0. Introduction

If a group $G$ is equal to its commutator subgroup $(G, G)$, it has a universal central extension

\[(0.1) \quad \pi \to E_G \to G\]

(C. Moore (1968) no. 1). Suppose that $G$ is the group $G(k)$, for $G$ a split simple simply-connected algebraic group over a field $k$, and let us forget about some small groups over small finite fields. Then R. Steinberg (1962) describes the central extension $E_G$ by generators and relations, while H. Matsumoto (1969) describes the kernel $\pi$ by generators and relations. Except for $G$ of type $C_n$ $(n \geq 1)$, $\pi$ is $k^* \otimes k^*/\langle u \otimes v | u + v = 1 \rangle$, the group we now call $K_2(k)$, following J. Milnor (1971). For $G$ of any type, including $C_n$, the following remains true: the adjoint group $G^{\text{ad}}(k)$ acts on $G(k)$, hence on the corresponding central extension (0.1), and the group of coinvariants $\pi_{G^{\text{ad}}(k)}$ is $K_2(k)$. The universal central extension hence gives rise to a central extension

\[(0.2) \quad K_2(k) \to G(k) \to G(k)\]

The central extension (0.2) is canonically an extension by $K_2(k)$: the action on $G(k)$ of the group of automorphisms of $G$ (and in particular of $G^{\text{ad}}(k)$) lifts to an action

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on $G(k)$, trivial on $K_2(k)$. We will call (0.2) Matsumoto’s central extension. Notation: the universal symbol $\langle , \rangle$: $k^* \times k^* \to K_2(k)$ maps $u, v$ to the image of $u \otimes v$ in $K_2(k)$.

Suppose now that $k$ is a global field, i.e. a number field or the field of rational functions of a curve over a finite field. Let $\mu$ be the group of roots of unity in $k$. Matsumoto’s central extensions, for $k$ and its completions $k_v$, then give rise to a topological central extension of the adelic group $G(\mathbb{A})$ by $\mu$, and to a splitting of this central extension over $G(k)$:

$$G(k) \xrightarrow{\mu} G(\mathbb{A}) \xrightarrow{\mu} G(\mathbb{A}).$$

See C. Moore (1968).

In this paper, we consider reductive groups $G$ over $k$, where reductive is meant to imply connected. We classify central extensions of $G(k)$ by $K_2(k)$ which, roughly speaking, are functorial in $k$. More precisely, we define objects called central extensions of $G$ by $K_2$, which give rise to a central extension of $G(k)$ by $K_2(k)$ and, for $k$ a global field, to a diagram (0.3). We classify the central extensions of $G$ by $K_2$, see below, and we determine their functoriality in $G$ and $k$. We hope that for $k$ a global field this will prove useful in the study of “metaplectic” automorphic forms, that is, the harmonic analysis of functions on $G(\mathbb{A})/G(k)$.

Notation. — For $X$ a scheme, the $K_n(X)$ are Quillen’s higher $K$-theory groups of the exact category of vector bundles on $X$. For $U$ open in $X$, $U \mapsto K_n(U)$ is a presheaf on $X$. We denote by $K_n$ the associated sheaf.

An algebraic group $G$ over a field $k$ defines a sheaf $S \mapsto G(S) = \text{Hom}_{\text{Spec}(k)}(S, G)$ on the big Zariski site $\text{Spec}(k)_{\text{Zar}}$ (see 1.4) of $\text{Spec}(k)$. A central extension of $G$ by $K_2$ is a central extension, on $\text{Spec}(k)_{\text{Zar}}$, of the sheaf of groups $G$ by the sheaf of abelian groups $K_2$. By SGA7 (I, exp. VII, no. 1), it can be viewed as a $K_2$-torsor $P$ on $G$, provided with a multiplicative structure: an isomorphism $m: \text{pr}_1^*P + \text{pr}_2^*P \to \mu^*P$ of $K_2$-torsors on $G \times G$, for $\mu: G \times G \to G$ the group law, obeying an associativity condition on $G \times G \times G$.

The following alternate description, to be explained in §8, played for us a useful heuristic role. It is valid when $k$ is infinite and $G$, as an algebraic variety over $k$, is connected and unirational. The face maps of the simplicial classifying space $BG$, that is, the simplicial scheme $G^\Delta/G$, induces homomorphisms of fields of rational functions

$$k \cong k(G) \cong k(G \times G) \cong k(G \times G \times G) \ldots$$
Applying the functor $K_2$ and taking the usual alternating sum of face maps, one obtains

$$K_2^0 k \to K_2^0 k(G) \to K_2^0 k(G \times G) \to K_2^0 k(G \times G \times G) \to \ldots$$

This complex *incarnates* in degree 2 the category of central extensions of $G$ by $K_2$. This means that: (a) a 2-cocycle $c \in K_2^0 k(G \times G)$ defines a central extension $E(c)$ of $G$ by $K_2$ and any isomorphism class of central extensions can be obtained in this way; (b) if $c_0$ and $c_1$ are two 2-cocycles, a 1-cochain $b$ such that $c_1 - c_0 = db$ defines an isomorphism $E(b)$: $E(c_0) \to E(c_1)$, and this construction is a bijection

$$\{ b | c_1 - c_0 = db \} \overset{\sim}{\longrightarrow} \text{Isom}(E(c_0), E(c_1)) ;$$

(c) obvious compatibilities hold. The idea to go from the cocycle $c \in K_2^0 k(G \times G)$ to the corresponding extension of $G(k)$ by $K_2(k)$ is as follows. One can write $c$ as a finite sum $\sum n_i \{ f_i, g_i \}$ with $f_i, g_i \in k(G \times G)^\ast$. The $f_i$ and $g_i$ are all regular on some dense open subset $U$ of $G \times G$ and

$$c(x, y) = \sum n_i \{ f_i(x, y), g_i(x, y) \}$$

is defined for $(x, y) \in U$. Because $c$ is a cocycle, $c(x, y)$ obeys the cocycle condition $c(y, z) - c(y, x) + c(x, z) - c(x, y) = 0$ for $(x, y, z)$ in some dense open subset of $G \times G \times G$. That such a generic cocycle suffices to define a group extension is similar to Weil's theorem that a birational group law determines an algebraic group (SGA3, t2, XVIII 3.7). We were also inspired by Mackey's theorem that for separable locally compact topological groups, borelian 2-cocycles define topological extensions (G. W. Mackey (1957)).

**Example.** — For $G$ split simple simply-connected, Matsumoto’s central extension carries a set theoretic section for which the cocycle describing the extension is expressed in terms of universal symbols. On some open subset $U$ of $G \times G$, this cocycle is given by a formula (0.4). As this formula continues to define a cocycle after extension of scalars from $k$ to $k(G \times G \times G)$, the element $\sum n_i \{ f_i, g_i \} \in K_2^0 k(G \times G)$ satisfies the cocycle condition in $K_2^0 k(G \times G \times G)$: Matsumoto’s central extension of $G(k)$ by $K_2^0 (k)$ comes from an extension of $G$ by $K_2$.

We now sketch the classification of the central extensions of a reductive group $G$ by $K_2$. If $G$ is simply-connected, the central extensions of $G$ by $K_2$ have no nontrivial automorphisms. In contrast, it can happen that $G(k)$ is not its own commutator subgroup and that the group of automorphisms $\text{Hom}(G(k), K_2(k))$ of a central extension of $G(k)$ by $K_2(k)$ is not trivial. If $G$ is in addition simple (over $k$), the group of isomorphism classes of central extensions of $G$ by $K_2$ is $\mathbb{Z}$. Corresponding to $1 \in \mathbb{Z}$, we have a canonical central extension of $G$ by $K_2$. It induces a canonical central...
extension of $G(k)$ by $K_2(k)$, well-defined up to unique isomorphism. Being canonical, it is acted upon by the group of automorphisms of $G$, and in particular by $G^\text{ad}(k)$. For $G$ split, the central extension of $G(k)$ by $K_2(k)$ obtained is the one constructed by Matsumoto.

Fix a maximal torus $T$ in $G$, and a Galois extension $k'$ of $k$ over which $T$ splits. Let $Y$ be the dual of the character group of $T$ over $k'_0$. It is acted upon by $\text{Gal}(k'/k)$ and by the Weyl group $W$. Rather than to say that central extensions of $G$ by $K_2$ are classified by integers, it is better to say that they are classified by the integer-valued quadratic forms $Q$ on $Y$ invariant by $W$ and by $\text{Gal}(k'/k)$. The classification then remains valid for $G$ semi-simple simply-connected, and is functorial in $G$ and $k$. For $G$ simple, invariant quadratic forms correspond to integers by $Q \mapsto Q(\alpha^\vee)$, for $\alpha^\vee$ the (short) coroot corresponding to a long root $\alpha$.

Let $G$ be a reductive group over $k$. The group of automorphisms of any central extension of $G$ by $K_2$ is $\text{Hom}(G, K_2)$. The $\text{Hom}$ group is in the category of sheaves on the big Zariski site of $k$. It is the subgroup $H^0(G, K_2)^\text{prim}$ of primitive elements of $H^0(G, K_2)$. It is also the subgroup of elements $b$ of $K_2(k(G))$ for which $pr_2^*b - \mu^*b + pr_1^*b = 0$ in $K_2(k(G \times G))$. Example: if $G$ is the quotient of a reductive group $G_1$ by a central $\mu_n$, Kummer theory attaches to the $\mu_n$-torsor $G_1$ over $G$ a section $f$ of $\mathbb{G}_a^*/\mathbb{G}_a$ on $G$ and for $\zeta$ a root of unity of $k$ of order dividing $n$, $b := \{\zeta, f\}$ belongs to $H^0(G, K_2)^\text{prim}$. Other example: $\text{Hom}(\mathbb{G}_m, K_2) = k^*/g$, with $a \in k^*$ giving rise to $x \mapsto \{a, x\}$.

Because of those automorphisms, the classification problem we consider, for $G$ reductive, is not that of determining the set of isomorphism classes of central extensions by $K_2$. It is that of determining the category of central extensions by $K_2$. This turns out to be an easier problem. It means: to define an equivalence from the category of central extensions of $G$ by $K_2$ to a more down-to-earth category $\mathcal{E}$.

Suppose first that $G$ is a torus $T$. Fix a Galois extension $k'$ of $k$ over which $T$ splits and let $Y$ be the dual of the character group of $T$ over $k'$. The category $\mathcal{E}$ is here the category of pairs $(Q, \mathcal{E})$, where $Q$ is a $\text{Gal}(k'/k)$-invariant integer-valued quadratic form on $Y$ and $\mathcal{E}$ a $\text{Gal}(k'/k)$-equivariant central extension of $Y$ by $k'^*$, for which (0.5) below holds. Let $B$ be the bilinear form associated to $Q$: $B(y_1, y_2) := Q(y_1 + y_2) - Q(y_1) - Q(y_2)$. The condition is that the commutator of liftings $\bar{y}_1, \bar{y}_2$ to $\mathcal{E}$ of $y_1, y_2 \in Y$ should be given by

$$\bar{y}_1, \bar{y}_2 = (-1)^{B(y_1, y_2)} \in k'^*.$$

In $\mathcal{E}$, there can be a morphism from $(Q', \mathcal{E}')$ to $(Q'', \mathcal{E}'')$ only if $Q' = Q''$. If $Q' = Q''$, morphisms are isomorphisms of equivariant central extensions from $\mathcal{E}'$ to $\mathcal{E}''$. 
Let $T'$ be the torus over $k$, split over $k'$, whose character group is $Y$. It follows from the equivalence of the category of central extensions of $T$ by $K_2$ with $\mathscr{C}$ that

$$T'(k) \xrightarrow{\sim} \text{Hom}(T, K_2).$$

Suppose that $T$ is a maximal torus in a semi-simple simply-connected group $G$. A Weyl group and $\text{Gal}(k'/k)$-invariant integer-valued quadratic form $Q_0$ on $Y$ defines a central extension $E_0$ of $G$ by $K_2$. Restricting $E_0$ to $T$, we obtain a central extension $E$ of $T$ by $K_2$, classified by a pair $(Q, \mathscr{C})$. One has $Q = Q_0$. In 11.7 we will describe $\mathscr{C}$ in terms of the root spaces decomposition of the Lie algebra of $G$ over $k'$, relative to the action of $T$.

The classification for general reductive groups is a mixture of the torus and simply-connected cases. Let $G$ be a reductive group over $k$, with maximal torus $T$ split over $k'$. Our main result, a particular case of Theorem 7.2, is:

**Theorem.** — The category of central extensions of $G$ by $K_2$ is naturally equivalent to the category of triples $(Q, \mathscr{C}, \varphi)$ as follows: $Q$ is a Weyl and Galois invariant integer-valued quadratic form on $Y$; $\mathscr{C}$ is a Galois equivariant central extension of $Y$ by $k'^*$, obeying (0.5). Let $f : G_{sc} \rightarrow G$ be the simply connected covering of the derived group of $G$, $T_{sc} := f^{-1}(T)$ and $Y_{sc} \subset Y$ be the dual character group of $T_{sc}$. The form $Q$ induces $Q_{sc}$ on $Y_{sc}$, from which we get a Galois-equivariant central extension $\mathscr{C}_{sc}$ of $Y_{sc}$ by $k'^*$, and $\varphi$ is a Galois equivariant morphism from $\mathscr{C}_{sc}$ to $\mathscr{C}$ making the diagram

$$
\begin{array}{ccc}
 k'^* & \longrightarrow & \mathscr{C}_{sc} \\
 \| & & \downarrow \varphi \\
 k'^* & \longrightarrow & \mathscr{C} \\
\end{array}
$$

commute.

Let $H$ be the group of multiplicative type over $k$, split over $k'$, with character group $Y/Y_{sc}$. It follows from the equivalence of the category of central extensions of $G$ by $K_2$ with $\mathscr{C}$ that

$$H(\ell) \xrightarrow{\sim} \text{Hom}(G, K_2).$$

For $X$ connected and smooth over $k$, pointed by $x \in X(\ell)$, $K_2$-torsors $P$ over $X$, given with a trivialization of $x^*P$, obey Galois descent. This results from the Quillen resolution of $K_2$ and from B. Kahn (1993) (corollary 1 to theorem 3.1).

Galois descent reduces the classification problem for central extensions by $K_2$ to the split case. It matters here that the classification problem considered is that of describing a category, not just a set of isomorphism classes. The split case is handled using the Bruhat decomposition and known results on cohomology with coefficients in $K_2$ for affine spaces or multiplicative group bundles (Quillen (1973), Sherman (1979)).
In 1978, the second named author constructed a canonical central extension of \( G(k) \) by \( K_2(k) \), for \( G \) absolutely simple and simply-connected, under the assumption that \( G \) is rational as an algebraic variety over \( k \). This construction was exposed in a seminar at the IHES in 1977-1978. It used the results of Tate on the structure of \( K_2(k(T)) \), and its consequence that \( H^4(\text{Gal}(k'/k), \, K_2(k'(T)/K_2(k')) = 0. \) In retrospect, one sees that this was a substitute for Galois descent.

Let \( n \) be an integer invertible in \( k \). For \( k \) a separable closure of \( k \), let us write simply \( \text{H}^2(k, \mathbb{Z}/n(2)) \). By Tate (1976) one has a map

\[
K_2(k)/nK_2(k) \rightarrow \text{H}^2(k, \mathbb{Z}/n(2)).
\]

A central extension by \( K_2(k) \) hence provides one by \( \text{H}^2(k, \mathbb{Z}/n(2)) \). For \( G \) simply-connected, Deligne (1996) attaches a central extension of \( G(k) \) by \( \text{H}^2(k, \mathbb{Z}/n(2)) \) to each étale cohomology class in \( \text{H}^4(BG \mod Br, \mathbb{Z}/n(2)) \). We will show that the central extension of \( G(k) \) by \( \text{H}^2(k, \mathbb{Z}/n(2)) \) deduced from the canonical central extension by \( K_2(k) \) is canonically isomorphic to that of Deligne (1996), applied to a natural generator of \( \text{H}^4(BG \mod Br, \mathbb{Z}/n(2)) \).

In this introduction, we have emphasized the case of a ground field \( k \). In the text, we consider a more general base \( S \) as well. This is useful to handle actions of an algebraic group \( H \) on an extension of \( G \) by \( K_2 \). Cf. 1.7. For the ease of exposition, we often present the ground field case first, and explain later the changes required to work over \( S \). Our reliance on the Quillen resolution of \( K_2 \) usually forces us to assume that \( S \) is regular of finite type over a field. Using Bloch (1986), one could also handle the case where \( S \) is smooth over the spectrum of a discrete valuation ring.

We now review the successive sections of the paper. In § 1, we define central extensions of \( G \) by \( K_2 \), review from SGA7 their description as multiplicative torsors and we explain how to compute at the cocycle level. In § 2, we prove the Galois descent theorem for pointed \( K_2 \)-torsors. We also consider a relative version, for a base \( S \) which is regular of finite type over a field, with Galois descent generalized to descent for étale surjective maps.

In § 3, 4, 5 and 6, we consider split groups. We successively consider tori, semi-simple simply-connected groups, semi-simple groups and reductive groups. Our results partially duplicate those of § 3 and 4 of Esnault et al. (1998).

In § 7, we put together § 2 and 6 to classify the central extensions of not necessarily split reductive groups by \( K_2 \). In § 8, we explain the formalism of generic cocycles. In § 9, we compare the central extensions constructed with those of Deligne (1996). In § 10, we explain how, over global fields, to construct diagram (0.3). The central extensions of \( Y \) by \( k^* \) alluded to above are made explicit in § 11. A number of examples are collected in § 12.

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0.N. Notation and terminology

0.N.1. We will systematically use the following notation:

- \( T \): an algebraic torus over a base scheme;
- \( X \): the character group \( \text{Hom}(T, \mathbb{G}_m) \) of \( T \);
- \( Y \): the dual \( \text{Hom}(\mathbb{G}_m, T) \) of \( X \).

If \( T \) is a split torus over a field \( k \); \( T \) isomorphic to \( \mathbb{G}_m^n \) for some \( n \), one can think of \( X \) as simply being the free abelian group \( \text{Hom}(T, \mathbb{G}_m) \). For \( T \) a split torus group scheme over a base \( S \), \( X \) is a locally constant sheaf of free abelian groups on \( S \). For \( T \) not necessarily split, one should use the étale topology: for \( T \) over \( S \), \( X \) is a locally constant sheaf of free abelian groups on the étale site of \( S \). For \( S \) the spectrum of a field \( k \), and \( k' \) a Galois extension of \( k \) on which \( T \) splits, one also writes \( X \) for the free abelian group \( \text{Hom}(T_{k'}, \mathbb{G}_m) \) with the natural action of \( \text{Gal}(k'/k) \), which carries the same information.

If a torus is denoted by \( T \), with some decoration, for instance the subscript \( _{sc} \), its character and dual character groups will be \( X \) and \( Y \), with the same decoration.

0.N.2. When considering a reductive group \( G \), we will reserve the notation \( T \) for a maximal torus of \( G \), and we denote by

- \( N(T) \): the normalizer of \( T \);
- \( W \): the Weyl group \( N(T)/T \).

Over a field, and for \( T \) split, one can think of \( W \) as simply being a finite group. Over \( S \), or for \( T \) not necessarily split, it is a locally constant sheaf of groups on the étale site of \( S \).

0.N.3. When considering a quadratic form \( Q \), we will write \( B \) for the associated bilinear form

\[
B(x, y) := Q(x + y) - Q(x) - Q(y).
\]

The construction \( Q \mapsto B \) is bijective, from integer-valued quadratic forms on a free abelian group \( Y \) to even symmetric bilinear forms on \( Y \).

0.N.4. Let

\[
A \to E \to G
\]
be a central extension of a group $G$ by an abelian group $A$. For $\gamma$ in $G$, we will denote by $\text{int}[\gamma]$, or $\text{int}_\gamma$, the inner automorphism of $E$ defined by any lifting $\tilde\gamma$ of $\gamma$ to $E$:

$$\text{(1)} \quad \text{int}[\gamma]: x \mapsto \tilde\gamma x \tilde\gamma^{-1}.$$ 

For $\gamma$ and $\delta$ in $G$, we will denote by $(\gamma, \delta)$ the commutator in $E$ of liftings $\tilde\gamma$ of $\gamma$ and $\tilde\delta$ of $\delta$:

$$\text{(2)} \quad (\gamma, \delta) := \tilde\gamma \tilde\delta \tilde\gamma^{-1} \tilde\delta^{-1} = \text{int}_\gamma(\tilde\delta) \tilde\delta^{-1}.$$ 

Both (1) and (2) are independent of the choices of liftings.

If a group $\Delta$ acts on the central extension $E$, inducing the trivial action on $A$, for $\gamma$ in $G$ and $\delta$ in $\Delta$, we will write

$$\text{(3)} \quad (\delta, \gamma) := \delta \tilde\gamma \delta^{-1} \gamma^{-1}$$

for $\tilde\gamma$ any lifting of $\gamma$. For the action (1) of $G$ on $E$, we recover (2).

The same constructions make sense for a sheaf of central extensions, if liftings are replaced by local liftings.

0.N.5. The group law in $K_n$ will be written additively, and the product map will be denoted by a dot. For $f$ in the multiplicative group, $\{f\}$ is its image in $K_1$; one defines $\{f, g\} := \{f\} \cdot \{g\}$ in $K_2$.

0.N.6. For $K$ a cochain complex and $n$ an integer, $\tau_{\leq n} K$ (resp. $\tau_{> n} K$) is the subcomplex (resp. quotient complex) of $K$ defined by

$$\text{(4)} \quad \begin{align*}
(\tau_{\leq n} K)^i &= 0 & \text{resp. } (\tau_{> n} K)^i &= K^i \\
&= \text{cycles in } K^i & &= \text{K^i/coboundaries} \\
&= K^i & & i = n \\
&= 0 & & i < n.
\end{align*}$$

For $n \leq m$, $\tau_{[n, m]} K := \tau_{> n} \tau_{\leq m}$ K has as cohomology groups $H^i$ those of $K$ for $n \leq i \leq m$, 0 otherwise.

0.N.7. A \emph{strict simplicial} object of a category $C$ is a contravariant functor with values in $C$ from the category of the finite ordered sets $\Delta_n$ ($n \geq 0$) and injective increasing maps. Similarly for \emph{strict cosimplicial} face maps are given, but not degeneracies.
1. Central extensions, torsors and cocycles

1.1. If $A$ is a sheaf of groups over some space, or site, $S$, an $A$-torsor (also called “principal homogeneous space”) on $S$ is a sheaf $P$ on which $A$ acts on the right and which locally on $S$ is isomorphic to the following standard model: $A$, with the action of $A$ on itself by right translations. If $P_i$ is an $A_i$-torsor ($i = 1, 2$), $P_1 \times P_2$ is a $A_1 \times A_2$-torsor. A morphism $f : A \to B$ induces a functor $f$ (also called “pushing by $f$”) from $A$-torsors to $B$-torsors; one has a map of sheaves $f : P \to f(P)$ obeying $f(pa) = f(p)f(a)$ and this characterizes $f(P)$ up to unique isomorphism. Special case: for $A$ commutative, and for the morphism $+ : A \times A \to A$, one obtains the bifunctor “addition of $A$-torsors”.

1.2. We recall from SGA7 (II, exp. VII, no. 1) Grothendieck’s description of central extensions in terms of torsors. We first consider the set-theoretic case. Let $A \to E \to G$ be a central extension of a group $G$ by an abelian group $A$. The set $E$, with its projection $p$ to $G$, and the action $e \mapsto ea$ of $A$ on $E$, is then an $A$-torsor on $G$ (viewed as a discrete topological space). The product map $E \times E \to E$ sits in a commutative diagram

$$
\begin{array}{ccc}
E \times E & \longrightarrow & E \\
\downarrow & & \downarrow \\
G \times G & \longrightarrow & G
\end{array}
$$

where $\mu$ is the product map of $G$. The top line can be reinterpreted as corresponding to a morphism of sets mapping to $G \times G$:

$$m_0 : \text{pr}_1^*E \times_{G \times G} \text{pr}_2^*E \to \mu^*E.$$ 

This map is such that $m_0(a_1a_2) = m_0(a_1a_2)$, hence defines an isomorphism of $A$-torsors over $G \times G$ 

(1.2.1) 

$$m : \text{pr}_1^*E + \text{pr}_2^*E \to \mu^*E.$$ 

If $\mu_{123} = \mu_{12} = \mu_{23} : G \times G \times G \to G$ is the triple product map, the associativity of the multiplication of $E$ is expressed by the commutativity of the following diagram of maps deduced from $m$ between $A$-torsors on $G \times G$:

$$
\begin{array}{ccc}
\text{pr}_1^*E + \text{pr}_2^*E + \text{pr}_3^*E & \longrightarrow & \text{pr}_{12}^*\mu^*E + \text{pr}_3^*E \\
\downarrow & & \downarrow \\
\text{pr}_1^*E + \text{pr}_2^*\mu^*E & \longrightarrow & \mu_{123}^*E.
\end{array}
$$

(1.2.2)

From the central extension $E$ we have deduced a multiplicative $A$-torsor on $G$, that is, an $A$-torsor $E$ provided with a morphism (1.2.1) of $A$-torsors on $G \times G$ for which the
diagram (1.2.2) is commutative. This construction is an equivalence from the category of central extensions of $G$ by $A$ to the category of multiplicative $A$-torsors on $G$.

1.3. As explained in SGA7 loc. cit., the natural framework for this construction is to work in any category of sheaves $T$, not just in the category of sets (sheaves over the space with one point). Here, “sheaf” means sheaf over some site $\mathcal{S}$, not just the case of sheaves on a topological space: $T$ is any topos (SGA 4 IV).

For $X$ in $T$, one can “think of $X$ as being a space”, i.e. consider the topos $T/X$ of pairs $(F,f)$ with $F$ in $T$ and $f$ a map from $F$ to $X$. In the case of sheaves on a topological space $S$, this amounts to looking at a sheaf $X$ on $S$ as being a topological space mapped to $S$ by a local homeomorphism. One has a morphism of topos $T/X \to T$.

Let $G$ be a sheaf of groups and $A$ be a sheaf of abelian groups. For $n = 1, 2, 3$, let us “think of $G^n$ as being a space”, and let us keep writing $A$ for the pull-back of $A$ to $G^n$. The same construction as in 1.2 then defines an equivalence from the category of central extensions of $G$ by $A$ to the category of multiplicative $A$-torsors on $G$, that is $A$-torsors on $G$ provided with a multiplicative structure (1.2.1) on $G^2$ obeying (1.2.2) on $G^3$.

1.4. A big Zariski site $S_{\text{Zar}}$ of a scheme $S$ is a full subcategory $\mathcal{C}$ of the category of schemes over $S$, with the Zariski topology. If $X$ is in $\mathcal{C}$, any open subscheme of $X$ should be in $\mathcal{C}$ too, and $\mathcal{C}$ should be large enough to contain the schemes over $S$ we are interested in. For us, the precise choice of $S_{\text{Zar}}$ will be immaterial. To fix ideas, we take for $S_{\text{Zar}}$ the category of schemes of finite type over $S$.

For $A$ a sheaf on $S_{\text{Zar}}$ and for $X$ in $S_{\text{Zar}}$, define the sheaf $A_X$ on the topological space $X$ by $A_X(U) := A(U)$. This construction identifies sheaves on $S_{\text{Zar}}$ with systems $(A_X)$ of sheaves on the topological spaces $X$, for $X$ in $S_{\text{Zar}}$, contravariant in $X$, and with $A_U = (A_X$ restricted to $U)$ for $U$ open in $X$: a morphism $f: X \to Y$ in $S_{\text{Zar}}$ defines $\varphi(f): f^*A_Y \to A_X$, with a compatibility for composition of morphisms, and $\varphi(f)$ is an isomorphism if $f$ is an open embedding.

The Zariski sheaves $K_n$ form such a system. We denote by $K_n$ the corresponding sheaf on $S_{\text{Zar}}$. For $n = 0$, it is the constant sheaf $\mathbf{Z}$. For $n = 1$, it is the sheaf $\mathcal{O}^\times$.

We now unravel the description 1.3 of central extensions of sheaves in the case of sheaves over $S_{\text{Zar}}$, when $G$ is representable.

For $Y$ in $S_{\text{Zar}}$, the representable functor $h_Y: T \mapsto Y(T) := \text{Hom}_S(T, Y)$ is a sheaf. To “think of $h_Y$ as being a space”, as in 1.3, simply means to consider the topos of sheaves over $Y_{\text{Zar}}$. If $A$ is a sheaf of abelian groups on $S_{\text{Zar}}$, its “pull-back to $h_Y$, thought of as a space”, is simply the sheaf on $Y_{\text{Zar}}$ whose sections on $T \to Y$ is $A(T)$. A further simplification is that the restriction from the big to the small Zariski site induces an equivalence of categories from $A$-torsors on $S_{\text{Zar}}$ to $A_{\text{S}}$-torsors on $S$. 
Let $G$ be a group scheme of finite type over $S$. What matters is that the fiber powers $G^n$ of $G$ over $S$ are in $S_{\text{Zar}}$. By 1.3, central extensions of $h_G$ by $A$ can be described as multiplicative $A$-torsors. In terms of the system of Zariski sheaves $A_X$, the description can be rephrased as follows. Notation: for $f: X \to Y$ in $S_{\text{Zar}}$ and $P$ a $A_Y$-torsor on $Y$, we write $f^*P$ for the $A_X$-torsor on $X$ obtained by pushing by $f^*A_Y \to A_X$ the $f^*A_Y$-torsor on $X$ inverse image of $P$. With this notation, a multiplicative $A$-torsor is an $A_G$-torsor on $G$, provided with a morphism (1.2.1) of $A_{G\times G}$-torsors on $G \times G$ for which the associativity diagram (1.2.2) is a commutative diagram of $A_{G\times G\times G}$-torsors on $G \times G \times G$. A section over $T$ of the corresponding central extension $\mathcal{E}$ of $G$ by $A$ is a pair $(g, e): g \in G(T): T \to G$ and $e \in \Gamma(T, g^*E)$. Attaching to $g$ the isomorphism class of the $A_T$-torsor $e^*E$ on $T$, one obtains an exact sequence

$$1 \to H^0(T, A_T) \to \mathcal{E}(T) \to G(T) \to H^1(T, A_T).$$

By abuse of language, we will often say “central extension of $G$ by $A$” for “central extension of $h_G$ by $A$”.

**Remark.** — Suppose that $S$ is the spectrum of a field $k$ and write $A(k)$ for $A(S)$. If $\mathcal{E}$ is the central extension of $G$ by $A$ corresponding to a multiplicative $A$-torsor $E$, (1.4.1) for $T = S$ reduces to

$$1 \to A(k) \to \mathcal{E}(k) \to G(k) \to 1,$$

making $\mathcal{E}(k) := \mathcal{E}(S)$ a central extension of $G(k)$ by $A(k)$.

Our strategy to construct central extensions of $G(k)$ by $K_2(k)$ will be to apply this construction to central extensions of $G$ by $K_2$.

1.5. **Variant.** — Let $X$ be a scheme in $S_{\text{Zar}}$, given with a section $e$. A pointed $A$-torsor on $(X, e)$ is a $A_X$-torsor $P$ on $X$, provided with a trivialization, on $S$, of the $A_S$-torsor $e^*P$.

If $G$ is the trivial group scheme over $S$: $G$ reduced to its neutral section, i.e. $G \simeq S$, a multiplicative structure on a $G$-torsor $E$ is an isomorphism $E \times E \simeq E$. It amounts to the data of a trivialization of $E$.

For a group scheme $G$, with neutral section $e$, it follows by restriction to $e$ that a multiplicative torsor on $G$ is automatically pointed. Multiplicative torsors could be redefined as pointed torsors on $G$, provided with a morphism (1.2.1) of pointed torsors on $G \times_S G$, obeying an associativity (1.2.2) on $G \times_S G \times_S G$.

1.6. Let $E$ be a multiplicative $A$-torsor on a group scheme of finite type $G$ over $S$. An automorphism $\sigma$ of $G$ transforms $E$ into a new multiplicative $A$-torsor $\sigma(E)$ on $G$. If $\sigma$ is inner, i.e. is $\text{int}[g]$ for some $g$ in $G(S)$, then $E$ and $\sigma(E)$ are isomorphic. This
is clear if \( E \) is interpreted as a central extension \( \mathcal{E} \): the construction (0.N.4) (1) lifts \( \text{int}[g] \) to an automorphism of \( \mathcal{E} \) trivial on \( A \).

Let \( G \) and \( H \) be group schemes of finite type over \( S \), and let \( \rho \) be an action of \( H \) on \( G \).

