COXETER GROUPS, SALEM NUMBERS AND THE HILBERT METRIC

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CONTENTS

1. Introduction

The shortest loop traced out by a billiard ball in an acute triangle is the pedal subtriangle, connecting the feet of the altitudes.

In this paper we prove a similar result for loops in the fundamental polyhedron of a Coxeter group W, and use it to study the spectral radius $\lambda(w)$, $w \in W$ for the geometric action of W. In particular we prove:

Theorem **1.1.** — *Let* (W, S) *be a Coxeter system and let* $w \in W$. *Then either* $\lambda(w) = 1$ *or* $\lambda(w) \geq \lambda_{Lehmer} \approx 1.1762808$.

Here λ_{Lehmer} denotes *Lehmer's number*, a root of the polynomial

$$1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10} \tag{1.1}$$

and the smallest known Salem number.

Billiards. — Recall that a *Coxeter system* (W, S) is a group W with a finite generating set $S = \{s_1, \ldots, s_n\}$, subject only to the relations $(s_i s_j)^{m_{ij}} = 1$, where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. The permuted products

$$s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n} \in W, \quad \sigma \in S_n,$$

are the *Coxeter elements* of (W, S). We say $w \in W$ is *essential* if it is not conjugate into any subgroup $W_I \subset W$ generated by a proper subset $I \subset S$.

Research supported in part by the NSF.

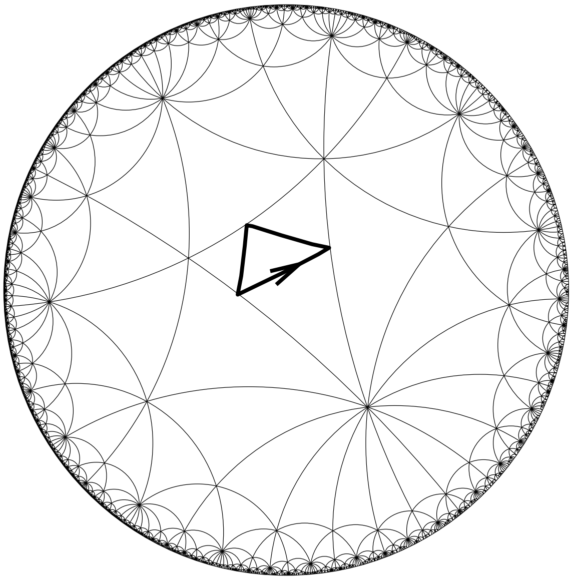


FIG. 1. — The shortest billiard loop in the (3, 4, 7)-triangle

The Coxeter group W acts naturally by reflections on $V \cong \mathbf{R}^S$, preserving an inner product $B(v, v')$. Let $\lambda(w)$ denote the spectral radius of $w|V$. When $\lambda(w) > 1$, it is also an eigenvalue of w. We will show (§4):

Theorem **1.2.** — *Let* (W, S) *be a Coxeter system and let* $w \in W$ *be essential. Then we have* $\lambda(w) \geq \inf_{S_n} \lambda(s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n})$.

Here is the relation to billiards. In the case of a hyperbolic Coxeter system (when (V, B) has signature $(p, 1)$), the orbifold $Y = \mathbf{H}^p/W$ is a convex polyhedron bounded by mirrors meeting in acute angles. Closed geodesics on Y can be visualized as loops traced out by billiards in this polyhedron. The hyperbolic length of the geodesic in the homotopy class of $w \in \pi_1(Y) = W$ is given by $\log \lambda(w)$. Thus the theorem states that the essential billiard loops in Y are no shorter than the shortest Coxeter element.

As a special (elementary) case, the shortest billiard loop in the (p, q, r)-triangle in $\mathbf{H}^2$ is the pedal subtriangle representing $w = s_1 s_2 s_3$; see Figure 1.

The Hilbert metric on the Tits cone. — To prove Theorem 1.2 for higher-rank Coxeter groups (signature (p, q), $q \geq 2$), we need a generalization of hyperbolic space. A natural geometry in this case is provided by the Hilbert metric on the Tits cone.

The Hilbert distance on the interior of a convex cone K is given in terms of the cross-ratio by $d_K(x, y) = (1/2) \inf \log[a, x, y, b]$, where the infimum is over all segments $[a, b]$ in K containing $[x, y]$; it is a metric when K contains no line. We will show (§2) that the translation length of a linear map T preserving K satisfies

$$\inf_{x \in K^\circ} d_K(x, Tx) = \log \lambda(T),$$

provided $\lambda(T) = \lambda(T^{-1})$.

The Tits cone $W \cdot F \subset V^*$ is the orbit, under the dual action of W, of a simplicial cone F which forms a fundamental domain for W. When (W, S) is hyperbolic or higher-rank, $K = \overline{W \cdot F}$ contains no line, so d_K is a metric. At the same time, $\log \lambda(w)$ is the translation length of w in the Hilbert metric on K, so this geometry can be used to study eigenvalues.

We propose $(\mathbf{P}K^\circ, d_K)$ as a natural generalization of the Klein model for projective space to higher-rank Coxeter groups (§3). Once this geometry is in place, the proof of Theorem 1.2 is based on the fact that a loop representing an essential element w must touch all the faces of the fundamental domain F (§4).

The bicolored eigenvalue. — Next we give a succinct lower bound for the spectral radius of Coxeter elements.

The Coxeter diagram D of (W, S) is the weighted graph whose vertices are the set S, and whose edges of weight m_{ij} join s_i to s_j when $m_{ij} \geq 3$. If D is a tree (such as one of the familiar spherical diagrams A_n, B_n, D_n or E_n), then the Coxeter elements $w \in W$ range in a single conjugacy class. When D has cycles, however, many different conjugacy classes (and different values of $\lambda(w)$) can arise.

When every cycle has even order (so D is bipartite), a special role is played by the *bicolored* Coxeter elements. These are defined by $w = \prod S_1 \prod S_2$, where $S = S_1 \sqcup S_2$ is a two-coloring of the vertices of D.

All bicolored Coxeter elements are conjugate. The value of $\lambda(w)$ they share can be computed directly, as follows. Let $\alpha(W, S)$ be the leading eigenvalue of the *adjacency matrix* of (W, S), defined by $A_{ij} = 2\cos(\pi/m_{ij})$ for $i \neq j$ and $A_{ii} = 0$. Let $\beta = \beta(W, S) \geq 1$ be the largest root of the equation

$$\beta + \beta^{-1} + 2 = \alpha(W, S)^2,$$

provided $\alpha(W, S) \geq 2$. Set $\beta(W, S) = 1$ if $\alpha(W, S) < 2$. Then $\lambda(w) = \beta(W, S)$ for all bicolored Coxeter elements.

