > r \) such that, for any norm \( || \cdot || \) on \( \Phi(\mathbb{C}) \), given a compact set \( C \subset \Phi(\mathbb{R}) \) one has for \( \Re(s) (= \Re(\omega)) \in C \)

\[
\hat{\psi}(g, \omega) \ll (1 + ||s||)^{-B}.
\]

moreover \( \hat{\psi}(\cdot, \omega) = 0 \) for \( \omega \) lying in all but finitely many elements of \( \Omega_0 \backslash \Omega \).

d) \( \int_{(0)} \) indicates that the integral is taken over the product of lines \( \Re(s) = \sigma \) given the usual orientation.

Before we continue, some remarks on this rather lengthy definition are in order. The fundamental part of the definition is given in \( i) \) and \( ii) \). Parts \( a) \) and \( b) \) of \( iii) \) state that \( \psi \) is a Fourier transform with respect to \( H^*_A \hat{H}^*_n, \hat{A}, \hat{Z}_A, \hat{Z}_A \) and \( c) \) imposes fairly stringent regularity conditions which, although they could be relaxed, suffice for our purposes. Note that, if the Haar measures are suitably chosen

\[
\hat{\psi}(g, \omega) = \int_{H^*_A \hat{H}^*_n, \hat{A}} (\omega H_A)^{-1} (k) \psi(hg) \, dh.
\]

Given \( \psi \in \Psi \) we define

\[
E(g, \psi) = \sum_{\gamma \in \Omega} \psi(\gamma g);
\]

under the assumptions above this converges absolutely and locally uniformly. One has, by \( iii) \) above,

\[
E(g, \psi) = (2\pi)^{-(r-1)} \int_{(0)} E(g, \omega, \psi) \, ds
\]

where \( \sigma \) satisfies \( \sigma(\omega) + \sigma > \rho \). Moreover, if \( \chi \) is unitary, as we shall henceforth assume, then

\[
\int_{\mathbb{R}^r, \hat{\mathbb{Z}}_A, \hat{\mathbb{Z}}_A} |E(g, \psi)|^2 \, dg < \infty.
\]
Let \( \Psi \) be the space of \( \hat{\phi}(g, \omega) \) as above. We define for \( \hat{\psi}_1, \hat{\psi}_2 \in \Psi \)

\[
[\hat{\psi}_1, \hat{\psi}_2](\omega) = \int_{B_\mathbb{C} \backslash \mathbb{Z}_\mathbb{C}} \int_{N_{x, x}} \hat{\psi}_1(\gamma \eta \omega n, \omega \hat{\psi}_2(\eta \omega n, \omega^{-1}) \ dn \ dn.
\]

Then the Maass-Selberg relation is, in this case, the inner-product formula

\[
\int_{G_\mathbb{C} \backslash \mathbb{Z}_\mathbb{C}} \frac{E(g, \psi_1) E(g, \psi_2)}{g} \ dg = c_1 \sum_{\omega \in \mathbb{W}} \sum (2\pi i)^{-r-1} \int_{(\omega)} \hat{\psi}_1(\lambda, \lambda) \hat{\psi}_2(\lambda, \lambda) \ (\omega \omega) \ ds
\]

where \( c_1 \) is a constant depending on the choice of Haar measure and \(- (\sigma(\omega) + \sigma + \rho)\) is dominant. This is a variant of [29] Lemma 4.6 and is proved in the same way.

The right-hand side of the equation above thus defines an inner product which gives \( \Psi \) the structure of a pre-Hilbert space.

Let now \( \omega \) satisfy \( \omega = \omega \ (w \in \mathbb{W}) \) and let \( \omega \) be unitary; for example, if \( \omega^1 \) is as in the statement of the theorem then

\[
\omega = \omega^1 \mu^{-1/n}
\]
satisfies these conditions. Let \( \Psi(\omega) \) be the subspace of those \( \psi \) of the form

\[
\psi(g) = (2\pi i)^{-r-1} \int_{(\omega)} \hat{\psi}(g, \omega \omega) \ ds.
\]

Recall that \( \langle , \rangle \) denotes the Killing form on \( \Phi(\mathbb{C}) \).

We let \( \Psi^\text{el} \) be the Hilbert space completion of \( \Psi \), and \( \Psi(\omega)^\text{el} \) the closure of \( \Psi(\omega) \) in it. We define an operator

\[
\Delta : \Psi(\omega) \rightarrow \Psi(\omega)^\text{el}
\]

by

\[
(\Delta \psi)^\text{el} (g, \omega \omega) = \langle s, s \rangle \hat{\psi}(g, \omega \omega)
\]

which is a symmetric operator with an essentially self-adjoint extension.

Let now \( L^2(G_\mathbb{C} \backslash \tilde{G}_\mathbb{C}, \chi) \) be the space of functions \( f \) on \( \tilde{G}_\mathbb{C} \) which satisfy

\[
f(\gamma g z) = \chi(z) f(g) \quad (\gamma \in G_\mathbb{C}, z \in \tilde{Z}_\mathbb{C})
\]

and such that

\[
\|f\|^2 = \int_{G_\mathbb{C} \backslash \tilde{G}_\mathbb{C}} |f(g)|^2 \ dg
\]
is finite. Let, for \( f_1, f_2 \in L^2(G_\mathbb{C} \backslash \tilde{G}_\mathbb{C}, \chi) \) be the inner product

\[
(f_1, f_2) = \int_{G_\mathbb{C} \backslash \tilde{G}_\mathbb{C}} f_1(g) \overline{f_2(g)} \ dg.
\]

Thus \( \Phi \) (the Eisenstein series function) can be regarded as a continuous map \( \Phi : \Psi \rightarrow L^2(G_\mathbb{C} \backslash \tilde{G}_\mathbb{C}, \chi) \). Let \( L_\omega \) be the image, under \( \Phi \), of \( \Psi(\omega) \); then \( \Delta \) induces a map (which will also be denoted by \( \Delta \)) from \( L_\omega \) to its completion \( L_\omega^\text{el} \) in \( L^2(G_\mathbb{C} \backslash \tilde{G}_\mathbb{C}, \chi) \).

Let \( \Theta(\omega^3) \) be the subspace of \( L^2(G_\mathbb{C} \backslash \tilde{G}_\mathbb{C}, \chi) \) (where \( \chi = \omega^1 \mid \tilde{Z}_\mathbb{C} \) and \( \omega^3 = \omega^1 \mu^{-1/n} \)) generated by the \( \theta(g, f_0) \ (f_0 \in F(\omega^3)) \).
Let $\mathcal{E}$ be the spectral resolution of $\Delta$ on $L^1$. Then the argument of Langlands is based on the following facts:

1. \( (E(\psi), \theta(f_0)) = \mathcal{E}_1[\hat{\psi}, \tilde{\psi}_0(f_0)](\omega \mu^I) \)

where $\mathcal{E}_1$ is as above. This is proved by the same techniques as are used to prove the Maass-Selberg relations.

From this it follows that

\[ \mathcal{E}_0 = \mathcal{E}(a) - \mathcal{E}(a -) \quad (a = \frac{1}{n^2} \langle \rho, \rho \rangle). \]

2. For a certain constant $\mathcal{E}_2$

\[ (\mathcal{E}_0, E(\psi_1), \mathcal{E}_0, E(\psi_2)) = \mathcal{E}_2[\hat{\psi}_1, \tilde{\psi}_2](\omega \mu^I). \]

This is proved by a transcription of the argument on pp. 145-148 of [30], using the version of the Maass-Selberg relations above instead of the corresponding result used by Langlands.

Now we choose $\psi_0$ such that

\[ \hat{\psi}_0(\omega \mu^I) = \check{f}_0. \]

Then, combining 1) and 2) we see that

\[ (\mathcal{E}_0, E(\psi_1), \mathcal{E}_0, E(\psi_0) - (\mathcal{E}_2 / \mathcal{E}_1) \theta(f_0)) = 0 \]

and hence

\[ \mathcal{E}_0(\mathcal{E}_0) = (\mathcal{E}_2 / \mathcal{E}_1) \theta(f_0) \]

and, in particular, the range of $\mathcal{E}_0$ is precisely $\Theta(\omega^I)$. Also

\[ (E(\psi), \theta(f_0)) = \mathcal{E}_0(\psi), \theta(f_0)) \]

\[ = (\mathcal{E}_2 / \mathcal{E}_1) (\theta(\psi(\omega^{I'})), \theta(f_0)) \]

and so, writing $f_0' = \hat{\psi}(\omega^{I'})$ we have, by 1),

\[ (\theta(f_0'), \theta(f_0)) = \mathcal{E}_2(\mathcal{E}_1) \mathcal{E}_0[f_0', \tilde{\psi}_0(f_0)](\omega \mu^I) \]

which is the assertion of the theorem.

### II.2. The Main Theorems

We are now in a position to state and prove our main results. These involve the construction of an automorphic representation $(\pi_0, V_0)$ of $\hat{G}_A$, which we shall identify locally.
We first remark that if \((\pi, V)\) is an admissible irreducible representation of \(G_A\) then there exist representations \((\pi^v, V^v)\) of \(G^v\) and, for almost all \(v\), a \(K_v\)-invariant \(\xi_v^v \in V^v\) such that \((\pi, V)\) is the restricted tensor product (over \(C[\mu_\infty(k)]\)) of the \((\pi^v, V^v)\) with respect to the family \((\xi_v^v)\). Compare [23], Proposition 9.1. The \((\pi^v, V^v)\) are irreducible and admissible.

Let \(\omega\) be as in § II. 1, and suppose that \(\omega\) is exceptional (which we could express as \(\omega_\alpha = \mu_\alpha\) for all \(\alpha \in \Phi\)). Then \(\theta(f_\omega)\) is defined. Let \(V_\omega\) be the space spanned by the \(\theta(f_\omega)\) as \(f_\omega\) runs through \(\mathcal{F}_\omega(\omega)\), and let \(\pi_\omega\) be the representation of \(G_A\) by right translation on \(V_\omega\). Since \(\theta(\gamma g, f_\omega) = \theta(g, f_\omega)\) (\(\gamma \in G^\omega\)) the representation \((\pi_\omega, V_\omega)\) is automorphic. (One sees easily that \(\theta(g, f_\omega)\) is slowly increasing, and even that \((\pi_\omega, V_\omega)\) is unitary when \(\omega^{1/n}\) is unitary (by Theorem II. 1.4).)

We write \(\omega = \otimes_v \omega_v\), where the tensor product is with respect to \(C[\mu_\infty(k)]\). Then it is possible to describe the local structure of \((\pi_\omega, V_\omega)\).

**Theorem II. 2. 1.** — With the notations above

\[
(\pi_\omega, V_\omega) \cong \bigotimes_v \mathcal{O}_{\mu_\omega(k)}(\pi_{\omega_v}, V_\omega(\omega_v))
\]

where \((\pi_{\omega_v}, V_\omega(\omega_v)) \cong \text{Im}(I_{\omega_v} : V(\omega_v) \rightarrow V(\omega_v^\ast))\) is the representation discussed in § I. 2.