1.7. Construction. — We will construct an equivalence of categories from the category of multiplicative \( A \)-torsors \( E_0 \) on the semi-direct product \( H \ltimes G \) to the category of triples \((a)\) \((b)\) \((c)\) as follows: \((a)\) a multiplicative \( A \)-torsor \( E \) on \( G \); \((b)\) a multiplicative \( A \)-torsor \( F \) on \( H \); \((c)\) an action of \( H \) on \((G, E)\) lifting the action of \( H \) on \( G \).

Here are equivalent definitions \((1)\) \((2)\) \((3)\) of what an action \((c)\) is.

\(1)\) For any \( T \) of finite type over \( S \), let \((G_T, E_T)\) be deduced from \((G, E)\) by base change to \( T \). The group \( H(T) \) acts on \( G_T \). Data \((c)\) is a lifting of this action \( \rho \) to an action \( \tilde{\rho} \) of \( H(T) \) on \((G_T, E_T)\), functorial in \( T \).

If \( H \) is smooth over \( S \), we have a variant

\(1')\) Same as \(1)\), except that \( T \) is restricted to be smooth over \( S \).

The definition \((2)\) is \((1)\), in a universal case:

\(2)\) Let \( h_0 \in H(H) = \text{Hom}(H, H) \) be the identity map of \( H \). The group \( H(H) \) acts on \( G_H \). Data \((c)\) is a lifting of the automorphism \( \rho(h_0) \) of \( G_H \) to an automorphism \( \rho(h_0) \) of \((G_H, E_H)\). If \( \mu: H \times H \to H \) is the multiplication map, it is required that

\[ \mu^{*}(\rho(h_0))^\sim \] = \( \text{pr}_1^{*}(\rho(h_0)) \text{pr}_2^{*}(\rho(h_0)) \) over \( H \times H \).

One goes from \(1)\) or \(1')\) to \(2)\) by taking \( \rho(h_0) = \tilde{\rho}(h_0) \). For any \( h \in H(T) \), \( h \) is the pull-back of \( h_0 \in H(H) \) by \( h: T \to H \) and one has \( \tilde{\rho}(h) = h^*(\rho(h_0)) \): the lifted action is determined by \( \rho(h_0) \), and the condition on \( H \times H \) in \(2)\) is what is needed for \( h \mapsto h^*(\rho(h_0)) \) to be an action.

We wrote \((2)\) to make clear the equivalence of \((1)\) and \((1')\) when \( H \) is smooth over \( S \). It is in order to construct actions in the sense of \(1)\) or \(1')\) that we will systematically be considering multiplicative torsors not only on algebraic groups over a field, but also for group schemes.

\(3)\) The multiplicative torsor \( E \) corresponds by 1.4 to a central extension \( \mathcal{E} \) of the sheaf of groups \( h_G \) on \( S_{\text{Zar}} \) by the sheaf of abelian groups \( A \). The sheaf of groups \( h_H \) acts on \( h_G \). An action \((c)\) is an action of \( h_H \) on \( \mathcal{E} \), lifting the action of \( h_H \) on \( h_G \) and trivial on \( A \).

To construct 1.7, we will use the interpretation 1.4 of multiplicative torsors as central extensions, and the interpretation \((3)\) of data \((c)\). Its equivalence with interpretation \((1)\) is easily checked. To simplify notations, we write \( G, H, H \ltimes G \) for \( h_G, h_H, h_{H \ltimes G} \).

Let \( E_0 \) be a central extension of \( H \ltimes G \) by \( A \). Let \( E \) (resp. \( F \)) be the inverse image of \( G \) (resp. \( H \)) in \( E_0 \). It is a central extension of \( G \) (resp. \( H \)) by \( A \). The central
extension $F \subset E_0$ acts on $E \subset E_0$ by conjugation. The action is trivial on $A \subset F \subset E_0$ hence factors through an action of $H$ on $E$, which lifts the action of $H$ on $G$. This gives data (a) (b) (c), from which $E_0$ is recovered as the quotient of the semi-direct product of $F \ltimes E$, a central extension of $H \ltimes G$ by $A \times A$, by the antidiagonal copy of $A$ in $A \times A$.

1.8. For any central extension $\mathcal{E}$ of $G$ by $A$, if $g$ and $h$ in $G$ commute, the lifted commutator (0.N.4) (2) is in $A$: on the subscheme $C \subset G \times G$ of commuting pairs of elements, the lifted commutator is a morphism of sheaves from $C$ to $A$, i.e. is a section

\[(1.8.1) \quad \text{comm} \in H^0(C, A).\]

If $G$ is commutative, then $C = G \times G$ and the section (1.8.1) of $A$ over $G \times G$ is bimultiplicative.

More generally, if an algebraic group $H$ acts on $G$, and if this action is lifted to an action on $\mathcal{E}$, trivial on $A$, we have a lifted commutator (0.N.4) (3) from $H \times G$ to $\mathcal{E}$. If $C \subset H \times G$ is the subscheme of pairs $(h, g)$ for which $h$ fixes $g$, this lifted commutator defines

\[(1.8.2) \quad \text{comm} \in H^0(C, A),\]

and if $H$ acts trivially on $G$, the section (1.8.2) of $A$ over $H \times G$ is bimultiplicative.

1.9. To go back and forth between cohomological computations and categorical interpretations, we will rely on the following construction of SGA4 XVIII 1.4. Let $K$ be a complex and $n$ an integer. Let $A \to B$ be the truncation $\tau_{[n-1,n]} K$ of $K$ (0.N.5): one has $A := K_{n-1}/\text{coboundaries}$ and $B := n\text{-cocycles}$. We define the category $\mathcal{C}_n(K)$ as the category of $n\text{-cocycles}$, a morphism from $b_0$ to $b_1$ being $a \in A$ such that $b_1 - b_0 = da$. Composition of morphisms is given by addition. Addition of $n\text{-cocycles}$ defines an addition law for objects, turning $\mathcal{C}_n(K)$ into a strictly commutative Picard category. The group of isomorphism classes of objects of $\mathcal{C}_n(K)$ is $H^n(K)$, and the automorphism group of any object is $H^{n-1}(K)$. A morphism of complexes $f : K \to L$ induces a functor $f : \mathcal{C}_n(K) \to \mathcal{C}_n(L)$, and if the morphism $f$ induces isomorphisms on $H^s$ and $H^{s-1}$, the functor $f$ is an equivalence.

We will say that $K$ incarnates a category $\mathcal{C}$ in degree $n$, if we are given an equivalence (usually: an equivalence of commutative Picard categories) from $\mathcal{C}_n(K)$ to $\mathcal{C}$. When no $n$ is mentioned, $n = 1$ is assumed.

The heuristics for using this definition is as follows. Let $\mathcal{C}$ be a strictly commutative Picard category. Let $h^1(\mathcal{C})$ be the group of its isomorphism classes of objects and $h^0(\mathcal{C})$ be the group of automorphisms of any object. Suppose that $h^1(\mathcal{C})$ and $h^0(\mathcal{C})$ can be interpreted as cohomology groups $H^s$ and $H^{s-1}$. Then, if $K$ is any complex which naturally computes the cohomology in question, the complex $K$ should
incarnate $C$ in degree $n$. The construction of the equivalence of $C_n(K)$ with $C$ will mimic the construction of the isomorphism of $H^i(K)$ with $h^i(C)$.

**Examples.** — (i) Let $\mathcal{F}$ be a sheaf of abelian groups on a space $S$ and let $\mathcal{F}^*$ be a resolution of $\mathcal{F}$, such that the complex $\Gamma(S, \mathcal{F}^*)$ computes the cohomology of $\mathcal{F}$ in degree 0 and 1: the canonical maps $H^i(S, \mathcal{F}^*) \to H^i(S, \mathcal{F})$ are isomorphisms for $i = 0$ and 1. Then, $\Gamma(S, \mathcal{F}^*)$ incarnates the category of $\mathcal{F}$-torsors on $S$. The equivalence $C(\Gamma(S, \mathcal{F}^*)) \to (\mathcal{F}$-torsors) is as follows: a 1-cocycle $c$ gives rise to the $\mathcal{F}$-torsor $P(c)$ of local sections $s$ of $\mathcal{F}$ with $ds = c$, and if $c_1 - c_0 = df$, $f$ defines the isomorphism $s \mapsto s + f: P(c_0) \to P(c_1)$.

(ii) Suppose that a group $G$ acts on $(S, \mathcal{F})$ and that the resolution $\mathcal{F}^*$ is equivariant. The group $G$ then acts on $\Gamma(S, \mathcal{F}^*)$ and we can for each $i$ form the standard complex $C^i(G, \Gamma(S, \mathcal{F}^*))$ computing $H^i(G, \Gamma(S, \mathcal{F}^*))$. These complexes organize into a double complex, whose component in bidegree $(p, q)$ is the group $C^q(G, \Gamma(S, \mathcal{F}^p))$ of $q$-cochains with values in $\Gamma(S, \mathcal{F}^p)$; it is the group of maps $G^q \to \Gamma(S, \mathcal{F}^p)$. The associated simple complex incarnates the category of equivariant $\mathcal{F}$-torsors. If $(c, r)$ is a 1-cocycle:

$$
\begin{array}{ccc}
* & \gamma & * \\
\gamma & c & \ldots \\
\gamma & \gamma & \gamma
\end{array}
$$

$c$ is a 1-cocycle in $\Gamma(S, \mathcal{F}^*)$ (vanishing of $d(c, r)$ at $*, \gamma$), $r(\gamma)$ gives an isomorphism between the corresponding torsor and its transform by $\gamma$ (vanishing at $*, \gamma$) and $r(\gamma)$ is a group action (vanishing at $*, \gamma$). A 0-cochain $f$ gives an equivariant morphism of torsors.

Standard operations on complexes can be given categorical meaning.

**Examples.** — (iii) Let $f: K \to L$ be a morphism of complexes. We view $f: K \to L$ as a double complex, with $K$ (resp. $L$) in second degree 0 (resp. 1) and form the associated simple complex $s(K \to L)$. This shifted mapping cone construction gives rise to a long exact sequence of cohomology

$$
\ldots \to H^i(s(K \to L)) \to H^i(K) \to H^i(L) \to \ldots
$$

Suppose that $K^i = L^i = 0$ for $i < 0$. Then, $s(K \to L)$ incarnates the category of objects of $C(K)$ given with a trivialization (= isomorphism with the object 0) of its image in $C(L)$. Indeed, a 1-cocycle is a 1-cocycle $k^i$ of $K$ given with $k^0 \in L^0$ such that $dk^0 = f(k^1)$.

(iv) We now view $f: K \to L$ as a double complex with $K$ in second degree $-1$ and $L$ in second degree 0. The long exact sequence of cohomology is

$$
\ldots \to H^i(K) \to H^i(L) \to H^i(s(K \to L)) \to \ldots
$$
Suppose that $K^i = L^i = 0$ for $i < 0$, and that $H^i(K)$ injects into $H^i(L)$ for $i = 0$ and 2. By the long exact sequence, $H^1(L)$ maps onto $H^1(s(K \to L))$: the category $\mathcal{C}(s(K \to L))$ is equivalent to its subcategory with objects the 1-cocycles of $L$. This category has as objects the objects of $\mathcal{C}(L)$, a morphism from $P$ to $Q$ being an isomorphism class of pairs $(A, a)$ with $A$ an object of $\mathcal{C}(K)$ and $a$ a morphism from $P + f(A)$ to $Q$. The assumption on $H^0$ ensures that pairs $(A, a)$ have no automorphisms. If $H^1(K) = 0$, morphisms can more simply be described as classes modulo $H^0(K)$ of morphisms in $\mathcal{C}(L)$.

(v) Let $K \xrightarrow{f} L \xrightarrow{g} M$ be morphisms of complexes with zero composite. They give rise to functors $\mathcal{C}(K) \to \mathcal{C}(L) \to \mathcal{C}(M)$ and to an isomorphism of the composite functor with the constant functor with value the object 0. We view $K \to L \to M$ as a double complex, with $K$ (resp. $L$, $M$) in second degree 0 (resp. 1, 2). We assume that the cohomology of $K$, $L$ and $M$ is zero in degree $i < 0$. The category $\mathcal{C}(s(K \to L \to M))$ does not change (up to a natural equivalence) if $K$, $L$, $M$ are replaced by their quasi-isomorphic truncation $\tau > 0$ (0.N.6) and, using this remark, one checks that it is the category of objects $P$ of $\mathcal{C}(K)$, given with a trivialization $0 \to f(P)$ in $\mathcal{C}(L)$ whose image $0 \to g(f(P))$ in $\mathcal{C}(M)$ is the identity automorphism of the object zero of $\mathcal{C}(L)$.

In our applications, we will start from a cosimplicial system of complexes $K(\rho)$. We won’t need degeneracies and so could start from a strict cosimplicial system (0.N.7). We assume that $H^i(K(\rho) = 0$ for $i < 0$.

For each $n$ one can consider $K(n) \to K(n + 1) \to K(n + 2)$, taking as morphisms of complexes the usual $\sum(-1)^i \partial_i$. The $\mathcal{C}(K(\rho))$ form a strict cosimplicial system of Picard categories. For $n = 0$ (resp. $n = 1$), $\mathcal{C}(s(K(n) \to K(n + 1) \to K(n + 2)))$ has the following interpretation.

Case $n = 0$: Objects $P$ of $\mathcal{C}(K(0))$, given with an isomorphism $f: \partial_0 (P) \to \partial_1 (P)$ in $\mathcal{C}(K(1))$ obeying a cocycle condition in $\mathcal{C}(K(2))$: for $i_a$ ($a = 0, 1, 2$) the three face maps from $\Delta_0$ to $\Delta_2$, commutativity of the diagram

\[
\begin{array}{ccc}
\partial_0(f) & \to & \partial_1(f) \\
\downarrow & & \downarrow \\
\partial_2(f) & \to & \partial_3(f)
\end{array}
\]

Case $n = 1$: Objects $P$ of $\mathcal{C}(K(1))$ given with a multiplicativity isomorphism $\partial_0(P) + \partial_2(P) \to \partial_1(P) \to \partial_3(P)$ in $\mathcal{C}(K(2))$ obeying an associativity condition in $\mathcal{C}(K(3))$.

(vi) The case $n = 1$ above will be met in the following context, explaining the terminology. Let $\tilde{\mathcal{H}}$ be a sheaf on the big Zariski site $S_{zar}$, and $G$ be a group scheme of finite type over $S$. In the inhomogeneous notation, the simplicial classifying space $BG = (G^{\Delta^\infty}/G)$ of $G$ is

\[
\cdots G^3 \equiv G^2 \equiv G \equiv e,
\]
the three maps $\partial_0$, $\partial_1$, $\partial_2$ from $G^2$ to $G$ being $\text{pr}_2$, $\mu$ and $\text{pr}_1$. Suppose that on each $G^s$ we are given a resolution $\mathcal{H}^*$ of $\mathcal{H}^*_{G^s}$, contravariant in the face maps, and which computes $H^i(G^s, \mathcal{H}^*)$ for $i = 0$ or $1$. By the example (i), $\Gamma(G^s, \mathcal{H}^*)$ incarnates the category of $\mathcal{H}^*_G$-torsors on $G^s$. Applying example (v) to

$$\Gamma(G, \mathcal{H}^*) \to \Gamma(G^2, \mathcal{H}^*) \to \Gamma(G^3, \mathcal{H}^*),$$

one obtains a complex which incarnates the category of multiplicative torsors on $G$.

This complex can be replaced by the subcomplex of simplicial degree $\geq 1$ of $s(\Gamma(G^*, \mathcal{H}^*))$: the complex

$$s(\Gamma(G^*, \mathcal{H}^*)) \quad \text{for} \quad n \geq 1$$

incarnates multiplicative torsors in degree 2. Let us write $e$ for the trivial group scheme $G^0$. For each $n$, $\Gamma(e, \mathcal{H}^*)$ maps to $\Gamma(G^s, \mathcal{H}^*)$, defining a cosimplicial map from $\Gamma((B^e)_*, \mathcal{H}^*)$ to $\Gamma((BG)_*, \mathcal{H}^*)$. The composite

$$s(\Gamma(G^s, \mathcal{H}^*)) \quad \text{for} \quad n \geq 1 \to s(\Gamma((BG)_*, \mathcal{H}^*)) \to s(s(\Gamma((B^e)_*, \mathcal{H}^*)) \to s(\Gamma((BG)_*, \mathcal{H}^*))$$

is a quasi-isomorphism. It follows that

$$s(s(\Gamma((B^e)_*, \mathcal{H}^*)) \to s(\Gamma((BG)_*, \mathcal{H}^*))$$

which computes the relative cohomology of $BG$ mod $B^e$ with values in $\mathcal{H}$, also incarnates the category of multiplicative $\mathcal{H}^*$-torsors in degree 2.

(vii) With notation as in example (vi), if $H^i(G^s, \mathcal{H}^*) = 0$ for $i > 0$, we can, on each $G^s$, take for $\mathcal{H}^*$ the complex reduced to $\mathcal{H}^*_G$ in degree 0. We find that the category of multiplicative torsors on $G$ is incarnated by the complex

$$\Gamma(G^s, \mathcal{H}^*) \quad \text{for} \quad n \geq 1.$$  

This is the familiar description of central extensions by 2-cocycles.

One could as well use the complex

$$\tilde{H}^0(G^s, \mathcal{H}), \quad d = \sum (-1)^i \partial_i,$$

where $\tilde{H}^0$ is the direct summand $\Gamma(G^s, \mathcal{H})/\Gamma(e, \mathcal{H})$ of $\Gamma(G^s, \mathcal{H})$.

1.10. The constructions 1.9 can be sheafified. Let $K$ be a complex of sheaves on $S$ and $n$ be an integer. As in 1.9, consider the truncation $\tau_{[a-1, a]}(K)$: $A \to B$ of $K$. Define $\mathcal{C}_K$ to be the stack on $S$ having as objects over $U$ the $A$-torsors $P$ over $U$ provided with a trivialization of the $B$-torsor $d(P)$. It is a strictly commutative Picard stack. For $S$ a point, i.e. for a complex of abelian groups, one recovers a category incarnated by $K$ in degree $n$: to the cocycle $b \in B$ attach the trivial $A$-torsor $A$ with
the trivialization $b$ of the trivial $B$-torsor $d(A)$. Indeed, over a point any $A$-torsor is isomorphic to the trivial one.

The analogue of 1.9, Example (i), is that for $f: X \to S$, $\mathcal{F}$ a sheaf on $X$, and $\mathcal{F}'$ a resolution of $\mathcal{F}$ such that $f_*\mathcal{F}'$ computes $R^if_*\mathcal{F}$ in degree 0 and 1, $f_*\mathcal{F}'$ incarnates the stack on $S$ having as objects over $U$ the $\mathcal{F}$-torsors on $f^{-1}(U)$.

2. Descent for $K_2$-torsors

The main result of this section is Theorem 2.2, which will allow us to reduce the construction of multiplicative $K_2$-torsors on connected reductive groups to the case of split groups. From 2.6 on, we prove a relative variant of 2.2. This variant will not be needed for our main results, or rather for their restriction to the case of reductive groups over fields.

Our main tool is the following theorem of Colliot-Thélène and Suslin, see B. Kahn (1993) (corollary 1 to theorem 3.1):

**Theorem 2.1.** — Let $X$ be a smooth geometrically irreducible variety over $k$. Assume that $X$ has a $k$-rational point $e$. If $k'$ is a finite Galois extension of $k$, with Galois group $\Gamma$, and $X' := X \otimes_A k'$, one has, denoting by $k(X)$ and $k(X')$ the fields of rational functions on $X$ and $X'$

\[(2.1.1) \quad K_2(k(X))/K_2(k) \xrightarrow{\sim} [K_2(k(X'))/K_2(k')]^\Gamma\]

and

\[(2.1.2) \quad H^1(\Gamma, K_2(k(X'))/K_2(k')) = 0.\]

In characteristic zero, (2.1.1) is Colliot-Thélène (1983) Th. 1. In any characteristic, it is in Suslin (1987), 3.6 and 5.8. The vanishing (2.1.2) can be deduced from Merkurjev-Suslin (1983) 14.1. The point of view of B. Kahn (1993) is that the Lichtenbaum complex $\Gamma[2]$ has good étale localization properties and that theorem 2.1 is a corollary of these. See also B. Kahn (1996). Under the assumptions of 2.1, one also knows that $K_2(k)$ injects as a direct summand in $K_2(X)$.

The group $\Gamma$ acts semi-linearly on $(X', \epsilon') = (X, \epsilon) \times_{\Spec(k)} \Spec(k')$. A (Galois) descent data on a pointed $K_2$-torsor $P'$ on $(X', \epsilon')$ is an action of $\Gamma$ on $P'$, covering the action of $\Gamma$ on $(X', \epsilon')$.

**Theorem 2.2.** — Under the assumptions of 2.1, the pull-back functor is an equivalence from

(a) the category of pointed $K_2$-torsors on $(X, \epsilon)$, to

(b) the category of pointed $K_2$-torsors on $(X', \epsilon')$, provided with a descent data.

**Proof:** — The category of $K_n$-torsors on $X$ is incarnated by the global sections of the Quillen resolution $G^*_u$ of $K_n$ by flabby sheaves on $X$. Let $X^0$ be the set of points
of codimension $i$ of $X$. One has
\[ G_i = \bigoplus_{x \in X^0} i_* K_{n-i}(k(x)). \]

For $n = 2$, the complex of global sections is
\[ K_2(k(X)) \to \bigoplus_{x \in X^1} k(x)^* \to \bigoplus_{x \in X^2} \mathbb{Z}. \]

We want to use the resolution $G_n^*$ of $K_n$ to construct a complex incarnating the category of pointed $K_n$-torsors. The fact that the resolution $G_n^*$, while contravariant for smooth maps, is not contravariant for closed embeddings, creates a difficulty. We go around it as follows. The point $e$, viewed as a section of $f : X \to \text{Spec}(k)$, defines an equivalence of the category of $K_n$-torsors on $X$ with the product of the categories of pointed torsors on $(X, e)$ and the category of $K_n(k)$-torsors. This allows us to view the category of pointed torsors as a quotient category, rather than a subcategory, of the category of $K_n$-torsors: it is naturally equivalent to the category of relative torsors having as objects the $K_n$-torsors on $X$, a map from $P$ to $Q$ being a class mod $K_n(k)$ of maps of torsors from $P$ to $Q$. The inverse of the forgetting functor from pointed torsors to relative torsors is the functor $P \mapsto P - f^* e^* P$.

By 1.9, Example (iv), the category of relative $K_2$-torsors on $X$ is incarnated by the complex
\[ (2.2.1) \quad K_2(k(X))/K_2(k) \to \bigoplus_{x \in X^1} k(x)^* \to \bigoplus_{x \in X^2} \mathbb{Z}. \]

By a variant of 1.9, Example (ii), the category of relative $K_2$-torsors on $X'$, provided with a descent data, is incarnated by the simple complex associated to the double complex
\[ (2.2.2) \quad C^*(\Gamma, K_2(k(X'))/K_2(k')) \to C^*\left( \bigoplus_{x \in X^1} k(x)^* \right) \to C^*\left( \bigoplus_{x \in X^2} \mathbb{Z} \right). \]

These incarnations reduce 2.2 to the
\[ \text{Lemma 2.3.} \quad \text{The natural map of complexes (2.2.1) \to s(2.2.2) induces an isomorphism on } H^0 \text{ and } H^1. \]

\[ \text{Proof.} \quad \text{More precisely, we will check that the cone on the natural map of complexes (2.2.1) \to s(2.2.2) has vanishing } H^{-1}, H^0 \text{ and } H^1. \text{ This cone is the associated simple complex of the double complex (2.2.2), coaugmented by (2.2.1) in second (= group cohomology) degree } -1. \text{ It vanishes outside of } 0 \leq p \leq 2, q \geq -1. \]

The $p = 0$ column is $C^*(\Gamma, K_2(k(X'))/K_2(k'))$, coaugmented by $K_2(k(X))/K_2(k)$ in degree $-1$. By 2.1, its cohomology vanishes in degree $-1$, 0 and 1.
The $p = 1$ column is the sum over $x$ in $X^{(1)}$ of the $C^* (\Gamma, (k(x) \otimes k')^\ast)$, coaugmented by $k(x)^\ast$ in degree $-1$. As $k(x)^\ast \xrightarrow{\sim} (k(x) \otimes k')^\ast \Gamma$, its cohomology vanishes in degree $-1$ and $0$.

The $p = 2$ column is the sum over $x$ in $X^{(2)}$ of the $C^* (\Gamma, \mathbb{Z}^{I(x)})$, where $k(x)^\ast \otimes k_0$ is a product of fields indexed by $I(x)$, coaugmented by $\mathbb{Z}$ in degree $-1$. As the diagonal map from $\mathbb{Z}$ to $\mathbb{Z}^{I(x)}$ is injective, its cohomology vanishes in degree $-1$.

The spectral sequence for the filtration by $p$ gives the required vanishing.

If we apply 2.2 to the group of automorphisms of an object (resp. to the set of isomorphism classes of objects), we obtain

**Corollary 2.4.** — Under the assumptions of 2.1,

(i) $H^0(X, \mathbb{K}_2)/\mathbb{K}_2(k) \xrightarrow{\sim} (H^0(X', \mathbb{K}_2)/\mathbb{K}_2(k'))^\Gamma$;

(ii) one has an exact sequence

$$0 \rightarrow H^1(\Gamma, H^0(X', \mathbb{K}_2)/\mathbb{K}_2(k')) \rightarrow H^1(X, \mathbb{K}_2)$$

$$\rightarrow H^1(X', \mathbb{K}_2)^\Gamma \rightarrow H^2(\Gamma, H^0(X', \mathbb{K}_2)/\mathbb{K}_2(k'))$$.

**Remark 2.5.** — (i) If further $\mathbb{K}_2(k') \xrightarrow{\sim} H^0(X', \mathbb{K}_2)$, then $\mathbb{K}_2(k) \xrightarrow{\sim} H^0(X, \mathbb{K}_2)$ and $H^1(X, \mathbb{K}_2) \xrightarrow{\sim} H^1(X', \mathbb{K}_2)^\Gamma$. This results from 2.4.

(ii) Suppose that $X$ is open and dense in a smooth geometrically irreducible variety $X_1$ over $k$, and that $X_1$ has a $k$-rational point. Then 2.2 remains valid, with the same proof, if “pointed $\mathbb{K}_2$-torsor” is replaced by “relative $\mathbb{K}_2$-torsor”.

**2.6.** Let $f: X \rightarrow S$ be a smooth morphism, with a section $e$. We assume that

(2.6.1) $S$ is regular and of finite type over some field $k$;

(2.6.2) For each generic point $\eta$ of $S$ (there is one for each connected component of $S$), the generic fiber $X_\eta$ is geometrically irreducible.

Assumption (2.6.1) is to have Quillen resolutions. Assumption (2.6.2) is to have 2.1 for $X_\eta/\eta$.

Let $\omega: S_0 \rightarrow S$ be an étale surjective map. We define $S_n$ to be the $(n+1)!$ fiber power $(S_0/S)^{S_n}$ of $S_0$ over $S$. Let $(X_n, e_n)$ be the pull-back of $(X, e)$ over $S_n$, $X_n := X \times_{S_0} S_n$ is also the $(n+1)!$ fiber power $(X_0/X)^{S_n}$ of $X_0$ over $X$. The $X_n$ $(n \geq 0)$ form a simplicial scheme, augmented to $X_{-1} := X$.

A descent data for a $\mathbb{K}_2$-torsor $P_0$ on $X_0$ is an isomorphism between its two inverse images on $X_1$, obeying a compatibility condition on $X_2$. If $Y$ maps to $X$, and if this map can be factorized through $X_0$, one can choose a factorization

$$Y \xrightarrow{a} X_0 \xrightarrow{a_0} X$$
and take the pull-back \( a^*P_0 \). If \( Y \xrightarrow{b} X_0 \xrightarrow{\pi} X \) is another factorization, \( (a, b) \) is a map from \( Y \) to \( X_1 \) and the descent data gives an isomorphism between \( a^*P_0 \) and \( b^*P_0 \). The compatibility condition ensures that for three factorizations \( a, b \) and \( c \), the diagram of isomorphisms

\[
\begin{array}{ccc}
a^*P & \xrightarrow{\sim} & b^*P \\
\downarrow & & \downarrow \\
c^*P
\end{array}
\]

is commutative. More precisely, the compatibility condition is this commutativity, in the universal case where \( Y = X_2 \) and \( a, b, c \) are the three projections of \( X_2 \) to \( X_0 \).

This can be repeated for pointed torsors relative to the sections \( e_n \) of \( X_n/S_n \).

**Theorem 2.7.** — Under the assumptions of 2.6, the pull-back functor is an equivalence from
(a) the category of pointed \( K_2 \)-torsors on \((X, e)\), to
(b) the category of pointed \( K_2 \)-torsors on \((X_0, e_0)\), provided with a descent data.

**Proof.** — If \( S \) is the spectrum of a field \( k \), 2.7 is little more than a rephrasing of 2.2. Indeed, if \( k' \) is a Galois extension of \( k \), with Galois group \( \Gamma \), the morphism \( k' \otimes_k k \to k' \Gamma : x \otimes y \mapsto (x \sigma(y))_{\sigma \in \Gamma} \) is an isomorphism. More generally, if \( S_0 = \text{Spec}(k') \), \( S_n \) is a disjoint union of copies of \( S_0 \), indexed by \( \Gamma^n \). Using those isomorphisms, one sees that descent data for \( S_0 \to S \) are the same thing as Galois descent data. A general \( S_0 \) is the spectrum of an étale algebra \( A \) over \( k \), product of finite separable extensions \( k_i \) of \( k \).

Let \( \mathcal{C}(A/k) \) be the full subcategory of the category of finite separable extensions of \( k \), consisting of the \( k' \) for which \( k \to k' \) admits a factorization through \( A \), i.e. through one of the \( k_i \). Pointed torsors with descent data on \((X_0, e_0)\) can be reinterpreted as the data for each \( k' \) in \( \mathcal{C}(A/k) \) of a pointed toror \( P[k'] \) on \((X', e') := (X, e) \otimes_k k' \), compatibly with base change \( k' \to k' \). Let \( K \) be a Galois extension of \( k \) in \( \mathcal{C}(A/k) \). Restricting \( P[k] \) from \( \mathcal{C}(A/k) \) to \( \mathcal{C}(K/k) \), one obtains a descent data for \( \text{Spec}(K) \to \text{Spec}(k) \). By 2.2, it comes from \( P \) on \((X, e)\), unique up to unique isomorphism. The isomorphism of \( P[k'] \) with \( P[k] \) for \( k' \) in \( \mathcal{C}(K/k) \) extends uniquely to all \( k' \) in \( \mathcal{C}(A/k) \). If \( K' \) is a composed extension of \( k' \) and \( K \), this results from the fully faithful part of 2.2, applied to the Galois extension \( K' \to k' \). We are here repeating Giraud (1964) 6.25. For \( S \) the spectrum of a field and under the assumptions of 2.5 (ii), the same argument proves 2.7 for relative \( K_2 \)-torsors.

We now prove 2.7. As in the proof of 2.2, the section \( e \) decomposes the category of \( K_2 \)-torsors on \( X \) into the product of the categories of pointed \( K_2 \)-torsors on \((X, e)\) and the category of \( K_2 \)-torsors on \( S \). If we view the category of pointed \( K_2 \)-torsors as a quotient category of the category of \( K_2 \)-torsors on \( X \), it naturally identifies with the
category of relative torsors, with objects the $K_2$-torsors on $X$, a map $P \to Q$ being an isomorphism class of pairs $(A, a)$, $A$ a $K_2$-torsor on $S$ and $a$ a morphism from $P + f^*A$ to $Q$.

Let us view the morphism of complexes

$$f^*: \Gamma(S, G^*_2) \to \Gamma(X, G^*_2)$$

as a double complex, with $\Gamma(X, G^*_2)$ in second degree 0 and $\Gamma(S, G^*_2)$ in second degree $-1$. By 1.9, Example (iv), we have:

**Lemma 2.8.** — The category of relative $K_2$-torsors on $X$ is incarnated by $s(2.7.1)$.

2.9. The complexes $s(2.7.1)$ for the $X_q/S_q$

$$s(\Gamma(S_q, G^*_2) \to \Gamma(X_q, G^*_2))$$

form a semi-simplicial system of complexes. Applying 1.9 Example (v) (case $n = 0$) to it, we see that the associated simple complex incarnates the category of relative (or, equivalently, pointed) torsors on $X_0$, given with a descent data. Those incarnations reduce 2.7 to

**Lemma 2.10.** — The natural map of complexes $s(2.7.1) \to s(2.9.1)$ induces an isomorphism on $H^0$ and $H^1$.

More precisely, we will check that this map induces an isomorphism on $H^0$ and $H^1$ and a monomorphism on $H^2$, i.e. that its cone has vanishing $H^i$ for $i \leq 1$. Let us denote by $p$ the degree in the complex $G^*_2$. Filtering by $p$ and using the corresponding long exact sequences of cohomology, one is reduced to check for $p = 0, 1, 2$ that the morphism of complexes

$$\Gamma(S, G^*_2) \to \Gamma(X, G^*_2), \text{ in degree } -1 \text{ and } 0) \to s(\Gamma(S_q, G^*_2) \to \Gamma(X_q, G^*_2))$$

induces an isomorphism in degree $i \leq 1 - p$ and a monomorphism in degree $2 - p$.