The above definition of the *bicolored eigenvalue* $\beta(W, S)$ makes sense for any Coxeter system, bipartite or not. We will show in §5:

Theorem **1.3.** — *For any Coxeter system* (W, S)*, we have*

$$\inf_{S_n} \lambda(s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n}) \geq \beta(W, S).$$

In particular the bicolored Coxeter elements, when they exist, minimize $\lambda(s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n})$.

In the hyperbolic and higher-rank cases, it is easy to see that $\beta(W, S) > 1$; thus every Coxeter element has infinite order. The same conclusion is well-known to hold in the affine case, so we obtain:

Corollary **1.4.** — *A Coxeter group* W *is infinite iff every Coxeter element* $s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n} \in W$ *has infinite order.*

This Corollary was first established in [How].

Minimal hyperbolic diagrams. — There is a natural partial ordering on Coxeter systems that is conveniently visualized in terms of diagrams: we write $(W', S') \geq (W, S)$ if the diagram D' is obtained from D by adding more vertices and edges and/or increasing their weights. A useful feature of the invariant $\beta(W, S)$ is that it is a monotone function: we have

$$(W', S') \geq (W, S) \implies \beta(W', S') \geq \beta(W, S),$$

by elementary properties of positive matrices.

Now suppose $w \in W'$ satisfies $\lambda(w) > 1$. Then (W', S') has indefinite signature, and therefore it dominates a minimal hyperbolic Coxeter system (W, S). In §6 we will show:

Theorem **1.5.** — *There are 38 minimal hyperbolic Coxeter systems, and among these we have* $\inf \beta(W, S) = \lambda_{Lehmer}$.

By monotonicity of β, we have

$$\lambda(w) \geq \beta(W', S') \geq \beta(W, S) \geq \lambda_{\text{Lehmer}},$$

completing the proof of Theorem 1.1.

Small Salem numbers. — The results above suggest using $\beta(W, S)$ as a measure of the complexity of a Coxeter system. We conclude in §7 with a few connections between the simplest Coxeter systems and small Salem and Pisot numbers.

Let $Y_{a,b,c}$ denote the Coxeter system whose diagram is a tree with 3 branches of lengths a, b and c, joined at a single node. For example $E_8 = Y_{2,3,5}$. We will show:

- The smallest Salem numbers of degrees 6, 8 and 10 coincide with $\lambda(w)$ for the Coxeter elements of $Y_{3,3,4}$, $Y_{2,4,5}$ and $Y_{2,3,7}$. (These are the hyperbolic versions of the exceptional spherical Coxeter systems E_6, E_7 and E_8.)
- In particular $\lambda_{\text{Lehmer}} = \lambda(w)$ for the Coxeter elements of $Y_{2,3,7}$.
- The set of all irreducible Coxeter systems with $\beta(W, S) < \lambda_{\text{Pisot}}$ consists exactly of $Y_{2,4,5}$ and $Y_{2,3,n}$, $n \geq 7$. Here $\lambda_{\text{Pisot}} \approx 1.324717$ is the smallest Pisot number; it satisfies $x^3 = x + 1$.

- The infimum of $\beta(W, S)$ over all higher-rank Coxeter systems coincides with λ_{Pisot}.
- There are exactly 6 Salem numbers < 1.3 that arise as eigenvalues in Coxeter groups. Five of these arise from the Coxeter elements of $Y_{2,3,n}$, $7 \leq n \leq 11$. (On the other hand, there are in all 47 known Salem numbers less than 1. 3.)
- The second smallest known Salem number, $\lambda \approx 1.188368$, arises as $\lambda(g)$ for $g \in O^+(II_{17,1})$, but does not arise as $\lambda(w)$ for any w in the index two Coxeter group $W \subset O^+(II_{17,1})$. Here $II_{17,1}$ denotes the unique even unimodular lattice of signature (17, 1).

At the end of §7 we connect the study of $\beta(W, S)$ to the many known results on the leading eigenvalues of graphs.

Notes and references. — E. Hironaka showed that Lehmer's number is the smallest of an infinite family of Salem numbers that arise as roots of Alexander polynomials of certain pretzel knots [Hir]. We observed that these Salem numbers are also the leading eigenvalues of Coxeter elements for the diagrams $Y_{p_1,\ldots,p_n}$, and were led to formulate Theorem 1.1.

It is conjectured that λ_{Lehmer} is the smallest Salem number, and more generally that it has minimal Mahler measure among all algebraic integers (other than roots of unity). This conjecture is confirmed by Theorem 1.1 for those algebraic integers $\lambda(w)$ that arise via Coxeter groups. Many Salem numbers can also be realized as eigenvalues of automorphisms of even, unimodular lattices [GM], but it is unknown if λ_{Lehmer} is a lower bound for the Salem numbers that arise in this way. See [GH] for a recent survey on this topic.

Basic references for Coxeter groups include [Bou], [Ha1] and [Hum]. See [A'C], [BLM], [Co] and [How] for related work on eigenvalues of Coxeter elements. Connections between Salem numbers and growth-rates of reflection groups in $\mathbf{H}^2$ and $\mathbf{H}^3$ are studied in [FP], [Fl], [CW] and [Par]. The pedal triangle is discussed in [RT, §5]. (The existence of billiard loops is an open question for obtuse triangles; see [HH].)

For the convenience of the reader, we have included short proofs of key results from the literature and a summary of the needed background on Coxeter groups.

I would like to thank D. Allcock, B. Gross and E. Hironaka for many informative and useful discussions.

2. Translation length in the Hilbert metric

Let V be a finite-dimensional real vector space. Let $K \subset V$ be a closed, convex cone, such that the interior K° of K is nonempty and K contains no line. Let $T : V \to V$ be a linear automorphism of V with $T(K) = K$, and let $\lambda(T)$ denote the spectral radius of T.

In this section we introduce the Hilbert metric d_K on K° and study the *translation length*

$$\delta(T, K) = \inf_{x \in K^\circ} d_K(x, Tx).$$

We will show:

Theorem **2.1** *(Hilbert length). — The translation length of* T *in the Hilbert metric satisfies*

$$\frac{1}{2} \log \max(\lambda_+, \lambda_-, \lambda_+\lambda_-) \quad \le \quad \delta(T, K) \quad \le \quad \log \max(\lambda_+, \lambda_-),$$

where $\lambda_\pm = \lambda(T^{\pm 1})$.

Corollary **2.2.** — *The translation length is given by* $\delta(T, K) = \log \lambda(T)$ *provided* $\lambda(T) = \lambda(T^{-1})$.