**Proof.** — This is an immediate consequence of Theorem II. 1.4.

We come now to the discussion of Whittaker models of these representations. Let \(e_\omega\) be a non-trivial character of \(k^*_A\), trivial on \(k\). We can write \(e_\omega = \otimes v \omega_v^*\). We define the character \(e\) on \(N^*_A\) by \(e(n) = e_\omega(\sum_{1 \leq i < r} \rho(n)_{i, i+1})\), and an analogous character \(e_v\) on \(N^*_v\). Let us denote the \(v\)-th factor of \((\pi_\omega, V_\omega)\) by \((\pi_{\omega_v}, V_\omega(\omega_v))\). Recall that a Whittaker model of \(V_{\omega_v}\) is a linear map \(\lambda_v : V_{\omega_v} \rightarrow C\) such that \(\lambda_v(\pi_v(n) v) = e(n) \lambda_v(v)\). We let \(\text{Wh}_v\) be the space of such functionals. At almost all places we have a preferred vector \(\omega_v^\ast\) of \(V_{\omega_v}\), as was described above.

Now, the same definition holds for \((\pi_\omega, V_\omega)\). However we can construct a Whittaker functional \(\lambda_A\) on \(V_\omega\) as follows:

\[
\lambda_A(\theta(f_\omega)) = \int_{N_v^*_A \setminus N_v^*} \theta(n, f_\omega) \bar{e}(n) \, dn.
\]

The space \(\mathcal{F}_\omega(\omega_v)\) can be regarded as the restricted tensor product of the \(V(\omega_v^\ast)\). Thus we can form

\[
\bigotimes_v \mathcal{O}_{\mu_v(k)} V(\omega_v^\ast) \xrightarrow{\text{factors through } I_{\omega_v}} F_\omega(\omega) \xrightarrow{\theta} V_\omega \xrightarrow{\lambda_A} C.
\]

This is a Whittaker functional on the left-hand space which factors through \(I_{\omega_v}\). Let \(S\) be a finite set of places of \(k\) containing all the archimedean places and all the places \(v\) with \(|\pi_v| < 1\) and such that if \(v \notin S\) then \(V(\omega_v^\ast)\) has a \(K_v\)-invariant vector. We can assume that \(\omega_v^\ast(v \notin S)\) is canonical in the sense of § I. 1.
Let $V(S)$ be the subspace of $\bigotimes_{\sigma \in S} V(\omega_{\sigma})$ which is the image of 
\[(\bigotimes_{\sigma \in S} V(\omega_{\sigma})) \otimes (\bigotimes_{\sigma \notin S} v_{\sigma}),\]
where $v_{\sigma} \in V(\omega_{\sigma})$ is the $K_{\sigma}$-invariant vector with $v_{\sigma}(I) = 1$, in $V(\omega_{\sigma})$. All the tensor products over $V(\omega_{\sigma})$ are to be taken with respect to $C[\mu_{n}(b)]$.

The space $V(S)$ is a $\hat{G}_{A}(S)$-representation and $\lambda_{A}$ is a Whittaker functional on $V(S)$. Let $\hat{H}_{A}(S) = \hat{H}_{A} \cap \hat{G}_{A}(S)$ and $\hat{H}_{A^{+}}(S) = \hat{H}_{A} \cap \hat{G}_{A}(S)$. Then from § I.3 we see that there exists a function $c_{S}$ on $\hat{H}_{A}(S)$ satisfying
\[c_{S}(\gamma h) = (\omega_{\sigma}, \mu_{A})(\gamma)^{-1} c_{S}(h)\]
if $\gamma \in \hat{H}_{A^{+}}(S)$ and $h \in \hat{H}_{A}(S)$, such that
\[\lambda_{A}(\theta(J(f_{\sigma}))) = \sum_{\gamma \in \hat{H}_{A^{+}}(S)} c_{S}(\gamma) \prod_{\sigma \in S} \langle \lambda_{\gamma, \sigma}, f_{\sigma} \rangle\]
where $(\gamma_{\sigma})_{\sigma \in S}$ projects to $\gamma$ in $\hat{H}_{A}(S)$. Here $\lambda_{\sigma}$ is as defined in § I.3. Moreover as $J$ factors through $I_{\omega_{\sigma}}$ we have that $c_{S}$ has to satisfy various relations. Let $v \in S$, $v$ non-archimedean. Then there is a projection $a$ of $\hat{H}_{A^{+}} \times \prod_{\sigma} \hat{H}_{A^{\sigma}}$ into $\hat{H}_{A}(S)$.

Let $\eta \in \hat{H}_{A}(S)$; then we require that for each simple reflection $s$ the following holds:
\[\sum_{\eta \in \hat{H}_{A^{+}} \backslash \hat{H}_{A}} c_{S}(a(\eta \times \gamma)) \tau(\theta_{\sigma}, s, \eta, \eta') = 0 \quad (*)
\]
where $\tau$ is as in § I.3. Moreover since
\[\langle \lambda_{\lambda}, v_{0, \eta} \rangle = 1
\]
and
\[\langle \lambda_{\eta}, v_{0, \sigma} \rangle = 0 \quad (\eta \notin \hat{H}_{A^{+}})
\]
for almost all $\sigma$ it follows that if $S$ is large enough and if $S' \supset S$ then
\[c_{S} = c_{S'}|\hat{H}_{A}(S).
\]
Thus the family of $c_{S}$ defines a function $c$ on $\hat{H}_{A}$.

In this sense we can write
\[\lambda_{A}(\theta(J(f_{\sigma}))) = \sum_{\gamma \in \hat{H}_{A^{+}} \backslash \hat{H}_{A}} c(\gamma) \prod_{\sigma} \langle \lambda_{\gamma, \sigma}, f_{\sigma} \rangle.
\]

Let $U(\omega)$ be the space of functions $f: \hat{H}_{A} \rightarrow C$ which satisfy
\[f(\eta h) = (\omega_{\sigma}, \mu_{A})(\eta)^{-1} f(h)
\]
when $\eta \in \hat{H}_{A^{+}}$. Let $U^{0}(\omega)$ be the subspace satisfying the relations $(\ast)$ adumbrated above. Then we have shown:
Theorem II.2.2. — There exists $c \in U^0(\omega)$ so that, when $f_0 \in \mathcal{F}_0(\omega)$ is such that $f_*= \bigotimes_v f_{*,v}$ then
\[
\int_{\mathcal{N}_*^A \setminus \mathcal{N}^*_{*,A}} \tilde{e}(n) \theta(n, f_0) \, dn = \sum_{\gamma \in \mathcal{H}_A \setminus \mathcal{H}^*} c(\gamma) \cdot \prod_v \langle \lambda_{\gamma_0}, f_{*,v} \rangle.
\]

Remark. — This result is particularly strong if we know that $U^0(\omega)$ is 1-dimensional. We see that this is so precisely when $\dim(\text{Wh}(V_0(\omega))) = 1$ for all $v$. By Corollary I.3.6 this is so if $k$ is a function-field and
\[
r = n \quad \text{or} \quad r = n - 1 \quad \text{and} \quad 2(c + 1) \equiv 0 \pmod{n}.
\]

We shall see in Theorem II.2.5 that the same conclusion holds when $k$ is a number field.

Note that we have not shown that $c \neq 0$. If $r > n$ then, as $\dim(\text{Wh}(V_0(\omega))) = 0$ when $|n|_v = 1$, $c = 0$. However, we shall also show in Theorem II.2.5 that, if $r \leq n$, then $c \neq 0$.

Theorem II.2.2 is one of the major results of this work. In principle the space $U^0(\omega)$ can be fully described by the methods of §1.3. When $n = 2$ the theorem has content only when $r = 2$; in this case it is easy to see that $(\pi_0, V_0)$ is the Weil representation $r_k$ discussed in [111]. When $n = 3$, $r = 2$ we obtain the results of [6] and [37].

There is another formulation of this theorem which is frequently useful. We observe that, for $o \in \mu_\omega \Omega^0_+ \setminus \Omega^0_+$ and $f_0 \in \mathcal{F}_0(\Omega)$,
\[
\int_{\mathcal{N}_*^A \setminus \mathcal{N}^*_{*,A}} \tilde{e}(n) \theta(n, o, f_0) \, dn = \int_{\mathcal{N}_*^A \setminus \mathcal{N}^*_{*,A}} \tilde{e}(n) \sum_{\gamma \in \mathcal{H}_A \setminus \mathcal{H}^*} f_0(\gamma n) \, dn.
\]
Using the non-degeneracy of $\theta$ and the Bruhat decomposition the right-hand side can be written as
\[
\int_{\mathcal{N}_*^A} \tilde{e}(n) f_0(o n) \, dn.
\]
If we write this in terms of $f_*= \bigotimes_v f_{*,v}$ we see that it is
\[
\sum_{\gamma \in \mathcal{H}_A \setminus \mathcal{H}^*} \prod_v \int_{\mathcal{N}_*^A} \tilde{e}(n) f_{*,v}(\gamma n) \, dn.
\]
This we write as $\Psi(\omega, f_*)$ which is essentially a Dirichlet series in several variables whose coefficients involve Gauss sums. Now define
\[
\Psi(\omega, f_*) = \prod_{a > 0} L(\|_A \omega^a) \Psi^*(\omega, f_*).
\]
Also define $I'_w$ by
\[
I'_w(\bigotimes_v f_{*,v}) = \bigotimes_v I'_w(f_{*,v})
\]
where $I'_w$ was defined in §II.1. Then one has:
Theorem II.2.3. — $\Psi(\omega, f_\sigma)$ can be continued to a meromorphic function in $\omega$ whose singularities in $\Omega_+$ lie in $\bigcup_{\sigma > 0} \{ \omega \in \Omega_+ : \omega_\sigma = || \mid_A \}$. It satisfies the functional equations

$$\Psi(\omega, f_\sigma) = \Psi(\omega_0, L'_\sigma f_\sigma) \quad (w \in W).$$

Moreover, there exists a constant $a > 0$ so that if $f_\sigma = \bigotimes f_{w_\sigma}$ then one has

$$\lim_{\omega \to \omega_0} (\Psi(\omega, f_\sigma) (\prod_{\sigma > 0} L(\omega_\sigma^{-1}))) = a \cdot \sum_{\eta \in \tilde{\Pi}_\sigma \setminus \tilde{\Pi}_A} e(\eta) \prod_{\sigma > 0} \langle \lambda_\eta, f_{w_\sigma} \rangle.$$

Proof. — The analytic properties of $\Psi(\omega, f_\sigma)$ follow directly from the corresponding properties of the Eisenstein series given in Theorem II.1.3. The description of the singularity follows from Theorem 11.2.2. The value of $a$ is

$$Z_2(2)^{-1} Z_3(2)^{-2} Z_4(4)^{-3} \ldots Z_n(r).$$

When $k$ is a number field this formulation has the unattractive feature that the function $\Psi(\omega, f_\sigma)$ involves factors which are Whittaker functions associated with the archimedean places. Since there are transcendental functions of a not particularly well understood type, and since they can be factored out, it is desirable to do this.

We assume now that $k$ is a totally imaginary number field. We let

$$\omega_\infty = \prod_{\nu \in \infty} \omega_\nu o_{\nu}, \quad \mu_\infty = \prod_{\nu \in \infty} \mu_\nu$$

which are quasicharacters of $\Pi H_\nu$. We define

$$\Psi'_\tau(\omega, f_\sigma) = \sum_{\eta \in \tilde{\Pi}_\sigma \setminus \tilde{\Pi}_A} \left\{ \prod_{\nu \in \infty} \int_{N_\nu \tau} \bar{\eta}_\nu(n) f_{w_\sigma}(\gamma w_\nu n) \, dn \right\} \omega_\infty \mu_\infty(\bar{\rho}_\eta(\eta))$$

and

$$\Psi'_\tau(\omega, f_\sigma) = \prod_{\sigma > 0} L_\sigma(\mid \mid_A \omega_\sigma^2) \Psi'_\tau(\omega, f_\sigma).$$

The function $\Psi'_\tau(\omega, f_\sigma)$ is a Dirichlet series and it appears to be a fundamental function in this connection. Its coefficients are Gauss sums.