Case $p = 0$. We may and shall assume $S$ connected. If $\eta$ is the generic point of $S$, making the base change by $\eta \to S$ does not change (2.10.1). Indeed, the points of codimension 0 of the $X_q$ and $S_q$ are above $\eta$. We hence may and shall assume that $S$ is the spectrum of a field $k$.

In (2.10.1)$_{p=0}$, because $X/S$ has a section, $\Gamma(S, G^*_2) = K_2(k)$ injects into $\Gamma(X, G^*_2) = K_2(k(X))$ and similarly for $X_q/S_q$. This allows to replace (2.10.1)$_{p=0}$ by

$$\Gamma(X, G^*_2)/\Gamma(S, G^*_2) \text{ in degree } 0) \to s(\Gamma(X_q, G^*_2)/\Gamma(S_q, G^*_2)).$$

Both complexes are now in degree $\geq 0$, and injectivity on $H^2$ is trivial. It suffices to prove isomorphism on $H^0$ and $H^1$. 

For $S$ the spectrum of a field, we already know that $2.7$ holds, i.e. that $2.10$ holds. If one uses relative, rather than pointed, $K_2$-torsors, this remains true if $X$ is replaced by any non empty open subset $U$. Taking the inductive limit in $U$ in $s(2.7.1) \to s(2.9.1)$, one obtains $(2.10.1)$ and the case $p = 0$ follows.

Case $p = 1$ or $2$. Taking the cone on $(2.10.1)$, one obtains the simple complex associated to the coaugmented cosimplicial system of complexes

$$\Gamma(S_q, G_2^p) \to \Gamma(X_q, G_2^p)$$

in degree $-1$ and $0$ if $q > -1$.

To show that it is acyclic in degree $\leq 1 - p$, it suffices to check that the coaugmented cosimplicial complexes $\Gamma(S_q, G_2^p)$ and $\Gamma(X_q, G_2^p)$ are acyclic in degree $\leq 2 - p$ (resp. $1 - p$). Both statements are instances of the following lemma, applied to $S_0 \to S$ or to $X_0 \to X$.

**Lemma 2.11.** — For $S$ regular and $u : S_0 \to S$ étale and surjective, if $p = 1$ or $2$, the coaugmented cosimplicial complex $\Gamma((S_0/S)^{\Lambda^p}, G_2^p)$ is acyclic in degree $\leq 2 - p$.

**Proof.** — Fix $p = 1$ or $2$. The points of codimension $p$ of $(S_0/S)^{\Lambda^p}$ are just the points of $(S_0/S)^{A^p}$ over a point of codimension $p$ of $S$, and the coaugmented cosimplicial complex $\Gamma((S_0/S)^{\Lambda^p}, G_2^p)$ is the sum over $s \in S^{(0)}$ of the following complexes: for $F = u^{-1}(s)$ and $F^{A^p}$ a power over $s$, the complex

$$K_{2-p}(F^{A^p}) \quad (q \geq -1)$$

This is the étale cohomology coaugmented Čech complex, for the étale covering $F \to s$, and the sheaf $\mathcal{O}^*$ if $p = 1$, the sheaf $\mathbb{Z}$ if $p = 2$. For $p = 1$, Hilbert 90 gives acyclicity in degree $\leq 1$. For $p = 2$, the sheaf property gives acyclicity in degree $\leq 0$.

### 3. Split tori

Our aim in this section is to compute the category of multiplicative $K_2$-torsors on a split torus. We also treat the case of split unipotent groups.

**3.1.** Let $S$ be regular and of finite type over a field $k$. If $\phi : A^1_S \to S$ is the affine line over $S$, Sherman (1979) proves that

$$p_*K_i = K_j \quad \text{and} \quad R^i p_*K_j = 0 \quad \text{for} \quad i > 0 \quad \text{for} \quad p : A^1_S \to S.$$ (3.1.1)

If we remove the 0-section, we obtain the multiplicative group $\phi : G_{mS} \to S$ over $S$. Sherman (1979) proves that

$$p_*K_i = K_i \oplus K_{j-1} \quad \text{and} \quad R^i p_*K_j = 0 \quad \text{for} \quad i > 0 \quad \text{for} \quad p : G_{mS} \to S.$$ (3.1.2)
The maps, from right to left, in (3.1.1) and (3.1.2) are as follows. The component $K_j \rightarrow p_* K_j$ expresses the contravariance of $K_j$. For 3.1.2, the component $K_j \rightarrow p_* K_j$ comes from the contravariance and from the multiplicative structure on $K_j$: the coordinate $u$ on $G_{ns}$ is an invertible function, hence defines a class $\{u\}$ in $H^0(G_{ns}, K_1)$ and the map is $x \mapsto \{u\}.p^*(x)$.

Let $U$ be a split unipotent group scheme over $S$: locally over $S$, $U$ is an iterated extension of additive groups. If $p: P \rightarrow S$ is a $U$-torsor over $S$, an iterated application of (3.1.1) shows that the functor $p_{\ast}$ is an equivalence from the category of $K_2$-torsors on $S$ to the category of $K_2$-torsors on $P$. If $G$ is a group scheme over $S$, extension of $G$ smooth over $S$ by $U$, $G$ is a $U$-torsor on $G$ and $p: G \rightarrow G = G/U$ induces an equivalence from $K_2$-torsors on $G$ to $K_2$-torsors on $G$. The same applies to multiplicative $K_2$-torsors. We conclude:

**Proposition 3.2.** — If $S$ is regular of finite type over a field, and if the group scheme $G$ over $S$ is an extension of $G$ smooth over $S$ by a split unipotent group scheme $U$, the pull back by $p: G \rightarrow G = G/U$ is an equivalence from the category of central extensions of $G$ by $K_2$ to the category of central extensions of $G$ by $K_2$.

For $\overline{G}$ the trivial group scheme $e$, 3.2 tells that a central extension of a split unipotent group scheme by $K_2$ is trivial, and admits a unique trivialization.

**3.3.** Let $p: T \rightarrow S$ be a split torus over $S$ and $X, Y$ be as in (0.N.1). If we choose an isomorphism of $Y$ with $Z^n$, we obtain $T \cong G^n_{ns}$. Using induction on $n$ we deduce from (3.1.2) that $R^ip_* K_j = 0$ for $i > 0$ and we obtain a description of $p_* K_j$. Using the relation $\{u, u\} = \{u, -1\}$ in $K_2$, the description of $p_* K_j$ can be made intrinsic:

**Lemma 3.3.1.** — The sheaf of graded rings $p_* K_\ast$ on $S$ is generated by $K_\ast$ and by $X$ in degree 1, with the only relations that

(i) $X \rightarrow p_* K_\ast$ is additive;

(ii) for $x$ in $X$, $x.x = x.\{1\}$.

**3.4.** If we filter $p_* K_\ast$ by “how many factors in $X$”, we obtain an increasing filtration $V$ with

$$
\text{Gr}^V_m p_* K_\ast = ^m X \otimes K_{\ast-m}.
$$

The bottom piece $V_0(p_* K_\ast) = K_\ast$ is a direct factor: the pullback by the 0-section $e$ is a retraction to the embedding of $K_\ast$ in $p_* K_\ast$. The kernel of $e^*$ is the reduced direct image $\tilde{p}_* K_\ast$.

The translation by $t \in T(S)$ transforms $x \in X$, viewed as a section of $p_* K_1$, into $x - x(t)$, where $x(t) \in H^0(S, K_1)$ is in $V^0$. It follows that a translation respects $V$ and
acts trivially on \( {\text{Gr}}^Y_{p_*K_*} \). If \( q: Q \to S \) is a \( T \)-torsor on \( S \), we still have \( R^i\eta_*K_* = 0 \) for \( i > 0 \). Moreover, we have a filtration \( V \) on \( q_*K_* \), for which

\[ \text{Gr}_{r}^Y_{p_*K_*} = \Lambda X \otimes p_*K_{* - r}. \]

As those claims are local on \( S \), it suffices to consider the case where \( Q \) is trivial. We leave this case to the reader.

Special cases: for \( K_1 \), the filtration \( V \) on \( p_*K_1 \) reduces to the exact sequence

\[ 1 \to C^* \to p_*C^*_T \to X \to 1, \]

and the reduced direct image \( \tilde{p}_*K_1 \) is just \( X \). For \( K_2 \), \( p_*K_2 \) has three graded pieces: \( K_2, X \otimes K_1 \) and \( \Lambda X \). The extension \( \tilde{p}_*K_2 \) of \( \Lambda X \) by \( X \otimes K_1 \) can be described as follows.

\textit{Construction 3.5.} — Under the assumptions of 3.1 and 3.3, the sheaf \( \tilde{p}_*K_2 \) is the sheaf of pairs \( (A, q) \), with \( A \) an alternating form on \( Y \) and \( q \) a map of sheaves \( Y \to C^*_S \) such that

\[ q(y) + z) = (-1)^{|A|} \cdot q(z). \tag{3.5.1} \]

In 3.5 and below, “form” means integer-valued form.

Using 3.3, one sees that \( \tilde{p}_*K_2 \) is generated by \( X \otimes X \) and by \( X \otimes C^*_T \), with the relations \( x \otimes x = x \otimes \{-1\} \).

The following interpretations will be useful:

\( X \otimes C^*_S \): morphisms from \( Y \) to \( C^*_S \); \( x \otimes f \mapsto \text{morphism } f^{(x)} \).

\( X \otimes X \): the group of not necessarily symmetric bilinear forms \( F(y_1, y_2) \) on \( Y \):

\[ x_1 \otimes x_2 \mapsto \text{form } x_1(y_1)x_2(y_2) - x_2(y_1)x_1(y_2). \]

\( \Lambda X \): the group of alternating forms on \( Y \): \( x_1 \otimes x_2 \mapsto \text{form } x_1(y_1)x_2(y_2) - x_2(y_1)x_1(y_2) \).

\( \text{Sym}^2(X) \): the group of quadratic forms \( Q(y) \) on \( Y \): \( x_1 \otimes x_2 \mapsto \text{form } x_1(y_1)x_2(y_2) \).

We will also meet \( \Gamma^2X \), the component of degree 2 of the divided power algebra \( \Gamma X \). It is the target of the universal quadratic map with source \( X \): \( \gamma_2: X \to \Gamma^2X \). It is also the symmetric part of \( X \otimes X \), with \( \gamma_2(x) = x \otimes x \).

\textit{Lemma 3.5.2.} — The map \( R: X \to (X \otimes X) \oplus (X \otimes C^*_S) \): \( x \mapsto x \otimes x - x \otimes \{-1\} \) is quadratic, i.e. \( R(x_1 + x_2) - R(x_1) - R(x_2) \) is biadditive and \( R(nx) = n^2R(x) \).

\textit{Proof:} — The summand \( x \otimes x \) is clearly quadratic. For the summand \( x \otimes \{-1\} \), one observes that it takes values in elements of order 2, and that \( n^2 \equiv n (\text{mod } 2) \).
By 3.5.2, the map $x \mapsto x \otimes x + x \otimes \{-1\}$ factors through $\Gamma^g(X)$ and $\overline{p}_*K_2$ is the cokernel of

$$(3.5.3) \quad \Gamma^g(X) \to X \otimes X \oplus X \otimes \mathcal{O}_S^\times := \gamma_2(x) \mapsto x \otimes x - x \otimes \{-1\}.$$

To $C + F$ in $X \otimes X \oplus X \otimes \mathcal{O}_S^\times$, we attach $(A, q)$ defined as follows: with $C$ and $F$ interpreted as above,

$$A(y, z) := C(y, z) - C(z, y)$$
$$q(y) := F(y).(-1)^{C(y, z)}.$$

Condition (3.5.1) holds for this $(A, q)$. To check that this construction identifies the sheaf of pairs $(A, q)$ with the cokernel of (3.5.3), one only needs to stare at the diagram

$$
\begin{array}{cccccc}
0 & \to & \Gamma^g(X) & \to & X \otimes X & \to & \mathcal{O}_S^\times \to 0 \\
\mathbb{Z} & \to & \text{coker} \Gamma^g(X) & \to & \text{coker} X \otimes X & \to & \text{coker} \mathcal{O}_S^\times \\
0 & \to & X \otimes \mathcal{O}_S^\times & \to & X \otimes \mathcal{O}_S^\times & \to & 0
\end{array}
$$

and conclude that the middle row is a short exact sequence.

3.6. A section $s$ of $K_2$ on a group scheme $G$ over $S$ is \textit{multiplicative} if

$$\mu^*(s) = \text{pr}_1^*(s) + \text{pr}_2^*(s).$$

In the language of 1.4, this means that $s$ is a morphism of sheaves of groups, from $G$ to $K_2$.

For group schemes $G_1$ and $G_2$, a section $s$ of $K_2$ on $G_1 \times_S G_2$ is \textit{bimultiplicative} if it is multiplicative in $G_1$ as well as in $G_2$: for the group schemes $G_1 \times_S G_2$ over $G_1$ and $G_1 \times_S G_2$ over $G_1$.

\textbf{Corollary 3.7.} — (i) Under the assumptions of 3.1 and 3.3, the group of multiplicative sections of $K_2$ over $T$ is $H^0(S, X \otimes \mathcal{O}_S^\times)$.

(ii) For split tori $T_1$, $T_2$ over $S$, a section of $K_2$ over $T_1 \times_S T_2$ is bimultiplicative as soon as it has a trivial restriction to $T_1 \times_S e$ and $e \times_S T_2$. The group of bimultiplicative sections is $H^0(S, X_1 \otimes X_2)$. 

Proof. — The description 3.5 is functorial in $T$. It follows that multiplicative sections correspond to pairs $(A, q)$ with $A = 0$. By (3.5.1), $q$ is then an homomorphism from $Y$ to $\mathbb{C}_S^*$, i.e. a section of $X \otimes \mathbb{C}_S^*$.

Sections with trivial restrictions to $T_1 \times_S e$ and $e \times_S T_2$ correspond to pairs $(A, q)$ with trivial restrictions to $Y_1$ and $Y_2$: $A$ reduces to a pairing between $Y_1$ and $Y_2$ and $A$ determines $q$. The sheaf over $S$ of such sections is identified to $X_{/\mathcal{O}_S}$ by $x_1 \otimes x_2 \mapsto \{\text{pr}^*_1(x_1), \text{pr}^*_2(x_2)\}$, and they are bimultiplicative.

3.8. Remark (cf. S. Bloch (1978)). — On the big Zariski site of $S$ regular of finite type over a field $k$, 3.7 can be interpreted as

\begin{align}
&\text{(3.8.1)} \quad \text{Hom}(\mathbb{G}_m, K_2) = H^0(S, \mathbb{C}_S^*) \\
&\text{(3.8.2)} \quad \text{Hom}(\mathbb{G}_m \otimes \mathbb{Z}, \mathbb{G}_m, K_2) = \mathbb{Z}.
\end{align}

On the big Zariski site of $S$, taken in the smooth sense (the category of $T$ smooth over $S$), this can be repeated locally, giving

\begin{align}
&\text{(3.8.3)} \quad \check{\text{Hom}}(\mathbb{G}_m, K_2) = K_1 \\
&\text{(3.8.4)} \quad \check{\text{Hom}}(\mathbb{G}_m \otimes \mathbb{Z}, \mathbb{G}_m, K_2) = K_0.
\end{align}

3.9. If $S$ is the spectrum of a field, it results from 1.9 Example (vii) and from the vanishing of $\check{H}^i(T, K_2)$ for $i > 0$ that the category of central extension of $T$ by $K_2$ is incarnated in degree 2 by the complex $\check{H}^0(T^p, K_2)$. The cohomology of the complex $\check{H}^0(T^p, K_2)$ is computed in Esnault et al. (1998) 4.6:

\begin{align}
&\text{(3.9.1)} \quad H^n(\check{H}^0(T^p, K_2)) = \text{Sym}^n(X) \otimes K_{m-n}(k).
\end{align}

For $m = 2$, one deduces from (3.9.1) that the complex $\check{H}^0(T^p, K_2)$ has cohomology only in degree 1 and 2, with

\begin{align}
&\text{(3.9.2)} \quad H^1(\check{H}^0(T^p, K_2)) = X \otimes k^* \\
&H^2(\check{H}^0(T^p, K_2)) = \text{Sym}^2 X.
\end{align}

We will need to know how the isomorphisms (3.9.2) are obtained at the level of cocycles.

Let $L$ be the cosimplicial group dual to the simplicial group $BY$. The component of degree $n$ of $L$ is the dual $X^n$ of $Y^n$; it is the character group of the component $T^n$ of $BT$. The coface maps are the

$$ \partial_i : f(y_1, ..., y_n) \longmapsto f(y_1, ..., y_i + y_{i+1}, ..., y_{n+1}), $$

suitably modified for the extreme values 0 and $n + 1$ of $i$. The codegeneracies are

$$ s_i : f(y_1, ..., y_n) \longmapsto f(y_1, ..., y_{i-1}, 0, y_i, ..., y_{n-1}). $$

By 3.4, 3.5, the cosimplicial group $\check{H}^0(T^n, K_2)$ is an extension of $L \otimes k^*$. 
As a differential complex, $L$ is homotopic to the subcomplex $\cap \ker(s_i)$ of non-degenerate cochains. This subcomplex is reduced to $X$ in degree 1. The cohomology of $L \otimes k^*$ is given by the Künneth formula: it is concentrated in degree 1, with $H^1(L \otimes k^*) = X \otimes k^*$. The subcomplex of non-degenerate cochains of $\wedge L$ is concentrated in degree 1 and 2. In degree 1, it is $\wedge X$. In degree 2, it is $\ker(\wedge(X \otimes X) \to \wedge X \otimes \wedge X)$, identified with $X \otimes X$: with the identification of (3.5), a bilinear form $C \in X \otimes X$ on $Y$ corresponds to the alternating form $C(y_1, z_2) - C(z_1, y_2)$ on $Y \otimes Y$. The differential maps $A$ in $\wedge X$ to the alternating form $(A(y_1, z_2)) \mapsto -(A(y_1, z_2) - A(z_1, y_2))$ on $Y \otimes Y$, identified with $-A$ in $X \otimes X$. The cohomology group $H^2 = \text{coker}(\wedge X \to X \otimes X)$ is $\text{Sym}^2(X)$, with $C \mapsto$ quadratic form $C(y, y)$ on $Y$.

The formula (3.9.2) now follows from the long exact sequence of cohomology. A bilinear form $C$ on $Y$, identified with an element of $X \otimes X$, defines a 2-cocycle in the complex $H^0(T^*, K_2)$ by

$$x_1 \otimes x_2 \longmapsto \{\text{pr}_1^* x_1, \text{pr}_2^* x_2\}$$

and the corresponding cohomology class is the quadratic form $C(y, y)$ on $Y$.

3.10. Let $E$ be a multiplicative $K_2$-torsor on $T$. By 3.9, for some bilinear form $C$ on $Y$, $E$ is isomorphic to the trivial torsor, with the multiplicative structure given by the cocycle image of $C$ by (3.9.3). The group of isomorphism classes of multiplicative $K_2$-torsors is $\text{Sym}^2(X)$, and the class of $E$ is given by the quadratic form $C(y, y)$ on $Y$.

Let us extend the scalars from $k$ to $k[t, t^{-1}]$. By 1.4, we deduce from $E$ a central extension (1.4.1), on $\text{Spec} k[t, t^{-1}]$, of the sheaf of local sections of $T$ by $K_2$. As $H^1(\text{Spec} k[t, t^{-1}], K_2) = 0$ (3.1.2), taking global sections, we obtain a central extension

$$H^0(\text{Spec}(k[t, t^{-1}]), K_2) \to \mathcal{E}_0 \to T(k[t, t^{-1}]).$$

We take the pull back of this central extension by

$$Y \to T(k[t, t^{-1}]) = k[t, t^{-1}]^* \otimes Y: \ y \mapsto t \otimes y,$n

and the push out by

$$H^0(\text{Spec}(k[t, t^{-1}], K_2) =_{\text{3.1.2}} K_2(k) \oplus k^* \to k^*,$n

obtaining a central extension

$$k^* \to \mathcal{E} \to Y.$$

**Proposition 3.11.** — The construction 3.10 is an equivalence of categories from the category of multiplicative $K_2$-torsors on $T$ to the category of pairs $(Q, \mathcal{E})$, where $Q$ is a quadratic form on
$Y$ and $E$ is a central extension of $Y$ by $k^*$ for which the commutator map \(0.N.A\) \((2)\): \(Y \times Y \to k^*\), is given by

\[(3.11.1) \quad (y, z) = (-1)^{B(y, z)}.
\]

In \((3.11.1)\), $B$ is the bilinear form associated to $Q (0.N.3)$.

**Proof.** — We first check that \((3.11.1)\) holds for the central extension \((3.10.4)\). Let $C$ be a (non-necessarily symmetric) bilinear form on $Y$ such that $Q(y) = C(y, y)$. The multiplicative torsor $E$ is then isomorphic to the trivial torsor, with multiplicative structure given by the cocycle image of $C$ by \((3.9.3)\). The extension \((3.10.4)\) is then given by the cocycle

\[c(y, z) = C(y, z) \text{(projection to } k^* \text{ of } \{t, t\}.\)

Here, $\{t, t\}$ is the product in $K_2$. It equals $\{t, -1\}$, with projection $-1$ in $k^*$, and $c(y, z) = (-1)^{C(y, z)}$ from which \((3.11.1)\) follows. One has indeed

\[B(y, z) = C(y, z) + C(z, y) \equiv C(y, z) - C(z, y) \pmod 2.
\]

If $Q = 0$, a central extension for which \((3.11.1)\) holds is commutative. As it is an extension of the free abelian group $Y$ by $k^*$, it is trivial. It follows that the isomorphism class of $(Q, E)$ only depends on $Q$, and the construction 3.10 induces a bijection on isomorphism classes of objects.

It remains to see that it induces a bijection on automorphism groups. By 3.7 \((i)\), the automorphism group of a multiplicative torsor $E$ is $X \otimes k^* = \text{Hom}(Y, k^*)$, with $x \otimes f$ acting as the addition of $\{x, f\}$. The automorphism group of a central extension of $Y$ by $k^*$ is $\text{Hom}(Y, k^*)$ as well, and we leave it to the reader to check that the functoriality map is the identity.

**Remark 3.12.** — In 3.10, instead of extending scalars from $k$ to the ring $k[t, t^{-1}]$, we could have extended scalars to the field $k(\{t\})$, obtaining from $E$ instead of \((3.10.1)\) a central extension

\[K_2 k(\{t\}) \to E_1 \to T(k(\{t\}))
\]

We still have a morphism

\[(3.12.1) \quad Y \to T(k(\{t\})) = k(\{t\})^* \otimes Y : y \mapsto t \otimes y
\]

and the tame symbol defines

\[(3.12.2) \quad K_2 k(\{t\}) \to k^*.
\]

Pushing by \((3.12.2)\) and pulling back by \((3.12.1)\), we obtain a central extension of $Y$ by $k^*$, canonically isomorphic to the central extension \((3.10.4)\).
Proposition 3.13. — Let \( E \) be a central extension of \( T \) by \( K_2 \), that is a multiplicative \( K_2 \)-torsor on \( T \) (1.4). Let \( Q \) be the corresponding quadratic form on \( Y \), and let \( B \) be the associated bilinear form. The corresponding commutator \( \text{comm} \) of (1.8.1), in \( H^0(T \times T, K_2) \), is bimultiplicative. It is the image of \( B \), by the identification 3.7 (ii).

Proof. — Choose \( C \) in \( X \) whose image in \( H^0(T \times T, K_2) \) is a cocycle \( c \) defining \( E \). One has \( C(y, y) = Q(y) \). The commutator map (1.8.1) is \( (t_1, t_2) \mapsto c(t_1, t_2) - c(t_2, t_1) \). The product \( K_1 \times K_1 \to K_2 \) being anticommutative, the corresponding section of \( K_2 \) over \( T \times T \) is the image of \( C + tC \), where \( tC(y_1, y_2) = C(y_2, y_1) \). It remains to observe that \( B = C + tC \).

Corollary 3.14. — If \( B = \sum b_x x \otimes x^i \), the commutator for the central extension
\[
K_2(k) \to E(k) \to T(k)
\]
is given by \( (t_1, t_2) \mapsto \sum b_x \{x(t_1), x^i(t_2)\} \).

3.15. We now return to the case of a split torus \( T \) over \( S \) regular of finite type over a field. Let \( p \) be the projection of \( T^n \) to \( S \). Ambiguity on \( n \) will be avoided by writing \( p_*(T^n, K_2) \) for the direct image of \( K_2 \) from \( T^n \) to \( S \). Let \( \widetilde{p}_*(T^n, K_2) \) be the reduced direct image \( p_*(T^n, K_2)/K_2 \). Applying the variant 1.10 of 1.9 Example (vii), one deduces from the vanishing (3.1.2) that the complex of sheaves \( \widetilde{p}_*(T^n, K_2) \) on \( S \) incarnates in degree 2 the stack
\[
U \mapsto \text{category of multiplicative } K_2\text{-torsors on the split torus } T_U = p^{-1}(U) \text{ over } U.
\]
As in 3.9, the local classification of \( K_2 \)-torsors on \( T \) is by quadratic forms on \( Y \): locally on \( S \), a multiplicative \( K_2 \)-torsor \( E \) on \( T \) defines a quadratic form on \( Y \) and \( E' \), \( E'' \) are locally isomorphic if and only if the corresponding quadratic forms are equal. As in 3.10, a multiplicative \( K_2 \)-torsor on \( T \) gives rise to a central extension, on \( S \), of the constant sheaf \( Y \) by \( p_* K_2 \), for \( p : G_{m_S} \to S \), and from this we get a central extension of \( Y \) by \( C^*_S \). As in 3.11, one has

Theorem 3.16. — For \( T \) a split torus over \( S \) regular of finite type over a field \( k \), the construction 3.15 induces an equivalence of commutative Picard categories from

(i) the category of multiplicative \( K_2 \)-torsors on \( T \), to

(ii) the category of pairs \((Q, E)\): \( Q \) a quadratic form on the sheaf \( Y \) and \( E \) a central extension of \( Y \) by the sheaf \( C^*_S \), for which, as in 3.11,
\[
(y, z) = (-1)^{R_T, y} z.
\]

The commutator (1.8.1) defined by a multiplicative \( K_2 \)-torsor is given as in 3.13.
4. Split semi-simple simply-connected groups

For $G$ split semi-simple simply-connected over a field $k$, with split maximal torus $T$ and Weyl group $W$, we will prove in this section results which, via the formalism of 1.9, amounts to $H^i(G, K_2) = 0$ for $i \neq 1$, $H^2(BG, K_2) = 0$ for $i \neq 2$, $H^2(BG, K_2) \cong H^2(G, K_2)$ and $H^2(BG, K_2) \cong H^2(BT, K_2)^W$, the group of $W$-invariant quadratic forms on the dual character group $Y$ of $T$. We could have quoted those results from Esnault et al. (1996) (3.2, 4.7, 4.8). Our proofs are similar, but closer to the cocycle level and at places more elementary. In loc. cit. the computations use the projection $G \rightarrow G/T$, the fact that $G/T$ admits a stratification where the strata are affine spaces, and the fact that $G$ can be viewed as a bundle over $G/T$, with fibers vector spaces minus coordinates hyperplanes. This leads to the spectral sequence loc. cit. (3.7) which presumably agrees with the spectral sequence of our complex (4.3.2), filtered by (4.3.5). At least, the $E_1$-terms and the abutment are the same. Our method is to use the Schubert cell decomposition of $G/T$ (which proves the cellularity of $G/T$), and to use not the projection $G \rightarrow G/T$, but rather the decomposition of $G$ into the inverse images of the Schubert cells, i.e. the Bruhat decomposition of $G$. The reader may prefer to skip our proofs, obtain the relative version 4.8 from loc. cit. 3.22, look at 4.7 and 4.9 as categorical interpretations, via 1.9, and resume reading at 4.10.

4.1. Let $G$ be a split reductive group over a field $k$. We fix a split maximal torus $T$ of $G$. We will use the notation $X, Y, W$ of (0.N.1) and (0.N.2), as well as the following:

- $\Phi$: the set of roots (a subset of $X$). For $\alpha$ a root, we denote by $\alpha^\vee$ the corresponding coroot; $\alpha^\vee$ is in $Y$ and $\langle \alpha, \alpha^\vee \rangle = 2$;
- $U_\alpha$: the corresponding radical subgroup; it is isomorphic to $G_\alpha$ and $T$ acts on it by $\alpha$;
- $s_\alpha$: the corresponding reflection in $W$;
- $S_\alpha$: the corresponding $SL(2)$ or $PGL(2)$ subgroup of $G$, generated by $\alpha^\vee(G_\alpha)$, $U_\alpha$ and $U_{-\alpha}$.

The walls are the hyperplanes $\langle x, \alpha^\vee \rangle = 0$ of $X \otimes R$. For $w \in W$, we will write $\hat{w}$ for a lifting of $w$ in the normalizer $N(T)$ of $T$.

We fix a Borel subgroup $B$ containing $T$. We denote by

- $\Phi^+$: the corresponding set of positive roots: a root $\alpha$ is in $\Phi^+$ if and only if $U_\alpha$ is in the unipotent radical $U$ of $B$;
- $I$: the corresponding system of simple roots. A simple root $i \in I$ will often be written $\alpha_i$;
- $\ell(w)$: the length of $w \in W$, relative to the system of generators $s_i$ ($i \in I$);
- $\Phi^-$: the complement of $\Phi^+$ in $\Phi$. It is also $-\Phi^+$;
- $C_0$: the fundamental chamber in $X \otimes R$, defined by $\langle x, \alpha^\vee \rangle \geq 0$ for $\alpha$ positive.
For \( w \in W \), let \( U_{\Phi^{+} \cap w \Phi^{-}} \) be the group generated by the \( U_{\alpha} \) with \( \alpha \in \Phi^{+} \cap w \Phi^{-} \). As a scheme, it is isomorphic to \( A^{n} \) for some \( n \). Indeed, for any order on \( \Phi^{+} \cap w \Phi^{-} \), the product map is an isomorphism from \( \prod_{\alpha \in \Phi^{+} \cap w \Phi^{-}} U_{\alpha} \) to \( U_{\Phi^{+} \cap w \Phi^{-}} \).

The product map

\[(4.1.1)\quad U_{\Phi^{+} \cap w \Phi^{-}} \times w T \times U \to BwB\]

is an isomorphism. The map

\[(4.1.2)\quad p: BwB \to w T: \quad p = pr_{2} \circ (4.1.1)^{-1}\]

is the unique retraction of \( BwB \) to \( w T \) such that \( p(u\dot{w}v) = iw t \) for \( u \) and \( v \) in \( U \), \( t \) in \( T \) and \( \dot{w} \) any lifting of \( w \) in \( N(T) \). By 3.1,

\[(4.1.3)\quad H^{i}(BwB, K) = 0 \quad \text{for} \quad i > 0\]

and \( p: BwB \to w T \) induces an isomorphism

\[(4.1.4)\quad H^{0}(w T, K) \sim H^{0}(BwB, K).\]

The length \( \ell(w) \) of \( w \) is \( |\Phi^{+} \cap w \Phi^{-}| \): it is the number of walls separating the fundamental chamber \( C_{0} \) from its transform \( w(C_{0}) \). By (4.1.1), one has

\[(4.1.5)\quad \dim BwB = \ell(w) + \dim B.\]

The Bruhat decomposition \( G = \bigcup BwB \) is a stratification of \( G \): the \( BwB \) are disjoint and the closure of a stratum \( BwB \) is the union of the strata \( Bw_{1}B \) with \( w_{1} \leq w \) in Bruhat’s order. The \( Bw_{1}B \) of codimension one in the closure of \( BwB \) are obtained as follows: for some simple root \( i \),

\[(4.1.6)\quad w = w' s_{i} w'' \quad \text{with} \quad \ell(w) = \ell(w') + 1 + \ell(w''), \quad \text{and} \quad w_{1} = w'w'' \quad \text{with} \quad \ell(w_{1}) = \ell(w') + \ell(w'') = \ell(w) - 1.\]

If (4.1.6) holds, then

\[(4.1.7)\quad BwB \cup Bw_{1}B = Bw'S_{i}w''B\]

is a smooth subscheme of \( G \).