The Hilbert metric. — Let $K \subset V$ be a closed convex set containing no line. Let $[x, y] \subset K$ denote the segment joining $x, y \in K$. The cross-ratio of 4 collinear points is given by

$$[a, x, y, b] = \frac{|y - a|}{|y - b|} \cdot \frac{|x - b|}{|x - a|}$$

for any norm $|\cdot|$ on V.

The *Hilbert metric* on K° is defined by

$$d_K(x, y) = \frac{1}{2} \inf \log[a, x, y, b],$$

where the infimum is over all $a, b \in K$ such that $[x, y]$ lies in the interior of $[a, b]$ with the same orientation. Compare [H], [Ha2], [Bus, p. 105], [BK, IV.28]. It is easy to see that d_K induces the usual topology on K°.

Examples. — Let $K = [A, B] \subset V = \mathbf{R}$; then d_K coincides with the Riemannian metric

$$\frac{(B - A)\,|dx|}{2|x - A||x - B|}.$$

More generally, if $K \subset \mathbf{R}^n$ is the closed unit ball, then K° coincides with the Klein model for hyperbolic space $\mathbf{H}^n$, and the Hilbert metric agrees with the hyperbolic metric of constant curvature -1. (The factor of $\frac{1}{2}$ in the definition of d_K compensates for the transition between the Poincaré and Klein models.)

Straightness. — Since the Hilbert metric restricts to a Riemannian metric on any segment in K°, we have the following crucial straightness property:

$$d_K(x, y) + d_K(y, z) = d_K(x, z) \tag{2.1}$$

for any $y \in [x, z]$.

Contraction principle. — Linear maps ϕ are contracting for the Hilbert metric: that is, if $\phi : V \to V'$ is a linear map that sends K into K′, then we have

$$d_{K'}(\phi(x), \phi(y)) \leq d_K(x, y).$$

In this respect the Hilbert metric behaves like the Poincaré metric from complex analysis.

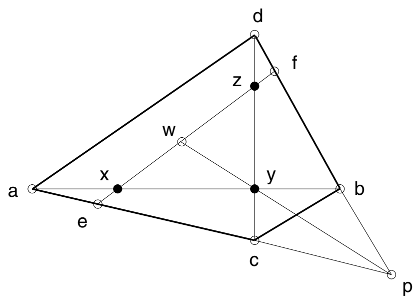


FIG. 2. — The triangle inequality

Triangle inequality. — We sketch a proof that d_K is a metric. Since K contains no line, $d_K(x, y) = 0$ iff $x = y$. The triangle inequality is verified by Figure 2. By convexity, given $x, y, z \in K^\circ$, K contains at least the quadrilateral L whose diagonals are the maximal segments $[a, b]$ and $[c, d]$ containing $[x, y]$ and $[y, z]$. Let $[e, f]$ denote the maximal segment through $[x, z]$ in L, and let p be the intersection of the lines through $[a, c]$ and $[b, d]$. (If these lines are parallel we take p at infinity.) Projection from p sends y to a point w in $[x, z]$. Since projection between lines preserves cross-ratios, we see that $[a, x, y, b] = [e, x, w, f]$ and thus

$$d_K(x, w) \leq d_L(x, w) = d_K(x, y).$$

Similarly $d_K(w, z) \leq d_K(y, z)$. Finally from the straightness property we obtain

$$d_K(x, z) = d_K(x, w) + d_K(w, z) \leq d_K(x, y) + d_K(y, z).$$

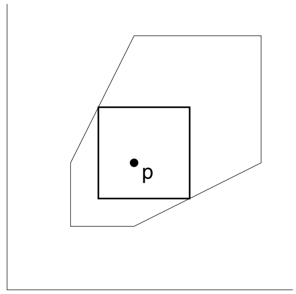


FIG. 3. — Balls of radius $(\log 2)/2$ centered at $p \in \mathbf{R}^2_+$ in the balanced metric (inner) and the Hilbert metric (outer)

The balanced metric. — Here is a variant of the Hilbert metric with useful properties. A segment $[x, y] \subset \mathrm{K}$ *extends by* $\alpha > 0$ if the segment with the same center but $(1 + 2\alpha)$-times longer, namely

$$[a, b] = [x + \alpha(x - y), y + \alpha(y - x)],$$

is also contained in K.

The *balanced metric* on K° is defined by

$$d^*_{\mathrm{K}}(x, y) = \inf\{\log(1 + \alpha^{-1}) \ : \ [x, y] \text{ extends by } \alpha.\}.$$

The proof of the triangle inequality is similar to the case of the Hilbert metric (see [BW, Lemma 2]).

The balanced metric is simply the Hilbert metric subject to the condition that $[a, b]$ has the same center as $[x, y]$. Thus it enjoys the same contraction principle. Noting that $[-\alpha, 0, 1, 1 + \alpha] = (1 + \alpha^{-1})^2$ we have:

$$(\mathbf{2.2}) \qquad \frac{1}{2}\, d^*_{\mathrm{K}}(x, y) \le d_{\mathrm{K}}(x, y) \le d^*_{\mathrm{K}}(x, y).$$

One advantage of the balanced metric is the product formula:

$$d^*_{\mathrm{K}_1 \times \mathrm{K}_2}((x_1, y_1), (x_2, y_2)) = \max\ \left(d^*_{\mathrm{K}_1}(x_1, y_1), d^*_{\mathrm{K}_2}(x_2, y_2)\right),$$

which makes it suitable for proofs by induction. For example, a Hilbert ball in $\mathrm{K} = \mathbf{R}^2_+$ is a hexagon, while a balanced ball is a square (Figure 3).

A disadvantage of the balanced metric is that the straightness property (2.1) fails.

Translation length. — We now assume, as in the beginning of this section, that $\mathrm{K} \subset \mathrm{V}$ is a closed convex cone containing no line, and $\mathrm{K}^\circ \neq \emptyset$.

Let $\mathrm{T} : \mathrm{V} \to \mathrm{V}$ be a linear map such that $\mathrm{T}(\mathrm{K}) = \mathrm{K}$. Then T induces an isometry of K° in both the balanced and Hilbert metrics. Let

$$\delta^*(\mathrm{T}, \mathrm{K}) = \inf_{x \in \mathrm{K}^\circ} d^*_{\mathrm{K}}(x, \mathrm{T}x).$$

Concentrating first on the balanced metric, we will show:

Theorem **2.3** *(Balanced length). — The translation length of* T *in the balanced metric is given by* $\delta^*(\mathrm{T},\mathrm{K}) = \log\max\,(\lambda(\mathrm{T}),\lambda(\mathrm{T}^{-1}))$.

Eigenvectors in K. — As a first step in the proof, we show:

Theorem **2.4.** — *Let* $\mathrm{T}:\mathrm{V}\to\mathrm{V}$ *be a linear map satisfying* $\mathrm{T}(\mathrm{K})=\mathrm{K}$. *Then* $\lambda(\mathrm{T})>0$ *is an eigenvalue of* T, *with a corresponding eigenvector* $v\in\mathrm{K}$.