Let now $\omega_\lambda$ be a quasicharacter of $\tilde{H}_\nu$; we define

$$\Gamma_\nu(\omega_\lambda) = \prod_{\sigma > 0} \prod_{j=1}^{n-1} L_\sigma(\omega_{1, \sigma} \mid \mid A^{(n)}).$$

We now observe that if we make use of the Gauss-Legendre multiplication formula, viz.,

$$\Gamma(n) = n^{n-\frac{1}{2}}(2\pi)^{-\frac{(n-1)n}{2}} \Gamma(s) \Gamma \left( s + \frac{1}{n} \right) \ldots \Gamma \left( s + \frac{n-1}{n} \right).$$
where $\Gamma$ is the usual gamma-function, then we can verify that the formula of Theorem 1.6.6 can be expressed as
\[
\prod_{\alpha > 0} L_\omega(\omega_\alpha) \langle \lambda, I', f \rangle = \prod_{\alpha > 0} e^{(\omega_\alpha)^{n-1}} \omega(\eta_\alpha) \omega(\eta_\alpha) \prod_{\alpha > 0} \frac{\Gamma_{\omega_\alpha}(\omega_\alpha)}{\Gamma_{\omega_\alpha}(\omega_\alpha)} \prod_{\alpha > 0} L_\omega(\omega_\alpha) \langle \lambda, f \omega_\alpha \rangle
\]
where $f_\omega \in V(\omega)$ and $\eta_\alpha$ is as in Theorem 1.6.6.

We now let $k$ be as above and
\[
\Gamma_{\omega}(\omega) = \prod_{\omega \neq 0} \Gamma_{\omega}(\omega).
\]

Then if $f_\omega$ is of the form $\bigotimes f_\omega$, we see that the following corollary holds.

**Corollary II.2.4.** — Let $k$ be a totally imaginary number field. Then $\Gamma_{\omega}(\omega)$ can be continued to a meromorphic function in $\omega$. It satisfies the functional equations
\[
\Gamma_{\omega}(\omega) \Psi_{\omega}(\omega, I', f) = \prod_{\omega \neq 0} \{ e^{(\omega_\alpha)^{n-1}} \omega(\eta_\alpha) \omega(\eta_\alpha) \} \cdot \Gamma_{\omega}(\omega) \Psi_{\omega}(\omega, f).
\]
The singularities of $\Gamma_{\omega}(\omega) \Psi_{\omega}(\omega, f_\omega)$ in $\Omega_\omega$ lie in the set $\bigcup_{\alpha > 0} \{ \omega \in \Omega_\omega : \omega_\alpha = || \omega || \}$.

**Proof.** — The only statement which remains to be proved is that the singularities are as described. To do this, it suffices to consider $\sigma(\omega)$ lying in the closure of the dominant Weyl chamber in $\Phi(\mathbb{R})$. Thus, in this region we have to show that the functional (when $F = C$)
\[
f_\omega \rightarrow \Gamma_{\omega}(\omega)^{-1} \prod_{\alpha > 0} L_\omega(\omega_\alpha) \langle \lambda, f \rangle
\]
on $V(\omega)$ is non-zero. For this see [44], § 2.

The following theorem closes our investigation.

**Theorem II.2.5.** — a) Suppose that for each place $v$ of $k$ for which $|n|_v = 1$ one has $\dim(\text{Wh}(V_{0,v})) > 0$. Then $\sigma \neq 0$.

b) Suppose that for each place $v$ of $k$ for which $|n|_v = 1$ one has $\dim(\text{Wh}(V_{0,v})) = 1$. Then one has $\dim(\text{Wh}(V_{0,v})) = 1$ for every place $v$ of $k$.

**Proof.** — This is based on the ideas of [13]. We must first make some remarks about representations of local groups, concentrating on those over non-archimedean fields. Thus we fix a non-archimedean local field $F$ and consider an exceptional representation $(\pi_0, \sigma_0)$ of $\mathcal{G}$. 122
Let $P$ be the subgroup of $G$ defined by

$$P = \{ g \in G : (0,0, \ldots, 0, 1) g = (0, \ldots, 0, 1) \}$$

and let $\overline{P} = P^{-1}(P)$. Let $Z^0$ be the centre of $G$ and $Z^0 = \rho^{-1}(Z^0)$. Let $\epsilon$ be a non-degenerate character of $N^*_e$. Let us now regard $V_\theta$ as a $P$-representation. Then there exists a filtration

$$V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_{r-1} \supset 0$$

of $V_\theta$ as a $\overline{P} \times_{\mu_n(F)} \hat{Z}^0$-module where

$$V_{r-1} = \text{ind}_{\eta|^* \times \mu_n(F)}^{\eta \times \mu_n(F)} \epsilon \otimes_{\mu_n(F)} r(V_\theta)$$

and

$$r(V_\theta) = V_\theta / \langle \pi_\theta(n) v - \epsilon(n) v | v \in V_\theta, n \in N_e^* \rangle.$$

This is analogous to [3] 3.5. Here $r(V_\theta)$ is to be regarded as a $C[\mu_n(F)]$-module. The group $\overline{P}$ acts on this through the first and $\hat{Z}^0$ through the second factor. From the discussion of § 1.3 it follows that $r(V_\theta)$ is finite dimensional, and is non-zero if $\text{Wh}(V_\theta) \neq \{0\}$.

Let us observe that $V_{r-1}$ is cuspidal in the sense that if $N^* \neq \{1\}$ is a unipotent subgroup of $P$ and is also the unipotent radical of a parabolic subgroup of $G$ then, letting $N^*_t$ be the lift of $N^*_t$ to $P$,

$$V_{r-1} / \langle v - \pi_\theta(n) v | v \in V_{r-1}, n \in N^*_t \rangle = \{0\}.$$

If $F$ is archimedean similar considerations can be applied—see [45]. These we shall pass over in silence. Note however that it follows from our assumptions, Theorem I.3.5 and Theorem I.6.5 that

$$\dim(\text{Wh}(V_{\theta,e})) = 1$$

for archimedean $e$.

Let us now return to the global situation. We shall reformulate assertion $a)$ in a suitable fashion. Let $S$ be a finite set of places which

$a)$ contains all archimedean places,

$b)$ contains at least one non-archimedean place, and

$c)$ is sufficiently large in a sense to be made precise later.

For $v \notin S$ let $K_v$ be a compact open subgroup of $G_v$ over which $\rho_v$ splits, and such $K_v = \text{GL}_r(r_v)$ if $|n|_v = 1$. If $|n|_v < 1$ we assume $K_v$ has the form

$$K_v = \{ k \in \text{GL}_r(r_v) : k \equiv 1 \pmod{p_v^m} \}$$

where $p_v$ is the maximal ideal of $r_v$, and $m_v$ is some suitable integer. Let $G^8 = \prod_{v \notin S} G_v$, the restricted topological product with respect to this family. Let $K^8 = \prod_{v \notin S} K_v$. 

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Consider the diagonal embedding $G_s \hookrightarrow G^S$ and let $\Gamma_S = G_s \cap K^S$. Let now $G_S = \Pi_{s \in S} G_s$ and
\[
\tilde{G}_S = \left( \prod_{s \in S} \tilde{G}_s \right)/\langle i_s(\zeta) \cdot i_{s'}(\zeta)^{-1} \mid s, s' \in S, \zeta \in \mu_n(k) \rangle.
\]

Let $\kappa^* : K^S \rightarrow \tilde{G}_A$ and $s_s : G_s \rightarrow \tilde{G}_A$ be the liftings which we have discussed. Let $\tilde{G}_A(S) \subset \tilde{G}_A$ be the subgroup $\tilde{G}_S \times (K^S)^*$ discussed in § 0.2 where $(K^S)^* = \kappa^*(K^S)$. Let $s_T : \Gamma_S \rightarrow \tilde{G}_S$ be the map defined by $s_T \times \kappa^* = s_s$ in $\tilde{G}_S \times (K^S)^* = \tilde{G}_A(S)$.

Now let $f$ be a function on $\tilde{G}_A$ such that
\begin{enumerate}
  \item \(f(\gamma g) = f(g) \quad (\gamma \in G_s^S, \text{ and})\)
  \item \(f(g k) = f(g) \quad (k \in (K^S)^*)\).
\end{enumerate}

Let $\alpha_s : \tilde{G}_S \rightarrow \tilde{G}_A$ be the composite
\[
\tilde{G}_S \rightarrow \tilde{G}_S \times (K^S)^* \rightarrow \tilde{G}_A
\]
where the first map is the injection into the first factor. Let $f_S$ be the function on $\tilde{G}_S$ defined by $f_S(g) = f(\alpha_s(g))$; then $f_S$ satisfies
\begin{enumerate}
  \item \(f_S(s_T(\gamma) g) = f_S(g) \quad (\gamma \in \Gamma_S)\).
\end{enumerate}

Conversely, given $f_S$ satisfying $a')$ we can construct $f$ on $\tilde{G}_A$ satisfying $a)$ and $b)$, and which has support $G_s^S, \tilde{G}_A(S)$. Moreover it follows from Kneser's Strong Approximation Theorem [24] that if $S$ is large enough then
\[
G_s^S, \tilde{G}_A(S) = \tilde{G}_A;
\]
if this is so then $f$ is uniquely determined by $f_S$ and $a), b)$. We shall henceforth demand that $S$ is so large that this holds, and also that for each $w \in S$ one has $|n|_w = 1$.

The theory developed above for $\tilde{G}_A, G_s^S$ can now be developed without essential change for the pair $\tilde{G}_S, \Gamma_S \quad (= : s_T(\Gamma_S))$. Indeed this new theory is nothing other than the theory of $(K^S)^*$-invariant vectors of the original theory.

Denote by $r^S$ the ring of $S$-integers of $k$ and let
\[
r^S = \{ x \in r^S : \text{ord}_r(x) \geq m \quad \text{if} \quad |n|_w < 1 \}.
\]

Let $e_0 = \prod_{s \in S} e_{0_s}$ be a character of $\prod_{s \in S} k_s$ which is trivial on $r^S$. Let $D$ be the fractional ideal of $r^S$ defined by
\[
D^{-1} = \{ x \in k : e_0(xy) = 1 \quad \text{for all} \quad y \in r^S \}.
\]

We shall assume that $S$ is so large that $r^S$ is a principal ideal domain. We can then write $D$ as $\langle \delta \rangle$, $\delta \in k^\times$.

For $d = (d_{1,2}, d_{2,3}, \ldots, d_{r-1,r})$
with $d_{i,i+1} \in D^{-1} - \{ 0 \}$ \(1 \leq i < r\)
we set $e_d(n) = e_0(\sum_{1 \leq i < r} d_{i,i+1} n_{i,i+1})$
where $n = (n_{i,i}) \in N_+ (\prod_{s \in S} k_s)$.
We regard $\varepsilon_d$ also as a character of $(N^*_+) = (N^*_+)^\wedge \bar{G}_g$. We write $\varepsilon$ for $\varepsilon_d$ with $d = (1, 1, \ldots, 1)$.