Fix a lifting \( \dot{w} \) of \( w \) in \( N(T) \). Each character \( x \) of \( T \) gives rise to the function \( x(\dot{w}^{-1}g) \) on \( w T \). Pulling back by (4.1.2), we obtain an invertible function \( x_{\dot{w}} \) on \( BwB \). Let us extend \( x \) to a character of \( B \) by \( x(tu) = x(t) \) for \( t \) in \( T \) and \( u \) in \( U \). The invertible function \( x_{\dot{w}} \) is characterized by the equivariance condition

\[(4.1.8)\quad x_{\dot{w}}(b'gb'') = [x(\dot{w})](b')x_{\dot{w}}(g)s(b'')\]
and the normalization \(x_\bar{w}(\bar{w}) = 1\). Changing the lifting \(\bar{w}\) of \(w\) changes \(x_\bar{w}\) by a multiplicative constant.

If \(w\) and \(w_1\) are as in (4.1.6), \(B_0B\) is a smooth divisor in the closure of \(B_0B\) and \(x_\bar{w}\) has a valuation along \(B_0B\). If the valuation is zero, \(x_\bar{w}\) extends to an invertible function on \(B_0B \cup B_0B\), which satisfies the equivariance condition (4.1.8).

**Lemma 4.2.** (i) With the notation above, the valuation of \(x_\bar{w}\) along \(B_0B\) is given by

\[
\nu(x_\bar{w}) = \left\langle x, w''(\alpha_i) \right\rangle.
\]

(ii) If \(\nu(x_\bar{w}) = 0\), let \(x_\bar{w}|B_0B\) be the restriction to \(B_0B\) of the extension of \(x_\bar{w}\) to \(B_0B \cup B_0B\). Then, \(x_\bar{w}|B_0B\) is a constant multiple of \(x_{\bar{w}_1}\).

(iii) If further \(w = \bar{w}'s_i\bar{w}''\) with \(s_i \in S\), and \(\bar{w}_1 = \bar{w}'\bar{w}''\), then

\[
x_\bar{w}|B_0B = x_{\bar{w}_1}.
\]

For the proof (by reduction to the case of SL(2)) we refer the reader to Demazure (1974).

Writing \(w_1 = \bar{w}(w''^{-1}s_i\bar{w}'')\), one can rewrite (4.2.1) as follows: \(w_1 = \bar{w}s\), for \(s\) a reflection in \(W\), and if \(\alpha\) is the corresponding positive root, then

\[
\nu(x_\bar{w}) = \left\langle x, \alpha^* \right\rangle.
\]

4.3. The Bruhat decomposition induces a filtration \(F\) on any sheaf \(\mathcal{F}\) on \(G\), with \(F^p\mathcal{F}\) the sheaf of sections with support in the union of the \(B_\omega B\) of codimension \(\geq p\). For the Quillen complex \(G^*_j\), if \(W^p\) is the set of \(\omega \in W\) for which \(B_\omega B\) is of codimension \(p\), we get

\[
Gr^*_p\Gamma(G, G^*_j) = \bigoplus_{\omega \in W^p} \Gamma(B_\omega B, G^*_j)[-p].
\]

The Quillen complexes in the second member of (4.3.1) are the Quillen complexes of the \(B_\omega B\). As \(F^pG^*_j\) vanishes in degree \(n < p\), \(H^pGr^*_p\Gamma(G, G^*_j)\) maps to \(Gr^*_p\Gamma(G, G^*_j) = F^p\Gamma(G, G^*_j) \subset \Gamma(G, G^*_j)\); the \(H^pGr^*_p\Gamma(G, G^*_j)\) form in this way a subcomplex of \(\Gamma(G, G^*_j)\).

By (4.1.3), \(\Gamma(B_\omega B, G^*_{j-p})\) is a resolution of \(H^0(B_\omega B, K^1_{j-p})\). It follows that the inclusion in \(\Gamma(G, G^*_j)\) of the subcomplex

\[
H^pGr^*_p\Gamma(G, G^*_j) = \bigoplus_{\omega \in W^p} H^0(B_\omega B, K^1_{j-p})
\]

is a quasi-isomorphism. If \(\eta(\omega)\) is the generic point of \(B_\omega B\), \(H^0(B_\omega B, K^1_{j-p})\) maps to \(\Gamma(G, G^*_j)\) by the restriction map to \(K^1_{j-p}(k(\eta(\omega)))\).
The only nonzero components of the differential of the subcomplex (4.3.2) of $\Gamma(G, G_j)$ are morphisms

\[(4.3.3) \quad d^w_w : H^0(B_w B, K_{j-p}) \to H^0(B_w B, K_{j-p-1})\]

for $w$ and $w_1$ as in (4.1.6). The local ring of $B_w B \cup B_{w_1} B$ at $\eta(w_1)$ is a valuation ring $V(w_1, w)$ with special point $\eta(w_1)$ and generic point $\eta(w)$. The differential $d^w_w$ is induced by the corresponding residue map for $K$-groups:

\[(4.3.4) \quad H^0(B_w B, K_{j-p}) \xrightarrow{d_w^w} H^0(B_w B, K_{j-p-1}) \xrightarrow{\text{Res}} K_{j-p-1}(k(\eta(w_1))).\]

The commutative diagram (4.3.4) and the following facts suffice to compute $d_w^w$:

(i) the residue $K_1(k(\eta(w_1)))! \to K_0(k(\eta(w_1)))! \to Z$ is the valuation;

(ii) the residue map is right linear over $K_{j-p}(V(w_1, w))$.

Indeed, by (4.1.4), any element of $H^0(B_w B, K_j)$ is a sum of $K$-theory products $f_1 \cdots f_p \cdot \alpha$, where $\alpha$ is in $K_{j-\ell}(k)$, the $f_i$ are invertible functions of the form $x_0$ on $B_w B$ and where, except for $f_1$, each $f_i$ has valuation zero along $B_{w_1} B$, hence extends to $B_w B \cup B_{w_1} B$. We transport to $H^0(B_w B, K_{j-p})$, by the isomorphism (4.1.4), the increasing filtration $V$ of $H^0(V, K_{j-p})$ defined in 3.4. Using (i) (ii), one checks that $d^w_w$ maps $V_n$ to $V_{n-1}$. We filter the complex (4.3.2) by the shifted filtration:

\[(4.3.5) \quad \text{in degree } p, \text{ (V shifted)}: V_n = V_{n-p}.\]

By (3.4.1),

\[(4.3.6) \quad \text{Gr}^{' \text{V shifted}}_n((4.3.2)) = \bigoplus_{w \in W^p} n \wedge X \otimes K_{j-n}(k).\]

For $w$ and $w_1$ as in (4.1.6), the component $\text{Gr}^{' \text{V shifted}}_n(d^w_w)$ of the differential is given as follows: if $x_1 \wedge \ldots \wedge x_{j-p} \otimes \alpha$ in $n \wedge X \otimes K_{j-n}(k)$ is such that for $i \geq 2$ the $x_i$ on $B_w B$ have valuation zero along $B_{w_1} B$, then it is mapped to $\langle x_1, u^{p-1}(\alpha^\nu) \rangle \cdot x_2 \wedge \ldots \wedge x_{j-p} \otimes \alpha$. In other words: one contracts with the linear form $\langle , u^{p-1}(\alpha^\nu) \rangle$ on $X$.

**4.4.** For $j = 2$, the $\text{Gr}^{' \text{V shifted}}_n((4.3.2))$ are

\[(4.4.1) \quad n = 0 : \quad K_2(k);\]

\[n = 1 : \quad X \otimes k^* \to \bigoplus_{u \in W^1} k^*;\]

\[n = 2 : \quad 2 \wedge X \to \bigoplus_{u \in W^2} X \to \bigoplus_{u \in W^2} Z.\]
As $W^0$ is reduced to the longest element $w_0$ of $W$, we dispensed with indicating a sum over $W^0$. The set $W^1$ consists of the $w_0 s_i$, and $W^2$ consists of the $w_0 s_i s_j$ for $i \neq j$. On $W^0 \cup W^1 \cup W^2$, the Bruhat order is generated by the inequalities $w_0 s_i \leq w_0$ and $w_0 s_i s_j \leq w_0 s_i$, $w_0 s_j$ for $i \neq j$.

The maps $d_{w_0 i}^0$ and $d_{w_0 j}^0$ are contraction with $\alpha_i^\vee$. The map $d_{w_0 j}^0$ is contraction with $s_j(\alpha_i^\vee)$ (cf. (4.2.3)).

Let $Y_{sc} \subset Y$ be generated by the coroots. It is a free $\mathbb{Z}$-module with basis the simple coroots $\alpha_i^\vee$. Let $X_{sc}$ be the dual of $Y_{sc}$. We identify $X_{sc}$ with $\bigoplus \mathbb{Z}$ by $x \mapsto (x(\alpha_i^\vee)$ at $w_0 s_j)$. With this identification, the complex (4.4.1)$_n=1$ becomes

$$(4.4.2) \quad X \otimes k^* \to X_{sc} \otimes k^*,$$

and can be identified with the restriction map from $\text{Hom}(Y, k^*)$ to $\text{Hom}(Y_{sc}, k^*)$. The complex (4.4.1)$_n=2$ becomes

$$(4.4.3) \quad \bigwedge X \to X_{sc} \otimes X \to \bigoplus \mathbb{Z}.$$

The first map is restriction, from alternating forms on $Y$ to bilinear forms pairing $Y_{sc} \subset Y$ and $Y$. The set $W^2$ is the set of $w_0 s_i s_j$ for $i$ and $j$ distinct simple roots, and the only equalities among the $w_0 s_i s_j$ is that $w_0 s_i s_j = w_0 s_j s_i$ if the simple roots $i$ and $j$ are orthogonal. The $w_0 s_i s_j$ component of second map in (4.4.3) is

$$(4.4.4) \quad C \mapsto C(\alpha_i^\vee, \alpha_j^\vee) + C(\alpha_i^\vee, s_j(\alpha_j^\vee)).$$

The first term comes from $d_{w_0 i}^0$, the second from $d_{w_0 j}^0$. One can express the right side of (4.4.4) in terms of the quadratic form $Q(y) := C(y, y)$ on $Y_{sc}$ and its associated bilinear form $B$:

$$(4.4.5) \quad C(\alpha_i^\vee, \alpha_j^\vee) + C(\alpha_i^\vee, s_j(\alpha_j^\vee)) = C(\alpha_i^\vee, \alpha_j^\vee) + C(\alpha_i^\vee, \alpha_j^\vee - \alpha_j(\alpha_j^\vee)\alpha_i^\vee)
\quad = B(\alpha_i^\vee, \alpha_j^\vee) - \alpha_j(\alpha_j^\vee)Q(\alpha_i^\vee).$$

Lemma 4.5. — $C$ is in the kernel of the second map (4.4.3) if and only if the quadratic form $Q(y) = C(y, y)$ on $Y_{sc}$ is $W$-invariant.

Proof. — A quadratic form $Q$, with associated bilinear form $B$, is invariant by $s_j$ if and only if

$$Q(s_j(y)) - Q(y) = Q(y - \alpha_j(y)\alpha_j^\vee) - Q(y) = B(y, -\alpha_j(y)\alpha_j^\vee) + Q(\alpha_j(y)\alpha_j^\vee)$$
$$= -\alpha_j(\alpha_j(y)\alpha_j^\vee) - \alpha_j(y)Q(\alpha_j^\vee))$$

vanishes identically in $y$. As the linear form $\alpha_j(y)$ is not identically zero, this holds if and only if

$$(4.5.1) \quad B(y, \alpha_j^\vee) - \alpha_j(y)Q(\alpha_j^\vee)$$

is in $C(\alpha_i^\vee, s_j(\alpha_j^\vee))$.
vanishes identically in \( y \). As (4.5.1) is linear in \( y \) and vanishes for \( y = \alpha_j \), \( Q \) is invariant by \( s_j \) if and only if (4.4.5) vanishes for all \( i \neq j \) and \( Q \) is \( W \)-invariant if and only if it is in the kernel of (4.4.3).

Until the end of this section, we assume that \( G \) is semi-simple simply-connected, i.e. that \( Y = Y_{\text{sc}} \).

**Proposition 4.6.** — If \( G \) is simply-connected the complex (4.4.1)\(_{n=1} \) is acyclic, and (4.4.1)\(_{n=2} \) has cohomology only in degree 1. In the model (4.4.3), the morphism \( C \mapsto C(y, y) \) identifies the \( H^1 \) of (4.4.1)\(_{n=2} \) with the group of \( W \)-invariant quadratic forms on \( Y \).

As a consequence,
\[
\begin{align*}
(4.6.1) & \quad H^0(G, K_2) = K_2(k), \text{ hence } \tilde{H}^0(G, K_2) = 0; \\
(4.6.2) & \quad H^1(G, K_2) = \{ \text{\( W \)-invariant quadratic forms on } Y \}; \\
(4.6.3) & \quad H^i(G, K_2) = 0 \text{ for } i \geq 2.
\end{align*}
\]

**Proof.** — Acyclicity of (4.4.1)\(_{n=1} \) is clear from the description (4.4.2). For (4.4.1)\(_{n=2} \), we use the description (4.4.3), with \( X_{\text{sc}} = X \). The cokernel of the map \( \wedge^2 X \to X \otimes X \) is the group \( \text{Sym}^2(X) \) of quadratic forms on \( Y = Y_{\text{sc}} \). It hence follows from 4.5 that the \( H^1 \) of (4.4.3) is the group of \( W \)-invariant quadratic forms on \( Y \). It remains to check the surjectivity of the map
\[
(4.6.4) \quad \text{Sym}^2(X) \to \bigoplus_{w \in W} \mathbb{Z}: Q \mapsto (B(\alpha_i^\vee, \alpha_j^\vee) - \alpha_j^w(\alpha_i^\vee)Q(\alpha_j^\vee)) \text{ at } w_0s_isj.
\]

The surjectivity of (4.6.4) amounts to the vanishing of \( H^2(G, K_2) \), the Chow group \( \text{CH}^2(G) \). In Esnault et al. (1998) 3.20, this is deduced from \( K_0(G) = 0 \), proved by Levine (1993) using the known topological \( K \) groups of the corresponding complex group. We give below an elementary verification. The result will not be needed in the rest of this article.

The Dynkin diagram of \( G \) is the graph \( I \), with \( i \) and \( j \) joined by an edge if \( i \) and \( j \) are not orthogonal, plus additional information. To the decomposition of \( I \) into connected components corresponds a decomposition of \( (Y, W) \), and the invariant quadratic forms are the sums of invariant quadratic forms on the summands. On each summand, there is only one up to a factor. For each component \( I^j \), either

(a) They are two lengths of roots and \( I^j \) is linear, with the short roots on one side and the long roots on the other. We order \( I^j \) linearly, starting from a long root.

(b) All roots have the same length and \( I^j \) is a tree. We order \( I^j \) in such a way that each \( i \in I^j \), except the first, is joined to a unique previous root.

We will also use

(c) If simple roots \( \alpha \) and \( \beta \) are not orthogonal and \( \text{length } (\alpha) \geq \text{length } (\beta) \), then \( \beta(\alpha^\vee) = -1 \).
This can be checked case by case in rank 2, or one may observe that for any W-invariant positive inner product \((\alpha, \alpha)\), the negative integer \(\beta(\alpha) = \frac{2(B(\alpha), \alpha)}{(\alpha, \alpha)}\) is \(< 2\) in absolute value.

We identify \(W(2)Z\) with the systems of integers \((d_{ij}) (i, j) \in I\) for which \(d_{ij} = d_{ji}\) whenever \(i \leq j\), that is, \(s_{ij} = s_{ji}\). We will show that for any such system \((d_{ij})\), and any family of integers \(d_{\alpha}\), there is a unique \(Q\) in \(\text{Sym}^2(X)\) for which \(B(\alpha, \alpha) = d_{ij}\) and \(Q(\alpha_1, \lambda) = d_{\lambda}\), for \(\alpha_1, \lambda\) the first element of \(I\):

\[
\begin{align*}
B(\alpha, \alpha) - \alpha_i(\alpha)Q(\alpha) &= d_{ij} \\
Q(\alpha_1, \lambda) &= d_{\lambda}, \text{ for } \alpha_1, \lambda 
\end{align*}
\]

We take as coordinates for \(Q\) the integers \(Q(\alpha_1, \lambda)\) and \(B(\alpha, \alpha) = B(\alpha, \alpha)\) for which \(i \leq j\). If \(i\) is orthogonal to \(j\), the first equation reduces to \(B(\alpha, \alpha) = d_{ij}\). We now consider the remaining equations, as well as the remaining \(B(\alpha, \alpha)\), and the \(Q(\alpha)\). The problem breaks into similar problems, one for each connected component \(I\). On \(I\), ordered as \(\alpha_1, ..., \alpha_\ell\), the second equation tells us to start with \(Q(\alpha_1, \lambda) = d_{\lambda}\).

For each \(j > 2\), \(\alpha_i\) is not orthogonal to \(\alpha_j\) for a unique \(i < j\), and by (c) \(\alpha_i(\alpha) = -1\). The equation

\[
B(\alpha, \alpha) - \alpha_i(\alpha)Q(\alpha) = d_{ij}
\]

determines \(B(\alpha, \alpha)\) from the previous \(Q(\alpha)\), while the equation

\[
B(\alpha, \alpha) - \alpha_i(\alpha)Q(\alpha) = d_{ij}
\]

determines \(Q(\alpha)\) from \(B(\alpha, \alpha)\). This proves inductively existence and unicity of the solution: (4.6.4) is surjective, and there is a unique W-invariant quadratic form with prescribed \(Q(H_\lambda)\).

**Theorem 4.7.** — For \(G\) split and simply-connected, pointed \(K_2\)-torsors on \(G\) have no nontrivial automorphism. They admit a unique multiplicative structure. They are classified by W-invariant quadratic forms on \(Y\).

**Proof.** — The first statement amounts to (4.6.1). The last statement expresses (4.6.2).

A cartesian power \(G^\circ\) of \(G\) is again split and simply-connected, with maximal split torus \(T^\circ\) and Borel subgroup \(B^\circ\). The Bruhat decompositions of \(G^\circ\) is the product of the Bruhat decomposition of the factors \(G\) and it follows that the classification 4.6 of (pointed) torsors by quadratic forms is functorial for the projections from \(G^\circ\) to \(G\).

As a consequence, the morphism

\[
\langle pr^* \rangle: H^1(G, K_2^\circ) \to H^1(G^\circ, K_2)
\]
is an isomorphism. If \( \text{inj}_i \) is the embedding in \( G^n \) of the \( i \)th factor \( G \), the morphism

\[
(\text{inj}_i^*): H^1(G^n, K_2) \to H^1(G, K_2)^n
\]

is a retraction of 4.7.1, hence an inverse to (4.7.1): \( K_2 \)-torsors on \( G^n \) are determined by their restrictions to each factor \( G \). If \( P \) is a pointed torsor on \( G \), it follows that \( \mu^*P \) is isomorphic to \( \text{pr}_1^*P + \text{pr}_2^*P \). The unique isomorphism of pointed torsors

\[
\text{pr}_1^*P + \text{pr}_2^*P \to \mu^*P
\]

is a multiplicative structure: one has to check the equality of two isomorphisms of pointed \( K_2 \)-torsors on \( G \times G \times G \), and such torsors have no nontrivial automorphisms.

4.8. For \( S \) regular of finite type over a field and \( G \) a split reductive group over \( S \), with split maximal torus \( T \) and Borel subgroup \( B \) containing \( T \), the arguments used in this section can be repeated, when sheafified over \( S \). If \( a \) (resp. \( a_0 \)) is the projection of \( G \) (resp. \( BxB \)) to \( S \), the complex \( a_*G_j^* \) is quasi-isomorphic to the subcomplex

\[
\bigoplus_{w \in W^0} a_{w_*}K_j^* - p
\]

which, for \( j = 2 \), can be filtered with associated graded \( \text{Gr}_n \) the following complexes of sheaves on \( S \)

\[
\begin{align*}
    n = 0 : & \quad K_2^*; \\
    n = 1 : & \quad X \otimes C_S^* \to \bigoplus_{w \in W^{(1)}} C_S^*; \\
    n = 2 : & \quad \wedge X \to \bigoplus_{w \in W^{(1)}} X \to \bigoplus_{w \in W^{(2)}} Z.
\end{align*}
\]

For \( G \) simply-connected the proof of 4.6 gives

\[
\begin{align*}
    (4.8.3) & \quad a_*K_2 = K_2, \text{ hence } \tilde{a}_*K_2 = 0 \text{ for the reduced direct image; }
    \\
    (4.8.4) & \quad R^1a_*K_2 = \text{the locally constant sheaf of } W\text{-invariant forms on } Y; \\
    (4.8.5) & \quad R^ia_*K_2 = 0 \text{ for } i \geq 2.
\end{align*}
\]

As in 4.7, it follows that pointed \( K_2 \)-torsors on \( G \) have no nontrivial automorphisms, have a unique multiplicative structure, and are classified by (4.8.4).

If \( (T', B') \) is another choice of a split maximal torus and of a Borel subgroup containing it, locally on \( S \), \( (T, B) \) and \( (T', B') \) are conjugate by some inner automorphism \( \text{int}(g) \) and the isomorphism \( T \to T' \) induced by \( \text{int}(g) \) does not depend on \( g \). The dual character groups \( Y \) and \( Y' \) (they are rather locally constant sheaves on \( S \)) are hence canonically isomorphic.

**Lemma 4.8.6.** — The quadratic forms on \( Y \) and \( Y' \) attached to a multiplicative \( K_2 \)-torsor on \( G \) correspond to each other by the isomorphism of \( Y \) with \( Y' \).
Proof. — The question is local on \( S \), so that we may assume that \((T, B)\) and \((T', B')\) are conjugate by some \( \text{int}(g), g \in G(S) \). An inner automorphism transforms a multiplicative torsor into an isomorphic one (1.6). The claim follows by transport of structures.

We now return to the case of a ground field \( k \) and to the notation 4.1.

Compatibility 4.9. — Assume that \( G \) is simply-connected. Let \( E \) be a multiplicative \( K_2 \)-torsor on \( G \), and let \( E_T \) be its restriction to \( T \). The quadratic form 3.11 describing the isomorphism class of the multiplicative \( K_2 \)-torsor \( E_T \) on \( T \) is the same as the quadratic form 4.7.

Proof. — We will use the interpretation 1.4 of multiplicative \( K_2 \)-torsors as central extensions of \( G \) by \( K_2 \), viewed as sheaves on a big Zariski site \( \text{Spec}(k)_{\text{Zar}} \). As \( G \) is smooth over \( k \), we may and shall take as big Zariski site the category of schemes smooth over \( k \), with the Zariski topology. We denote by \( E_T \) the extension of \( T \) by \( K_2 \) induced by the extension \( E \) of \( G \) by \( K_2 \).

For \( S \) in \( \text{Spec}(k)_{\text{Zar}} \), the product law of \( E(S) \) induces a product map

\[
E(S) \times E_T(S) \to E(S); \quad (\tilde{g}, \tilde{t}) \mapsto \tilde{g} \tilde{t}
\]

and the right multiplication by \( \tilde{t} \) is an automorphism of \((G, E)\) with \( E \) viewed as a torsor over \( G \). More precisely, the image \( t \) of \( \tilde{t} \) in \( T(S) \) acts by a right translation \( r_t \) on \( G_S = G \times S \), and the right multiplication \( r_t \) by \( \tilde{t} \) lifts this action of \( t \) on \( G_S \) to an automorphism of \((G_S, E_S)\). By (4.8.3), any other lifting differs from \( r_t \) by an element of \( H^0(S, K_2) \), hence is also of the form \( r_{\tilde{t}} \) for \( \tilde{t} \) in \( E_T(S) \) projecting to \( t \) in \( T(S) \). This construction identifies \( E_T(S) \) with the set of pairs \( (t, \tau) \), with \( t \) in \( T(S) \) and \( \tau \) a lifting of \( r_t \). The lifting \( \tau \) is an isomorphism \( E \to r_{\tilde{t}} E \), and the product in \( E_T \) becomes

\[
(l', \tau')(l'', \tau'') = (ll'', r_{\tilde{t}}(\tau'') \circ \tau').
\]

The complex (4.3.2) incarnates, for \( j = 2 \), the category of \( K_2 \)-torsors on \( G \). Suppose that \( E \) is isomorphic to the \( K_2 \)-torsor defined by the cocycle \((f_i)\), where \( f_i \) is an invertible function on \( B_{\text{et}, S} B \). It is the torsor whose sections on \( V \subset G \) are the sections of \( K_2 \) on \( B_{\text{et}, B} \cap V \), with residue \( f_i \) along \( B_{\text{et}, S} B \cap V \). Let \( f_i \) be the character of \( T \) such that \( f_i(gf) = f_i(g)f_i(t) \).

Fix \( S \), and \( t \in T(S) \). We will now tacitly work over \( S \). The pullback torsor \( r_{\tilde{t}}(E) \) is given by the cocycle \((r_{\tilde{t}}(f_i)) = (f_i(gf_i(t)) \) in the complex (4.8.1). The \( K_2 \)-torsors \( E \) and \( r_{\tilde{t}}(E) \) are both trivialized over \( B_{\text{et}, B} \). Suppose that \( \sigma \) is a section of \( K_2 \) over \( B_{\text{et}, B} \) with residue \( f_i(t) \) along \( B_{\text{et}, S} B \). Note that \( f_i(t) \) is a function on \( S \), and that we continued to write \( f_i(t) \) for its pullback to \( B_{\text{et}, S} B \). Then, addition of \( \sigma \) defines an isomorphism of
\textbf{K}_2\text{-torsors } r : E \to \tau_\ell^*(E). To construct \( \sigma \), we start by choosing liftings \( \tilde{w}_i \) of \( w_0 \) in \( N(T) \). By abuse of notation, for \( x \) in \( X \), let us write \( x_{\tilde{w}_i} \) for \( x_{\tilde{w}_i} \). If \( x^i \) is the basis of \( X \) dual to the basis \( \alpha_i^\vee \) of \( Y \), with the notations of 4.1, we may take

\[
\sigma = \sum x_{\tilde{w}_i}^i \cdot \mathcal{J}_i(\ell).
\]

This \( \sigma \) depends on \( t \), functorially in \( S \), and \( t \mapsto \sigma(t) \) defines a section \( t \mapsto (t, \tau(t)) \) of \( E_T \) over \( T \). The central extension \( E_T \) by \( \text{K}_2 \) is given by the cocycle \( \epsilon(\ell', \ell') \) such that

\[
(x_{\tilde{w}_i}^i, \mathcal{J}_i(\ell)) = (x_{\tilde{w}_i}^i, \mathcal{J}_i(\ell) + \epsilon(\ell', \ell')).
\]

And as \( r_\tau(\tau(\ell)) \circ \tau(\ell) \) is, on \( \text{B}_{\tau_\ell} \), the addition of

\[
x_{\tilde{w}_i}^i \cdot \mathcal{J}_i(\ell') + x_{\tilde{w}_i}^i \cdot \mathcal{J}_i(\ell') = x_{\tilde{w}_i}^i \cdot \mathcal{J}_i(\ell') + x_{\tilde{w}_i}^i \cdot \mathcal{J}_i(\ell'),
\]

the cocycle \( \epsilon(\ell', \ell') \) is \( \sum x_{\tilde{w}_i}^i \cdot \mathcal{J}_i(\ell') \). It is defined (3.9.3) by the bilinear form \( C = \sum \alpha_i^\vee \cdot \mathcal{J}_i \) in \( X \otimes X \). As a bilinear form on \( Y \), it is characterized by \( C(\alpha_i^\vee, y) = \mathcal{J}_i(y) \), and the quadratic form \( C(y, y) \) of 3.9 coincides with the quadratic form of 4.7.

\textbf{4.10.} Let \( G_{\text{ad}} \) be the adjoint group of \( G \) and \( T_{\text{ad}} \) be the maximal split torus of \( G_{\text{ad}} \) image of \( T \). The group \( G_{\text{ad}} \) acts on \( G \) (adjoint action). Multiplicative \( \text{K}_2\text{-torsors} \) on \( G \) have no nontrivial automorphism and their classification is discrete. It follows that if \( E \) is a multiplicative \( \text{K}_2\text{-torsor} \) on \( G \), the action of \( G_{\text{ad}} \) on \( G \) extends uniquely to an action of \( G_{\text{ad}} \) on \( (G, E) \). Let us make this argument explicit. If \( G_S \) is deduced from \( G \) by the base change \( S \to \text{Spec}(\mathbb{k}) \), any \( g \in G_{\text{ad}}(S) \) defines an automorphism of \( G_S \). We take \( S = G_{\text{ad}} \) and for \( g \) the identity map of \( G_{\text{ad}} \). Let \( G(E) \) be the transform of \( E \) by \( g \); a multiplicative \( \text{K}_2\text{-torsor} \) on \( G_{\text{ad}} \). It is classified by a \( W \)-invariant quadratic form on the constant sheaf \( Y \) on \( G_{\text{ad}} \). As \( G_{\text{ad}} \) is connected, this form is constant and \( g(E) \) is isomorphic to (the pullback to \( G_{\text{ad}} \) of) \( E \). The isomorphism is unique by 4.3. It is the sought-for action.

\textbf{4.11.} We pass to the language 1.4 of central extensions by \( \text{K}_2 \). As \( T_{\text{ad}} \) fixes \( T \), the commutator section (1.8.2) is a bimultiplicative section

\[
\text{comm} \in H^0(T_{\text{ad}} \times T, \text{K}_2).
\]

By 3.7 (ii), it corresponds to an element of \( X_{\text{ad}} \otimes X \), or equivalently to a bilinear pairing between \( X_{\text{ad}} \) and \( Y \).

The pullback of (4.11.1) to \( T \times T \) is induced by the commutator section (1.8.1). It can be computed using 4.9 and 3.14. This fixes (4.11.1) uniquely:
Proposition 4.12. — Suppose that $E$ corresponds by 4.7 to the $W$-invariant quadratic form $Q$ on $Y$. Then, the associated bilinear form $B$ extends to a pairing $B_1$ between $Y_{\text{ad}}$ and $Y$ and, with the identifications of 3.7 (ii), the bimultiplicative section (4.11.1) of $K_2$ corresponds to $B_1$.

4.13. Fix $E$ as in 4.10 to 4.12. Taking $k$-points, we deduce from $E$ a central extension

\[(4.13.1) \quad K_2(k) \to E(k) \to G(k)\]

as in (1.4). The action 4.10 of $G_{\text{ad}}$ on $E$ induces an action of $G_{\text{ad}}(k)$ on this central extension. By restriction to $T_{\text{ad}}(k)$ and $T(k)$, the corresponding “commutator” (0.1.4) (3): $G_{\text{ad}}(k) \times G(k) \to E(k)$ induces a bimultiplicative map

\[(4.13.2) \quad T_{\text{ad}}(k) \times T(k) \to K_2(k).\]

We deduce from 4.12 that


\[(Y_{\text{ad}} \otimes k') \times (Y \otimes k') \to K_2(k)\]

is given by $(y_1 \otimes a, y \otimes b) \mapsto B_1(y_1, y)\{a, b\}$.

Proposition 4.15. — Suppose that $G$ is simple and that $Q$ takes the value 1 on short coroots. Then the central extension (4.13.1) is Matsumoto’s central extension (0.2).

Proof. — The group $G(k)$ is its own commutator subgroup (except for small finite fields $k$, for which $K_2(k)$ is trivial anyway). It follows that a central extension $E$ of $G(k)$ by an abelian group $A$ has no nontrivial automorphism (trivial on $G(k)$ and $A$). Suppose that the action of $G_{\text{ad}}(k)$ on $G(k)$ extends to $E$, the action being trivial on $A$. As explained in Deligne (1996) 3.8, the central extension $E$ is then determined up to unique isomorphism by the “commutator” pairing

\[T_{\text{ad}}(k) \times T(k) \to A.\]

We leave it to the reader to compare 4.14 with the “commutator” for Matsumoto’s extension.

4.16. Remark. — We keep the notations of 4.1, with $G$ simply-connected. The choice of a Borel subgroup $B$ is not needed.

By 4.7, a $W$-invariant quadratic form $Q$ on $Y$ defines a canonical multiplicative $K_2$-torsor $E$ on $G$. By restriction to $T$, we obtain a multiplicative $K_2$-torsor $E_T$ on $T$. 

Proof. —
well defined up to unique isomorphism, and whose isomorphism class is given by $Q$. It defines a central extension

$$k^* \rightarrow \mathcal{E} \rightarrow Y$$

with commutator map $(-1)^{[y,z]}$. In 11.7, we will describe this central extension in terms of $Q$ and of the root space decomposition of the Lie algebra of $G$.

5. Split semi-simple groups

We keep the notation of 4.1.