Proof. — Since $\mathrm{T}(\mathrm{K}) = \mathrm{K}$, T is invertible. Using the generalized eigenspace decomposition of T on $\mathrm{V}\otimes\mathbf{C}$, we obtain a T-invariant splitting $\mathrm{V}=\mathrm{X}\oplus\mathrm{Y}$ such that the spectrum of $(\mathrm{T}|\mathrm{X})$ lies on the circle $|z| = \lambda(\mathrm{T})$, while $\lambda(\mathrm{T}|\mathrm{Y}) < \lambda(\mathrm{T})$.

Choose a norm $|\cdot|$ on V. Since $\mathrm{K}^\circ \neq \emptyset$, there exists a vector $u = (x,y) \in \mathrm{K}$ with $x\neq 0$. Then we have

$$|\mathrm{T}^n(x)| \asymp \lambda(\mathrm{T})^n|x| \gg |\mathrm{T}^n(y)|$$

as $n\to\infty$. It follows that any accumulation point w of the sequence $\mathrm{T}^n(u)/|\mathrm{T}^n(u)|$ lies in $\mathrm{K}\cap\mathrm{X}$. Since K is a cone, the entire ray $\mathbf{R}_+\cdot w$ is also contained in $\mathrm{K}\cap\mathrm{X}$.

The set of all rays contained in $\mathrm{K}\cap\mathrm{X}$ determines a nonempty, T-invariant subset $\mathbf{P}(\mathrm{K}\cap\mathrm{X})\subset\mathbf{P}\mathrm{X}$. Since K is closed, convex and contains no line, $\mathbf{P}(\mathrm{K}\cap\mathrm{X})$ is a compact, convex disk. Therefore $\mathrm{T}:\mathbf{P}(\mathrm{K}\cap\mathrm{X})\to\mathbf{P}(\mathrm{K}\cap\mathrm{X})$ has a fixed point $[v]$. We have $\mathrm{T}v = \alpha v$ and $|\alpha| = \lambda(\mathrm{T})$ since $v\in\mathrm{X}$. Since K contains no line we have $\alpha>0$ and thus $\alpha=\lambda$. □

Corollary **2.5.** — *We have* $\delta^*(\mathrm{T},\mathrm{K}) \geq \log\max\,(\lambda(\mathrm{T}),\lambda(\mathrm{T}^{-1}))$.

Proof. — Let $\lambda=\lambda(\mathrm{T})$. Let $\mathrm{T}^*:\mathrm{V}^*\to\mathrm{V}^*$ be the dual of T, and let $\mathrm{K}^* = \{f\in\mathrm{V}^* : f(\mathrm{K})\geq 0\}$ be the dual cone to K. Since the interior of K is nonempty and K contains no line, the same properties obtain for K^*.

By the preceding Theorem, there is a nonzero $f\in\mathrm{K}^*$ such that $\mathrm{T}^*(f) = \lambda f$. Then $f:\mathrm{V}\to\mathbf{R}$ satisfies:

$$f(\mathrm{T}v) = \lambda f(v) \quad \text{and} \quad f(\mathrm{K})\subset[0,\infty).$$

By the contraction principle, $\delta^*(\mathrm{T},\mathrm{K})$ is bounded below by the translation distance $\delta^*(\mathrm{T}',\mathrm{K}') = |\log\lambda|$ of $\mathrm{T}'(x) = \lambda x$ on $\mathrm{V}' = \mathbf{R}$ with $\mathrm{K}' = [0,\infty)$. Applying the same reasoning to T^{-1} gives the Corollary. □

For the reverse inequality it is convenient to prove a slightly more general statement that allows K to contain a line.

Lemma **2.6.** — *Let* $\mathrm{K}\subset\mathrm{V}$ *be a closed convex cone with* $\mathrm{K}^\circ\neq\emptyset$. *Let* $\mathrm{T}:\mathrm{V}\to\mathrm{V}$ *be an automorphism preserving* K, *and suppose*

$$1+\alpha^{-1} > \max\,(\lambda(\mathrm{T}),\lambda(\mathrm{T}^{-1})).$$

Then there exists an $x\in\mathrm{K}^\circ$ *such that* $[x,\mathrm{T}x]$ *extends by* α.

Proof. — The proof will be by induction on $\dim V$. If $\dim V = 0$, the Lemma holds for all values of $\alpha > 0$ by taking $x = 0$.

Now suppose that $\dim V > 0$ and that the Lemma has been established for all vector spaces with $\dim V' < \dim V$. Observe that any T-invariant subspace $S \subset V$ of positive dimension yields a quotient map $f : V \to V' = V/S$, and an automorphism T' of V' making the diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ V' & \xrightarrow{T'} & V' \end{array}$$

commute. Since the spectrum of T' is contained in that of T, and $\dim V' < \dim V$, the Lemma provides an x' in $(K')^\circ = f(K^\circ)$ such that $[x', T'(x')]$ extends by α. Lifting to V, we obtain $y, z \in K^\circ$ such that $[y, z]$ extends by α and $T(y) = z + s$ for some $s \in S$.

We apply this observation in two ways to complete the proof. First, suppose $K \subset V$ contains a line. Let S be the maximal subspace contained in K. Then $[y, z]$ and $[y, z+s]$ extend by the same amount, so $[y, T(y)]$ extends by α and we are done.

Second, suppose $K \subset V$ contains no line. It is convenient to assume that $\lambda = \lambda(T) \geq \lambda(T^{-1})$ (if not, replace T by its inverse); then $\lambda \geq 1$. By Theorem 2.4 there is an eigenvector $v \in K$ such that $Tv = \lambda v$. (If v were in K° we could finish the proof by taking $x = v$; but frequently v lies in ∂K.)

Let S be the subspace $\mathbf{R} \cdot v \subset V$. By the observation above, we have $y, z \in K^\circ$ such that $[y, z]$ extends by α and $T(y) = z + mv$ for some $m \in \mathbf{R}$.

Let $x = y + Mv$ where $M \gg 0$. Since y lies in K° and $\mathbf{R}_+ \cdot v \subset K$, we have $x \in K^\circ$. We claim that for M sufficiently large, $[x, Tx]$ extends by α. To see this, we compute:

$$\begin{aligned} x + \alpha(x - Tx) &= y + \alpha(y - T(y)) + M(v + \alpha(v - T(v))) \\ &= y + \alpha(y - z) - mv + M(1 + \alpha - \alpha\lambda)v \\ &= y + \alpha(y - z) + (M\beta - m)v \end{aligned}$$

where the coefficient $\beta = 1 + \alpha - \alpha\lambda$ is positive by our assumption that $1 + \alpha^{-1} > \lambda$. Therefore we have

$$(M\beta - m)v \in \mathbf{R}_+ \cdot v \subset K$$

when M is large enough. On the other hand, $y + \alpha(y - z)$ lies in K since $[y, z]$ extends by α. Since $K + K \subset K$, we have $x + \alpha(x - Tx) \in K$.