Now, if $v \in S$ let $T_v \subseteq V_{0,s}$ be the sub-space $V_{r-1}$ discussed above, $T_v = \bigotimes_{v \in S} T_v$, the product being taken with respect to $C[\mu_v(k)]$. Let now $\bar{\mathcal{P}}_g \subseteq \bar{G}_g$ be the group whose local factors are the $p_v^{-1}(P(k_v))$ ($v \in S$). Let, for $g \in \bar{G}_g$, $\beta_g : \bar{\mathcal{P}}_g \rightarrow \bar{G}_g$; $p \rightarrow p^g$ be an embedding of $\bar{\mathcal{P}}_g$ in $\bar{G}_g$. Let

$$T(g) = \pi_g(g^{-1}) (T_g).$$

Then $T(g) \subseteq V_{0,\bar{g}}$ is a $\beta_g(\bar{\mathcal{P}}_g)$-module. Both $\beta_g(\bar{\mathcal{P}}_g)$ and $T(g)$ depend only on the class of $g$ in $\bar{\mathcal{P}}_g \mathcal{Z}_0 \backslash \bar{G}_g$, where the notations used should be self-explanatory.

Let $\theta : V_{0,\bar{g}} \rightarrow \mathcal{C}$ be the $\Gamma^*_\mathcal{R}$-invariant linear form given by the theory of Eisenstein series. We shall show that if

$$\int_{N^*_+ \cap R \backslash N^*_+, \bar{g}} \varepsilon_d(n) \theta(\pi_0(n)f) \, dn = 0$$

for all $d$ as above and for all $f \in V_{0,\bar{g}}$, then $\theta | T(\gamma) = \{0\}$ for all $\gamma \in \Gamma^*_\mathcal{R}$. If this is so then, as $\Gamma^*_\mathcal{R} \bar{\mathcal{P}}_g \mathcal{Z}_0$ is dense in $\bar{G}_g$, we deduce that $\theta | T(g) = \{0\}$ for all $g \in \bar{G}_g$ and hence $\theta$ is zero on the space spanned by all the $T(g)$ ($g \in \bar{G}_g$). But this latter space is $\bar{G}_g$-invariant and hence $\theta = 0$. But this contradicts the fact that

$$\int_{N^*_+ \cap R \backslash N^*_+, \bar{g}} \theta(\pi_0(n)f) \, dn$$

is not identically zero.

Hence it will follow that there exists at least one $d$ and one $f \in V_{0,\bar{g}}$ such that

$$\int_{N^*_+ \cap R \backslash N^*_+, \bar{g}} \varepsilon_d(n) \theta(\pi_0(n)f) \, dn \neq 0.$$

On translating this back into terms of the group $\bar{G}_\mathcal{A}$ one sees that the "$\varepsilon$-th Fourier coefficient" is not identically zero. This is what we had to prove.

It therefore remains to prove that, given the assumption above, $\theta | T(\gamma) = \{0\}$ for all $\gamma \in \Gamma^*_\mathcal{R}$. It will suffice to show that $\theta | T(I) = \{0\}$ since $\theta$ is $\Gamma^*_\mathcal{R}$-invariant. To do this consider $f \in T(I)$; as $S$ contains at least one non-archimedean place, $f$ is then cuspidal with respect to any unipotent subgroup of $\bar{\mathcal{P}}$. The function $p \mapsto \theta(\pi_0(p)f)$ can then be expanded in a "Fourier series"

$$\theta(f) = \sum_d \sum_{p \in (N^*_+ \cap R \backslash I^+_\mathcal{R})} \int_{(N^*_+ \cap R \backslash N^*_+, \bar{g})} \varepsilon_d(n) \theta(\pi_0(npf)) \, dn$$

by following step for step the proof of [40] Theorem 1. Here we have assumed that the measure on $N^*_+, \bar{g}$ is so normalised that the measure of $(N^*_+ \cap \Gamma^*_\mathcal{R}) \backslash N^*_+, \bar{g}$ is one.
But, by assumption, all the terms in this sum are zero. Hence $\theta(f) = 0$, as we required. This completes the proof of $a$.

We shall now prove $b$. Let $S$ be the finite set of places

$$S = \{ v : v \text{ a place of } k, \ |\nu|_v < 1 \}.$$ 

Under the assumptions made the group $\mathbb{Z}_v^*$ is abelian for all $v$. Moreover, since $\text{Wh}(V_{0,v})$ is one-dimensional when $v \notin S$ we see that there exists a quasicharacter $\chi'_v$ of $\mathbb{Z}_v^*$ so that $\mathbb{Z}_v^*$ acts on $\text{Wh}(V_{0,v})$ by $\chi'_v$. It follows easily from $a)$ that the $\chi'_v$ are local components of a quasicharacter $\chi'$ of $\mathbb{Z}_A^*$ trivial on $\mathbb{Z}_v^*$.

We can factor

$$\mathcal{G}_A = \mathcal{G}_S \times \mathcal{G}^S$$

$$\mathcal{G}_A^\theta = \mathcal{G}_S \times \mathcal{Z}^\theta$$

with the products being taken over $\mu_n(k)$. Likewise one has

$$V_\theta = V_{0,8} \otimes V^8$$

where the tensor product is with respect to $\mathbb{C}[\mu_n(k)]$.

For a non-archimedean place $w$ of $k$ define $T_w$ as above and form

$$V^\text{eup} = \sum_{w \notin S} (T_w \otimes V^8 \uparrow (w)).$$

Again the argument of [40] Theorem 1 yields the expansion

$$\theta(I, J(x \otimes y)) = \sum_{\gamma \in \mathcal{N}/\mathcal{P}} \lambda_\gamma(\gamma(x \otimes y)).$$

If we now choose $\lambda^S$ to be a generator of $\text{Wh}(V^8)$ then we see that there exists a $\lambda^S \in \text{Wh}(V^S)$ such that for $x' \in V^S$, $y' \in V^8$ one has

$$\lambda_\gamma(x' \otimes y') = \lambda^S(x') \lambda^8(y').$$

The local results recalled at the beginning of the proof of this theorem can easily be extended to the semi-local case and we obtain an embedding

$$j: \text{ind}^{\mathbb{P}_G}_{\mathbb{P}_G \times \mu_n(k)}(e \times e) \times \text{Wh}(V^S) \rightarrow V^S$$

where we have extended some of our notations in what we hope is a self-evident fashion. Here $\text{Wh}(V^S)'$ is the dual space to $\text{Wh}(V^S)$. We shall suppose that the induced representation here is realised by functions on $\mathbb{P}_G$ in the usual way. If $\lambda \in \text{Wh}(V^S)$ then one has for $f$ in the induced representation and for $l \in \text{Wh}(V^S)'

$$\lambda(j(f \otimes l)) = f(I) \langle \lambda, l \rangle.$$
Then there exists \( l \in \text{Wh}(V_{\theta})' \), \( l \neq 0 \) such that \( \langle \lambda_{\theta}, l \rangle = 0 \). It follows that there exists a \( \mathbb{P} \mathcal{Z}_{\theta} \)-subspace \( U_{\theta} \) of \( V_{\theta, \theta} \) so that
\[
\lambda_{\theta}(u) = 0 \quad (u \in U_{\theta}).
\]

We now construct \( U_{\lambda} = U_{\theta} \otimes V_{\text{esup}}^{\mathbb{A}} \subset V_{\theta} \); it is a \( \mathbb{P} \mathcal{Z}_{\lambda} \)-subspace. From the Fourier expansion given above we also have
\[
\theta(1, J(u)) = 0 \quad (u \in U_{\lambda}).
\]

For \( u \in U_{\lambda} \) one can therefore regard
\[
g \mapsto \theta(g, J(u))
\]
as a function on \( G_{\lambda} \setminus \mathcal{G}_{\lambda} \) which is zero on the subset \( P_{\lambda} \mathbb{Z}_{\lambda} \setminus \mathbb{P} \mathcal{Z}_{\lambda} \). This subset is dense in the sense above and it follows that for \( u \in U_{\lambda} \) one has \( \theta(g, J(u)) = 0 \) for all \( g \in \mathcal{G}_{\lambda} \). Thus \( U_{\lambda} \) generates a proper \( \mathcal{G}_{\lambda} \)-subspace of \( V_{\theta} \). This is however impossible since \( V_{\theta} \) is irreducible. This shows that the assumption that \( \dim(\text{Wh}(V_{\theta})) > 1 \) is untenable. Thus
\[
\dim(\text{Wh}(V_{\theta})) = 1
\]
by \( a \), and the assertion \( b \) follows at once.

We record here a corollary of this theorem, an immediate consequence of it and Theorem I.3.5.

**Corollary II.2.6.** — Let \( F \) be a non-archimedean local field with \( \text{Card}(\mu_{n}(k)) = n \). With the notations of Theorem I.3.5 one has
\[
a) \text{ if } N > 0 \text{ then } \dim(\text{Wh}(V_{\theta}(\omega'))) > 0,
b) \text{ if } N = 1 \text{ then } \dim(\text{Wh}(V_{\theta}(\omega''))) = 1.
\]

**Remarks.** — 1) We are indebted to I. I. Piatetski-Shapiro for pointing out Theorem II.2.5 \( b \) and Corollary II.2.6 \( b \) and their proofs to us.

2) Examples based on Theorem I.5.3 and [7] show that one does not always have
\[
\dim(\text{Wh}(V_{\theta}(\omega))) = N.
\]

3) This corollary completes the local theory in an essential point. Its significance for the global theory is that it shows \( c \) to be determined up to a constant by local considerations. Whereas it is possible to give a purely local proof of Corollary II.2.6 \( a \) we have not found one for Corollary II.2.6 \( b \). Another approach to this question can be given by the techniques of \( \S \) I.5 and a generalisation of the Shimura Correspondence. We hope to return to this in a later publication.

**II.3. Examples**

In this section we give some examples to show how the theorems of the previous section relate to more familiar concepts. We restrict our attention to the case \( r = 2 \). What we shall do is to choose \( (f, \omega) \) conveniently and to make Corollary II.2.4 explicit.
Let \( k \) be a totally imaginary number field. Let \( \mathfrak{A} \) be an integral ideal of \( k \) such that if \( v \) is a place of \( k \) for which \( |n_v| < 1 \) holds then \( |\mathfrak{A}|_v < 1 \) also holds. We shall also assume that for each \( v \) for which \( |\mathfrak{A}|_v < 1 \) holds we have furthermore
\[
(1 + \mathfrak{A}r_v) \subset k_v^{\times n}.
\]
Let
\[
S(\mathfrak{A}) = \{v : \text{a place of } k, |\mathfrak{A}|_v < 1\}.
\]
If \( v \in S(\mathfrak{A}) \) we let
\[
L_v(\mathfrak{A}) = \{k \in \text{GL}_n(r_v) : k = I(\mathfrak{A}r_v)\}
\]
and we let \( L_\mathfrak{A}(A) \) be the Kubota lift of this to \( \widehat{G} \) ([26], Theorem 2). For convenience we shall take \( c = 0 \).