5.1. Let $(f_w)_{w \in W(i)}$ be a 1-cochain in the complex (4.3.2) for $j = 2$. Each $f_w$ is an invertible function on $B \times B$. Write $f_i$ for $f_{w_0}$, and, as in the proof of 4.9, let $\tilde{f}_i$ be the character of $T$ such that $f_i(g) = f_i(g)f_i(t)$. Whether $(f_w)$ is a cocycle depends only on its image in the quotient (4.4.1) of the complex (4.3.2). As in 4.4, we identify this image with a bilinear form $C_{\alpha, y}$ pairing $Y_{sc}$ and $Y$. The form $C$ is defined by $C(\alpha, y) = D(f_{w_0}(g, g))$. By 4.5, $(f_w)$ is a cocycle if and only if the quadratic form $C(y, y)$ on $Y_{sc}$ is $W$-invariant.

Until the end of this section, we assume that $G$ is semi-simple. This means that $Y_{sc}$ is of finite index in $Y$: $Y_{sc} \sim \rightarrow Y \rightarrow \rightarrow Y$. Dually, $X_{sc} \sim \rightarrow \rightarrow X_{sc}$ and $X_{sc}$ is finite index in $X_{sc}$.

5.2. Lemma. — Suppose that $(f_w)$ is a cocycle and that $E$ is the corresponding $K_2$-torsor.

The following conditions are then equivalent:

(i) the inverse image of $E$ by $pr_1: G \times T \rightarrow G$ is isomorphic to its inverse image by the multiplication map $\mu$;

(ii) the inverse image of $E$ by $pr_2: T \times G \rightarrow G$ is isomorphic to its inverse image by the multiplication map $\mu$;

(iii) the bilinear pairing $C$ of $Y_{sc}$ with $Y$ extends to a bilinear pairing of $Y$ with itself.

If the $K_2$-torsor $E$ on $G$ admits a multiplicative structure, the torsors $\mu^*E$ and $pr_1^*(E) + pr_2^*(E|_T)$ on $G \times T$ are isomorphic. As, by 3.3, the restriction of $E$ to $T$ is a trivial torsor, 5.2(i) then holds. The main result of this section is a converse to be proven in 5.7.

Proof (i)$\Leftrightarrow$(iii). — The right action of $T$ respects the Bruhat stratification. The difference $pr_1^*E - \mu^*E$ is given by the difference of the inverse image cocycles. We have $f_i(g)/f_i(g) = \tilde{f}_i(t)$ and this difference is the cocycle

$$pr_2^*(\tilde{f}_i) \quad \text{on} \quad (B \times B) \times T.$$
We have to show it is a coboundary if and only if (iii) holds.

The form of (5.2.1) makes it convenient to view $G \times T$ as a group scheme over $T$, and the complex (4.3.2) for $G \times T$ as the complex of global sections of the complex (4.8.1) of sheaves on $T$. This complex is filtered by the filtration $V$ of 3.4, shifted as in (4.3.5). The associated graded is given by (4.8.2), where for basis $S$ one takes $T$. The cocycle (5.2.1) is in $V^\text{shifted}_1$. In $Gr_{V^\text{shifted}_2}$, $\wedge^2 X$ injects into $\bigoplus_{w \in W_1} X = X_{sc} \otimes X$. It follows that (5.2.1) is a coboundary if and only if it is a coboundary in $\Gamma(T, V^\text{shifted}_1(4.8.1))$.

The open stratum $(B_{w_0} \times T)$ has over $T$ a global section $w_0$. It follows that the subcomplex $V^\text{shifted}_0(4.8.1)$ of (4.8.1), reduced to $K_2$ in degree 0, is a direct summand, and the cocycle (5.2.1) is a coboundary if and only if it is one in $\Gamma(T, Gr_{V^\text{shifted}_1}(4.8.1))$.

The complex of sheaves $Gr_{V^\text{shifted}_1}(4.8.1)$ is

\begin{equation}
X \otimes C^*_T \rightarrow \bigoplus_{w \in W_1} C^*_T = X_{sc} \otimes C^*_T.
\end{equation}

Evaluation at the origin splits $\Gamma(T, (5.2.2))$ into two direct factors: $X \otimes k^* \rightarrow X_{sc} \otimes k^*$, and

\begin{equation}
X \otimes X \rightarrow X_{sc} \otimes X.
\end{equation}

Our cocycle (5.2.1) lives in the direct factor (5.2.3). It is the element $C$ of $X_{sc} \otimes X$ and is a coboundary if and only if (iii) holds.

(ii)$\iff$(iii) We proceed in the same way. This time,

\[ f_i(g)/f_i(g) = \overline{f_i(w_0 s_i^{-1} w_0 s_i)} \]

and we find that the relevant cocycle is a coboundary if and only if the pairing $C_1$ defined by

\[ C_1(\alpha, \gamma) = \left< f_i(w_0 s_i^{-1} \gamma) \right> \]

extends. As we are in the semi-simple case, a pairing between $Y_{sc}$ and $Y$ can be extended to $Y \times Y$ if and only if its restriction to $Y_{sc} \times Y_{sc}$ can. For the restrictions to $Y_{sc} \times Y_{sc}$, we have

\[ C_1(\alpha, \gamma) = \left< f_i, s_i^{-1} w_0^{-1} \gamma \right> = C(\alpha, s_i^{-1} w_0^{-1} \gamma). \]

By 5.1, the quadratic form $C(\gamma, \gamma)$ is $W$-invariant. This invariance is equivalent to the vanishing of (4.4.4) for $i \neq j$. As (4.4.4) vanishes identically for $i = j$, we have

\begin{equation}
C_1(\alpha, \gamma, \gamma) = -C(\gamma, \alpha). \tag{5.2.4}
\end{equation}
One concludes by observing that the pairing $-C(\omega_0^{-1}y, z)$ extends to $Y \times Y$ if and only if $C(z, y)$ does.

**Lemma 5.3.** — If $G$ is a product $G_1 \times G_2$, with $T = T_1 \times T_2$, the conditions of (i) (ii) (iii) 5.2 are equivalent to

(iv) the inverse image of $E$ by $\text{pr}_1 : T_1 \times G \times T_2 \rightarrow G$ is isomorphic to its inverse image by the product map.

**Proof.** — We have $Y = Y_1 \times Y_2$ and the condition (iv) translates, as in the proof of 5.2, into the condition that the pairing between $Y_{sc}$ and $Y$ defined by

$$
\begin{align*}
C_0(\alpha_i^*, y_1) &= \langle f_i, y_1 \rangle \\
C_0(\alpha_i^*, y_2) &= \langle f_i, (\omega_0^{-1}y) \rangle
\end{align*}
$$

for $y_1 \in Y_1$ and $y_2 \in Y_2$ extends to a pairing between $Y$ and itself. We decompose $C_0$ into four pairings $C_{0ij}$, with $C_{0ij}$ between $Y_{i,sc}$ and $Y_j$. The pairing $C$ decomposes similarly, and we have

$$
\begin{align*}
C_{0ij} &= C_{ij} & \text{if } j &= 1 \\
C_{012}(y, z) &= C_{12}(y, \omega_0^{-1}z) \\
C_{022}(y, z) &= -C_{22}(\omega_0^{-1}z, y).
\end{align*}
$$

The second line comes from $s_i$ invariance of $\omega_0 z$ for $\alpha_i^*$ in $Y_1$ and $z$ in $Y_2$. The third is as in 5.2.4. It follows that each $C_{0ij}$ extends if and only if $C_{ij}$ does, proving 5.3.

**Lemma 5.4.** — For $G = G_1 \times G_2$ as in 5.3, the functor of restriction to $G_1$ and $G_2$ is an equivalence from

(i) the category of pointed $K_2$-torsors on $G$, for which the equivalent conditions of 5.3 hold,

(ii) the category of pairs $(E_1, E_2)$ of pointed $K_2$-torsors on $G_1$ and $G_2$, for which the equivalent conditions of 5.2 hold.

**Proof.** — We first show that if 5.2 (iii) holds for a $K_2$-torsor $E$ on $G_1 \times G_2$, then $E$ is isomorphic to a sum of pullbacks $\text{pr}_1^*E_1 + \text{pr}_2^*E_2$.

The decomposition $G = G_1 \times G_2$ induces decompositions $T = T_1 \times T_2$, $Y = Y_1 \times Y_2$, $W = W_1 \times W_2$, $B = B_1 \times B_2$, $I = I_1 \sqcup I_2$. We have $\omega_0 = (\omega_{01}, \omega_{02})$ and we choose lifting $\hat{\omega}_0$ of $\omega_0$ in $N(T) \subset G_i$.

Let $(f_\omega)$ be a cocycle in the complex (4.3.2) defining $E$. We write $f_i$ for $f_{\omega_0 i}$. For $i \in I_1$, $f_i$ is an invertible function on

$$B_i \omega_{0i} B = B_1 \omega_{01i} B_1 \times B_2 \omega_{02i} B_2 \subset G_1 \times G_2.$$
Let us restrict \( f_i \) to \( B_{1\omega_{01}}B_1 \) by \( g \mapsto (g, \omega_{02}) \). This corresponds to restricting \( E \) to \( G_1 \times \{\omega_{02}\} \). Letting \( i \) run over \( I_1 \), we obtain a cocycle in the complex (4.3.2) for \( G_1 \), for which 5.2 (iii) holds. It corresponds to a \( K_2 \)-torsor \( E_1 \) on \( G_1 \). Let us do the same for \( i \in I_2 \), obtaining \( E_2 \) on \( G_2 \). The torsor \( E = pr_1^*E_1 - pr_2^*E_2 \) is described by the cocycle

\[
\begin{align*}
\hat{\nu}_i &= \frac{f_i(g_1, g_2)}{f_i(g_1, \omega_{02})} & \text{for } i \in I_1 \\
\hat{\nu}_i &= \frac{f_i(g_1, g_2)}{f_i(\omega_{01}, g_2)} & \text{for } i \in I_2.
\end{align*}
\]

Let us show that the cocycle \( \hat{\nu} \) is a coboundary. We have

\[
\begin{align*}
\hat{\nu}_i(g_1, \omega_{02}) &= 1 & \text{for } i \in I_1, \\
\hat{\nu}_i(\omega_{01}, g_2) &= 1 & \text{for } i \in I_2,
\end{align*}
\]

and the equivalent conditions of 5.4 hold for \( \hat{\nu} \).

By (5.4.1), \( \hat{\nu}_i \) is determined by the \( \hat{\nu} \), which are encoded in \( C^0 H \in X_{sc} \otimes X \). The conditions (5.4.1) translate into \( C^0 \) having trivial restrictions to \( Y_{1sc} \times Y_1 \) and \( Y_{2sc} \times Y_2 \). By 5.2 (iii), \( C^0 \) extends to \( Y \times Y \): it is the sum of pairings \( C_{12}^0 \) and \( C_{21}^0 \) between \( Y_1 \) and \( Y_2 \) and \( Y_1 \) and \( Y_2 \), respectively. The \( W \)-invariance of \( C^0(y, y) \), and the vanishing of \( C^0(y, y) \) for \( y \) in \( Y_1 \) or \( Y_2 \), forces \( C^0(y, y) \) to vanish: one has

\[
C^0_{12}(y_1, y_2) = -C^0_{21}(y_2, y_1).
\]

Let us write \( C^0_{12} \in X_1 \otimes X_2 \) as a sum \( \sum \alpha \beta + \sum \beta \alpha \). The cocycle \( \nu \) is then the coboundary of

\[
\sum \{pr_1^* \left( x_1^\alpha \right) - pr_2^* \left( x_2^\beta \right) \}
\]

on \( B_{\omega_{01}}B \).

Restricting to \( G_1 \) or \( G_2 \), one sees that \( K_2 \)-torsors \( pr_1^*E_1 + pr_2^*E_2 \) and \( pr_1^*E_1' + pr_2^*E_2' \) on \( G_1 \times G_2 \) are isomorphic if and only if \( E_1 \) is isomorphic to \( E_1' \) and \( E_2 \) to \( E_2' \). It remains to check that

\[
\text{Aut}(E_1) \times \text{Aut}(E_2) \cong \text{Aut}(pr_1^*E_1 \times pr_2^*E_2),
\]

i.e. that

\[
\tilde{H}^0(G_1, K_2) \oplus \tilde{H}^0(G_2, K_2) \cong \tilde{H}^0(G_1 \times G_2, K_2).
\]

In the semi-simple case, the first map (4.4.1) is injective, and by the rewriting (4.4.2) of (4.4.1) at \( i = 2 \), one has

\[
\tilde{H}^0(G) = \text{Ker}(X \otimes k' \rightarrow X_{sc} \otimes k').
\]

By (5.4.1), \( \hat{\nu}_i \) is determined by the \( \hat{\nu} \), which are encoded in \( C^0 H \in X_{sc} \otimes X \). The conditions (5.4.1) translate into \( C^0 \) having trivial restrictions to \( Y_{1sc} \times Y_1 \) and \( Y_{2sc} \times Y_2 \). By 5.2 (iii), \( C^0 \) extends to \( Y \times Y \): it is the sum of pairings \( C^0_{12} \) and \( C^0_{21} \) between \( Y_1 \) and \( Y_2 \) and \( Y_2 \) and \( Y_1 \), respectively. The \( W \)-invariance of \( C^0(y, y) \), and the vanishing of \( C^0(y, y) \) for \( y \) in \( Y_1 \) or \( Y_2 \), forces \( C^0(y, y) \) to vanish: one has

\[
C^0_{12}(y_1, y_2) = -C^0_{21}(y_2, y_1).
\]

Let us write \( C^0_{12} \in X_1 \otimes X_2 \) as a sum \( \sum \alpha \beta \). The cocycle \( \nu \) is then the coboundary of

\[
\sum \{pr_1^* \left( x_1^\alpha \right) - pr_2^* \left( x_2^\beta \right) \}
\]

on \( B_{\omega_{01}}B \).

Restricting to \( G_1 \) or \( G_2 \), one sees that \( K_2 \)-torsors \( pr_1^*E_1 + pr_2^*E_2 \) and \( pr_1^*E_1' + pr_2^*E_2' \) on \( G_1 \times G_2 \) are isomorphic if and only if \( E_1 \) is isomorphic to \( E_1' \) and \( E_2 \) to \( E_2' \). It remains to check that

\[
\text{Aut}(E_1) \times \text{Aut}(E_2) \cong \text{Aut}(pr_1^*E_1 \times pr_2^*E_2),
\]

i.e. that

\[
\tilde{H}^0(G_1, K_2) \oplus \tilde{H}^0(G_2, K_2) \cong \tilde{H}^0(G_1 \times G_2, K_2).
\]

In the semi-simple case, the first map (4.4.1) is injective, and by the rewriting (4.4.2) of (4.4.1) at \( i = 2 \), one has

\[
\tilde{H}^0(G) = \text{Ker}(X \otimes k' \rightarrow X_{sc} \otimes k').
\]
The required decomposition follows, proving 5.4.

**Proposition 5.5.** — If \( G \) is split semi-simple, a pointed \( K_2 \)-torsor \( E \) on \( G \) has at most one multiplicative structure. It has one if and only if the equivalent conditions of 5.2 hold.

**Proof.** — We have already shown, in the comment after 5.2, that if \( E \) has a multiplicative structure, 5.2 (i) holds.

Conversely, suppose that 5.2 (i) (ii) hold. Using that \( t_1(gh)t_2 = (t_1g)(ht_2) \), one sees that 5.3(iv) holds for \( \mu^1(E) \) on \( G \times G \). Applying 5.4 one finds that there is a unique isomorphism

\[
m : \text{pr}_1^*E + \text{pr}_2^*E \to \mu^1E
\]

which extends the obvious isomorphisms on \( G \times \{e\} \) and \( \{e\} \times G \). It is the only possible multiplicative structure and it is a multiplicative structure, as the two isomorphisms (1.1.2) deduced from \( m \) on \( G \times G \times G \):

\[
\text{pr}_1^*E + \text{pr}_2^*E + \text{pr}_3^*E \to \mu_{123}E
\]

have the same restriction to \( G \times \{e\} \times \{e\} \), \( \{e\} \times G \times \{e\} \) and \( \{e\} \times \{e\} \times G \) and, by (5.4), are characterized by those three restrictions.

5.6. The arguments leading to 5.5 extend to the case of \( G \) split semi-simple over a base \( S \) regular of finite type over a field. In 5.2, that (iii) is equivalent to (i) or (ii) holds locally on \( S \), and similarly for 5.3. The Lemma 5.4 holds for torsors satisfying locally on \( S \) the equivalent conditions of 5.2, and from this we deduce as in 5.5 that

**Proposition 5.7.** — If \( G \) is split semi-simple over \( S \) regular of finite type over a field, a pointed \( K_2 \)-torsor on \( G \) has at most one multiplicative structure. It has one if and only if the locally equivalent conditions of 5.2 hold locally on \( S \).

### 6. Split reductive groups

6.1. Let \( S \) be regular of finite type over a field \( k \) and let \( G \) be a split reductive group scheme over \( S \). We fix a split maximal torus \( T \). Let \( G_{\text{der}} \) be the commutator subgroup of \( G \), \( G_{\text{sc}} \) be its simply-connected covering and \( G_{\text{ad}} \) be the adjoint group:

\[
G_{\text{sc}} \to G_{\text{der}} \to G \to G_{\text{ad}}.
\]

The inverse images of \( T \) in \( G_{\text{der}} \) and \( G_{\text{sc}} \), and its image in \( G_{\text{ad}} \), are split maximal tori in the respective reductive groups. Notation: \( T_{\text{der}}, T_{\text{sc}}, T_{\text{ad}} \). With the notation (0.N.1), (0.N.2), the morphisms \( T_{\text{sc}} \to T_{\text{der}} \to T \to T_{\text{ad}} \) induce \( W \)-equivariant morphisms \( Y_{\text{sc}} \to Y_{\text{der}} \to Y \to Y_{\text{ad}} \) and \( X_{\text{sc}} \leftarrow X_{\text{der}} \leftarrow X \leftarrow X_{\text{ad}} \). The group \( Y_{\text{sc}} \) (resp. \( X_{\text{ad}} \)) is the
subgroup of \( Y \) (resp. \( X \)) generated by the coroots (resp. roots). The group \( Y_{\text{der}} \) is \( Y_{\text{sc}} \otimes \mathbb{Q} \cap Y \), and \( Y_{\text{sc}} \) and \( Y_{\text{der}} \) are subgroups of finite index in \( Y_{\text{ad}} \).

If \( E \) is a multiplicative \( K_2 \)-torsor on \( G \), its restriction to \( T \) is a multiplicative torsor \( E_T \) on \( T \). By the equivalence of categories 3.16, \( E_T \) gives \( (Q, \mathcal{E}) \), with \( Q \) a quadratic form on \( Y \) and \( \mathcal{E} \) a central extension, on \( S \), of \( Y \) by \( \mathcal{E}_{\mathfrak{g}} \), obeying (3.11.1).

Locally on \( S \), any \( w \in W \) can be lifted to \( G \) and, by 1.6, the automorphism \( w \) of \( T \) preserves the isomorphism class of \( E_T \). It follows by transport of structure that the quadratic form \( Q \) is \( W \)-invariant.

The pullback \( E_{\text{sc}} \) of \( E \) to \( G_{\text{sc}} \) is a multiplicative torsor on \( G_{\text{sc}} \). By 4.9, the corresponding quadratic form 4.7 on \( Y_{\text{sc}} \) is the restriction \( Q_{\text{sc}} \) of \( Q \) to \( Y_{\text{sc}} \). The multiplicative torsor \( E_{\text{sc}} \) is determined up to unique isomorphism by \( Q_{\text{sc}} \). Its restriction to \( T_{\text{sc}} \) gives rise to a central extension 3.15 of sheaves on \( S \)

\[
(6.1.1) \quad \mathcal{E}_{\mathfrak{g}} \to \mathcal{E}_{\text{sc}} \to Y_{\text{sc}}
\]

depending only on \( Q_{\text{sc}} \). As \( E_{\text{sc}} \) restricted to \( T_{\text{sc}} \) is the pullback to \( T_{\text{sc}} \) of \( E_T \), we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}_{\mathfrak{g}}^* & \to & \mathcal{E}_{\text{sc}} \\
\downarrow & & \downarrow \\
\mathcal{E}_{\mathfrak{g}} & \to & Y_{\text{sc}}
\end{array}
\]

\[
(6.1.2)
\]

**Theorem 6.2.** — Let \( G \) be a split reductive group scheme over \( S \) regular of finite type over a field. The construction 6.1 is an equivalence of categories from

(i) the category of multiplicative \( K_2 \)-torsors \( E \) on \( G \), to

(ii) the category of triples \( (Q, \mathcal{E}, \phi) \), where

(a) \( Q \) is a \( W \)-invariant quadratic form on \( Y \). By 4.16, the restriction of \( Q \) to \( Y_{\text{sc}} \) defines a central extension \( \mathcal{E}_{\text{sc}} \) of \( Y_{\text{sc}} \) by \( \mathcal{E}_{\mathfrak{g}}^* \).

(b) \( \mathcal{E} \) is a central extension of \( Y \) by \( \mathcal{E}_{\mathfrak{g}}^* \), obeying (3.11.1),

(c) \( \phi: \mathcal{E}_{\text{sc}} \to \mathcal{E} \) makes the diagram (6.12) commute.
compatible with \( \varphi' \) and \( \varphi'' \): commutativity of the diagram

\[
\begin{array}{ccc}
E_{sc} & \longrightarrow & E_{sc} \\
\downarrow \varphi' & & \downarrow \varphi'' \\
E' & \longrightarrow & E''.
\end{array}
\]

6.3. Proof of 6.2, when \( G \) is semi-simple. — The group \( G \) is assumed to be semi-simple and for the ease of exposition, we begin by assuming that \( S \) is the spectrum of a field \( k \).

We first show that any \( W \)-invariant \( \mathbb{Z} \)-valued quadratic form \( Q \) on \( Y \) comes from some multiplicative \( K_2 \)-torsor on \( G \). There exists a possibly nonsymmetric \( \mathbb{Z} \)-valued bilinear form \( C \) on \( Y \), such that \( C(y, y) = Q(y) \). We have \( C \in X \otimes X \subset X_{sc} \otimes X \) and there exists a 1-cochain \( (f_a) \) as in 5.1 giving rise to \( C \). By 5.1, based on 4.5, this cochain is a cocycle. Let \( E \) be the corresponding \( K_2 \)-torsor, and choose a trivialization of \( E \) at \( e \in G \). By 5.5, the pointed torsor \( E \) has a unique multiplicative structure. Applying 4.9 to the restriction of the multiplicative torsor \( E \) to \( G_{sc} \), we obtain that the quadratic form 6.1 attached to \( E \) is \( Q \).

As any \( W \)-invariant quadratic form on \( Y \) comes from some multiplicative \( K_2 \)-torsor, it suffices to prove 6.2 for the full subcategories of the categories described in (i) and (ii) given by the condition \( Q = 0 \). Those categories can be incarnated as follows. For (i), if \( E \) is defined by a cocycle as in 5.1, giving rise to \( C \) in \( X \otimes X \) by 5.5, the condition \( Q = 0 \) means that \( C \) is in the image of \( \wedge X \). The category is incarnated by (4.4.1)\(_{a=1} \), i.e. by the complex (4.4.2):

\[
(6.3.1) \quad X \otimes k^* \to X_{sc} \otimes k^*.
\]

For (ii), we have to consider the category of pairs \( (E, \varphi) \), where \( E \) is a commutative central extension of \( Y \) by \( k^* \), and \( \varphi \) is splitting of \( E \) over \( Y_{sc} \). The extension \( E \) is commutative because \( Q = 0 \). This category is incarnated by (6.3.1) as well: an element \( f \in X_{sc} \otimes k^* = \text{Hom}(Y_{sc}, k^*) \) defines the following object \( [f] \): the trivial extension \( k^* \times Y \) of \( Y \) by \( k^* \), with the splitting \( f \) on \( Y_{sc} \). An element \( g \) of \( X \otimes k^* = \text{Hom}(Y, k^*) \) defines an isomorphism from \( [f] \) to \( [fg] \).

We leave it to the reader to check that the functor 6.2 is incarnated by the identity automorphism of (6.3.1).

Over a base \( S \), one similarly finds that any \( Q \) can be locally obtained, and local incarnations of the subcategory of 6.2(ii) where \( Q = 0 \) are given by

\[
(6.3.2) \quad X \otimes C^*_S \to X_{sc} \otimes C^*_S,
\]

6.4. Proof of 6.2 (general case).

The quotient \( Y/Y_{der} \) is torsion free. It follows that \( Y_{der} \) is a direct factor in \( Y \). We choose a decomposition \( Y = Y_0 \times Y_{der} \). It gives us a product decomposition
We will apply 1.7 to the semi-direct product decomposition (6.4.1) of $G$. For this, even if we cared only for the case when $S$ is the spectrum of a field, we need the semi-simple case of 6.2 over a more general base, to describe actions of $T_0$ on a central extension of $G_{\text{der}}$ by $K_2$.

The inner action of $G$ on itself factors through $G_{\text{ad}}$. The action of $T_0$ on $G_{\text{der}}$ is hence the composite of $q$: $T_0 \subset T \rightarrow T_{\text{ad}}$, and of the action of $T_{\text{ad}} \subset G_{\text{ad}}$ on $G_{\text{der}}$.

We continue to write $q$ for the induced map (6.4.2) $q: Y_0 \subset Y \rightarrow Y_{\text{ad}}$.

The action of $T_0$ on $G_{\text{der}}$ fixes $T_{\text{der}}$.

6.5. Lemma. — Let $E_{\text{der}}$ be a multiplicative $K_2$-torsor on $G_{\text{der}}$, and let $Q_{\text{der}}$ be the corresponding quadratic form on $Y_{\text{der}}$.

(i) If the action of $T_0$ on $G_{\text{der}}$ can be lifted to $E_{\text{der}}$, the lifting is unique;

(ii) A lifting exists if and only if the quadratic form $Q_{\text{der}}$ can be extended to a $W$-invariant quadratic form on $Y$.

Proof. — Let $E_{\text{sc}}$ be the pullback of $E_{\text{der}}$ to $G_{\text{sc}}$. It is determined up to unique isomorphism by the restriction of the quadratic form $Q_{\text{der}}$ to $Y_{\text{sc}}$. To $E_{\text{sc}}$ corresponds a central extension (6.5.1) $C_{S}^{\ast} \xrightarrow{i} C_{\text{sc}}^{\ast} \xrightarrow{p} Y_{\text{sc}}$

on $S$. The sheaf of automorphisms of the central extension (6.5.1) is the torus $\text{Hom}_S(Y_{\text{sc}}, C_{S}^{\ast})$ with character group $Y_{\text{sc}}$. To $f: Y_{\text{sc}} \rightarrow C_{S}^{\ast}$ corresponds the automorphism $e \mapsto e.f(p(e))$ of $C_{\text{sc}}^{\ast}$. The action of $T_{\text{ad}}$ on $G_{\text{sc}}$ lifts uniquely to $E_{\text{sc}}$ and, as it fixes $T_{\text{sc}}$, it induces an action on the extension (6.5.1), that is, defines a morphism of tori from $T_{\text{ad}}$ to $\text{Hom}_S(Y_{\text{sc}}, C_{S}^{\ast})$. This morphism corresponds to a morphism of dual character sheaves, or equivalently to a pairing of $Y_{\text{ad}}$ with $Y_{\text{sc}}$. One deduces from 4.12 that this pairing is $B(y_{\text{ad}}, y_{\text{sc}})$ for $B$ the bilinear form associated to $Q_{\text{der}}$. The action of $T_0$ on (6.5.1), induced by that of $T_{\text{ad}}$, similarly corresponds to the pairing $B(q(y_0), y_{\text{sc}})$, with $q$ given by (6.4.2).

The multiplicative torsor $E_{\text{der}}$ gives rise to a central extension $E_{\text{der}}$ of $Y_{\text{der}}$ by $C_{S}^{\ast}$, to which (6.5.1) maps:

$$
\begin{array}{ccc}
C_{S}^{\ast} & \rightarrow & C_{\text{sc}}^{\ast} \\
\downarrow & & \downarrow \\
C_{S}^{\ast} & \rightarrow & E_{\text{der}} \\
\downarrow & & \downarrow \\
\rightarrow & & Y_{\text{der}}.
\end{array}
$$
By 6.3, actions of $T_0$ on $E_{\text{der}}$, lifting the action of $T_0$ on $G_{\text{der}}$, are identified with actions of $T_0$ on the central extension $E_{\text{sc}}$, extending the known action of $T_0$ on $E_{\text{sc}}$. Such actions correspond to extensions of the pairing $B(q(y_0), y_{\text{sc}})$ between $Y_0$ and $Y_{\text{sc}}$ to a pairing between $Y_0$ and $Y_{\text{der}}$. As $Y_{\text{sc}}$ is of finite index in $Y_{\text{der}}$, such an extension, if it exists, is unique. This proves (i), while (ii) is reduced to point (i) of the next lemma.

Lemma 6.6. — Let $Q_{\text{der}}$ be a $W$-invariant quadratic form on $Y_{\text{der}}$. By 4.12, the restriction to $Y_{\text{sc}}$ of the associated bilinear form extends to a pairing $B$ of $Y_{\text{ad}}$ with $Y_{\text{sc}}$.

(i) The form $Q_{\text{der}}$ extends to a $W$-invariant quadratic form on $Y$ if and only if $B(q(y_0), y_{\text{sc}})$ extends to a pairing between $Y_0$ and $Y_{\text{der}}$.

(ii) If $Q_{\text{der}}$ extends to a $W$-invariant quadratic form on $Y$, the $W$-invariant extensions $Q$ correspond one to one to quadratic forms on $Y_0$, by $Q \mapsto Q|Y_0$.

Proof. — After we tensor with $Q$, $Y \otimes Q$ is the direct sum of $Y_{\text{der}} \otimes Q$ and of $(Y \otimes Q)^W$. Invariant (rational) quadratic forms are the orthogonal direct sums of an invariant quadratic form on $Y_{\text{der}} \otimes Q$ and an arbitrary quadratic form on $(Y \otimes Q)^W$.

The extension $Q_1 := Q_{\text{der}} \oplus 0$ of $Q_{\text{der}}$, expressed in terms of the decomposition $Y = Y_0 \oplus Y_{\text{der}}$, is

\[ Q_1(y_0 + y_{\text{der}}) = B(q(y_0), y_{\text{der}}) + Q_{\text{der}}(y_{\text{der}}). \]

Any other $W$-invariant (rational) extension is the sum of $Q_1$ and of the pull back of a quadratic form on $Y \otimes Q/Y_{\text{der}} \otimes Q \sim Y_0 \otimes Q$. It has the form

\[ Q(y_0 + y_{\text{der}}) = Q_0(y_0) + B(q(y_0), y_{\text{der}}) + Q_{\text{der}}(y_{\text{der}}). \]

Integral extensions are given by the same formula: they exist if and only if $B(q(y_0), y_{\text{der}})$ is integral on $Y_0 \times Y_{\text{der}}$, and correspond one to one to $Q_0$.

6.7. We return to the proof of 6.2 for $G$, using the dictionary 1.4 between multiplicative torsors and central extensions. As explained in 1.7, the data of a central extension $E$ of the semi-direct product $G = T_0 \ltimes G_{\text{der}}$ by $K_2$ amounts to the data of

(a) its restriction $E_{\text{der}}$ to $G_{\text{der}}$;
(b) its restriction $E_0$ to $T_0$;
(c) an action of $T_0$ on the central extension $E_{\text{der}}$ of $G_{\text{der}}$ by $K_2$, lifting the action of $T_0$ on $G_{\text{der}}$.

In (c), the action of $t$ in $T_0$ is that of any lifting of $t$ to $E_0$. One recovers $E$ as a quotient of the semi-direct product $E_0 \ltimes E_{\text{der}}$.

We now translate (a) (b) (c), using 6.2 (semi-simple case), and 6.5. We get:
from (a): a $W$-invariant quadratic form $Q_{\text{der}}$ on $Y_{\text{der}}$, a central extension $\mathcal{E}_{\text{der}}$ of $Y_{\text{der}}$ by $\mathcal{E}_S^\ast$, obeying (3.11.1), and a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}_S^\ast & \longrightarrow & \mathcal{E}_{\text{sc}} \\
\downarrow & & \downarrow \\
\mathcal{E}^\ast & \longrightarrow & Y_{\text{sc}}
\end{array}
\]  

(6.7.1)

with $\mathcal{E}_{\text{sc}}$ defined by the restriction $Q_{\text{sc}}$ of $Q_{\text{der}}$ to $Y_{\text{sc}}$.

from (c): that $Q_{\text{der}}$ can be extended to a $W$-invariant form on $Y$. The action (c) then exists, and is unique.

from (b): a quadratic form $Q_0$ on $Y_0$, and a central extension

\[
\mathcal{E}_S^\ast \rightarrow \mathcal{E}_0 \rightarrow Y_0
\]

(6.7.2) 

obeying (3.11.1).