A similar argument shows $Tx + \alpha(Tx - x) \in K$. Thus $[x, Tx]$ extends by α. □

Proof of Theorem 2.3 (Balanced length). — By Corollary 2.5 we have

$$\delta^*(T, K) \geq \log \max (\lambda(T), \lambda(T^{-1})).$$

Since $d^*_K(x, Tx) \leq \log(1 + \alpha^{-1})$ when $[x, Tx]$ extends by α, the preceding Lemma provides the reverse inequality. □

Proof of Theorem 2.1 (Hilbert length). — Using the comparison (2.2) between the Hilbert metric and the balanced metric, the Theorem 2.3 immediately yields

$$\frac{1}{2} \log \max (\lambda_+, \lambda_-) \quad \leq \quad \delta(T, K) \quad \leq \quad \log \max (\lambda_+, \lambda_-)$$

where $\lambda_\pm = \lambda(T^{\pm 1})$.

When λ_+ and λ_- are both > 1, the lower bound can be strengthened to $\frac{1}{2} \log \lambda_+ \lambda_-$, as follows. Proceeding as in Corollary 2.5, there exist eigenvectors $f_\pm \in K^*$ such that $(T^*)^{\pm 1} f_\pm = \lambda_\pm f_\pm$. Define $f : V \to \mathbf{R}^2$ by $f(v) = (f_+(v), f_-(v))$, and $T' : \mathbf{R}^2 \to \mathbf{R}^2$ by $T'(x, y) = (\lambda_+ x, \lambda_-^{-1} y)$. Then $f \circ T = T' \circ f$, and $f(K) = K' = \mathbf{R}^2_+$. By the contraction principle, we have

$$\delta(T, K) \quad \geq \quad \delta(T', K') \quad = \quad \frac{1}{2} \log(\lambda_+ \lambda_-),$$

completing the proof. □

3. Coxeter groups and the Tits cone

This section summarizes geometric properties of Coxeter groups. Basic references for this material are [Bou] and [Hum]; see also [Vin1], [Ha1].

Our main interest will be hyperbolic and higher-rank Coxeter groups. For such groups, we observe that the Hilbert metric on the interior of the Tits cone K° is well-defined and invariant, and passes to the space of rays $\mathbf{P}K^\circ$. Thus $(\mathbf{P}K^\circ, d_K)$ serves as a generalization of the Klein model for hyperbolic space to the case of higher-rank Coxeter groups.

Coxeter systems. — Let W be a group generated by a finite set S, and let $m(s, t)$ denote the order of $st \in W$. Assume $m(s, s) = 1$ and $m(s, t) \geq 2$ for all $s \neq t$ in S.

The pair (W, S) is a *Coxeter system* if the generators S, together with the relations $(st)^{m(s,t)} = 1$ for all $s, t \in S$, give a presentation for W. Then W itself is a *Coxeter group*.

Let $V = \mathbf{R}^S$ be the real vector space with one basis element e_s for each $s \in S$. Define a symmetric bilinear form $B : V \times V \to \mathbf{R}$ by

$$B(e_s, e_t) = -2 \cos(\pi / m(s, t)).$$

There is a natural *geometric action* of W on V preserving the form B, given on the generators $s \in S$ by

$$s \cdot v = v - B(e_s, v)e_s;$$

that is, by letting s acts via reflection through the hyperplane normal to e_s. The representation $W \to O(V, B)$ is faithful.

The quadratic form $B(v, v)$ on V is equivalent over $\mathbf{R}$ to one of the standard forms

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

on $\mathbf{R}^n$; its *signature* is (p, q). The *radical* is defined by

$$\operatorname{rad}(V) = \{v \,:\, B(v, v') = 0 \ \forall v' \in V\};$$

it satisfies $\dim \operatorname{rad}(V) = n - p - q$.

Remark: when (st) has infinite order, one drops the relation $(st)^{m(s,t)} = 1$ and sets $B(e_s, e_t) = -2$.

Eigenvalues. — Let $\lambda(w)$ denote the spectral radius of $w \in W$ acting geometrically on V. Clearly $s|\operatorname{rad}(V) = I$ for all $s \in S$, so the same is true of w. Moreover B descends to a non-degenerate quadratic form on $V/\operatorname{rad}(V)$, preserved by w. It follows that $\det(\lambda I - w)$ is a reciprocal polynomial, and in particular that

$$\lambda(w) = \lambda(w^{-1}). \tag{3.1}$$

The Tits cone. — The Coxeter group W also acts naturally on the dual space V^*. The dual action is characterized by the equation

$$\langle w \cdot f, w \cdot v\rangle = \langle f, v\rangle,$$

where $\langle f, v\rangle$ denotes the natural pairing between $f \in V^*$ and $v \in V$. The spectral radii of $w|V$ and $w|V^*$ agree.

The *fundamental chamber* $F \subset V^*$ for (W, S) is defined by:

$$F = \{f \in V^* \,:\, \langle f, e_s\rangle \geq 0 \ \forall s \in S\}.$$

Passage to the dual space permits a uniform treatment of the geometric action even in the case where $\operatorname{rad}(V) \neq (0)$. For example, the chamber $F \subset V$ is always a cone on a simplex, while the region

$$\{v \,:\, B(v, e_s) \geq 0 \ \forall s \in S\} \subset V$$

need not be.

The *Tits cone* is the full orbit $W \cdot F$ of the fundamental chamber under the action of W. From [Bou, V.4] or [Hum, §5.13] we have:

Theorem **3.1.** — *The Tits cone* $W \cdot F$ *is convex, and* $w(F) = F$ *iff* $w = \mathrm{id}$.

Diagrams. — The *Coxeter diagram* of (W, S) is the weighted graph D whose vertices are S and whose edges of weight $m(s,t)$ join s to t whenever $m(s,t) \geq 3$. To make a picture of the diagram D, we draw single lines for edges of weight 3, double lines for edges of weight 4, and lines labeled by n for edges of weight $n \geq 5$.

A Coxeter system is *irreducible* if the action of W on $V/\mathrm{rad}(V)$ is irreducible; equivalently, if its Coxeter diagram is connected.

A general Coxeter system (W, S) reduces naturally into irreducible subsystems (W_i, S_i), such that $S = \sqcup S_i$ and $W = \prod W_i$. The geometric action of W on V is the product of the actions of W_i on V_i.