We shall next define the \( \widehat{H}_{n,v} \). If \( v \) is a finite place of \( k \), \( v \notin S(\mathfrak{A}) \), then we take \( \widehat{H}_{n,v} \) to be the standard group \( \widehat{H}_{n,v} \) where
\[
H_{n,v} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_j \in k_v^{\times}, \text{ord}_v(a_j) \equiv 0 \pmod{n} \quad (j = 1, 2) \right\}.
\]
If, however, \( v \in S(\mathfrak{A}) \) then we can take \( \widehat{H}_{n,v} \) to be \( \rho^{-1}(H_{n,v}) \) where
\[
H_{n,v} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_1 \in k_v^{\times n}, a_2 \in k_v^{\times} \right\}.
\]
That this is a maximal isotropic subgroup is special feature of the case \( r = 2 \).

For our applications the set \( S(\mathfrak{A}) \) can be taken to be large, and to simplify our discussion we shall assume that it is sufficiently large. Recall that we have already assumed that
\[
S(\mathfrak{A}) \supset \{v : |n|_v < 1\};
\]
we supplement this with:

(A) the ring of \( S(\mathfrak{A}) \)-integers is a principal ideal domain.

Let \( \varepsilon : \mu_n(k) \to \mathbb{C}^\times \) be an injective character. We shall next denote by \( \Omega_\varepsilon(\mathfrak{A}) \) the group of quasicharacters of \( \widehat{H}_{n,A} \), which restrict to \( \varepsilon \) on \( i(\mu_n(k)) \), and whose ramification is restricted by the following conditions on the local components \( \omega_v \) of an \( \omega \in \Omega_\varepsilon(\mathfrak{A}) \) at the non-archimedean places of \( k \):
\[
\omega_v | (\widehat{H}_{n,v} \cap \mathbb{K}_v) = 1 \quad (v \notin S(\mathfrak{A})),
\]
and
\[
\omega_v | (\widehat{H}_{n,v} \cap \mathbb{L}_n(\mathfrak{A})) = 1 \quad (v \in S(\mathfrak{A})).
\]

We are assuming here that \( \mathbb{K}_v = \text{GL}_n(r_v) \) and \( \mathbb{K}_v^\ast \) is the canonical lift of \( \mathbb{K}_v \), which is defined whenever \( v \) is a non-archimedean place of \( k \), \( v \notin S(\mathfrak{A}) \).

Note that if \( \omega \in \Omega_\varepsilon(\mathfrak{A}) \) then \( \omega s \in \Omega_\varepsilon(\mathfrak{A}) \) \( (s \in \Phi(\mathbb{C}) \cong \mathbb{C}) \).

Let \( U(\mathfrak{A}) \) be the group of units of the ring of \( S(\mathfrak{A})\)-integers. If \( S \) is any finite set of non-archimedean places of \( k \) let \( r^S \) be the ring of \( S \)-integers of \( k \).
We now observe that \( \omega \in \Omega_v(\mathfrak{A}) \) can be extended to \( \tilde{H}_{s,v} \) in such a fashion that
\[
\omega_{s,v} | K_v^* \cap \tilde{H}_{s,v} = 1 \quad (v \notin S(\mathfrak{A})),
\]
\[
\omega_{s,v} | L_v^*(\mathfrak{A}) \cap \tilde{H}_{s,v} = 1 \quad (v \in S(\mathfrak{A})),
\]
and
\[
\omega_{s} | H_v^* \cap \tilde{H}_{s,v} = 1.
\]

To see this we first note that
\[
\{ x \in k^x : \text{ord}_v(x) \equiv 0 \pmod{n} \quad (v \in S(\mathfrak{A})) \}
\]
is, by (A), \( k^{x^n} \cdot U_n(\mathfrak{A}) \), where
\[
U_n(\mathfrak{A}) = \{ u \in U(\mathfrak{A}) : u \in k_v^{x^n} \quad (v \in S(\mathfrak{A})) \}.
\]

Then
\[
\rho(H_v^* \cap \tilde{H}_{s,v}) = H_{n,k} \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in U_n(\mathfrak{A}), b \in U(\mathfrak{A}) \right).
\]

We seek therefore \( \omega_{s,v} \) on \( \tilde{H}_{s,v} \) such that
\[
\left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) \mapsto \prod_{v \in S(\mathfrak{A})} \omega_{s,v} \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right), \quad (a \in U_n(\mathfrak{A}), b \in U(\mathfrak{A}))
\]
where \( S_n(\mathfrak{A}) = S(\mathfrak{A}) \cup \{ v : v \text{ archimedean} \} \),

should be trivial. Since this quasi-character has trivial restriction to
\[
\left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in U^n(\mathfrak{A}) \right),
\]
where
\[
U^n(\mathfrak{A}) = \{ u \in U(\mathfrak{A}) : u \in k^{x^n} \};
\]
the existence of such \( \omega_{s,v} \) is easy to see. That \( \omega_{s,v} | L_v^*(\mathfrak{A}) \cap \tilde{H}_{s,v} = 1 \quad (v \in S(\mathfrak{A})) \) is automatic.

Now that we have constructed \( \omega \), we shall construct an element \( f^o_v \in V(\omega_v) \). This is \( \otimes f^o_v \), where

(i) if \( v \notin S(\mathfrak{A}) \) but \( v \) is non-archimedean then \( f^o_v \) is the element of \( V(\omega_v) \) which is \( K_v^* \)-invariant and such that \( f^o_v(I) = 1 \);
(ii) if \( v \in S(\mathfrak{A}) \) we let \( f^o_v \in V(\omega_v) \) be the \( L_v^*(\mathfrak{A}) \)-invariant vector with support \( \tilde{H}_{s,v} L_v^*(\mathfrak{A}) \omega_0 \) and such that \( f^o_v(\omega_0) = 1 \), and
(iii) if \( v \) is archimedean then \( f^o_v \) is arbitrary.

As in Corollary II.2.4 we shall remove the archimedean places from our discussion.

From \( f^o_v \) and \( h \in H_v^* \) we form \( f \) by
\[
f(g) = f^o_v(gk).
\]

We shall mainly have to consider \( h \in H_v^* \).

Let \( e_0 \) be a character of \( k_v \) such that the conductor of the local factor \( e_0 \) is \( r_v \) at all non-archimedean \( v \notin S(\mathfrak{A}) \). It follows from assumption (A) that such an \( e_0 \) can be
found. Let $D(\varepsilon_0)$, or simply $D$, be the fractional ideal of $k$ associated with the differential idele of $\varepsilon_0$ as in [50], VII-2.

Let $x$ be the unique positive root of $GL_2$.

If we define, for $\eta_0, h_0 \in \widehat{\mathbb{H}}_x$,

$$G_x(\omega_0, \eta_0, h_0) = \int_{S^1} \overline{x}(n) f(x, \eta_0, \omega_0, n) \, dn,$$

then we are to investigate

$$\Psi(\omega, f) = L_f(\| \|_A \omega_0) \cdot \sum_{\eta \in \mathbb{H}_x \cap \mathbb{H}_\omega} \omega_\infty \mu_\infty(\eta) \prod_{v \neq \infty} G_v(\omega_v, \eta_v),$$

where

$$\omega_\infty \mu_\infty(\eta) = \prod_{v \neq \infty} \omega_v \mu_v(\eta_v).$$

More precisely we shall show that $\Psi(\omega, f)$ is a Dirichlet series the coefficients of which are Gauss sums. In the obvious sense $\Psi(\omega, f)$ has an analytic continuation to $\Omega(\mathbb{A})$ as a meromorphic function. There is a pole at $\omega_0^* = \| \|_A$ and (using the isomorphism $G \cong \Phi(C); \ s \mapsto \kappa(s)$) the residue at $\omega$ with $(\omega')^* = \| \|_A$ is given by

$$\sum_{\eta \in \mathbb{H}_x \cap \mathbb{H}_\omega} \mathbf{c}(\eta) \cdot \prod_{v \neq \infty} \lambda_v f(x, \eta_v) \prod_{v \neq \infty} \omega_v \mu_v(\eta_v)$$

as in Corollary II.2.4, where $\mathbf{c} \in U_0(\omega)$.

To formulate the results we let

$$\psi_\infty(\omega, h) = \Psi(\omega, f),$$

let $\omega'$ be exceptional, i.e. $(\omega')^* = \| \|_A$, and

$$\rho_\infty(\omega', h) = \Re s \psi_\infty(\omega, h).$$

Let $\mathbb{H}_{f, A}$ (resp. $\mathbb{H}_{n, f, A}$) be the group for which

$$\mathbb{H}_{f, A} \cong \prod_{v \neq \infty} \mathbb{H}_v \times \mathbb{H}_{n, A} \ (\text{resp. } \mathbb{H}_{n, f, A} \cong \prod_{v \neq \infty} \mathbb{H}_{n, v} \times \mathbb{H}_{n, f, A})$$

where all the products are with respect to $\mu_\infty(k)$. Then $\mathbb{H}_{f, A}$ (resp. $\mathbb{H}_{n, f, A}$) is naturally a subgroup of $\mathbb{H}_{A}$ (resp. $\mathbb{H}_{n, A}$). In particular, we can make use of the restrictions of $\omega$ and $\mu_\infty$ to $\mathbb{H}_{n, f, A}$ as these are well-defined.

**Theorem II.3.1.** — There exist $c_v, c'_v \in \mathbb{R}_+^\times$ for each non-archimedean place $v$ of $k$ such that $c_v = c'_v = 1$ for $v \notin S(\mathbb{A})$ and

a) if, for one place $v$ of $k$ one has

$$|p(h_v)^*|_v > c_v,$$

then

$$\rho_\infty(\omega', h) = 0,$$

b) if $h^{(1)}, h^{(2)} \in \mathbb{H}_{f, A}, h^{(1)} h^{(2)} - 1 \in \mathbb{H}_{n, f, A}$ and

$$|p(h^{(j)})^*|_v \leq c_v \quad (j = 1, 2)$$

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for all non-archimedean places \( v \) of \( k \) then one has

\[
\varphi^*\omega|_A(h^{(1)} h^{(2)}-1)^{-1} \varphi_H(\omega', h^{(2)}) = \varphi_H(\omega', h^{(2)}).
\]

**Remark.** — All the other consequences of Theorems I.4.1 and I.4.2 can be deduced from this one, as we shall explain in more detail later. Note that the corresponding identity for the \( \psi_H(\omega, h) \) does not hold.

**Proof.** — Part \( a) \) of the theorem is an immediate consequence of Theorem I.4.1. Likewise part \( b) \) follows from Theorems I.4.1 and I.4.2.