By 6.6 (ii), there is a unique $W$-invariant quadratic form $Q$ on $Y$ with restrictions $Q_{\text{der}}$ and $Q_0$ to $Y_{\text{der}}$ and $Y_0$. There is also a central extension

\[
\mathcal{E}_S^\ast \rightarrow \mathcal{E} \rightarrow Y
\]

obeying (3.11.1) and inducing respectively (6.7.2) and the second line of (6.7.1) on $Y_0$ and $Y_{\text{der}}$. It is unique up to unique isomorphism, and sits in a commutative diagram deduced from (6.7.1)

\[
\begin{array}{ccc}
\mathcal{E}_S^\ast & \longrightarrow & \mathcal{E}_{\text{sc}} \\
\downarrow & & \downarrow \\
\mathcal{E}^\ast & \longrightarrow & Y_{\text{sc}}
\end{array}
\]  

(6.7.3)

The data (a) (b) (c) hence amounts to that of $Q$, $W$-invariant on $Y$, of a central extension $\mathcal{E}$ of $Y$ by $\mathcal{E}_S^\ast$, obeying (3.13.1), and of a commutative diagram (6.7.3). One easily checks that the construction (6.1), applied to the corresponding central extension (or rather, multiplicative $K_2$-torsor) give back $Q$, $\mathcal{E}$ and (6.7.3). This concludes the proof.

7. Reductive groups

7.1. Let $S$ be regular of finite type over a field $k$ and $G$ be a reductive group scheme over $S$. Locally for the étale topology of $S$, $G$ is split [SGA3 XXII 2.3]: there exists a surjective étale map $\kappa: S_0 \rightarrow S$ such that $G_0$ over $S_0$ deduced from $G$ by base change is a split reductive group over $S_0$. By 2.7, the change of base from $S$ to $S_0$ is an equivalence of categories from pointed $K_2$-torsors on $(\Omega, e)$ to pointed $K_2$-torsors on $(G_0, e)$ provided with a descent data. Applying this to the $G^\vee$, and using
that this equivalence of categories is compatible with pull-backs, one sees that giving a multiplicative $K_2$-torsor on $G$ amounts to giving one on $G_0$, and a descent data from $S_0$ to $S$.

Suppose $T$ is a maximal torus of $G$. As in 6.1, one deduces from $G$ semi-simple group schemes $G_{sc}$, $G_{der}$ and $G_{ad}$ over $S$, with maximal tori $T_{sc}$, $T_{der}$ and $T_{ad}$. With the notations (0.N.1), (0.N.2), let $Q$ be a $W$-invariant quadratic form on $Y$, and $Q_{sc}$ be the induced quadratic form on $Y_{sc}$. Locally on $S$ for the étale topology, $G$ and $T$, hence $G_{sc}$ and $T_{sc}$, are split, and by 4.7, amplified by 4.8, $Q_{sc}$ defines a multiplicative $K_2$-torsor $E_{sc}$ on $G_{sc}$. Locally on $S$ for the étale topology, more precisely after any étale change of base $S_0 \to S$ such that $T$ splits on $S_0$, the multiplicative torsor $E_{sc}$, restricted to $T_{sc}$, defines by 3.16 a central extension of Zariski sheaves on $S_0$

\[(7.1.1) \quad 1 \to C^* \to E_{sc} \to Y_{sc} \to 1,\]

obeying (3.11.1). It is functorial in $S_0$, hence defines locally on $S$ for the étale topology a central extension (7.1.1) of sheaves for the étale topology. To define a central extension of sheaves, it suffices to define it locally: we obtain a central extension (7.1.1) of étale sheaves on $S$ obeying (3.11.1).

We can now consider triples $(Q, E, \phi)$ as in 6.2, with all sheaves for the étale topology. For triples $(Q, E, \phi)$, étale descent is trivially valid: to give a triple $(Q, E, \phi)$ on $S$ amounts to giving one on $S_0$, and a descent data. By descent, one hence deduces from 6.2:

**Theorem 7.2.** — By étale descent, the equivalence of categories 6.2 induces an equivalence of categories from

(i) the category of multiplicative $K_2$-torsors on $G$, to

(ii) the category of triples $(Q, E, \phi)$ as in 6.2, with all sheaves taken in the étale topology.

**7.3. Special cases**

(i) If $G$ is semi-simple simply-connected, the data of $(Q, E, \phi)$ reduces to that of $Q$: multiplicative $K_2$-torsors on $G$ have no nontrivial automorphisms, and are classified by $W$-invariant quadratic forms $Q$ on $Y$.

(ii) For $G$ reductive, the group $\text{Hom}(G, K_2)$ of automorphisms of a central extension $E$ of $G$ by $K_2$ is the group of automorphisms for the corresponding $(E, \phi)$. For variable $S$, they form the étale sheaf $\mathcal{H}om(Y/Y_{sc}, C^*)$.

(iii) Suppose $S = \text{Spec}(k)$. A triple $(Q, E, \phi)$ determines a multiplicative $K_2$-torsor. This torsor in turn defines a central extension $E(k)$ of $G(k)$ by $K_2(k)$. Automorphisms of $(Q, E', \phi)$ induce automorphisms of the central extension $E(k)$.

Applying descent to 3.5 and 3.7, one obtains
7.4. Proposition. — Fix $S$ as in 7.1.

(i) For $p : T \to S$ a torus over $S$, $\tilde{p}_* K_2$ is the sheaf of pairs $(A, q)$ as in 3.5.

(ii) The group of multiplicative sections of $K_2$ over $T$ is $H^0(S_{\text{et}}, X \otimes \mathcal{O}_{S}^*)$.

(iii) For tori $T_1$ and $T_2$ over $S$, with sheaves of character groups $X_1$ and $X_2$, the group of bimultiplicative sections of $K_2$ over $T_1 \times_S T_2$ is $H^0(S_{\text{et}}, X_1 \otimes X_2)$.

7.5. Special case. — Suppose $S = \text{Spec}(k)$ and let $T_1$ and $T_2$ be tori over $S$. For $k_i$ a separable closure of $k$, the character groups of $T_1$ and $T_2$ are $\text{Gal}(k_i/k)$-modules $X_1$ and $X_2$. Bimultiplicative sections of $K_2$ are in bijection with the group of $\text{Gal}(k_i/k)$-invariants in $X_1 \otimes X_2$. Such an invariant element $B$ hence gives rise to a bimultiplicative map

\[ (7.5.1) \quad (\ , \ )_B : T_1(k) \times T_2(k) \to K_2(k). \]

This, completed by 7.6 below, answers the question 7.11 of Deligne (1996). Another, more natural, answer to this question has been communicated to us by B. Kahn. He systematically works in the étale topology, using the complex $\Gamma(2)$ of Lichtenbaum (1987).

7.6. Fix $G$ over $S$ as in 7.1, and a multiplicative $K_2$-torsor on $G$. We view it as an extension $E$ of $G$ by $K_2$, viewed as sheaves on a big Zariski site. If $C \in G \times_S G$ is the subscheme of commuting elements, the commutator map $G \times G \to E$ induces a morphism from $C$ to $K_2$, i.e. an element

\[ (7.6.1) \quad \text{comm} \in H^0(C, K_2). \]

For $T$ a maximal torus in $G$, the restriction of $\text{comm}$ to $T \times_S T$ is a bimultiplicative section of $K_2$. To determine which, one may first localize for the étale topology. Applying 3.13, one sees that it corresponds by 7.5 to the bilinear form $B$ associated to the quadratic form $Q$ attached to the multiplicative torsor we started with.

In particular, for $S = \text{Spec}(k)$, we have a central extension of $G(k)$ by $K_2(k)$ and on $T(k)$ the commutator map is (7.5.1).

Similarly, if $G$ is semi-simple simply-connected, we have an action of $G^{\text{ad}}$ on the multiplicative $K_2$-torsor $E$ corresponding to a $W$-invariant quadratic form $Q$. Let $\overline{C} \subset G^{\text{ad}} \times G$ be the subscheme of pairs $(g, h)$ such that $g$ fixes $h$. The scheme $C \subset G \times G$ is the inverse image of $\overline{C}$, and the section (7.6.1) comes from

\[ (7.6.2) \quad \text{comm} \in H^0(\overline{C}, K_2). \]

As previously, its restriction to $T^{\text{ad}} \times T$ is bimultiplicative. It is given by the unique pairing $B_1$ between $Y^{\text{ad}} \supset Y$ and $Y$ extending $B$. 

For \( S = \text{Spec}(k) \), \( G^\text{ad}(k) \) acts on the central extension of \( G(k) \) by \( \mathbb{K}_2(k) \) defined by \( Q \) and the corresponding commutator map

\[
T^\text{ad}(k) \times T(k) \to \mathbb{K}_2(k)
\]

is (7.5.1) for \( B_1 \).

8. Generic cocycles

8.1. Let \( k \) be a field and \( \mathcal{H} \) be a sheaf on the big Zariski site of \( \text{Spec}(k) \). For \( X \) a reduced and irreducible scheme of finite type over \( k \), with field of rational functions \( k(X) \), we define \( \mathcal{H}^0(k(X)) \) to be the fiber of \( \mathcal{H} \) at the generic point of \( X \). It is the inductive limit over \( U \) of \( \mathcal{H}(U) \), for \( U \) a nonempty open subscheme of \( X \). For \( \mathcal{H} = \mathbb{K}_2 \), we get \( \mathbb{K}_2(k(X)) \).

Let \( G \) be a connected algebraic group over \( k \). Each component \( G^p \) of the simplicial classifying space \( BG = (G^\Delta/G) \) is geometrically irreducible and the face maps \( @^i : G^p \to G^{p-1} \) map the generic point of \( G^p \) to the generic point of \( G^{p-1} \). The generic points of the components \( G^p \) of \( BG \) hence form a strict (0,N.7) simplicial system of spectrum of fields.

Applying the functor \( \mathcal{H} \) to the \( k(G^p) \), one obtains a strict cosimplicial abelian group. The complex of generic cochains is the corresponding complex

\[
\mathcal{H}(k(G^p)), \quad d = \sum (-1)^i \partial_i^p.
\]

8.2. Construction. — Assume that the field \( k \) is infinite, and that the connected algebraic group \( G \) over \( k \) is unirational. Then the category of multiplicative \( \mathcal{H}\)-torsors on \( G \) is incarnated in degree 2 by the complex of generic cochains.

The unirationality assumption holds if \( G \) is reductive (SGA3 XIV 6.7).

In cohomological degree one, the construction boils down to the following statement: the map “restriction to the generic point of \( G \)” is an isomorphism from the group of multiplicative sections of \( \mathcal{H} \) over \( G \) to the group of elements of \( \mathcal{H}^0(k(G)) \) for which \( s(g_1g_2) = s(g_1) + s(g_2) \) in \( \mathcal{H}(k(G \times G)) \).

As \( d = 0 \) on \( \mathcal{H}^0(G^0) \), the complex of generic cochains and the subcomplex \( (\mathcal{H}(G^p), \text{ for } p \geq 1) \) incarnate in degree 2 the same Picard category.

Proof. — Let us view \( \mathcal{H} \) as a functorial system of Zariski sheaves \( \mathcal{H}^0 \) on the schemes \( S \) in \( \text{Spec}(k)_{\text{Zar}} \) (1.4), and let us choose a functorial system of resolutions \( \mathcal{H}^q \), with each \( \mathcal{H}^q \) acyclic. Such systems of resolutions exist: one can use the canonical flabby resolutions of Godement (1958), or an injective resolution of the sheaf \( \mathcal{H} \) on \( \text{Spec}(k)_{\text{Zar}} \).
As we saw in 1.9 Example (vi), the category of multiplicative $\mathcal{H}$-torsors on $G$ is incarnated in degree 2 by the associated simple complex $s(\mathcal{T}(G^p, \mathcal{H}^s))$ for $p \geq 1$.

For each $p$, the fiber of $\mathcal{H}^s$ at the generic point of $G^p$ is a resolution of $\mathcal{T}(k(G^p))$ for $p > 1$.

For each $p$, the fiber of $k(G^p)$ at the generic point of $G^p$ is a resolution of $k(G^p)$.

It follows that the morphism of complexes

(8.2.1) $\mathcal{T}(k(G^p))$ for $p \geq 1 \rightarrow s(\mathcal{H}^s((k(G^p)))$ for $p \geq 1)$

is a quasi-isomorphism reducing 8.2 to

8.3. Proposition. — Under the assumptions of 8.2, the morphism of restriction to the generic points

(8.3.1) $s(\Gamma(G^p, \mathcal{H}^s))$, for $p \geq 1 \rightarrow s(\mathcal{T}(k(G^p)))$, for $p \geq 1)$

is a quasi-isomorphism.

As $G^0$ is reduced to a point, 8.3 is equivalent to:

$s(\Gamma(G^p, \mathcal{H}^s)) \rightarrow s(\mathcal{T}(k(G^p)))$

is a quasi-isomorphism. This is what we will prove.

Some variants of 8.2 hold with more generality. To prepare for them, the next lemma will be proven in the following more general setting: $S$ is a noetherian scheme, and $G$ is a group scheme over $S$, flat of finite type over $S$ and with connected fibers.

Fix $n$ and, for $0 \leq i \leq n$, let $U_i$ be an open subset of $G = G^h/G$. For $i > n$, define $U_i := G^i$.

Lemma 8.4. — Assume that for $0 \leq i \leq n$ the fibers of $U_i \rightarrow S$ are not empty. Then, there exists a system of open subsets $V_i \subset G_i$ $(i \geq 0)$ such that

(i) the fibers of $V_i \rightarrow S$ are not empty;

(ii) for $i \leq n$, $V_i \subset U_i$;

(iii) each face map $\bullet: G^i \rightarrow G^{i-1}$ maps $V_i$ to $V_{i-1}$;

(iv) for $i > 0$, $\partial_0: G^i \rightarrow G^{i-1}$ maps $V_i$ onto $V_{i-1}$;

(v) For $i > n$, $V_i$ is the intersection of the $\partial^{-1}(V_j)$ for $\partial$ an iterated face map and $j \leq n$.

Proof. — We will construct $V_i$ as follows:

(a) define $U'_i$ to be the intersection of the $\partial^{-1}(U_{j+1})$ $(j \geq 0)$;

(b) define $V_i$ as the intersection of all $\partial^{-1}(U'_j)$ for $0 \leq j \leq n$ and $\partial$ an iterated face map from $G'$ to $G'$. The identity map is considered an iterated face map.

As $U_i = G^i$ for $i > n$, it suffices in (a) to take the intersection for $i + j \leq n$, and $U'_i = G^i$ for $i > n$. As $G$ is flat over $S$, all face maps are open maps and the $U'_i$ are
open subsets. So are the $V_i$. That they satisfy (ii) (iii) (vi) is clear. The construction of the $V_i$ is compatible with passage to geometric fibers. To check (i) (iv), it hence suffices to prove (i), (iv) when $S$ of the form $\text{Spec}(k)$, for $k$ an algebraically closed field. We assume $S$ of this form.

We will use that a nonempty open subset of $G^i$ is dense (irreducibility of $G^i$). As a finite intersection of open dense subsets is still open and dense, this gives us first that the $U_{0i}$ are dense. Next that the $V_i$ are. This proves (i). By construction, one has

\[(8.4.1) \quad \varpi_0(U'_0) \supset U'_{i-1} \quad \text{for} \quad i > 0.\]

We have $G^i \equiv G^h/G$. Let $\tilde{U}'_{i}$ and $\tilde{V}_i$ be the inverse image of $U'_i$ and $V_i$ in $G^h$. We have to check that $\varpi_0$ maps $\tilde{V}_i(k)$ onto $\tilde{V}_{i-1}(k)$, i.e. that for $(g_1, ..., g_i)$ in $\tilde{V}_{i-1}(k)$ there exist $g_0$ such that $(g_0, g_1, ..., g_i)$ is in $\tilde{V}_i(k)$. The conditions on $g_0$ are that for any subsequence $j_0, ..., j_{i'}$ of $0, ..., i$, $(g_{j_0}, g_{j_1}, ..., g_{j_{i'}}) \in \tilde{U}'_{i'}(k)$. If $j_0 \neq 0$, this results from $(g_1, ..., g_i)$ being in $\tilde{V}_{i-1}(k)$. It remains to find $g_0$ such that for each subsequence $j_1, ..., j_{i'}$ of $1, ..., i$, $(g_0, g_{j_1}, ..., g_{j_{i'}})$ is in $\tilde{U}'_{i'}(k)$. If $i' = 0$, this holds for any $g_0$ by (i). For $i' > 0$, $(g_0, g_1, ..., g_i)$ is in $\tilde{U}_{i-1}(k)$ by (iii), and as $\tilde{V}_{i-1}(k) \subset \tilde{U}_{i-1}(k) \subset \varpi_0 \tilde{U}'_{i'}$ (iii) and (8.4.1)), it holds for $g_0$ in a nonempty open subset. The intersection of these nonempty open subsets is nonempty, proving (iv).

**Variant.** Suppose $S$ is the spectrum of a field $k$. If $G$ is not necessarily connected, 8.4 should be modified as follows. The $U_i$ should be assumed dense, not just nonempty. In (i), the $V_i$ should be Zariski dense too. In (iv), the fibers of $\varpi_0$: $V_i \to V_{i-1}$ should be dense in the fibers of $\varpi_0$: $G' \to G^{i-1}$.

In the proof, each time one has to take an image by $\varpi_0$ (or an iterated image, by some $\varpi_0^i$), it should be replaced by the following modified image: modified $\varpi_0(U)$ is the set of $x$ such that the trace of $U$ is dense in $\varpi_0^{-1}(x)$. In our setting, the modified image of an open set is open. Indeed, modified images are compatible with a finite extension of scalars $k'/k$: we may and shall assume that the geometric connected components $G^h$ of $G$ are defined over $k$. Our maps $\varpi_0$ are isomorphic to projections $\varphi: G \times X \to X$, and

\[\text{modified } \varphi(U) = \bigcap \varphi(U \cap (G^h \times X)).\]

**Lemma 8.5.** — Under the assumptions of 8.2, suppose that open subsets $V_i \subset G^i$ obey (i) (iii) (iv) of 8.4 (for $S = \text{Spec}(k)$). Let $\tilde{V}_i$ be the inverse image of $V_i$ in $G^h$. Then, for any scheme $T$ over $k$, any $i > 0$, and any finite family of morphisms $(g_{1i}, ..., g_{ki})$ from $T$ to $\tilde{V}_{i-1}$, locally on $T$ (for the Zariski topology) there exist $g_0$: $T \to G$ such that all $(g_0, g_{1i}, ..., g_{ki})$ are with values in $\tilde{V}_i$. 


Proof. — For each $\alpha$, $(g_0, g_{\alpha}^1, \ldots, g_{\alpha}^i)$ is in $\tilde{V}_i$ if and only if $g_0$, viewed as a section of $\text{pr}_1: G \times S \to S$, takes value in the open set $U^\alpha$, inverse image by $(g, s) \mapsto (g, g_{\alpha}^1, \ldots, g_{\alpha}^i): S \to G^\Delta$ of $\tilde{V}_i \subset G^\Delta$:

$$
\begin{array}{ccc}
U^\alpha & \hookrightarrow & G \times S \\
\downarrow & & \downarrow \\
\tilde{V}_i & \hookrightarrow & G^\Delta
\end{array}
$$

By assumption each $U^\alpha$ maps onto $S$. As $G$ is irreducible, so does their intersection $U$.

By the unirationality assumption on $G$, there exist an open subset $W$ of an affine space $A^N$ and a dominant morphism $f$ from $W$ to $G$. Let $W' \subset W \times S$ be the inverse image of $U$ in $W \times S$. It is an open subset of $A^N/S$, fiber by fiber not empty. The residue fields of $S$ being infinite, $W'/S$ has local sections. Their images by $f$ are the required $g_0$.

Lemma 8.6. — Under the same assumptions as in 8.5, the restriction map

$$(8.6.1) \quad s \Gamma(G^p, \mathcal{H}^q) \to s \Gamma(V^p, \mathcal{H}^q)$$

is a quasi-isomorphism.

Proof. — We will show that both sides of (8.6.1) have as cohomology groups the same Ext groups in the big Zariski site of $\text{Spec}(k)$. We identify each scheme of finite type $X$ over $k$ with the corresponding representable sheaf, and we let $Z[X]$ be the sheaf of abelian groups freely generated by $X$. If $G$ acts on $X$, $Z[X]$ is a sheaf of modules over the sheaf of rings $Z[G]$.

The $Z[\tilde{V}_i]$ form a complex of $Z[G]$-modules, augmented to $Z$. The augmentation $\varepsilon$ is $Z[\tilde{V}_0] = Z[G] \to Z$: $[g] \mapsto 1$. The complex $Z[\tilde{V}]$ is a resolution of $Z$. Indeed, a section $c$ of $Z[\tilde{V}_{i-1}]$ over $S$ is a finite linear combination $\Sigma_{a}^{g_0}(g_{i}^1, \ldots, g_{i}^a)$ and, by 8.5, there exists locally on $S$ a $g_0$ such that the $(g_0, g_{i}^1, \ldots, g_{i}^a)$ are in $\tilde{V}_i$. If $c$ is a cocycle (resp., for $i = 0$, if $c$ is in the kernel of $\varepsilon$), it is the coboundary of $\Sigma_{a}^{g_0}(g_{i}^1, \ldots, g_{i}^a)$.

For each $i$, the subscheme $g_0 = 1$ of $\tilde{V}_i$ maps isomorphically to $V_i$, and the action of $G$ gives an equivariant isomorphism

$$(8.6.2) \quad G \times V_i \simeq \tilde{V}_i.
$$

The isomorphism (8.6.2) induces an isomorphism

$$(8.6.3) \quad Z[\tilde{V}_i] = Z[G] \otimes Z Z[V_i].
$$

For any sheaf of $Z[G]$-modules $\mathcal{F}$, one hence has

$$(8.6.4) \quad \text{Hom}_{Z[G]}(Z[\tilde{V}_i], \mathcal{F}) = \text{Hom}_{Z}[Z[V_i], \mathcal{F}] = \Gamma(V_i, \mathcal{F}).$$
The functor “forgetting the \( \mathbb{Z}[G] \)-module structure” has an exact left adjoint: the extension of scalars functor from sheaves of \( \mathbb{Z} \)-modules to sheaves of \( \mathbb{Z}[G] \)-modules. Hence it transforms injectives into injectives, and, deriving (8.6.4) in \( \mathcal{F} \), one finds that
\[
\text{Ext}_{\mathbb{Z}[G]}^*(\mathbb{Z}[^i], \mathcal{F}) = H^*(V_i, \mathcal{F}).
\]
If \( \mathcal{F}^* \) is a resolution of \( \mathcal{F} \) by sheaves of \( \mathbb{Z}[G] \)-modules such that \( H^p(V_i, \mathcal{F}^0) = 0 \) for \( p > 0 \), it follows that the cohomology groups of the complex
\[
s \Gamma(V_p, \mathcal{F}^0)
\]
are the \( \text{Ext}_{\mathbb{Z}[G]}^i(\mathbb{Z}, \mathcal{F}) \). This applies to \( \mathcal{H}_i \) and its resolution \( \mathcal{H}_i^* \), with the trivial action of \( G \):
\[
H^s \Gamma(V_p, \mathcal{H}_i^*) = \text{Ext}_{\mathbb{Z}[G]}^i(\mathbb{Z}, \mathcal{H}_i^*).
\]
are quasi-isomorphic to the complex (1.9.2) computing the cohomology of \( H^*(BG, K) \).

8.7 Proof of 8.3. — Let \( \mathcal{F}^* \) be the set of systems of open subsets \( V_i \subset G^i \) obeying 8.4 (i) (iii), (iv) as well as (v) for some \( n \). We order it by inclusion: \( V'_i \subset V''_i \) if \( V_i \subset V'' \) for all \( i \). By 8.4, this order is filtering, and each \( V_i \) can be arbitrarily small. The morphism (8.3.1) is hence the inductive limit of the quasi-isomorphisms (8.6.1) and, as such, is a quasi-isomorphism.

Corollary 8.8. — Under the assumptions of 8.2, the complexes
\[
\mathcal{H}^*(k(G^p)) \quad (p \geq 1)
\]
as well as
\[
\text{coker}(\mathcal{H}^*(k) \rightarrow \mathcal{H}^*(k(G^p)))
\]
are quasi-isomorphic to the complex (1.9.2) computing the cohomology of \( H^*(BG, \mathcal{H}) \).

Proof. — The first quasi-isomorphism follows from 8.2 applied to \( G \) and to the trivial group.

For each \( p \), the map \( \mathcal{H}^*(k) \rightarrow \mathcal{H}^*(k(G^p)) \) is injective: indeed, \( \mathcal{H}^*(k(G^p)) \) is the inductive limit of the \( \mathcal{H}^*(U) \), for \( U \subset G^p \) open and nonempty, and each map \( \mathcal{H}^*(k) \rightarrow \mathcal{H}^*(U) \) is injective, because by unirationality of \( G \), \( U \) has a rational point. It follows that the complex (8.8.2) is quasi-isomorphic to (8.8.1).

8.9. Remark. — The proof SGA3 XIV 6.7 of the unirationality of reductive groups holds in a relative situation, for \( G \) over \( S \), provided that the torus \( G/G^{\text{der}} \) is
isotrivial. Isotriviality holds, for instance, if $S$ is normal. It follows that 8.6 holds for $G$ reductive over $S$, and $S$ normal with infinite residue fields.

**8.10. Remark.** — Let $G$ be a smooth connected linear algebraic group over an infinite field $k$. Assume that $G$ is unirational. Let $k'$ be a finite Galois extension of $k$.

We have seen that extension of scalars to $k'$ induces an equivalence of categories from

(a) the category of multiplicative $K_2$-torsors on $G$, to
(b) the category of multiplicative $K_2$-torsors on $G' = G \otimes_k k'$, provided with a (Galois) descent data.

The proof in 7.1 was based on 2.2. Using 8.2, it is possible to base it instead directly on 2.1. Indeed, the categories (a) (b) are incarnated in degree 2, respectively by the complex $K_2(k)(G)/K_2(k)$ and by the complex deduced from the complex $K_2(G'/k)/K_2(k')$ by applying the derived functor of invariants by $\text{Gal}(k'/k)$. By 2.1, they are quasi-isomorphic in degree $\leq 2$.

**8.11. Remark.** — Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are sheaves on the big Zariski site of $\text{Spec}(k)$ and that $u : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ induces isomorphisms $\mathcal{H}_1(k(X)) \rightarrow \mathcal{H}_2(k(X))$ for any scheme $X$ reduced irreducible and of finite type over $k$. For instance, one can take for $\mathcal{H}_1$ the Zariski sheaf $K_2^M$ associated to the presheaf of Milnor $K$-groups $C^*(U) \otimes C^*(U)/\langle u \otimes v | u + v = 1 \rangle$, and for $\mathcal{H}_2$ the sheaf $K_2$.

Under the assumptions of 8.2, it then follows from 8.2 that $u$ induces an equivalence from the category of multiplicative $\mathcal{H}_1$-torsors on $G$ to that of multiplicative $\mathcal{H}_2$-torsors.

**8.12.** Let $G$ be an algebraic group over a field $k$, with $G(k)$ Zariski dense in $G$.

The methods used to prove 8.2 apply as well to the study of central extensions of $G(k)$ by an abelian group $A$.

The category of central extensions of $G(k)$ by $A$ is incarnated in degree 2 in the standard complex $C^*(G(k), A)$ computing the cohomology $H(BG(k), A)$ of the $G(k)$-module $A$, with the trivial action of $G(k)$. The group $C^n(G(k), A)$ is the group $\mathcal{F}(G(k)^n, A)$ of $A$-valued functions on $G(k)^n$.

Let $C^n_{\text{gen}}(G(k), A)$ (generic cocycles) be the inductive limit of the groups $\mathcal{F}(U(k), A)$ of functions from $U(k)$ to $A$, for $U$ an open Zariski dense subset of $G^*$.

**Proposition 8.13.** — Under the assumptions and with the notations of 8.12, the morphism of complexes

\[
(8.13.1) \quad C^*(G(k), A) \rightarrow C^*_\text{gen}(G(k), A)
\]

is a quasi-isomorphism.
Proof. — For $V_i \subseteq G^i$ a system of open dense subsets of the $G^i$, obeying 8.4 (i) (iii) (iv) modified as in 8.4 variant, the morphism of complexes

$$\text{(8.13.2)} \quad C^\ast(G(k), A) \to \mathcal{F}(V_\ast(k), A)$$

is a quasi-isomorphism. This is proved as in 8.6, identifying the cohomology groups of both sides with the $\text{Ext}^i_{\mathbb{Z}[G(k)]}(\mathbb{Z}, A)$. Indeed, as in 8.6, if $\tilde{V}_i$ is the inverse image of $V_i$ in $G$, $\mathbb{Z}[^{\tilde{V}_i}(k)]$ is a resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules, because for any finite family $(g_1^0, ..., g_n^0)$ of elements of $\tilde{V}_i-1(k)$, there exists $g_0 \in G(k)$ such that the $(g_0, g_1^0, ..., g_n^0)$ are in $\tilde{V}_i(k)$, and

$$\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[^{\tilde{V}_i}(k)], A) = \mathcal{F}(V,(k), A).$$

Taking an inductive limit as in 8.7, one obtains (8.13.1) as an inductive limit of quasi-isomorphisms (8.13.2).

8.14. Under the assumptions of 8.12, it follows that the category of central extensions of $G(k)$ by $A$ is incarnated in degree 2 by the complex $C^\ast_{\text{gen}}(G(k), A)$ of generic cochains. In particular, a generic 2-cochain $\epsilon$ defines a central extension $E(\epsilon)$, unique up to unique isomorphisms. More precisely, $\epsilon$ is the image of a cocycle $\overline{\epsilon}$ in $C^\ast(G(k), A)$, defining $E(\overline{\epsilon})$ and if $\overline{\epsilon}$, $\overline{\epsilon}'$ are two such liftings of $\epsilon$, there exists a unique 1-cochain $\delta$, vanishing on some $U(k)$, with $U$ open and Zariski dense in $G$, with coboundary $\overline{\epsilon}' - \overline{\epsilon}$. It defines an isomorphism from $E(\overline{\epsilon}')$ to $E(\overline{\epsilon})$.

9. Comparison with Galois cohomology

9.1. Let $k$ be a field and $n$ an integer prime to the characteristic. We write $\mathbb{Z}/n(1)$ for the group of $n^{th}$ roots of unity in a separable closure $\overline{k}$ of $k$, $\mathbb{Z}/n(d)$ for its $d^{th}$ tensor power and $H^p(k, \ )$ for a Galois cohomology group $H^p(\text{Gal}(\overline{k}/k), \ )$. Let $\text{cl}$ be the Kummer coboundary

$$\text{cl}: k^* \to H^1(k, \mathbb{Z}/n(1)).$$

By Tate (1970), if $u + v = 1$, one has $\text{cl}(u) \cup \text{cl}(v) = 0$ in $H^2(k, \mathbb{Z}/n(2))$. As $K_2(k) = k^* \otimes k^* / (u \otimes v | u + v = 1)$, the morphism $u, v \mapsto \text{cl}(u) \cup \text{cl}(v)$ induces a morphism

$$\text{(9.1.1)} \quad K_2(k) \to H^2(k, \mathbb{Z}/n(2)).$$

More generally, if $n$ is invertible on a scheme $S$, $\mathbb{Z}/n(1)$ is the étale sheaf of $n^{th}$ root of 1 and $\mathbb{Z}/n(d)$ its $d^{th}$-tensor power. Let $\mathcal{E}$ be the morphism from the étale to the Zariski site of $S$, and let

$$\mathcal{H}^p(\mathbb{Z}/n(d)) := R^p \mathcal{E}_* \mathbb{Z}/n(d)$$
be the Zariski sheaf associated to the presheaf \( U \mapsto H^0_p(U, \mathbb{Z}/n(2)) \) (étale cohomology). We will mainly use the \( H^p_p(\mathbb{Z}/n(2)) \) and denote them simply \( H^p_p \).

For \( U \) affine, Soulé (1979) has defined Chern class maps \( c_2: K_2(U) \to H^2_p(U, \mathbb{Z}/n(2)) \). They define a morphism of sheaves

\[
(9.1.2) \quad K_2 \to H^2_p.
\]

By loc. cit. II 3, for \( S \) spectrum of a field \( k \), (9.1.2) reduces to (9.1.1) possibly up to a sign \( s \). If needed, we change the sign of (9.1.2) so that it agrees with (9.1.1). The sheaf \( H^2_p \) is contravariant in \( S \). By 1.7, it defines a sheaf on the big Zariski site \( S_{\text{Zar}} \) of \( S \), and the morphisms (9.1.2) define a morphism of sheaves on \( S_{\text{Zar}} \).