Classification by signature. — Assume the Coxeter system (W, S) is irreducible. Then (W, S) can be classified into one of 4 types according to the signature of (V, B). Letting $n = |S| = \dim V$, we say (W, S) is:

- *Spherical* if $\mathrm{sig}(V, B) = (n, 0)$;
- *Affine* if $\mathrm{sig}(V, B) = (n-1, 0)$;
- *Hyperbolic* if $\mathrm{sig}(V, B) = (p, 1)$; and
- *Higher-rank* if $\mathrm{sig}(V, B) = (p, q)$, $q \geq 2$.

This classification is conveniently approached via the *adjacency matrix*

$$A_{st} = (2I - B)(e_s, e_t) = \begin{cases} 2\cos(\pi/m(s,t)), & s \neq t, \\ 0 & s = t. \end{cases}$$

Let $\alpha = \alpha(W, S)$ denote the spectral radius of A. Since the smallest eigenvalue of the symmetric matrix B is $2 - \alpha(W, S)$, we find:

$$(W, S) \text{ is } \begin{cases} \text{spherical} & \text{if } \alpha(W, S) < 2, \\ \text{affine} & \text{if } \alpha(W, S) = 2, \text{ and} \\ \text{hyperbolic or higher-rank} & \text{if } \alpha(W, S) > 2. \end{cases}$$

Terminology. — The term 'adjacency matrix' comes from the case where $m(s,t) \leq 3$ for all s, t; then $A_{st} = 1$ if s and t are joined by an edge in the Coxeter diagram of (W, S), and $= 0$ otherwise. The term 'higher rank' is meant to remind one that the real Lie group $SO(p, q)$ has real rank ≥ 2 when $(p, q) \geq (2, 2)$. Note that in [Bou] and [Hum], the term 'hyperbolic' is used differently than here; these authors include the additional condition that $\mathrm{rad}(V) = (0)$ and $\mathbf{H}^p/W$ has finite volume.

Perron-Frobenius. — Since the Coxeter diagram of (W, S) is connected, the matrix A is one to which the Perron-Frobenius theory applies. That is, $\alpha = \alpha(W, S)$ is a simple eigenvalue of A, and there is a positive vector $v_0 = \sum a_s e_s$, $a_s > 0$ such that $Av_0 = \alpha v_0$ and $Bv_0 = (2 - \alpha)v_0$.

Now assume $\alpha \neq 2$ and let $f_0 \in V^*$ be the dual vector satisfying

$$\langle f_0, v\rangle = (2-\alpha)^{-1} B(v_0, v)$$

for all $v \in V$. Then clearly f_0 belongs to $\mathrm{rad}(V)^\perp$; that is, $\langle f_0, v\rangle = 0$ for all $v \in \mathrm{rad}(V)$. Moreover, we have $\langle f_0, e_s\rangle = a_s > 0$, so $f \in F^\circ$. This shows:

Proposition **3.2.** — *Except in the affine case,* F° *meets* $\mathrm{rad}(V)^\perp$.

Spherical, affine and hyperbolic groups. — The spherical and affine groups are well-understood; for example, their diagrams are classified. In the spherical case the Tits cone is all of V^* and W is finite. In the affine case the closure of the Tits cone is a half-space bounded by $\mathrm{rad}(V)^\perp$, and $W = W_0 \ltimes \mathbf{Z}^{n-1}$ with $|W_0| < \infty$. By considering the space of rays in the interior of the Tits cone, one obtains an isometric action of W on the sphere S^n or on the Euclidean space $\mathbf{R}^{n-1}$.

Hyperbolic and higher rank groups. — We will use the Hilbert metric on the Tits cone to obtain an isometric action in the hyperbolic and higher-rank cases. Let $K = \overline{W \cdot F}$.

Theorem **3.3.** — *Let* (W, S) *be a hyperbolic or higher-rank Coxeter system. Then the closure of the Tits cone* K *contains no line.*

Proof. — (From [Vin1, Lemma 15].) Let $X \subset V^*$ be the maximal subspace contained in K. For the sake of contradiction, suppose $X \neq (0)$. Then $X^\perp \subset V$ is a proper W-invariant subspace. By irreducibility of the action of W on $V/\mathrm{rad}(V)$, we have $X^\perp \subset \mathrm{rad}(V)$, and thus $X \supset \mathrm{rad}(V)^\perp$.

Since F° meets $\mathrm{rad}(V)^\perp$, there is an $f_0 \in \mathrm{rad}(V)^\perp$ and a neighborhood U of the origin in V^* such that

$$f_0 + U \subset F^\circ \subset \overline{W \cdot F} = K.$$

We also have $-f_0 \in X \subset K$. Since K is a convex cone, this implies

$$U = (-f_0) + f_0 + U \subset K,$$

and thus $K = V$. Therefore W is finite and (W, S) is spherical, a contradiction. □

Since K contains no line, the Hilbert metric d_K is well-defined and we have:

Corollary **3.4.** — *In the hyperbolic or higher-rank case, the Coxeter group* W *acts isometrically on* K° *in its Hilbert metric.*

Since $\lambda(w) = \lambda(w^{-1})$ ((3.1) above), Theorems 2.1 and 2.4 imply:

Corollary **3.5.** — *Let w belong to a hyperbolic or higher-rank Coxeter group. Then $\lambda(w) \geq 1$ is an eigenvalue of w, and*

$$\log \lambda(w) = \inf_{x \in K^\circ} d_K(x, w \cdot x).$$

Projective models. — The Hilbert determines a W-invariant metric on the space of rays

$$\mathbf{P}\mathrm{K}^\circ \subset \mathbf{P}\mathrm{V}^*,$$

because the cross-ratio is projectively invariant. The space $\mathbf{P}\mathrm{K}^\circ$ is isometric (via projection) to the affine slice $(\mathrm{K}^\circ \cap \mathrm{H}, d_\mathrm{K})$, for hyperplane $\mathrm{H} \subset \mathrm{V}$ with $\mathrm{H} \cap \mathrm{K}^\circ \neq \emptyset$ and $\mathrm{K} \cap \mathrm{H}$ compact.

In the case of hyperbolic Coxeter groups, W also acts isometrically on the Klein model for hyperbolic space,

$$\mathbf{H}^p = \mathbf{P}\mathrm{H} \subset \mathbf{P}\mathrm{V}^*,$$

where H is the image of the timelike cone $\mathrm{B}(v, v) < 0$ under the map $\mathrm{V} \to \mathrm{V}^*$ defined by B. In fact, when the radical is trivial and $\mathbf{H}^p/\mathrm{W}$ has finite volume, $\mathbf{H}^p$ coincides isometrically with $(\mathbf{P}\mathrm{K}^\circ, d_\mathrm{K})$. Thus we can regard $(\mathbf{P}\mathrm{K}^\circ, d_\mathrm{K})$ as a generalization of the Klein model for hyperbolic space to the infinite-volume and higher-rank cases.