Let \( \varepsilon_1 : \mu_n(k) \to \mathbb{C}^* \) be a homomorphism. Let \( c \in \mathbb{R}^{[\mathfrak{m}]} \), \( c \neq 0 \), and define the restricted Legendre symbol \( \left( \frac{d}{c} \right)_v \) by

\[
\left( \frac{d}{c} \right)_v = \prod_{v \mid c} (\varepsilon, d)_v
\]

for \( d \) coprime to \( c \) in \( \mathbb{R}^{[\mathfrak{m}]} \). Let, for \( x \in k \),

\[
\varepsilon_v(x) = \prod_{v \mid \infty} \varepsilon_v(x),
\]

where \( \varepsilon_v \) is the character of \( k_\mathfrak{A} \) defined above. Then we define, for \( X \in \mathbb{R}^{[\mathfrak{m}]} \)

\[
\mathfrak{g}_n(\varepsilon_1, X, c) = \sum_d \varepsilon_1 \left( \frac{d}{c} \right)_v \cdot \varepsilon_v \left( \frac{X d}{c} \right)
\]

where \( d \) is summed over a set of representatives for \( (\mathbb{R}^{[\mathfrak{m}]}/\mathbb{R}^{[\mathfrak{m}]}))^* \) which also satisfy

\[
|d|_v \leq |c| D^{-1} X^{-1} \quad (v \in S(\mathfrak{m})).
\]

We define the quasicharacter \( \omega^{(1)} \) of \( \prod_{v \in S_\mathfrak{m}(\mathfrak{m})} k_v^{\times n} \) by

\[
\omega^{(1)}(\varepsilon_v) = \prod_{v \in S_\mathfrak{m}(\mathfrak{m})} \omega_v \left[ s_v \begin{pmatrix} \varepsilon_v^{-1} & 0 \\ 0 & \varepsilon_v \end{pmatrix} \right].
\]

Regarding \( U_n(\mathfrak{m}) \) as a subgroup of \( \prod_{v \in S_\mathfrak{m}(\mathfrak{m})} k_v^{\times n} \) we see that \( \omega^{(1)}(u) = 1 \) \((u \in U_n(\mathfrak{m}))\).

Conversely it is easy to see that if a quasicharacter \( \varphi \) of \( \prod_{v \in S_\mathfrak{m}(\mathfrak{m})} k_v^{\times n} \) is given such that \( \varphi(u) = 1 \) \((u \in U_n(\mathfrak{m}))\), then there exists a quasicharacter \( \omega \in \Omega_n(\mathfrak{m}) \) of \( \mathbb{H}_n \mathfrak{A} \) trivial on \( \mathbb{H}_n^* \) such that \( \omega^{(1)} = \varphi \).

Let \( \varphi \) be as above and define, for \( X \in \mathbb{R}^{[\mathfrak{m}]} \), \( \varepsilon_1 : \mu_n(k) \to \mathbb{C}^* \) injective,

\[
\varphi^*_n(\varphi, \varepsilon_1, X) = \left( \sum_{v \in S_\mathfrak{m}(\mathfrak{m}), \varepsilon_1, X, c \varepsilon_v(c) \varphi(c)} \lambda \right) || \left[ \begin{array}{c} x_v \\ c_v \end{array} \right],
\]

where the sum is taken over \( c \in \mathbb{R}^{[\mathfrak{m}]} \) such that \( c \in k_v^\times \) for \( v \in S_\mathfrak{m}(\mathfrak{m}) \), and modulo \( U_n(\mathfrak{m}) \) (multiplicatively). Here \( \varphi^*_n \) is the Grössencharakter of \( k_\mathfrak{A} \) unramified outside \( S_\omega(\mathfrak{m}) \) and whose \( S_\omega(\mathfrak{m}) \)-factor is

\[
(x_v) \to \varphi((x_v^*)).
\]
Note that the series is well-defined, and, as one can easily verify, that it converges absolutely if \( \sigma(\varphi) > 3/2 \), where \( \sigma(\varphi) \in \mathbb{R} \) satisfies
\[
|\varphi(\zeta_n)| = \prod_{\nu} |\zeta_n|^{-\sigma(\varphi)} \quad (\zeta_n \in \prod_{\nu} k^\times_{\nu}).
\]
That \( \sigma(\varphi) \) exists follows from Dirichlet's Unit Theorem.

Observe that whereas \( \omega \) involves a choice of character \( \varepsilon : \mu_n(k) \to \mathbb{C}^\times \), the quasi-character \( \varphi \) involves no such choice.

It is convenient here to record a proposition which will be of use to us later. Let \( \mathcal{H}, \varphi \) be as above.

**Proposition II 3.2.** — Let \( p \) be a prime ideal of \( k \), coprime to \( \mathcal{H} \). Let \( \pi \in \mathcal{P}(\mathcal{H}) \) be such that \( \pi^\mathcal{P}(\mathcal{H}) = p \cdot \mathcal{P}(\mathcal{H}) \). Let \( v \) be the place of \( k \) associated with \( p \). Let \( \varphi' \) be the quasicharacter of
\[
\prod_{w \in \mathcal{S}_\infty(\mathcal{H})} k^\times_{w} \times k^\times_{v}
\]
which is \( \varphi \) on the first factor, is unramified on the second factor and is trivial on \( U^\times_{\mathcal{P}}(\mathcal{P}) \). Let \( X \in \mathcal{P}(\mathcal{H}) \) and let \( l = \text{ord}_p(X) \), and \( \chi_0 = X^{-1} \). Suppose that
(i) \( \pi^{l+1} \in k^\times_{\mathcal{P}} \quad (w \in \mathcal{S}_\infty(\mathcal{H})) \),
and (ii) the natural map \( U(\mathcal{H}) \to r^\times \) is surjective.

Suppose that \( \varepsilon_1 \) is injective. Let \( N \) be the absolute norm. Then we have
\[
\psi^0_\mathcal{H}(\varphi, \varepsilon_1, X) = \left( \frac{1 - N(\pi)^{-1} \varphi(\pi^n) - (1 - N(\pi)^{-1} ) (N(\pi^n) \varphi(\pi^n)^{l+1})^0(\mathcal{H})}{1 - N(\pi^n) \varphi(\pi^n)} \right) \psi^0_\mathcal{H}(\varphi', \varepsilon_1, X_0 \pi^l) + g(\varepsilon_1^{l+1}, X_0, \pi) \cdot N(\pi) \cdot \varphi(\pi^{l+1}) \cdot \psi^0_\mathcal{H}(\varphi', \varepsilon_1, X_0 \pi^{-i})
\]

**Proof.** — In the series defining \( \psi^0_\mathcal{H}(\varphi, \varepsilon_1, X) \) we replace \( c \) by \( \pi^k \cdot \varepsilon_0 \), where \( |\varepsilon_0|_\mathcal{P} = 1 \) and \( \varepsilon_0 \in \mathcal{P}(\mathcal{H}) \). It is summed over a set of representatives modulo \( U(\mathcal{H}) \). Since
\[
g(\varepsilon_1, X_0 \pi^k, \varepsilon_0) = \varepsilon_1 \left( \frac{\varepsilon_0}{\pi^k} \right)^{-1} \cdot \varepsilon_1 \left( \frac{\pi^k}{\varepsilon_0} \right)^{-1} \cdot g(\varepsilon_1, X_0 \pi^k, \varepsilon_0) \cdot g(\varepsilon_1, X_0 \pi^k, \varepsilon_0)
\]
and since, as we shall see, we may assume that
\[
\pi^k, \varepsilon_0 \in \prod_{w \in \mathcal{S}_\infty(\mathcal{H})} k^\times_{w}
\]
for otherwise the term which we are computing will be zero, we obtain from the reciprocity law
\[
g(\varepsilon_1, X, \pi^k \varepsilon_0) = g(\varepsilon_1, X_0 \pi^k, \varepsilon_0) \cdot g(\varepsilon_1, X_0 \pi^{-k}, \varepsilon_0).
\]
Then
\[ \psi^0_\mathfrak{M}(\varphi, e_1, X) = L_f(|| \varphi^0_{\mathfrak{M}}, \{ \sum_{k \geq 0} \sum_{\text{c}_0} g_\mathfrak{M}(e_1, X_0 \pi^k, \pi^k) \varphi(e_1, X_0 \pi^{l-\text{c}_0}) \varphi(\text{c}_0 \pi^k) \}. \]

Observe that
\[ g_\mathfrak{M}(e_1, X_0 \pi^k, \pi^k) = 0 \]
unless \( k \leq l \), \( k \equiv 0 \pmod{n} \) or \( k = l + 1 \). As
\[ c_0 \pi^k \in \prod_{w \in \mathfrak{S}_\mathfrak{M}(\mathfrak{M})} k_w^x \]
it follows from assumption (i) that
\[ c_0 \in \prod_{w \in \mathfrak{S}_\mathfrak{M}(\mathfrak{M})} k_w^x. \]

It is easy to verify that the summand does not change if \( c_0 \) is replaced by \( c_0 u \) \((u \in U_n(\mathfrak{M}))\); hence by assumption (ii) we can assume that
\[ c_0 \in r_v^x. \]
Thus the summation is now carried out over \( c_0 \in r_v^x \), \( c_0 \in \prod_{w \in \mathfrak{S}_\mathfrak{M}(\mathfrak{M})} k_w^x \times r_v^x \) and modulo
\[ \{ u \in U_n(\mathfrak{M}) : u \in r_v^x \} \]
multiplicatively. Since
\[ U_n(\mathfrak{M}) = \{ u \in U_n(\mathfrak{M}) : u \in r_v^x \}. \]
this set of elements is also a set of representatives for those \( c_1 \in r_v^x \), \( c_1 \in \prod_{w \in \mathfrak{S}_\mathfrak{M}(\mathfrak{M})} k_w^x \)
taken modulo \( U_n(\mathfrak{M}) \). Hence we obtain
\[ \psi^0_\mathfrak{M}(\varphi, e_1, X) = L_f(|| \varphi^0_{\mathfrak{M}}, \{ \sum_{k \geq 0} \sum_{\text{c}_0} g_\mathfrak{M}(e_1, X_0 \pi^k, \pi^k) \varphi(e_1, X_0 \pi^{l-\text{c}_0}) \varphi(\text{c}_0 \pi^k) \}. \]

However
\[ g_\mathfrak{M}(e_1, X_0 \pi^k, \pi^k) = \Phi(\pi^k) \quad (k \leq l, k \equiv 0 \pmod{n}), \]
\[ = N(\pi^l) \varphi_{\mathfrak{M}}(e_1 + 1, X_0, \pi) \quad (k = l + 1), \]
\[ = 0 \quad (\text{otherwise}), \]
where \( \Phi \) is the Euler totient function in \( k \). Hence the term in braces becomes
\[ \{ \sum_{0 \leq k \leq l} \Phi(\pi^k) \varphi(\pi^k) \} \varphi_{\mathfrak{M}}(\varphi', e_1, X_0 \pi^k) \]
\[ + \sum_{k \equiv 0 \pmod{n}} \sum_{\text{c}_0} g_\mathfrak{M}(e_1 + 1, X_0, \pi) N(\pi^k) \varphi(e_1 + 1) \varphi_{\mathfrak{M}}(\varphi', e_1, X_0 \pi^{n-1-\text{c}_0}). \]
On carrying out the summation this gives the expression of the statement.

Let \( D_\mathfrak{M} \) be the discriminant of \( k \). The central point in these computations is that
\[ \psi^0_\mathfrak{M}(\omega, k) \]
can be expressed in terms of \( \psi^0_\mathfrak{M} \), at least if \( k \in H^* \).
Theorem 11.3.3. — Suppose that $h \in \mathbb{H}_1$ and let $X = \rho(h)^n$. Let $\omega \in \Omega_1(\mathfrak{A})$ and let $\omega^{(1)}$ be defined as above. Then

$$
\psi_H(\omega, h) = [U(\mathfrak{A}) : U_n(\mathfrak{A})]. N(\mathfrak{A})^{-1}.|D_k|^{-1/2} (\omega \mu)(h)^{-1}\psi_H(\omega^{(1)}, e^{-1}, X)
$$

if $X \in \mathfrak{A}^{-1} D^{-1}$,

$$
= 0 \text{ otherwise.}
$$

The proof of this theorem is based on a fairly long computation, so that it will be more convenient for us to discuss its significance now and to postpone its proof to the end of this section.