9.2. Let \( G \) be a simply-connected absolutely simple algebraic group over a field \( k \). Let \( T \) be a maximal torus, \( Y \) be its cocharacter group (an étale sheaf over \( \text{Spec}(k) \), i.e. \( \text{a Gal}(\overline{k}/k)-\text{module} \)) and \( Q \) be the minimal Weyl group invariant positive quadratic form on \( Y \): one has \( Q(\alpha^\vee) = 1 \) for \( \alpha \) a long root. Applying 7.3(i), we obtain a multiplicative \( K_2 \)-torsor \( E \) on \( G \), hence a central extension of \( G(k) \) by \( K_2(k) \). Pushing it by (9.1.1), one obtains a central extension

\[
(9.2.1) \quad H^2_p(k, \mathbb{Z}/n(2)) \to E_m(k) \to G(k).
\]

Construction 9.3. — We construct an isomorphism of the central extension (9.2.1) with the opposite of the canonical central extension of Deligne (1996) 3.6.

For simplicity, we assume \( k \) infinite. If \( k \) is finite, both central extensions are by the trivial group.

Lemma 9.4. — Both central extensions are deduced, as in 1.4, from a multiplicative \( H^2_p \)-torsor on \( G \).

Proof. — For the central extension (9.2.1), the required multiplicative \( H^2_p \)-torsor is deduced from \( E \) by pushing by (9.1.2). To describe the other central extension, we have to review the construction of Deligne (1996).

The quadratic form \( Q \) defines an étale cohomology class \( c_Q \) in \( H^4(BG/\text{mod} B_r, \mathbb{Z}/n(2)) \). This \( H^4 \) is canonically isomorphic to \( \mathbb{Z}/n \), with \( c_Q \) as generator (loc. cit. §1). If \( \mathcal{B}^{\cdot} \) is a functorial acyclic resolution of \( \mathbb{Z}/n(2) \), for instance an injective resolution in the category of sheaves of \( \mathbb{Z}/n \)-modules on the big étale site of \( \text{Spec}(k) \), \( c_Q \) is given by a 4-cocycle in \( s\Gamma(G^2, \mathcal{B}^q_{\beta \geq 1}) \) with components \( \gamma_i \in \Gamma(G^{4-i}, \mathcal{B}^q) \) \((0 \leq i \leq 3)\).

The component \( \gamma_3 \) defines a class \([\gamma_3]\) in \( H^4_p(G, \mathbb{Z}/n(2)) \). This class is primitive. By loc. cit. 1.13, it vanishes on some open subset \( U \) of \( G \). Translating \( U \), one sees that \([\gamma_3]\) vanishes locally on \( G \) for the Zariski topology.
If $[\gamma_3]$ vanishes on $U \subset G$, there exists $b \in H^0(U, \mathcal{B}^2)$ such that $db = \gamma_3$. The possible $b$'s form a torsor under the group $Z^2$ of cocycles, and, pushing by $Z^2 \to H^2_* (U, \mathbb{Z}/n(2))$, one obtains a $H^2_* (U, \mathbb{Z}/n(2))$-torsor. In this way, $\gamma_3$ gives rise to a $\mathcal{H}^2$-torsor on $G$. The component $\gamma_2$ provides a multiplicative structure on this $\mathcal{H}^2$-torsor. The resulting multiplicative $\mathcal{H}^2$-torsor is the one promised by 9.4.

Here is another description of its isomorphism class, viewed as a class in $H^2(BG \text{ mod } B\mathcal{E}, \mathcal{H}^2)$. The projection $\varepsilon$ from étale to Zariski sites gives rise to a spectral sequence (9.4.1)

$$E_2^{pq} = H_p^0(BG \text{ mod } B\mathcal{E}, \mathcal{H}^q) \Rightarrow H_p^q(BG \text{ mod } B\mathcal{E}, \mathbb{Z}/n(2)).$$

**Lemma 9.5.** — In the spectral sequence (9.4.1),

1. $E_2^{0,q} = 0$
2. The morphism $H^1 \to E_2^{1,3}$ vanishes
3. The resulting morphism $H^4 \to E_2^{2,2} = H^2(BG \text{ mod } B\mathcal{E}, \mathcal{H}^2)$ carries $c_2$ to the class of the multiplicative $\mathcal{H}^2$-torsor we constructed.

**Proof.** — Let $\mathcal{B}^*$ be as in 9.4. Following Grothendieck (1957) 2.4, the spectral sequence (9.4.1) is obtained by viewing $\mathfrak{s}(\Gamma^{\varepsilon}(G^p, \mathcal{B}^q))$ for $p > 1, q > 0$ as the hypercohomology of $BG \text{ mod } B\mathcal{E}$ with coefficients in $\varepsilon_* \mathcal{B}^*$, and using the canonical filtration $\tau$ of $\varepsilon_* \mathcal{B}^*$. Functorially in $G^p$, one should take a filtered resolution $(\mathcal{B}^*_1, F)$ of $(\varepsilon_* \mathcal{B}^*, \tau)$, so that on each $G^p (p > 1)$, $Gr^F_1(\mathcal{B}^*_1)$ is a resolution of $\mathcal{H}^n$ with acyclic components. The morphism

$$\mathfrak{s}(\Gamma^{\varepsilon}(G^p, \mathcal{B}^q))_{p > 1} \Rightarrow \mathfrak{s}(\Gamma^{\varepsilon}(G^p, \mathcal{B}^q))_{p > 1}$$

is a quasi-isomorphism, and one uses on the target the filtration $F$: $E_2^{pq} = H^{pq} Gr^F_q = H^q sF\Gamma^{\varepsilon}(G^p, \mathcal{B}^q)$, resolution of $\mathcal{H}^q$ for $p > 1$.

The term $E_2^{0,q}$ vanishes, being the $H^0$ of a complex which starts in degree 1. If a class in $H^1$ is given by a cocycle $(\gamma_3)$, as in 9.4, the term $E_2^{1,3}$ is the primitive part of $H^0(G, \mathcal{H}^3)$, and the map $H^1 \to E_2^{1,3}$ is $c \mapsto$ class in $\mathcal{H}^3$ of $\gamma_3$. It vanishes (see 9.4) and we leave it to the reader to check that the resulting map $H^4 \to E_2^{2,2}$ attaches to $c$ the class of the multiplicative $\mathcal{H}^2$-torsor constructed as in 9.4.

**Lemma 9.6.** — (i) In the spectral sequence (9.4.1), the morphism

$$H^1 \to E_2^{2,2} = H^2(BG \text{ mod } B\mathcal{E}, \mathcal{H}^2)$$

(defined thanks to 9.5 (i) (ii)) is an isomorphism.

(ii) One has $E_2^{1,2} = H^1(BG \text{ mod } B\mathcal{E}, \mathcal{H}^2) = 0$: multiplicative $\mathcal{H}^2$-torsors on $G$ have no nontrivial automorphisms.
Proof. — We first prove the vanishing of $E_{pq}^{b}$ for $q = 0$ or 1. By the spectral sequence $$E_{1}^{r,s} = H^{r}(G \text{ mod } e, \mathcal{K}^{s}) \Rightarrow H^{r+s}(BG \text{ mod } Be, \mathcal{K}^{s}),$$ it suffices to show that for $G$ simply-connected, one has $$H^{q}(G \text{ mod } e, \mathcal{K}^{s}) = 0 \quad \text{for } q = 0, 1,$$ applied to the $G^{r}$, which are simply-connected, this gives $E_{1} = 0$.

For $q = 0$, the Zariski sheaf $\mathcal{K}^{0}$ is the constant sheaf with value $H^{0}(k, \mathbb{Z}/n(2))$. Its higher cohomology group vanishes, as well as all its reduced cohomology groups $H^{r}(G \text{ mod } e, \mathcal{K})$.

To handle the case $q = 1$, we use the spectral sequence $$E_{2}^{u,v} = H^{u}(G \text{ mod } e, \mathcal{K}^{v}) \Rightarrow H^{u+v}_{\text{et}}(G \text{ mod } e, \mathbb{Z}/n(2))$$ and what we know of its abutment to obtain $E_{2}^{1} = 0$. The case $q = 0$ gives us $E_{2}^{0,0} = 0$. By Bloch-Ogus (1974), $E_{2}^{u,v} = 0$ for $u > v$. The $E_{2}^{u,v}$ for $u, v > 0$ hence look as follows:

$$
\begin{array}{cccccc}
E_{0}^{01} & E_{1}^{11} & 0 & 0 & \cdots \\
E_{0}^{02} & E_{2}^{12} & \cdots & 0 & \cdots \\
E_{0}^{03} & E_{2}^{13} & 0 & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots \\
\end{array}
$$

The abutment $H^{n}_{\mathcal{K}}(G \text{ mod } e, \mathbb{Z}/n(2))$ vanishes for $n \leq 2$. Indeed, in the geometric case of an algebraically closed field $k$, this follows from similar results for complex groups (cf. Deligne (1996) 1.5, 1.7), and the general case follows from the geometric case by the Hochschild-Serre spectral sequence

$$H^{0}(k, H^{r}(G_{r} \text{ mod } e, \cdots)) \rightarrow H^{r+s}(G \text{ mod } e, \cdots).$$

a diagram chase in (9.6.1) shows that the vanishing of the abutment for $n \leq 2$ implies the vanishing of $E_{2}^{u,v}$ for $u + v \leq 2$. In particular, $E_{2}^{0,1} = E_{2}^{1,1} = 0$, as required.

By 9.5 (i), we know now that the $E_{2}^{b}$ of (9.4.1) look as follows:

$$
\begin{array}{cccccc}
0 & & & & & \\
0 & E_{2}^{13} & & & & \\
0 & E_{2}^{12} & E_{2}^{22} & & & \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
\end{array}
$$

As the abutment vanishes for $n \leq 3$, (Deligne (1996) 1.10), (i) and (ii) of 9.6 follow from 9.5 (ii) by a diagram chase in (9.6.2).
9.7. Proof of 9.3. — We will construct the isomorphism 9.3 as the image of an isomorphism between the multiplicative $\mathcal{H}^2$-torsors of 9.4. By 9.6 (ii), such an isomorphism, if it exists, is unique: we are left with showing that the two multiplicative $\mathcal{H}^2$-torsors of 9.4 are isomorphic. The group $H^2(BG \text{ mod } B\mathfrak{e}, \mathcal{H}^2)$ of isomorphism classes coincides with $H^1_{et}(BG \text{ mod } B\mathfrak{e}, \mathbb{Z}/n(2))$ (9.6 (i)). This group is $\mathbb{Z}/n$, independently of $k$. To prove the isomorphism of the two multiplicative $\mathcal{H}^2$-torsors of 9.4, it therefore suffices to do so after a field extension $k'/k$: we may and shall assume that $G$ is split.

The two $\mathcal{H}^2$-torsors to be compared have the same functoriality in $G$. As in loc. cit. 3.7, we can hence reduce the general case to that of $G = SL(2)$, and reduce the case of $SL(2)$ to that of $SL(3)$.

The multiplicative $\mathcal{H}^2$-torsors on $G$ can be detected as follows: extend scalars from $k$ to $k(U, V)$; consider the resulting central extension

$$1 \to H^2(Gal(k(U, V)/k(U, V)), \mathbb{Z}/n(2)) \to E \to G(k(U, V)) \to 1;$$

fix coroots $y_1, y_2$ in $Y$ with $B(y_1, y_2) = 1$, where $B$ is the bilinear form attached to $Q$; take the commutator

$$(y_1(U), y_2(V)) \in H^2(Gal(k(U, V)/k(U, V)), \mathbb{Z}/n(2)).$$

It will be a power of the Galois symbol $(U, V)$. This Galois symbol is of order $n$, as one sees by taking its residue along $U = 0$, and which power of $(U, V)$ we get tells us which $\mathcal{H}^2$-torsor we started from. To prove 9.2, it then suffices to compare the commutator formulae of 3.14 and of loc. cit. 3.5.

9.8. Let $T_1$ and $T_2$ be two tori over a field $k$. We let $Y_1$ and $Y_2$ be their groups of cocharacters and view them as Galois-modules. To a bilinear form $B: Y_1 \otimes Y_2 \to \mathbb{Z}$, supposed to be Galois invariant, we have attached a bimultiplicative map (7.5)

$$(9.8.1) \quad T_1(k) \times T_2(k) \to K_2(k).$$

Fix an integer $n$ prime to the characteristic of $k$. Composing (9.8.1) with the map (9.1.1): $K_2(k) \to H^2(k, \mathbb{Z}/n(2))$, we obtain

$$(9.8.2) \quad T_1(k) \times T_2(k) \to H^2(k, \mathbb{Z}/n(2)).$$

The short exact sequences of sheaves for the étale topology,

$$0 \to Y_i \otimes \mathbb{Z}/n(1) \to T_i \to T_i \to 1,$$

give rise to coboundaries

$$T_i(k) \to H^1(k, Y_i \otimes \mathbb{Z}/n(1)).$$
Taking a cup product, and applying $B$, we obtain

$$(9.8.3) \quad T_1(k) \times T_2(k) \to H^2(k, Y_1 \otimes Y_2 \otimes \mathbb{Z}/n(2)) \to H^2(k, \mathbb{Z}/n(2)).$$

**Proposition 9.9.** — The maps (9.8.2) and (9.8.3) are equal.

Both (9.8.2) and (9.8.3) come from a bimultiplicative section of $H^2$ on $T_1 \times T_2$. For (9.8.2): the map (9.8.1) comes from a bimultiplicative section of $K_2$, to which one applies (9.1.2): $K_2 \to H^2$. For (9.8.3): the same construction continues to make sense, using étale cohomology, over any base, and one applies it in the universal case: over the base $T_1 \times T_2$, for the universal $T_1 \times T_2$-points $pr_1$ and $pr_2$ of $T_1$ and $T_2$.

If $T_1$ and $T_2$ are split, it is clear from the definitions that those two bimultiplicative sections of $H^2$ on $T_1 \times T_2$ are equal. We reduce to the split case by using the

**Lemma 9.10.** — Let $H^0(T_1 \times T_2 \text{ mod } T_1, T_2, H^2)$ be the group of sections of $H^2$ with trivial restrictions to $T_1$ and $T_2$. If we extend scalars from $k$ to a separable closure $k'$ of $k$, this relative $H^0$ injects into the similar relative $H^0$ on $k'$.

**Proof.** — The inclusion and projections $T_i \to T_1 \times T_2 \to T_i$ give commuting idempotent endomorphisms $pr_i^* \text{ in } H^0(T_1 \times T_2, H^2)$, and the relative $H^0$ we consider is one of the four corresponding direct summands.

The projection from the étale site to the Zariski site gives a spectral sequence

$$E^{pq}_{2} = H^p(T_1 \times T_2 \text{ mod } T_1, T_2, H^q) \Rightarrow H^{p+q}(T_1 \times T_2 \text{ mod } T_1, T_2, \mathbb{Z}/n(2)).$$

By Bloch-Ogus (1974), $H^p(T_1 \times T_2, H^q)$ vanishes for $p > q$. So does the direct summand $E^{pq}_{2}$ and the $E^{pq}_{2}$ for $p, q \geq 0$ look as follows:

* * * 0  
* * 0 0  
* 0 0 0

At the abutment of the spectral sequence, $H^i = 0$ for $i < 2$, while $H^2$ is the group of Galois invariant bilinear forms from $Y_1 \times Y_2$ to $\mathbb{Z}/n$ (Deligne (1996) 3.1). From the spectral sequence, we get a short exact sequence

$$(9.10.1) \quad 0 \to E^{1,1}_2 \to H^2 \to E^{0,2}_2 \to 0.$$
The middle vertical arrow is injective. The map \( \Omega \) is injective as well: over \( k' \), choosing bases \( x_1, \ldots, x_N \) and \( y_1, \ldots, y_M \) of the characters group of \( T_1 \) and \( T_2 \), one can write any element of \( 'H^2 \) uniquely in the form \( \sum n_i \text{cl}(x_i) \cup \text{cl}(y_j) \) with \( n_i \) in \( \mathbb{Z}/n \). If \( K = k'(x_1, \ldots, x_N, y_1, \ldots, y_M) \) is the field of rational functions on \( T_1 \times T_2 \), the element is uniquely determined by its image in \( H^2(K, \mathbb{Z}/n(2)) \), as one sees by taking an iterated residue along \( x_i = 0 \), and then along \( y_j = 0 \).

Those two injectivities force \( E_{2,1}^{1,1} = 0 \), then \( E_{2,1}^{1,1} = 0 \) and finally the required injectivity of \( E_{2,0}^0 \rightarrow E_{2,0}^{0,2} \).

10. Local and global fields

10.1. Let \( F \) be a local field: \( F \) a finite extension of \( \mathbb{R}, \mathbb{Q}_p \) or \( \mathbb{F}_p((T)) \) for some \( p \).

The group \( K_2(F) \) is the target of the universal symbol \( \{ , \} \), where a symbol with values in an abelian group \( A \) is a bimultiplicative map \( F^* \times F^* \rightarrow A \), with \( (x,y) = 0 \) for \( x + y = 1 \). If one considers only locally constant symbols, there is also a universal one. Its target \( K_2^{\text{cont}}(F) \) is trivial if \( F \simeq \mathbb{C} \), and is the group \( \mu_F \) of roots of unity of \( F \) otherwise, with \( \{ , \} = \text{Hilbert symbol} \). See Moore (1968), Theorem 3.1.

Lemma 10.2. — Let \( X \) be a scheme of finite type over \( F \) and let \( s \) be a section of \( K_2 \) over \( X \). Evaluating \( s \) at each \( x \in X(F) \), one obtains a map \( s_1: X(F) \rightarrow K_2(F) \rightarrow K_2^{\text{cont}}(F) \). The map \( s_1 \) is locally constant.

Proof. — Let \( n \) be the order of \( \mu_F \). One has
\[
H^2(F, \mathbb{Z}/n(2)) = \mu_F,
\]
the universal locally constant symbol being the Galois symbol. Let \( s' \) be the image of \( s \) in \( H^0(X, \mathcal{M}^2) \) by (9.1.2). The map \( s_1 \) is obtained by evaluating \( s' \) at \( s \in X(F) \) and 10.2 reduces to Deligne (1996) 2.10.

By 10.2, a \( K_2 \)-torsor on \( X \) gives rise to a \( K_2^{\text{cont}}(F) \)-torsor on the topological space \( X(F) \). Applying this to multiplicative torsors, one obtains the

Construction 10.3. — Let \( E \) be a multiplicative \( K_2 \)-torsor on the algebraic group \( G \) over \( F \). The resulting central extension
\[
(10.3.1) \quad K_2^{\text{cont}}(F) \rightarrow \tilde{G}(F) \rightarrow G(F)
\]
deduced from 1.4 by pushing by \( K_2(F) \rightarrow K_2^{\text{cont}}(F) \) is a topological central extension of the locally compact group \( G \) by the finite (and discrete) group \( K_2^{\text{cont}}(F) \).

For \( G \) absolutely simple and simply-connected, the central extensions (10.3.1) coincide with the central extensions constructed in Deligne (1996). See §9 and loc.
cit. 5.4. For $E$ a generator of the group of central extensions of $G$ by $K_2$, Prasad-Rapinchuk (1996) 8.4 proves that if $F$ is nonarchimedian and $G$ isotropic, the central extension $(10.3.1)$ is a universal topological central extension of $G(F)$.

10.4. Let now $F$ be a global field. For each place $v$ of $F$, let $F_v$ be the completion of $F$ at $v$. Let $\mu_F$, or simply $\mu$, be the group of roots of unity of $F$. For each place $v$, if $v$ is not a complex place, let $\mu_v$ be the group of roots of unity in $F_v$, and let $[\mu_v: \mu]$ be the index of $\mu$ in $\mu_v$. For $v$ a complex place, define $\mu_v$ to be trivial. The reciprocity law for the Hilbert symbols says that for $x, y$ in $F^\times$, $(x, y)_v$ is in the kernel of the map

$$(\zeta_v) \mapsto \prod_v \zeta_v^{[\mu_v: \mu]}$$

from $\oplus \mu_v$ to $\mu$. By Moore (1968), Theorem 7.4 one has more precisely an exact sequence

$$(10.4.2) \quad K_2(F) \to \oplus_{v} K_2^{\text{cont}}(F_v)^{10.4.1} \to \mu \to 0.$$ 

From this, we will deduce that for $G$ a linear algebraic group over $F$ and $E$ a multiplicative $K_2$-torsor over $G$, one has a topological central extension of the adelic group $G(A)$ by $\mu$, canonically split over $G(F)$:

$$(10.4.3) \quad \begin{array}{ccc}
G(F) & \twoheadrightarrow & G(A) \\
\mu \hookrightarrow & & \downarrow \\
& & G(A) \\
& \rightarrow & G(A)
\end{array}$$

A little care is required because torsors on $G$ are for the Zariski topology, and that for $U \subset G$ an affine open subset, $U(A)$ is not in general open in $G(A)$. We will not work directly with $G(A)$, but rather obtain $(10.4.3)$ as an inductive limit of similar diagrams, relative to larger and larger finite sets of places of $F$.

10.5. If $F$ is a number field, let $\mathcal{O}_F$ be its ring of integers. For $S_1$ a finite set of closed points of $X := \text{Spec}(\mathcal{O}_F)$, identified with places of $F$, let $S := S_1 \cup \{\text{infinite places}\}$. If $F$ is a function field, let $X$ be the projective nonsingular curve of which it is the field of rational functions. We let $S = S_1$ be a finite set of closed points, identified with places. In both cases, we note $|X|$ the set of closed points of $X$.

Fix $S_1$ large enough so that $G$ is the generic fiber of a group scheme $G_{X-S_1}$ over $X - S_1$ and that the $K_2$-torsor $E$ comes from a multiplicative $K_2$-torsor $E_{X-S_1}$ on $G_{X-S_1}$. Fix $G_{X-S_1}$ and $E_{X-S_1}$. Two choices become isomorphic after restriction to some $X - S'_1$, and two isomorphisms between two choices become equal after restriction to some $X - S''_1$. The following lemma is analogous to Deligne (1996) 6.2.
Lemma 10.6. — Except for $F$ a function field and $S_1$ empty, one has

$$H^1(X - S_1, K_2) = 0.$$  

As $K_2$ of a finite field vanishes, we have an exact sequence of sheaves over $X$

$$0 \rightarrow K_2 \rightarrow \text{constant sheaf } K_2(F) \rightarrow \bigoplus_{x \in X} (i_x)_* k^*_x \rightarrow 0$$

with $i_x$ the inclusion of $x$ in $X$ and $k_x$ the residue field. From this, we get an exact sequence

$$K_2(F) \rightarrow \bigoplus_{x \in X - S_1} k^*_x \rightarrow H^1(X - S_1, K_2) \rightarrow 0.$$  

By (10.4.2), the image of $K_2(F)$ in the sum of the $k^*_x$ ($x \in X - S_1$) coincide with the image of $\text{Ker}(\oplus \mu_v \rightarrow \mu)$, where the sum runs over all noncomplex places. We have to show that, except for $F$ a function field and $S_1$ empty, the natural map

$$(10.6.1) \quad \text{Ker}(\oplus \mu_v \rightarrow \mu) \rightarrow \bigoplus_{x \in X - S_1} k^*_x$$

is onto. If $F$ is a function field and if $v \in S_1$, this results from the surjectivity of $\zeta \mapsto \zeta^{[\mu_v : \mu]}$, $k^*_v = \mu_v \rightarrow \mu$. If $F$ is a number field, for any prime $\ell$, let $v$ be a place of characteristic $\ell$. The map $\zeta \mapsto \zeta^{[\mu_v : \mu]}$ induces a surjection from $\text{Ker}(\mu_v \rightarrow k^*_v)$ to the $\ell$-primary component of $\mu$, and it follows that (10.6.1) is onto on the $\ell$-primary components. This holding for each $\ell$, 10.6 follows.

10.7. If $S_1$ is large enough so that the conclusion of 10.6 holds, we have a central extension

$$(10.7.1) \quad H^0(X - S_1, K_2) \rightarrow E_1 \rightarrow G(X - S_1).$$

For $v$ a place of $F$, it maps to the local central extension

$$(10.7.2) \quad \mu_v \rightarrow \tilde{G}(F_v) \rightarrow G(F_v).$$

For $v \in |X|$, the map factors through a central extension

$$K_2(C_v) \rightarrow \tilde{G}(C_v) \rightarrow G(C_v).$$

If $\mu_v$ is of order prime to the residue characteristic, the exact sequence

$$K_2(C_v) \rightarrow K_2(F_v) \rightarrow k^*_v$$
shows that $K_2(C_v)$ maps trivially to $\mu$. We obtain a trivialization of $G(C_v)$ over $G(C_v)$:

$$
\begin{array}{c}
K_2(C_v) \\
\downarrow^0
\end{array} 
\rightarrow 

\begin{array}{c}
G(C_v) \\
\downarrow
\end{array} 
\rightarrow 

\begin{array}{c}
G(C_v)
\end{array}

and the map (10.7.1) → (10.7.2) maps $E_1$ to a lifting of $G(C_v)$ in $\tilde{G}(C_v)$. Taking a product, we obtain

$$
\prod_{v \in S} \mu_v 
\rightarrow 

\prod_{v \in S} \tilde{G}(C_v) \times \prod_{v \in X - S_1} G(C_v) 
\rightarrow 

\prod_{v \in S} G(F_v) \times \prod_{v \in X - S_1} G(C_v)
$$

provided that for all $v$ in $X - S_1$, $\mu_v$ is of order prime to the residual characteristic. This holds for $S_1$ large enough. The first vertical map, composed with the reciprocity map $\prod_{v} [\mu_v: \mu]$ with values in $\mu$, vanishes. From 10.7.3 we hence get

$$
\begin{array}{c}
G(X - S_1)
\end{array}
\rightarrow 

\begin{array}{c}
\mu
\downarrow
\end{array} 
\rightarrow 

\begin{array}{c}
\prod_{v \in S} G(F_v) \times \prod_{v \in X - S_1} G(C_v)
\end{array}

The diagram 10.4.3 is the inductive limit of the diagrams 10.7.4.

The geometric analogue of those constructions will be considered in 12.8 to 12.15.

11. Associated central extensions of dual characters groups

If $G$ is a split simply connected group over a field $k$, with split maximal torus $T$, a Weyl group invariant integer-valued quadratic form $Q$ on the dual character group $Y$ of $T$ determines a central extension of $Y$ by $k^\times$. This central extension is an ingredient in the classification 6.2 of central extensions of split reductive groups by $K_2$. Our aim in this section is to describe it in terms of the root spaces decomposition of $\text{Lie}(G)$. The classification 6.2 depends on the choice of a split maximal torus in the split reductive group. We will explain also how the classifications for different maximal torus are related.

To simplify the exposition, we will work in this section over a field $k$.

11.1. We first consider the case when $G$ is isomorphic to $SL(2)$. Choose a split maximal torus $T$. Let $U^+$, $U^-$ be the root subgroups normalized by $T$. We write $\alpha_+$


and \( \alpha \) for the corresponding roots. In the constructions which follow, \( U^+ \) and \( U^- \) play symmetric roles: the constructions are invariant by the group of automorphisms of \((G, T)\), i.e. by the normalizer of \( T_{\text{ad}} \) in the adjoint group \( G_{\text{ad}} \).

Define \( U^{\pm*} := U^\pm - \{ e \} \) and \( N^1 := N(T) - T \). We will use the trijection of Tits (1966) 1.1 between \( U^{+*} \), \( U^{-*} \) and \( N^1 \). Trijection means; transitive system of isomorphisms. For the standard \( \text{SL}(2) \) and its torus of diagonal matrices, the triples in trijection are the

\[
\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}.
\]

The elements \( e^+, e^- \) and \( n \) are in trijection if and only if \( n = e^+ e^- e^+ \). One then has

\[
\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}.
\]

If \( e^+, e^- \) and \( n \) are in trijection, so are their inverses (take the inverses in (11.1.2)) and the conjugation by \( n \) exchanges \( e^+ \) and \( e^- \) (apply conjugation by \( n \) to (11.1.2) and observe that conjugation by \( n \) fixes \( n \) and exchanges \( U^+ \) and \( U^- \)). The square of any \( n \in N^1 \) is the central element \( \alpha_\cdot(-1) \), hence \( n^{-1} = \alpha_\cdot(-1)n \).

Let \( E \) be the extension of \( G \) by \( K_2 \) defined by a quadratic form \( Q \) on \( Y \). By 3.2, it is trivial on the unipotent subgroup \( U^+ \) (resp. \( U^- \)). We note \( u \mapsto \tilde{u} \) the unique trivializing section. We define a section of \( E \) over \( N^1 \) by \( \tilde{n} = \tilde{e}^\cdot \tilde{e}^- \cdot \tilde{e}^+ \), for \( n \) decomposed as in (11.1.2). Applying \( \text{int}[n] \), which fixes \( \tilde{n} \), one obtains that \( \tilde{n} = \tilde{e}^\cdot \tilde{e}^- \cdot \tilde{e}^+ \). Taking the inverse, one obtains \( \tilde{n}^{-1} = (n^{-1})^{-1} \).

Fix \( n \) in \( N^1 \). The right translation by \( n \) induces an isomorphism of schemes from \( T \) to \( N^1 \), and \( h \mapsto s[n][h] := (hn) \tilde{n}^{-1} \) is a section \( s[n] \) of \( E \) over \( T \), trivial at \( e \). One has

\[
\tilde{n}^{-2} = (n^{-1})^{-1} \tilde{n}^{-1} = (\alpha_\cdot(-1)n) \tilde{n}^{-1} = s[n](\alpha_\cdot(-1))
\]

Steinberg’s cocycle \( \epsilon \in H^0(T \times T, K_2) \) is the 2-cocycle defined by

\[
s[n][h_1]s[n][h_2] = s[n][h_1, h_2].c(h_1, h_2).
\]

Let us identify \( T \) with \( G_n \) and \( Y \) with \( Z \), using \( \alpha_\cdot \). The cocycle \( c(h_1, h_2) \) is trivial for \( h_1 = e \) or \( h_2 = e \). By 3.7 (ii), it is of the form \( c(h_1, h_2) = q, h_1, h_2 \) for some integer \( q \). Applying 3.13 and 4.9, one obtains that the quadratic form \( Q \) is \( qn^2 \):

\[
c(\alpha_\cdot(a), \alpha_\cdot(b)) = Q(\alpha_\cdot\{a, b\})
\]
The adjoint group \( G_{ad} \) acts on \( G \), with \( T_{ad} \) respecting \( T \). One has \( Y_{ad} = \frac{1}{2} Y \).

The group \( U^+ \) is naturally a vector space, with

\[
\left( \frac{1}{2} \alpha^\vee \right)(a)[e^+] = ae^+.
\]  

The action of \( G_{ad} \) on \( G \) extends uniquely to \( E \) (4.10). If \( \left( \frac{1}{2} \alpha^\vee \right)(a) \) transforms \( n \) into \( n' \), it will transform the section \( s[n] \) of \( E \) over \( T \) into \( s[n'] \). By 4.12,

\[
\left( 11.1.7 \right) s[n'][\theta] = s[n][\theta][Q(\alpha^\vee)](a, t).
\]

The central extension \( E \) of \( Y \) by \( k^* \) attached to \( Q \) is obtained as follows. One extends the scalars from \( k \) to \( k((\tau)) \), one obtains from \( E|_T \) a central extension of \( T(k((\tau))) \) by \( K_2 k((\tau)) \), one pulls it back by \( Y \) to \( T(k((\tau))) \): \( y \mapsto y(\tau) \) and pushes it out by the residue map \( \text{Res}: K_2 k((\tau)) \rightarrow k^* \). We normalized the residue map to be the tame symbol

\[
\left( 11.1.8 \right) \text{Res}_{f,g} = (-1)^{d(f)d(g)} e^{d(f)^{-1}(g)}(0).
\]

We choose to call the variable \( \tau \), not \( t \) as in 3.10 or 3.12, to avoid confusion with elements of \( T \).

For \( e^+, e^- \) and \( n \) in Tits trigation, we define \( [e^+] \) to be the image in \( E \) of \( s[n][\alpha^\vee(\tau)] \). Applying this definition to \( e^- \), \( e^+ \), \( n \), which are also in trigation, we get \( [e^-] = s[n][\alpha^-(\tau)] \). The element \( [e^+] \) (resp. \( [e^-] \)) of \( E \) is a lifting to \( E \) of the generator \( \alpha^\vee \) (resp. \( \alpha^- \)) of \( Y \). By (11.1.6), (11.1.7) and (11.1.8), one has

\[
\left( 11.1.9 \right) [ae^+] = a^{-Q(\alpha^\vee)}[e^+]
\]

in \( E \). By (11.1.4), (11.1.5) and (11.1.8), one has

\[
\left( 11.1.10 \right) [e^+][e^-] = (-1)^{Q(\alpha^\vee)}.
\]

As \( U^+ \) is a vector space, it can be identified with its Lie algebra \( u^+ \). By (11.1.9), the inverse image of \( \alpha^\vee \) in \( E \) can be identified with the complement of \( 0 \) in the \( N \)th tensor power of the line \( u^+ \), for \( N = -Q(H^+) \), by \( (e^*)^\otimes N \rightarrow [e^+] \). The central extension \( E \) of \( Y \) by \( k^* \) is characterized up to unique isomorphism as being a central extension of \( Y \) by \( k^* \), provided with maps \( \left[ \_ \right]: U^{**} \rightarrow \{ \text{liftings of } \alpha^\vee \} \) and \( U^{-*} \rightarrow \{ \text{liftings of } \alpha^- \} \), obeying (11.1.9), (11.1.10).