4. Coxeter elements

In this section we show Coxeter elements minimize translation length among all essential elements in W.

Coxeter elements. — Let (W, S) be a Coxeter system with $\mathrm{S} = \{s_1, \dots s_n\}$. We say $w \in \mathrm{W}$ is a *Coxeter element* if

$$w = s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n}$$

for some permutation $\sigma \in \mathrm{S}_n$.

Essential elements. — Let $\mathrm{W}_\mathrm{I} \subset \mathrm{W}$ denote the *parabolic subgroup* generated by $\mathrm{I} \subset \mathrm{S}$. Then $(\mathrm{W}_\mathrm{I}, \mathrm{I})$ is also a Coxeter system. An element $w \in \mathrm{W}$ is *peripheral* if it is conjugate into a proper parabolic subgroup $\mathrm{W}_\mathrm{I} \subset \mathrm{W}$, $\mathrm{I} \neq \mathrm{S}$; otherwise it is *essential*.

We will show:

Theorem **4.1.** — *Let* (W, S) *be a Coxeter system and let* $w \in \mathrm{W}$ *be essential. Then we have* $\lambda(w) \geq \inf_{\mathrm{S}_n} \lambda(s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n})$.

Loops in the fundamental chamber. — It is easy to see that Theorem 4.1 for general (W, S) follows from the irreducible case. In the spherical and affine cases, $\lambda(w) = 1$ for all $w \in \mathrm{W}$ and the Theorem is immediate.

Now assume (W, S) is hyperbolic or higher-rank. Then W acts isometrically and discretely on K°, with $\mathrm{X} = \mathrm{F} \cap \mathrm{K}^\circ$ as a fundamental domain. The natural projection map

$$\pi : \mathrm{K}^\circ \to \mathrm{X},$$

characterized by $x \in W \cdot \pi(x)$, is a covering map in the sense of orbifolds. By convexity, K° is contractible, and hence the orbifold fundamental group of X is W.

Let $\gamma : [0, 1] \to K^\circ$ be a piecewise linear path. Breaking the domain into intervals $[t_i, t_{i+1}]$ on which γ is linear, and setting $x_i = \gamma(t_i)$, we define its length by

$$L(\gamma) = \sum d_K(x_i, x_{i+1}).$$

The sum is independent of the choice of subdivision because of the triangle equality (2.1) for collinear points.

A *loop in* X is a piecewise-linear path $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1)$. A *lift* of γ is a path $\widetilde{\gamma} : [0, 1] \to K^\circ$ such that

$$\pi \circ \widetilde{\gamma} = \gamma.$$

In this case $\widetilde{\gamma}(1) = w \cdot \gamma(0)$ for some $w \in W$, and we say γ (or $\widetilde{\gamma}$) *represents* w. A given loop has many lifts, and thereby represents many elements in W.

General position. — The codimension-one faces of F are given by $F(s) = \{f \in F : f(s) = 0\}$, $s \in S$; the codimension-two faces, by $F(s) \cap F(t)$, $s \neq t$.

Let us say a piecewise-linear path $\gamma : [0, 1] \to X$ is in *general position* if it is disjoint from the codimension-two faces of F and meets the codimension-one faces at most in a finite set.

Proposition **4.2.** — *We have* $\log \lambda(w) = \inf L(\gamma)$ *over all loops* $\gamma : [0, 1] \to X$ *in general position representing* w.

Proof. — Let γ represent w via the lift $\widetilde{\gamma}$. By Corollary 3.5, $\log \lambda(w)$ is the minimal translation length of w in the Hilbert metric. Thus we have

$$L(\gamma) = L(\widetilde{\gamma}) \quad \geq \quad d_K(\widetilde{\gamma}(0), w \cdot \widetilde{\gamma}(0)) \quad \geq \quad \log \lambda(w).$$

Moreover, there exist $x_n \in K^\circ$ with

$$d_K(x_n, w \cdot x_n) \to \log \lambda(w).$$

Since the orbits of the codimension-one faces $W \cdot F(s)$ are nowhere dense, we can assume $\pi(x_n) \in F^\circ$.

Let $\widetilde{\gamma}_n$ denote the straight line from x_n to $w \cdot x_n$; then

$$L(\widetilde{\gamma}_n) \to \log \lambda(w).$$

Since the orbits $W \cdot (F(s) \cap F(t))$ of the codimension-two faces of F do not separate K°, we can modify $\widetilde{\gamma}_n$ slightly (introducing new bends if necessary but increasing its length by at most $1/n$) so that $\gamma_n = \pi \circ \widetilde{\gamma}_n$ is in general position. Then $L(\gamma_n) \to \log \lambda$ and γ_n represents w, completing the proof. □

Let γ be a loop in general position, and let $t_1 < t_2 < \ldots < t_m$ be the parameters such that $\gamma(t_i) \in \partial F$. Then we have $\gamma(t_i) \in F(g_i)$ for a unique $g_i \in S$. We say w is a *subword* of $g_1 g_2 \cdots g_m$ if we have $w = g_{i_1} g_{i_2} \cdots g_{i_k}$ for some indices $1 \le i_1 < i_2 < \cdots < i_k \le m$.

Proposition **4.3.** — *The loop γ represents w iff w is conjugate to a subword of* $g_1 g_2 \cdots g_m$.

Proof. — Define a lift $\widetilde{\gamma}$ of γ by

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, t_1], \\ g_1 g_2 \cdots g_k \cdot \gamma(t) & \text{if } t \in [t_k, t_{k+1}], \text{ and} \\ g_1 g_2 \cdots g_m \cdot \gamma(t) & \text{if } t \in [t_m, 1]. \end{cases}$$

Since $g_k \cdot \gamma(t_k) = \gamma(t_k)$, the definition is consistent and $\widetilde{\gamma}$ is continuous. Thus γ represents the full word $g_1 \cdots g_m$. By ignoring a subset of the (t_i)'s, we can similarly obtain a lift which represents any subword of $g_1 \cdots g_m$.

Now let $\widetilde{\gamma}$ be a lift with $\widetilde{\gamma}(0) \in X$. Then $\widetilde{\gamma}(t) = w(t) \cdot \gamma(t)$ for some $w(t) \in W$. The element $w(t)$ can only change when $\gamma(t)$ touches a face $F(g_i)$, and then only by composition on the right with g_i. Thus $\widetilde{\gamma}$ has the form above and therefore γ represents a subword of $g_1 \cdots g_m$.