The first conclusion that one can draw is that $\psi_H(\varphi, e, X) (e \text{ injective})$ has an analytic continuation to the group of all $\varphi$, that it has at most a simple pole in $\sigma(\varphi) > 1$ and this where $\varphi = \varphi_0$ with

$$
\varphi_0(y_\varphi) = \prod_{\omega \in \Delta(\mathfrak{A})} |x_\varphi|^{1 + \frac{1}{n}}.
$$

Let

$$
\rho^0(\varphi, e, X) = \text{Res}_{\varphi = \varphi_0} \psi_H(\varphi, e, X).
$$

From Proposition 11.3.2 we see that if $X, X_0, p, \pi, l$ are as in the enunciation of that proposition then one has

$$
\rho^0(\varphi, e, X) = \left[1 - N(\pi)\right]^{-\frac{1}{n}} \rho^0(\varphi, e, X) + \left\{\delta(\varepsilon^{l+1}, X_0, \pi) \rho^0(\varphi, e, X_0, \pi^{n^{-1} - 2}) N(\pi) \left[\frac{l+1}{n}\right]^{-1}
\right.

- N(\pi) \left[\frac{l}{n}\right]^{-1} \rho^0(\varphi, e, X)

$$

If we assume, as we may, that $\pi \in \mathfrak{A}$ then we can deduce from Theorem II.3.1 that $\rho^0(\varphi, e, X_0, \pi)$ is ultimately periodic with period $n$ in $l$ for large $l$, as $\pi^n \in k^\times \cap k_1^n$. Hence we can deduce that

$$
\rho^0(\varphi, e, X) = (1 + N(\pi)^{-1}). \rho^0(\varphi, e, X)
$$

and, if $0 \leq l \leq n - 1$, that

$$
\rho^0(\varphi, e, X_0, \pi) = \delta(\varepsilon^{l+1}, X_0, \pi) N(\pi) \left[\frac{l+1}{n}\right]^{-1} \rho^0(\varphi, e, X_\pi^{n^{-1} - 2}).
$$

These represent the arithmetic form of our results. The restrictions on $\pi$ represented by conditions (i) and (ii) of Proposition II.3.2 are quite stringent, but other variants can be given. Those quoted in [39], p. 180, are such; we shall not give a proof for those formulae here. Nevertheless the formal similarity between those derived here and those in [39] should convince the reader that the latter formulae are a consequence of the techniques developed here. The variations lie in the choice of $H^1_{x,w_0}$ and $f^0_{x,w}$ if $w \in S(\mathfrak{A})$, and these lead to variations on Proposition II.3.2.
Observe that, if $n = 3$, then
\[ \rho^0_{\Pi}(\varepsilon, X_0 \pi^3) = 0 \]
and if $\pi \in \prod_{\nu \in \mathcal{A}_0(\mathfrak{m})} k^\times_\nu$
\[ \rho^0_{\Pi}(\varepsilon, X_0 \pi) = g_{\Pi}(\varepsilon^{-1}, X_0, \pi) N(\pi)^{-\frac{g_3}{2}} \rho^0_{\Pi}(\varepsilon, X_0). \]
Moreover
\[ \rho^0_{\Pi}(\varepsilon, X_0 m^3) = \rho^0_{\Pi}(\varepsilon, X_0) \]
if $m \in \mathfrak{m}(\mathfrak{m})$, by Theorem II.3.1. Hence $\rho^0_{\Pi}(\varepsilon, X_0 \pi^k) (k \geq 0)$ is determined essentially by $\rho^0_{\Pi}(\varepsilon, X_0)$. This observation is that of Corollary II.2.4 in a different guise.

The arithmetic significance of these $\rho^0_{\Pi}(\varepsilon, X)$ for general $n$ has been discussed in [39].

The significance of these results with $n = 3$ for the construction of the "cubic theta function" in [37] should also be clear, since the "Fourier coefficients" of that function are essentially $p(\varepsilon, X)$. The results discussed here would not determine this function completely, but explain the "multiplicative" relations between different Fourier coefficients. In this connection the reader should also consult [6].

We turn finally to the proof of Theorem II.3.3. This is based on the evaluation of the $G_{\Pi}(\omega_{\nu}, \eta_{\nu}, h_{\nu})$ given by the following lemma. The additive measure on $k_{\nu}$ will be taken to be the self-dual measure with respect to $\varepsilon_{0, \nu}$.

**Lemma II.3.4.** a) Suppose that $\nu \in S(\mathfrak{M})$; then
\[ G_{\Pi}(\omega_{\nu}, \eta_{\nu}, h_{\nu}) = |\mathfrak{M}| \cdot |\mathfrak{D}|^{\frac{1}{12}} \mu_\nu(h_{\nu})^2 \xi(\omega_{\nu} \mu_\nu(\eta_{\nu} h_{\nu}^*) \]
if $\eta_{\nu} h_{\nu}^* \in \mathfrak{H}_{\nu, v}$ and
\[ |\mu(\nu)_{\nu}| \leq |\mathfrak{M}^{-1} \mathfrak{D}^{-1}| \nu \]
otherwise.

b) Suppose that $\nu$ is a non-archimedean place of $k_{\nu}, \nu \notin S(\mathfrak{M})$. Let $\pi_{\nu}$ be a uniformizer of $k_{\nu}$ and let $q_{\nu}$ be the modulus of $k_{\nu}$. Let
\[ \pi_{\nu} = s_{\nu} \begin{pmatrix} \pi_{\nu} & 0 \\ 0 & \pi_{\nu}^{-1} \end{pmatrix} \]
and let
\[ p(h_{\nu}) = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}. \]
Let $l = \text{ord}_{\nu}(h_2/h_1)$, and
\[ g_{\nu}^{(l)}(\pi_{\nu}) = \sum_{\varepsilon (\nu) \equiv 0 \mod \mathfrak{m}_{\nu}} \varepsilon((\pi_{\nu}, \varepsilon_{\nu}))^l \sigma_{\nu, \varepsilon}(\varepsilon/\pi_{\nu}). \]
Then
\[ G_{\Pi}(\omega_{\nu}, \eta_{\nu}, h_{\nu}) = (\omega_{\nu} \mu_\nu(\eta_{\nu} h_{\nu} \pi_{\nu}) g_{\nu}^{(l-1)}(\pi_{\nu}) \xi(\pi_{\nu}, h_{\nu}/h_1)_{\nu}^{-1} \xi(-1, h_{\nu})_{\nu} + (\omega_{\nu} \mu_\nu(\eta_{\nu} h_{\nu}^*) \mu_\nu(\varepsilon_{\nu})^2 \left( 1 - q_{\nu}^{-1} \omega_{\nu, \varepsilon}(\pi_{\nu}) - (1 - q_{\nu}^{-1}) \omega_{\nu, \varepsilon}(\pi_{\nu})^{1 + [\mathfrak{m}]} \right) \right) \]
if \( l \geq 0 \); if the argument of \( (\omega, \mu) \) in either term does not lie in \( \mathfrak{H}_s \), then that term is taken to be zero. If \( l < 0 \) then

\[
G_0(\omega_s, \eta_n, h_2) = 0.
\]

**Proof.** — Let us write \( \eta_n = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} i(\zeta) \) and \( h_n = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} i(\zeta') \).

Let \( n = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \). Then

\[
\eta_n w_0 nh_0 = \begin{pmatrix} 0 & \gamma_1 h_2 \\ \gamma_2 h_1 & 0 \end{pmatrix} i((\eta_1 h_2, h_1) \cdot \zeta') \begin{pmatrix} 0 & 1 \\ 1 & h_2 \zeta / h_1 \end{pmatrix}
\]

and this is equal to both

\[
\begin{pmatrix} 0 & \gamma_1 h_2 \\ \gamma_2 h_1 & 0 \end{pmatrix} i((\eta_1 h_2, h_1) \cdot \zeta') \begin{pmatrix} 0 & 1 \\ 1 & h_2 \zeta / h_1 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 0 & \gamma_1 h_2 \\ \gamma_2 h_1 & 0 \end{pmatrix} i((\eta_1 h_2, h_1) \cdot \zeta') \begin{pmatrix} 0 & 1 \\ 1 & h_2 \zeta / h_1 \end{pmatrix}
\]

\[\text{a)} \quad \text{In this case, using the Iwasawa decomposition given above, one sees that one must have}
\]

\[
\gamma_1 h_2 \in \mathfrak{h}_0
\]

and

\[
|h_2 \zeta / h_1| \leq |\mathfrak{A}|_s.
\]

The integral defining \( G_0(\omega_s, \eta_n, h_2) \) is then simply

\[
(\omega, \mu)_{\eta_n} \begin{pmatrix} \gamma_1 h_2 & 0 \\ 0 & \gamma_2 h_1 \end{pmatrix} i((\eta_1 h_2, h_1) \cdot \zeta') \int_{|\zeta'| \leq |\mathfrak{A}|_s |h_1|_o |h_2|_o} \bar{\epsilon}_{\eta_n}(\xi) \, d\xi.
\]

The integral here is non-zero if and only if

\[
|\mathfrak{A}|_s |h_1|/|h_2| \leq |D|_s^{-1};
\]

when this condition is satisfied the integral is

\[
|\mathfrak{A}|_s |h_1|/|h_2| \cdot |D|_s^{1/2}
\]

by [50] VII-2, Cor. 3 to Prop. 2. Since

\[
p(h)^a = h_1/h_2
\]

the statement in the lemma follows at once.

\[\text{b)} \quad \text{We split the integral into two parts, that over } \{\xi : |\zeta'| \leq |h_1|/|h_2|_o\} \text{ and that over } \{\xi : |\zeta'| > |h_1|/|h_2|_o\}. \text{ The first of these can be treated as we treated a}; \text{ we find}
\]

\[
(\omega_s, \mu)_{\eta_n} h_2^{a_n} \mu_2(h_2)^a \quad \text{if } l \geq 0.
\]
Now we turn to other integral, which is
\[ \int_{|\xi| > |h_1 h_2|} (\omega, \mu)_v \left( \begin{array}{cc} -\gamma_1 h_1 & 0 \\ 0 & \gamma_2 h_2 \end{array} \right) \frac{\xi}{\xi} \cdot \bar{\epsilon}_{v} \, d\xi \]
from the second of the two formulae above. To compute this integral we let \( \xi = \pi^{-i} x \) with \( t > -\text{ord}_{v}(h_1 h_2) \) and \( x \in r_v^{\times} \). Thus we obtain
\[ \sum_{t > -\text{ord}_{v}(h_1 h_2)} (\omega, \mu)_v \left( \begin{array}{cc} -\gamma_1 h_1 \pi_v^t & 0 \\ 0 & \gamma_2 h_2 \pi_v^{-t} \end{array} \right) \frac{\xi}{\xi} \cdot \bar{\epsilon}_{v} \cdot \epsilon_{v} \, d\xi \]
since \( \omega_{\ast, \ast} \) is unramified in the sense discussed above. Observe that with our conventions the summand is non-zero only when
\[ \text{ord}_{v}(\gamma_1 h_1) + t \equiv 0 \pmod{n} \]
and
\[ \text{ord}_{v}(\gamma_2 h_2) - t \equiv 0 \pmod{n}. \]
Moreover the integral over \( r_v^{\times} \) is non-zero only when \( t \leq 0, \text{ord}_{v}(\gamma_1 h_1) = 0 \pmod{n} \) or \( t = 1 \). In the former case it has the value \((1 - q_v^{-1})\), and in the latter
\[ q_v^{-1} g^{(j)}(\pi_v) \]
with \( J = \text{ord}_{v}(\gamma_1 h_1) \).