11.2. Suppose that \( G \) is a split semi-simple simply-connected algebraic group, with split maximal torus \( T \). We will use the notation of 4.1. For each root \( \alpha \), we denote by \( T_\alpha \) the maximal torus \( \alpha^\vee(G_\alpha) \) of the \( SL(2) \)-subgroup \( S_\alpha \), and define \( N_\alpha := N(T_\alpha) - T_\alpha \).
in $S_\alpha$. Let $E$ be the central extension of $G$ by $K_2$ defined by a quadratic form $Q$ on $Y$, and $E$ be the corresponding central extension of $Y$ by $k^\times$. Let us apply 11.1 to $S_\alpha$. We obtain a canonical section $n_7! \sim n_o$ of $E$ over $N_1\alpha$; the choice of $n_\alpha$ in $N_1\alpha$ defines a section $s[n_\alpha]$ of $E$ over $T\alpha$; $U_a := U_a - \{e\}$ maps to the inverse image of $\alpha^\gamma$ in $E$, by a map $[\ ]$, and for $e_\alpha$ in $U_a$, one has

$$[n_\alpha e_\alpha] = a^{-Q(e_\alpha)}. [e_\alpha]. \quad (11.2.1)$$

**Proposition 11.3.** — With the notations of 11.2, for $n_\alpha$ in $N_1\alpha$, for $t$ in $T$, and for $\tilde{t}$ a lifting of $t$ in $E$, one has

$$\int[n_\alpha](\tilde{t}) = \tilde{t}. s(n_\alpha)(\alpha^\gamma(\alpha(\tilde{t})^{-1})). \quad (11.3.1)$$

**Proof.** — One has

$$\int[n_\alpha](\tilde{t}) = \tilde{n_\alpha}\tilde{n_\alpha}^{-1} = \tilde{t}(\tilde{t}^{-1}n_\alpha\tilde{n_\alpha}^{-1}) = \tilde{t}(\int[\tilde{t}^{-1}]n_\alpha, \tilde{n_\alpha}^{-1}).$$

As $\int[\tilde{t}]$ respects $N_1\alpha$ and the canonical section of $E$ over $N_1\alpha$, the second factor is of the form $s[n_\alpha](a)$, for some $a$ in $T_\alpha$. The conjugation by $n_\alpha$ acts on $T$ as the Weyl group reflection $s_\alpha$, and $s_\alpha(t) = t.\alpha^\gamma(\alpha(t)^{-1})$. This gives $u = \alpha^\gamma(\alpha(t)^{-1})$ and (11.3.1).

We will not need the next proposition, which could be used to give a complete description of the restriction of $E$ to the normalizer $N(T)$ of $T$. We choose a simple root system $I$. For simple roots $\alpha$ and $\beta$, let $m_{\alpha\beta}$ be the order of $s_\alpha s_\beta$. We write $\prod(m; x, y)$ for a product $xyxy...$ with $m$ factors.

**Proposition 11.4.** — Let $\alpha$ and $\beta$ be distinct simple roots. For $n_\alpha$ in $N_1\alpha$, and $n_\beta$ in $N_1\beta$, one has in $E$ the braid group relation

$$\prod(m_{\alpha\beta}; \tilde{n_\alpha}, \tilde{n_\beta}) = \prod(m_{\alpha\beta}; \tilde{n_\beta}, \tilde{n_\alpha}). \quad (11.4.1)$$

**Proof.** — The proof is inspired by Tits (1966) 2.4. Define $\beta'$ to be $\beta$ when $m_{\alpha\beta}$ is even, $\alpha$ when $m_{\alpha\beta}$ is odd, and let $q$ and $q'$ be, respectively, the left and right sides of (11.4.1). One has

$$q^{-1} q' = q^{-q^{-1}} q^{-1}.$$  

By loc. cit, $q$ and $q'$ have the same image in $G$, i.e. $n_\beta = qn_\beta^{-1} q^{-1}$. As the system of sections $\tilde{n}$ of $E$ over the $N_1\alpha$ is respected by passage to the inverse and by conjugation by $N(T)$, one has $\tilde{n_\alpha} = q\tilde{n_\alpha}^{-1} q^{-1}$ and $q = q'$. 


Lemma 11.5. — With the notation of 11.2 and 4.1, $Y$ admits the following presentation. 
Generators: the coroots. Relations:

\[(11.5.1) \quad \alpha \beta = \beta \alpha.
\]

Proof. — The coroots form a root system $\Phi^*$, the dual of $\Phi$. By Bourbaki Lie VI §1 no. 6 corollary 2 to proposition 9, applied to $\Phi^*$, all relations among the coroots follow from those of the form $\alpha + \beta + \gamma = 0$. The first, i.e. the relations $(-\alpha) = -\alpha$, are the relations (11.5.1) for $\gamma = \alpha$. If $\alpha + \beta + \gamma = 0$, then $\alpha$, $\beta$, and $\gamma$ belong to a subroot system of type $A_2$, $B_2$, or $G_2$, and we leave it to the reader to check that in the roots systems $A_2$, $B_2$, and $G_2$, any relation among coroots is, up to permutation of $\alpha$, $\beta$, and $\gamma$, of the form $\alpha + \beta + (-\alpha) = 0$.

11.6. With notation as in 11.2, the central extension $E$ of $Y$ by $k^*$ is deduced from an extension of $T(k((\tau)))$ by $K_2(k((\tau)))$.

In $S_\alpha$, fix $e_\alpha$, $e_{-\alpha}$ and $n_\alpha$ in Tits trijection. For any root $\beta$ and for $e_\beta$ in $U^*_\beta$, one has by 11.3

\[
\int(n_\alpha)(e_\beta) = \int(n_\alpha)([e_\beta]) = [e_\beta].(\text{projection in } E \text{ of } s[n_\alpha](\alpha^{-\beta})(\tau\beta)).
\]

Applying (11.1.4), (11.1.5), one obtains

\[
(11.6.1) \quad \int(n_\alpha)(e_\beta) = [e_\beta][\epsilon_\alpha^{-\beta}(-1)^{\epsilon(N)\alpha^\tau}].
\]

For $\alpha = \beta$ and $\epsilon_\alpha = e_\beta$, this gives back (11.1.10).

Proposition 11.7. — Under the assumption of 11.2, the central extension $E$ of $Y$ by $k^*$ defined by $Q(4.16)$ is characterized up to unique isomorphism as being a central extension of $Y$ by $k^*$, provided for each root $\alpha$ with a map $[\ ]$ from $U^*_\alpha$ to the inverse image of $\alpha$, obeying (3.11.1) (11.2.1) and (11.6.1).

The unicity claim results from 11.5.

This description extends by Galois descent to the non split case.

11.8. Let $G$ be a split reductive group, with split maximal torus $T$. We will use the notations of 6.1. Let $E$ be a central extension of $G$ by $K_2$, and $E_{sc}$ its pullback to $G_{sc}$. We write $q$ for the projection from $G_{sc}$ to $G$, and from $E_{sc}$ to $E$. 


Proposition 11.9. — With the notations of 11.8, for \( n_\alpha \in N_1 \subset G_{sc} \) and for \( t \) in \( T \), lifted to \( \tilde{t} \) in \( E \), one has

\[
\text{int}[q(n_\alpha)](\tilde{t}) = \tilde{t}q(s[n_\alpha](\alpha^\vee(\alpha^\vee)^{-1})).
\]

Proof. — For \( g \) in \( G \), with image \( g_{\text{ad}} \) in \( G_{\text{ad}} \), and \( h \) in \( G_{sc} \), the commutator \((g, qh)\) in \( E \) is the image by \( q \) of the commutator \((g_{\text{ad}}, h)\) of 0.N.4 (3). Indeed, the difference between the two commutators is given by a section of \( K_2 \) on \( G \times G_{sc} \), vanishing for \( h = e \), and one applies (4.8.3) to \( a: G \times G_{sc} \to G \). One can then repeat the proof of 11.3.

11.10. The group \( G \) acts on \( E \) by inner automorphisms. The induced action of the normalizer \( N(T) \) of \( T \) respects \( T \), hence acts on the central extension \( \mathcal{E} \) of \( Y \) by \( k^* \) attached to \( E \). Here is how to describe the action of \( N(T) \) on \( \mathcal{E} \) in terms of the triple \((Q, \mathcal{E}, \phi)\) attached to \( E \), and of the description 11.7 of the central extension \( \mathcal{E}_{sc} \) of \( Y_{sc} \) by \( k^* \).

Proposition 11.11. — With the notation of 11.10, let \( B \) be the bilinear form associated to \( Q \).

(i) For \( y \) in \( Y \), the linear form \( B(y, \cdot) \) on \( Y \) can be identified with a character of \( T \). For \( t \) in \( T \), and \( \tilde{y} \) in \( E \) lifting \( y \), the inner automorphism \( \text{int}[t] \) acts on \( \mathcal{E} \) by

\[
\text{int}[t](\tilde{y}) = \tilde{y}B(y, t^{-1}).
\]

(ii) For \( e_\alpha, e_{-\alpha} \) and \( n_\alpha \) in Tits trijection,

\[
\text{int}[n_\alpha](\tilde{y}) = \tilde{y}\phi([e_\alpha], (-1)^{\varepsilon(-\alpha^\vee)}Q\alpha^\vee)).
\]

Proof. — The formula (11.11.1) follows from the commutator formula 3.14 for \((t, y(\tau))\). The formula (11.11.2) follows from (11.9.1) applied to any lifting \( \tilde{t} \) of \( y(\tau) \). Cf. the proof of (11.6.1).

11.12. As \( N(T) \) is generated by \( T \) and the images of the \( N_1 \), 11.11 determines the action of \( N(T) \) on \( \mathcal{E} \). If \( T' \) is another split maximal torus of \( G \), 6.2 gives an equivalence from the category of central extensions of \( G \) by \( K_2 \) with the category of triples \((Q, \mathcal{E}, \phi)\) for \( T \), or for \( T' \). The resulting equivalence

triples \((Q, \mathcal{E}, \phi)\) for \( T \to \) triples \((Q', \mathcal{E}', \phi')\) for \( T' \)

can be constructed directly as follows. Let \( P \) be the set of \( g \) in \( G(k) \) conjugating \( T \) to \( T' \). It is a \( N(T)(k) \)-torsor. Any \( g \) in \( P \) transforms \( Q \) into \( Q' \). The group \( N(T)(k) \) acts on
by 11.11, and $\mathcal{E}'$ is deduced from $\mathcal{E}$ by twisting by $P$. The same rule applies to $\mathcal{E}_{\infty}$, and $\phi'$ is obtained by twisting as well.

This can be repeated over a base $S$ regular of finite type over a field, and localized for the étale topology. It shows how the description 7.2 of central extensions of $G$ by $K_2$ by triples $(Q, \mathcal{E}, \phi)$ depends on the chosen maximal torus $T$.

12. Examples

12.1 $SL(2)$: Let $G$ be a semi-simple simply-connected algebraic group over a field $k$, and let $E$ be a multiplicative $K_2$-torsor on $G$. Arguing as in the proof of 4.9, one finds the following description for the central extension $E(k)$ of $G(k)$ by $K_2(k)$: it is the group of automorphisms of the pair (space $G$, $K_2$-torsor $E$ on $G$), acting on $G$ as a left translation.

Take $G = SL(2)$. Let $U$ be the unipotent subgroup of matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. The quotient $SL(2)/U$ is identified with the punctured affine plane $A^2 - \{0\}$ by $g \mapsto g.(1,0)$. The projection $p: SL(2) \to SL(2)/U = A^2 - \{0\}$ is a $U$-torsor. The pull-back functor $p^*$ is hence an equivalence of categories from $K_2$-torsors on $A^2$ to $K_2$-torsors on $SL(2)$. Applying 4.7, we conclude:

**Proposition 12.2.** Central extensions $E$ of $SL(2)$ by $K_2$ correspond one to one to isomorphism classes of $K_2$-torsors on $A^2 - \{0\}$. For the central extension $E$ corresponding to $P$, $E(k)$ is the group of $g$ in $SL(2, k)$ given with a lifting of their action on $A^2 - \{0\}$ to an action on $(A^2 - \{0\}, P)$.

The punctured affine plane $A^2 - \{0\}$ is covered by the two open sets $x \neq 0$ and $y \neq 0$. The group $H^1(A^2 - \{0\}, K_2)$ of isomorphism classes of $K_2$-torsors on $A^2 - \{0\}$ is $\mathbb{Z}$, generated by the Čech cocycle $\{x,y\}$ on the intersection $x \neq 0, y \neq 0$.

12.3. Let $G$ be a reductive group over $k$, with maximal torus $T$, and $\rho: G \to SL(V)$ be a unimodular representation of $G$. Let $E_0$ be the central extension of $SL(V)$ by $K_2$ corresponding to the quadratic form $Q_0$ taking the value 1 on each coroot. By pullback to $G$, it gives a central extension $E_V$ of $G$ by $K_2$. To compute the quadratic form $Q$ on $Y$ defined by $E_V$, we may first extend the scalars to a splitting field for $T$: we may and shall assume that $T$ is split.

In characteristic zero, one can identify $Y$ with a subgroup of the Lie algebra $t$ of $T$, by attaching to $y: G_m \to T$ the element $y(1)$ of $t$. The quadratic form $Q_0$ is then induced by the quadratic form $\frac{1}{2} \text{Tr}(x^2)$ on the Lie algebra $sl(V)$, and the quadratic form $Q$ is hence induced by the quadratic form $\frac{1}{2} \text{Tr}(\rho(x)^2)$ on the Lie algebra of $G$. 
In any characteristic, similar arguments show that
\[ Q(y) = \frac{1}{2} \sum \mu(y)^2, \]
where the sum runs over the weights \( \mu \) of \( V \), counted with their multiplicity.

Suppose \( G \) split simple and simply connected, fix a simple root system, let \( \alpha \) be the largest root and let \( \alpha' = \sum n_i \alpha' \) be the decomposition of the coroot \( \alpha' \) into simple coroots. When \( V \) is the adjoint representation, \( Q(\alpha') \) is twice the dual Coxeter number \( h' = 1 + \sum n_i \) of \( G \). One can case by case check that for any \( V \), each \( n_i \) divides \( Q(\alpha') \) and that for \( V \) nontrivial of the smallest possible dimension, \( Q(\alpha') \) is the l.c.m. of the \( n_i \).

12.4 PGL(\(n\)): Let \( T \) be the image in PGL(\(n\)) of the torus \( \Gamma_1 \) of diagonal matrices in GL(\(n\)). The cocharacter group \( Y \) of \( T \) is the quotient of \( Y_1 = \mathbb{Z}^n \) by the diagonal \( \mathbb{Z} \). The Weyl group invariant integer-valued quadratic forms \( Q \) on \( Y \) are the Weyl group invariant integer-valued quadratic forms on \( \mathbb{Z}^n \), which vanish on the diagonal \( \mathbb{Z} \). For \( n \) even, the minimal positive \( Q \) is
\[ Q(y) = n \sum y_i^2 - \left( \sum y_i \right)^2. \]
For \( n \) odd, it is \( 1/2 Q \). One has \( Q(\alpha') = 2n \). The dual Coxeter number is \( n \), and \( Q \) is obtained as in 12.3, for the adjoint representation.

12.5 GL(\(n\)). Let \( T \) be the split maximal torus of diagonal matrices in GL(\(n\)). On \( Y = \mathbb{Z}^n \), take the quadratic form
\[ Q(y) = \frac{1}{2} \left( \sum y_i^2 - \left( \sum y_i \right)^2 \right). \]
The construction of the §7 of Deligne (1996) agrees with those of §11. Applying loc. cit. 7.10, one finds that for \( G \) an inner form of GL(\(n\)), to give a central extension of \( G \) by \( K_2 \) corresponding to the form \( Q \) amounts to giving a representation \( V \) of \( G \), which is a form of the defining representation of GL(\(n\)). Such a representation exists only if \( G \) is isomorphic to GL(\(n\)).

12.6. Over \( \mathbb{R} \), any torus is a product of tori \( T \) of the following three types: \( \mathbb{G}_m \), \( U^1 \) and \( \prod \mathbb{G}_m \), with groups of points \( \mathbb{R}^* \), the circle group and \( \mathbb{C}^* \). The corresponding \( Y \) are \( \mathbb{Z} \) with the trivial action of Gal(\(\mathbb{C}/\mathbb{R}\)), \( \mathbb{Z} \) with the sign action: \( F_\infty(n) = -n \) and \( \mathbb{Z} \oplus \mathbb{Z} \), with \( F_\infty \) interchanging \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). For \( E \) a central extension of \( T \) by \( K_2 \), we will determine the resulting topological central extensions of the groups of real points by \( \mu_\mathbb{R} = \{ \pm 1 \} \) (10.3).
In the split case: \( T = G_m \), isomorphism classes of central extensions by \( K_2 \) correspond to quadratic forms \( Q \) on \( \mathbb{Z} \). For \( Q(n) = qn^2 \), the central extension is given by the cocycle \( (x, y)^\phi \), with \( (x, y) = -1 \) if \( x, y < 0 \) and \( (x, y) = 1 \) otherwise. It is trivial for \( q \) even, and for \( q \) odd, liftings of \(-1 \in \mathbb{R}^*\) have order 4.

For \( T = U^1 \) or \( \prod_{C/\mathbb{R}} G_m \), \( T(R) \) is connected and central extensions by \( \mu_R \) have no nontrivial automorphism. The group \( \text{Hom}(\pi_1, \mu_R) \) of their isomorphism classes is \( \mathbb{Z}/2 \), and the question is to determine which central extensions \( E \) of \( T \) by \( K_2 \) give the trivial (resp. nontrivial) central extension \( E(R) \) of \( T(R) \) by \( \mu_R \). The inclusion of \( U^1 \) in \( C \) induces an isomorphism on \( \pi_1 \), and this reduces the case of \( \prod_{C/\mathbb{R}} G_m \) to that of \( U^1 \).

Central extensions of \( T \) by \( K_2 \) are classified by pairs \( (Q, E) \) (7.2) and, for \( T = U^1 \), the group of isomorphism classes of pairs \( (Q, E) \) is \( \mathbb{Z} \times \mathbb{Z}/2 \). The first factor gives \( Q \): \( qn \) the form \( qn^2 \). The second factor gives \( E \): the central extension \( E \) is commutative, and equivariant extensions of \( \mathbb{Z} \), with the sign action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \), by \( C^* \) are classified by

\[ H^1(\mathbb{R}, C^* \text{ with } F_\infty \text{ acting by } z \mapsto z^{-1}) = \mathbb{Z}/2. \]

If \( u \) is a lifting in \( E \) of 1, \( \tilde{u} \) is a lifting of \(-1\), \( u\tilde{u} \) is in \( \mathbb{R}^* \subset C^* \) and the extension \( E \) is trivial if and only if \( u\tilde{u} > 0 \). We have to compute the homomorphism

\[ \mathbb{Z} \times \mathbb{Z}/2 \rightarrow \text{Hom}(\pi_1(U^1), \mu_R) = \mathbb{Z}/2; (Q, E) \mapsto \text{extension } E(R). \]

Let us embed \( U^1 \) in \( SL(2) \) (resp. \( SU(2) \)), and let \( E_0 \) be the central extension of \( SL(2) \) (resp. \( SU(2) \)) by \( K_2 \) corresponding to the quadratic form \( Q_0 \) taking the value 1 on coroots. For the induced central extension \( E \) of \( U^1 \), \( q = 1 \).

Case of \( SL(2) \): the action of \( SL(2, \mathbb{R}) \) on \( R^2 - \{0\} \) lifts to an action of \( E_0(R) \) on the double covering of \( R^2 - \{0\} \) (cf. 12.1), and \( E(R) \) is the nontrivial double covering of \( U^1 \). If \( e_0 \) is an eigenvector of \( U^1 \) acting on \( R^2 \otimes C \), \( e_0 \) and \( \tilde{e}_0 \) form a basis of \( R^2 \otimes C \). Take \( (e^+, e^-, n) \) to be the triple (11.1.1), in this basis. One has \( e^- = -\tilde{e}^+ \), where \( -\tilde{e}^+ \) refers to the vector space structure (11.1.6). For \( u = [e^+] \), it follows from 11.1.9 and 11.1.10 that \( u\tilde{u} = 1 \): the extension \( E \) is trivial.

Case of \( SU(2) \): As the topological group \( SU(2) \) is simply-connected, the central extension \( E_0(R) \) is trivial, and so is the central extension \( E(R) \) of \( U^1 \). It is not isomorphic to the central extension of \( U^1 \) obtained by embedding \( U^1 \) into \( SL(2, \mathbb{R}) \). The same must hold for the extension \( E \), which gives rise to it: the extension \( E \) must be nontrivial.

From those two examples, it follows that

**Proposition 12.7.** — The morphism (12.6.1) is

\[ (q, n) \mapsto q + n. \]
12.8. Let $K$ be a local field, by which we mean: the field of fractions of a complete discrete valuation ring. Let $V$ be the valuation ring, and $k$ be the residue field. Let $E$ be a central extension by $K_2$ of an algebraic group $G$ over $K$. Pushing the central extension $E(K)$ by the residue map $K_2(K) \to k^*$, we obtain a central extension of $G(K)$ by $k^*$, which we denote by $\tilde{G}(k)$:

\[(12.8.1) \quad k^* \to \tilde{G}(k) \to G(K).\]

Let $K'$ be an unramified Galois extension of $K$, with valuation ring $V'$ and residue field $k'$. The field extension $k'/k$ is Galois, with $\text{Gal}(K'/K) \stackrel{\sim}{\longrightarrow} \text{Gal}(k'/k)$. Let $G'$, $E'$ be deduced from $G$, $E$ by extension of scalars and write $\tilde{G}(k')$ for the resulting central extension (12.8.1). The determination of $\tilde{G}(k)$ is reduced to that of $\tilde{G}(k')$ by the

**Proposition 12.9.** — One has $\tilde{G}(k) \sim \tilde{G}'(k')^{\text{Gal}(K'/K)}$.

*Proof.* — As $k$ and $G(K)$ are the groups of invariants of $\text{Gal}(K'/K)$ acting on $k'$ and $G(K')$, respectively, this follows from the commutativity of the diagram

\[
\begin{array}{c}
1 \to k^* \to \tilde{G}(k) \to G(K) \to 1 \\
\downarrow \quad \downarrow \\
1 \to k'^* \to \tilde{G}(k') \to G(K') \to 1.
\end{array}
\]

12.10. Suppose that $K = k((t))$, and that $G$ is absolutely simple and simply-connected, and splits over some unramified extension $k'((t))$ of $K$. In this situation, Garland (1980) constructed a central extension of $G(K)$ by $k^*$. It agrees with the central extension 12.8.1, for $E$ defined by the quadratic form $Q$ taking the value 1 on small coroots. Indeed, both obey 12.9, and for $G$ split, Garland’s extension is derived from Matsumoto’s central extension by pushing by $K_2(K) \to k^*$.

Assume now that $G$ is the general fiber of a group scheme $G_V$ smooth over $V$, with special fiber $G_s$.

12.11. *Construction.* — From the central extension $E$ of $G$ by $K_2$ we will construct a central extension $\tilde{G}_s$ of $G_s$ by $G_m$ such that the central extension $G_V(V)^\sim$ restriction of (12.8.1) to $G_V(V)$ is the pull back of $\tilde{G}_s(k)$; commutativity of the diagram

\[
\begin{array}{c}
k^* \longrightarrow G_V(V)^\sim \longrightarrow G_V(V) \\
\| \quad \| \\
k^* \longrightarrow \tilde{G}_s(k) \longrightarrow G_s(k).
\end{array}
\]

(12.11.1)

In this construction, we will treat central extensions as multiplicative torsors.
Lemma 12.12. — Let $X_V$ be a smooth scheme over $V$, with general and special fibers $j : X_\eta \hookrightarrow X_V$ and $i : X_s \hookrightarrow X_V$. Let $E$ be a $K_2$-torsor on $X_\eta$, for which

(*) each point of $X_s$ has a neighborhood $U$ in $X$ such that $E$ admits a section over $U_\eta = U \cap X_\eta$.

To $E$, we will attach a $G_m$-torsor $\bar{E}$ over $X_s$, by a construction compatible with a pull back $Y_V \to X_V$, with the addition of torsors, and which for $X_V = V$ is simply pushing a $K_2(K)$-torsor by $\text{Res} : K_2(K) \to K_1(k) = k^\times$.

Proof. — The assumption (*) amounts to saying that $j_*E$ is a $j_*K_2$-torsor. The residue map in $K$-theory: $K_2(U \cap X_\eta) \to K_1(U \cap X_s)$ for $U \not\subseteq X$, induce a morphism of sheaves $j_*K_2 \to i_*K_1 = i_*G^\times$. Pushing $j_*K_2$ by this morphism, we obtain a $i_*G^\times$-torsor on $X$ or, what amounts to the same, a $G^\times$-torsor on $X_s$, i.e. a $G_m$-torsor over $X_s$. We leave the compatibilities to the reader.

Proof of 12.11. If condition (*) of 10.12 holds for $E$, the multiplicative structure of $E$ induces one on $\bar{E}$, which is the required multiplicative $G_m$-torsor. Its formation is compatible with an étale extension $V' \to V$ and, by Galois descent, it suffices to show that (*) holds after such an “extension of the residue field” in $V$.

The $K_2$-torsor $E$ is trivial over some Zariski dense open subset $U$ of $G$. Define $U_V := G_V - (G - U)$. The closure $(G - U)$ is flat, of relative dimension strictly smaller than that of $G$, and it follows that the special fiber $U_s$ of $U_V$ is dense in $G_s$. The property (*) holds at each point of $U_s$: it holds for $U_V$. By the multiplicativity of $E$. It also holds at the points of any translate of $U_s$ by an element $g$ of $G_s(k)$. Indeed, $V$ being henselian, $g$ can be lifted to $\tilde{g}$ in $G_V(V)$ and (*) holds for $\tilde{g}U_V$. After a finite extension of $k$, such translates cover $G_s$.

Questions 12.13. — (i) Suppose that $G$ is reductive, and that $E$ is given as in 7.2, for $T$ a maximally split maximal torus of $G$. Suppose that $G_\nu$ is given as in Bruhat Tit (1984) 4.6. It would be interesting to compute the central extension $G_\nu$ in that case, especially for $G_V(V)$ a maximal bounded subgroup of $G(K)$, given by a vertex of the building of $G$.

(ii) Suppose that $V = k[[t]]$ and that $G$ is a split reductive group. How to relate the point of view where central extensions of $G(K)$ by $k^\times$ are viewed as infinite dimensional groups of $k$, corresponding to affine Cartan matrices, with our point of view of central extensions by $K_2$, classified by 6.2?

(iii) For $V = k[[t]]$, not all natural central extensions by $k^\times$ are captured by 12.8. For example, if $G$ is the multiplicative group, $G(K) = K^\times$ acts projectively on the semi-infinite exterior algebra of $k(\langle \hat{t} \rangle)$, relative to $k[[t]]$, giving rise to a central extension $E$ of $K^\times$ by $k^\times$. For $x$ in $K^\times$, the line

$$[xk[[t]] : k[[t]]] = \det(xk[[t]]/\Lambda)\det(k[[t]]/\Lambda)^{-1}$$
is independent of \( \Lambda \), supposed to contain \( k[[t]] \) for \( N \) large enough and to be contained in \( k[[t]] \) as well as in \( xk[[t]] \). A lifting of \( x \) to \( E \) is the choice of a generator of \( [xk[[t]]: k[[t]]] \). The commutator \( 0.N.4(2) \) defined by \( E \) is, up to a sign, the tame symbol. See E. Arbarello, C. De Concini and V. Kac, The infinite wedge representation and the reciprocity law for algebraic curves, in: Theta functions, Proc. Symp. Pure Math, 49. 1 AMS (1987), p. 171-190. The square of this commutator is the commutator for the central extension (12.8.1) defined by the tame symbol used as a cocycle. We expect that “natural” central extensions of \( G(K) \) by \( k^* \) are attached to data as follows: a Weyl group and Galois group invariant integer-valued symmetric bilinear form \( B \) on \( Y \), even on \( Y_{sc} \); a central extension \( \mathcal{E} \) to \( Y \) by \( \mathbb{G}_m \), for which the commutator is

\[
(y_1, y_2) = (-1)^{|B(j_1, j_2)+\epsilon(j_1)\epsilon(j_2)}
\]

for \( \epsilon(y) \equiv B(y, y) \mod 2; \phi: \mathcal{E}_{sc} \to \mathcal{E} \) as in 6.2.

Remarks 12.14. — (i) 12.11 holds for any discrete valuation ring \( V \), complete or not: if \( \hat{V} \) is the completion of \( V \), and if \( G_{\hat{V}} \) is deduced from \( G_V \) by base change, the central extension \( G_{\hat{V}} \) attached by 12.11 to \( G_{\hat{V}} \) gives rise to a diagram 12.11.1 for \( G_V \).

(ii) The property \((\ast)\) of 12.12 holds, as shown by the proof of 12.11, as soon as \( V \) is henselian and \( G_{\hat{k}} \) Zariski dense in \( G \).

(iii) Suppose that \( V \) is henselian and essentially of finite type over a field. For \( j \) (resp. \( i \)) the inclusion of \( G \) (resp. \( G_i \)) in \( G_V \), Quillen resolution gives a short exact sequence of sheaves on \( G_V \)

\[
0 \to K_2 \to j_! K_2 \to i_* K_1 \to 0,
\]

and a trivialization of the central extension \( G_{\hat{V}} \) of 12.11 defines an extension of \( E \), as a multiplicative \( K_2 \)-torsor, over \( G_V \).

12.15. Let \( X \) be a projective and smooth curve over a field \( k \). Let \( K \) be its field of rational functions. For \( x \) a closed point of \( X \), let \( V_x \) be the completion of the local ring of \( X \) at \( x \), and \( K_x \) be its field of fractions, the completion of \( K \) at \( x \).

For \( G \) a linear algebraic group over \( K \), we write \( G(A) \) for the restricted product of the \( G(K_x) \). It is also the group of \( A \)-points of \( G \), for \( A = \prod K_x \).

Let \( E \) be a central extension of \( G \) by \( K_2 \). For some Zariski dense open subset \( U \) of \( X \), \( G \) extends as an affine group scheme of finite type \( G_U \) over \( U \), and \( E \) as a multiplicative \( K_2 \)-torsor over \( G_U \). By extension of scalars to \( K_x \), \( E \) defines central extensions (12.8.1) of \( G(K_X) \) by \( k^* \), split over \( G(V_x) \) for \( x \) in \( U \). For \( U_1 \subset U \) Zariski
dense, we get commutative diagrams

\[
\begin{array}{cccc}
\text{H}^0(U_1, K_2) & \rightarrow & \text{G}(U_1) & \rightarrow \\
\downarrow & & \downarrow & \downarrow \\
\prod_{x \not\in U_1} k(x)^* & \rightarrow & \prod_{x \not\in U_1} \text{G}(V_x) \times \prod_{x \not\in U_1} \text{K}(x) & \rightarrow \\
& & \prod_{x \not\in U_1} \text{G}(V_x) \times \prod_{x \not\in U_1} \text{K}(x) & .
\end{array}
\]

with inductive limit

\[
\begin{array}{cccc}
1 & \rightarrow & K_2(K) & \rightarrow \\
& & \downarrow & \downarrow \\
& & G(K) & \rightarrow \\
& & \downarrow & \downarrow \\
1 & \rightarrow & \oplus k(x)^* & \rightarrow \\
& & \downarrow & \downarrow \\
& & G(A) & \rightarrow \end{array}
\]

(12.15.1)

By the reciprocity law for tame symbols, the first vertical arrow has its image in the kernel of the product of the norm maps N: \(\oplus k(x)^* \rightarrow k^*\). Pushing by N, we hence get from (12.15.1) a central extension of \(G(A)\) by \(k^*\), canonically split over \(G(K)\):

\[
\begin{array}{ccc}
& & G(K) \\
& & \downarrow \\
k^* & \rightarrow & G(A) \rightarrow G(A)
\end{array}
\]

(12.15.2)

the geometric analogue of 10.4.3. For \(G\) simply-connected, it is also an analogue, in algebraic geometry, of the analytic constructions of Segal (1988) and Brylinski-McLaughlin (1994).

As in 12.13(iii), we don’t obtain all natural diagrams (12.15.2) by this method.

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