From these remarks it follows that our integral has become
\[ \sum_{-\text{ord}_{v}(h_1 h_2) < t \leq 0} (\omega, \mu)_v \left( \begin{array}{cc} -\gamma_1 h_1 \pi_v^t & 0 \\ 0 & \gamma_2 h_2 \pi_v^{-t} \end{array} \right) \frac{\xi}{\xi} \cdot \bar{\epsilon}_{v} \cdot \epsilon_{v} \, d\xi \]
under the assumption that \( l \geq 0 \). In the first term we replace \( \pi_v^t \) by \( h_1^{-1} h_2 \pi_v^{-u} \) where \( u \) is summed over
\[ -\text{ord}_{v}(h_1 h_2) \leq u < 0, \quad u \equiv 0 \pmod{n}. \]
The \( u \)-th summand is then
\[ (\omega, \mu)_v (\mu_v h_v^{u}) \cdot \omega_{\ast, \ast}^{u}(\pi_v)^{-u n} \cdot |h_1 h_2|^u (1 - q_v^{-1}). \]
The sum is now a geometric series, which is easily computed. Combined with the first integral it yields the second term in the quoted formula.
In the second term we observe that
\[ \text{ord}_n(\eta_1 h_1) \equiv -1 \pmod{n} \]
so that
\[ J \equiv -l - 1 \pmod{n}. \]

After a little rearrangement this yields the first term in the formula of the lemma.

Finally we observe that if \( l < o \) the integral is zero as in Theorem I.4.1.

We are now in a position to prove Theorem II.3.3. This we shall do by bringing both expressions \( \psi^\#(\omega, h) \) and \( \psi^\#(\varphi, e^{-1}, X) \) to forms which can be easily compared. Observe first that
\[ \psi^\#(\omega, h) = L_\eta \left( \left|_{\mathbb{A}} \omega^\# \right| \right) \sum_{\eta \in \mathbb{H}_n \setminus \mathbb{H}_0} \omega^{\#}_{\mathbb{A}}(\eta) \prod_{\tau \in \mathbb{R}} G_\kappa(\omega, s_\kappa(\eta), s_\kappa(h)) \]
and hence, by Lemma II.3.4, \( \psi^\#(\omega, h) = 0 \) unless \( X = \rho(h)^{\#} \in \mathbb{H}^{-1} \). Hence we can make this assumption henceforth.

Let us examine the factors in the product. The factors corresponding to \( v \in \mathbb{S}(\mathbb{A}) \) yield
\[ N(\mathbb{H})^{-1} \left| \mathbb{D}_n \right|^{-1/2} \prod_{v \in \mathbb{S}(\mathbb{A})} (\omega_v, \mu_v) (s_\kappa(\eta_v), s_\kappa(h_v)) \mu_v(h)^2. \]

It is now convenient to replace \( \eta \) by \( \eta_\#(h_v)^{-1}. \)

Suppose that \( \eta = \left( \begin{array}{cc} \eta_1 & 0 \\ 0 & \eta_2 \end{array} \right) \); we require that \( \eta_1 \in \prod_{v \in \mathbb{S}(\mathbb{A})} k_v^{\times n} \). Let \( \rho(h) = \left( \begin{array}{cc} h_1 & 0 \\ 0 & h_2 \end{array} \right). \)

We let \( \Sigma(\mathbb{A}) \) be the set of non-archimedean places of \( k \) not in \( \mathbb{S}(\mathbb{A}) \). Then the product of those \( G_\kappa(\omega_v, s_\kappa(\eta_v(h_v)^{-1}), s_\kappa(h_v)) \) can be taken to be the sum over all finite subsets \( \Sigma' \) of \( \Sigma(\mathbb{A}) \) of
\[ \prod_{v \in \Sigma'} (\omega_v, \mu_v) (s_\kappa(\eta(h_v)^{-1}), s_\kappa(h_v)) \mu_v(h)^2 \times \left\{ \frac{1 - q_v^{-1} \omega_v^n(\pi_v) - (1 - q_v^{-1}) \omega_v^n(\pi_v)^{\left[\pi_v^n\right] + 1}}{1 - \omega_v^n(\pi_v)} \right\}. \]

where \( l_v = \text{ord}_n(h_1/h_2). \)

In order that this be non-zero we require that
\[ \text{ord}_n(\eta_1) \equiv 0 \pmod{n} \]
\[ \text{ord}_n(\eta_2) \equiv 0 \pmod{n} \]
if \( v \in \Sigma(\mathbb{A}) - \Sigma', \) and
\[ \text{ord}_n(\eta_1 X) \equiv -1 \pmod{n} \]
\[ \text{ord}_n(\eta_2 X^{-1}) \equiv 1 \pmod{n}, \]
if $v \in \Sigma'$. Recall that $\eta_1$ and $\eta_2$ are to be chosen modulo $n$-th powers, and that each summand is unchanged if either of these is replaced by a multiple of itself by an $n$-th power. Because $\mathfrak{c}^{\mathfrak{m}}(\mathfrak{n})$ is a principal ideal domain we can choose $\eta_1$ to be $\epsilon^{-1}$ where

$$\text{ord}_e(c) = 0 \quad (v \in \Sigma - \Sigma')$$

$$= \text{ord}_e(X) + 1 \quad (v \in \Sigma').$$

This is then to be chosen modulo $\mathcal{U}_n(\mathfrak{A})$. Let $\eta_2 = cu$, where $u \in \mathcal{U}(\mathfrak{A})$, and $u$ is to be taken modulo $\mathcal{U}_n(\mathfrak{A})$. It is clear that if write $\eta$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} c^{-1} \\ 0 \end{pmatrix}$$

and noting that

$$\prod v (\omega_{\sigma}) = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$$

by the assumption made on $\omega_\sigma$ at the outset) the summation over $u$ can be carried out and yields the factor

$$[\mathcal{U}(\mathfrak{A}) : \mathcal{U}_n(\mathfrak{A})].$$

Combining these, and using the reciprocity law, we see that

$$\psi_n(\omega, h) = L_{\mathfrak{A}}(\| \| A \omega_\sigma^n, (\omega h) N(\mathfrak{A})^{-1} \| D_\mathfrak{A})^{-\frac{1}{2}} [\mathcal{U}(\mathfrak{A}) : \mathcal{U}_n(\mathfrak{A})]$$

$$\cdot \sum_{e, \Sigma'} \prod_{v \in S(e, \mathfrak{A})} \begin{pmatrix} c^{-1} \\ 0 \end{pmatrix} \Pi_{v \in \Sigma'} (\pi_v e^0 - 1) (\mathfrak{p}_v - 1) \omega_{\sigma, v}^{\sigma}(\pi_v - 1) \omega_{\sigma, v}^{\sigma}(\pi_v - 1) \Pi_{v \in \Sigma'} \left\{ \frac{1 - c_v^{-1} \omega_{\sigma, v}(\pi_v) - (1 - g_v^{-1}) \omega_{\sigma, v}(\pi_v)}{1 - \omega_{\sigma, v}(\pi_v)} \right\}. $$

Observe that here $\Sigma'$ can be characterized as $\{ v \in \Sigma(\mathfrak{A}) : \| v \| < 1 \}$, so that the summation may be taken to be over $c$ alone. The product over $S(e, \mathfrak{A})$ is $\omega^{\sigma}(e)$. In the product over $\Sigma'$ we can take $\pi_v = c/X$. Hence we obtain

$$L_{\mathfrak{A}}(\| \| A \omega_\sigma^n, N(\mathfrak{A})^{-1} \| D_\mathfrak{A})^{-\frac{1}{2}} [\mathcal{U}(\mathfrak{A}) : \mathcal{U}_n(\mathfrak{A})] N(\mathfrak{A})^{-1}$$

$$\cdot \sum_{e} \omega^{\sigma}(e). \prod_{v \in \Sigma'} \left\{ \frac{1 - c_v^{-1} \omega_{\sigma, v}(\pi_v) - (1 - g_v^{-1}) \omega_{\sigma, v}(\pi_v)}{1 - \omega_{\sigma, v}(\pi_v)} \right\}. $$

We shall now show that the sum over $e$ is $\sum_{v \in \mathfrak{A}} (\omega^{\sigma}(e)^{-1}, X) L_{\mathfrak{A}}(\| \| A \omega_\sigma^n)^{-1}$. First observe that if $e_1$ is injective $g_\sigma(e_1, X, c)$ is zero unless for each $v \in \Sigma(\mathfrak{A})$ one has either

$$\text{ord}_e(c) = 0 \pmod{n}, \quad |X/c|_v \leq 1$$

or

$$\text{ord}_e(X/c) = -1.$$
Thus, if we let
\[ \Sigma'(c) = \{ v \in \Sigma(\mathbb{H}) : \text{ord}_v(X/c) = -1 \} \]
we obtain by a familiar transformation
\[
g_\mathbb{H}(e^{-1}, X, c) = \prod_{v \in \Sigma(\mathbb{H})} g_v(e^{-b-1})(c/X) \cdot |X|_v^{-1} \cdot \prod_{v \in \Sigma(\mathbb{H}) - \Sigma'} \text{Card}(\tau_v(c)^*)
\]

Consider next a fixed set \( \Sigma' \) and
\[
\{ c : \Sigma'(c) = \Sigma' \}.
\]
If this set is non-empty there is an element \( c_0 \) in it with \( \text{ord}_v(c_0) = 0 \) for \( v \in \Sigma(\mathbb{H}) - \Sigma' \) since \( \mathbb{R}^\mathbb{H} \) is a principal ideal domain. Then the sum of \( g_\mathbb{H}(e^{-1}, X, c) \varphi(c) \) over this set is equal to
\[
\varphi(c_0) \prod_{v \in \Sigma'} g_v(e^{-b-1})(c_0/X) \cdot |X|_v^{-1} \times \prod_{w \in \Sigma(\mathbb{H}) - \Sigma'} \left( \frac{1 - g_w^{-1} \varphi(\pi_w)}{1 - g_w \varphi(\pi_w)} \right)
\]
where \( \pi_v \in k^\times \) is a uniformizer of \( k_v \). Now notice that
\[
\omega_{(1)}((c_v^*)) = \prod_{w \in \Sigma(\mathbb{H})} \omega_{n,v}(c_w)^{-1} |c_w|_w^{-n}
\]
and so
\[
\omega_{n,v}(\pi_v) = \omega_{(1)}(\pi_v) \cdot g_v^*.
\]
Finally observe that if \( c \) is as above
\[
\prod_{v \in \Sigma'} (-1, c)_v = 1.
\]
Hence, comparing the two expressions, we see that the identity asserted in the theorem is true. This completes the proof of the theorem.

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Department of Mathematics
Harvard University
1 Oxford St.
Cambridge, MA 02138
U.S.A.

Mathematisches Institut
der Universität
Bunsenstraße 3/5
D-3400 Göttingen
BRD

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