To complete the proof, observe that $\widetilde{\gamma}$ represents w iff $g \cdot \widetilde{\gamma}$ represents gwg^{-1}. □

Proof of Theorem 4.1. — As discussed above, the Theorem reduces to the case of a hyperbolic or higher-rank group.

Let $w \in W$ be an essential element of such a group. For any $\epsilon > 0$ we can find a loop $\gamma : [0, 1] \to X$ in general position such that γ represents w and $L(\gamma) \le \log \lambda(w) + \epsilon$.

Let $\gamma(t_i) \in F(g_i)$ be the points of γ that meet ∂F. Then w is conjugate to a subword of $g_1 \cdots g_m$. Since w is essential, every element of S must occur in the sequence (g_i). Thus $g_1 \cdots g_m$ also contains a Coxeter element w' as a subword, and therefore γ also represents w'. We then have

$$\lambda(w') \le L(\gamma) \le \log \lambda(w) + \epsilon,$$

and the proof is completed by letting $\epsilon \to 0$. □

Hyperbolic groups. — For hyperbolic Coxeter systems, the proof above can also be carried through using hyperbolic space $\mathbf{H}^p$ in place of $\mathbf{P}K^\circ$.

Question. — Are all Coxeter elements essential?

5. Bipartite Coxeter diagrams

Let (W, S) be a Coxeter system. In this section we introduce the *bicolored eigenvalue* $\beta(W, S) \geq 1$ and prove it controls the eigenvalues of all Coxeter elements. We will show:

Theorem **5.1.** — *Any Coxeter system* (W, S) *satisfies*

$$\inf_{S_n} \lambda(s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n}) \quad \geq \quad \beta(W, S).$$

Equality holds if the Coxeter diagram of (W, S) *is bipartite.*

Corollary **5.2.** — *We have* $\lambda(w) \geq \beta(W, S)$ *for all essential* $w \in W$.

Bicolored Coxeter elements. — When the Coxeter diagram D of (W, S) is a tree (or forest), the Coxeter elements range in a single conjugacy class in W [Hum, §3.16].

When D contains cycles, in general several conjugacy classes occur. However, when all the cycles in D are of even order, there is still a special class of Coxeter elements that are unique up to conjugacy.

To define these, let us say a partition $S = S_1 \sqcup S_2$ of the vertices of D is a *two-coloring* if all edges of D lead from S_1 to S_2. A two-coloring exists iff all cycles in D are of even order. In the terminology of graph-theory, the diagram D is *bipartite*.

Let D admit a two-coloring $S = S_1 \sqcup S_2$. Since there are no edges between elements $s, t \in S_i$, we have $(st)^2 = 1$. Thus S_i generates an abelian subgroup of W, isomorphic to $(\mathbf{Z}/2)^{|S_i|}$. The product σ_i of the elements of S_i is independent of the choice of ordering and satisfies $\sigma_i^2 = 1$.

We refer to $w = \sigma_1\sigma_2$ as a *bicolored* Coxeter element. Its conjugacy class is independent of the choice of two-coloring. In fact, if (W, S) is irreducible then its bicolored Coxeter element is unique up to $w \mapsto w^{-1}$, since the two-coloring of a connected diagram is unique up to $(S_1, S_2) \mapsto (S_2, S_1)$.

As noted in [A'C] and [BLM], the spectrum of the bicolored Coxeter elements is determined by the spectrum of A_{st}. In particular we have:

Proposition **5.3.** — *Let w be a bicolored Coxeter element for* (W, S). *Then the spectrum of w is contained in* $S^1 \cup \mathbf{R}_+$, *and the eigenvalue(s) maximizing* $\operatorname{Re}\lambda$ *satisfy*

$$2 + \lambda + \lambda^{-1} = \alpha(W, S)^2. \qquad (\mathbf{5.1})$$

Proof. — It is easy to check that the adjacency matrix determines an operator $A : V \to V$ satisfying $A = \sigma_1 + \sigma_2$, where $w = \sigma_1\sigma_2$. Thus

$$A^2 = 2 + \sigma_1\sigma_2 + \sigma_2\sigma_1 = 2 + w + w^{-1}.$$

The spectrum of w is therefore the preimage of the spectrum of A^2 under $\lambda \mapsto 2 + \lambda + \lambda^{-1}$. Since A is symmetric, the spectrum of A^2 lies in the interval $[0, \alpha(W, S)^2]$, and the Proposition follows. □

The bicolored eigenvalue. — Motivated by equation (5.1), we define the *bicolored eigenvalue* $\beta(W, S)$ as the unique root $\beta \geq 1$ of the equation

$$2 + \beta + \beta^{-1} = \alpha(W, S)^2,$$

provided $\alpha(W, S) \geq 2$. For $\alpha(W, S) < 2$ we set $\beta(W, S) = 1$. In the first case, (W, S) has a hyperbolic or higher-rank component; in the second, all components are affine or spherical. In the first case $\lambda(w)$ is an eigenvalue of w, showing:

Corollary **5.4.** — *We have* $\lambda(w) = \beta(W, S)$ *for all bicolored Coxeter elements.*

Proof of Theorem 5.1. — Assume (W, S) is hyperbolic or of higher-rank; the Theorem easily reduces to this case. Let $\alpha = \alpha(W, S) > 2$.

Let $w = s_1 \cdots s_n$ be a Coxeter element in W. We will write vectors $v \in V$ as $v = \sum v_i e_i$, $e_i = e_{s_i}$, and write $v \geq v'$ to mean $v_i \geq v'_i$ for all i. Since $s_k \cdot v = v - B(v, e_k)e_k$, and $B = 2I - A$, we have:

$$(\mathbf{5.2}) \qquad (s_k \cdot v)_i = \begin{cases} (A \cdot v)_i - v_i & \text{if } k = i, \\ v_i & \text{otherwise.} \end{cases}$$

Note that $(A \cdot v)_i$ only depends on v_j, $j \neq i$.

Let $v > 0$ be a Perron-Frobenius eigenvector for A; it satisfies $Av = \alpha v$. To prove the Theorem, it suffices to show

$$(\mathbf{5.3}) \qquad (w + w^{-1})(v) \geq (\alpha^2 - 2)v,$$

since this equation implies

$$\lambda(w) + \lambda(w)^{-1} \geq \lambda(w + w^{-1}) \geq \alpha^2 - 2$$

and thus $\lambda(w) \geq \beta(W, S)$.

To prove (5.3) first note that $v' \geq v$ implies $Av' \geq Av$. Since $\alpha - 1 \geq 1$, we have $(s_n \cdot v)_n = (\alpha - 1)v_n \geq v_n$, and thus $s_n \cdot v \geq v$. By induction, we have the inequalities

$$\begin{aligned} s_k s_{k+1} \cdots s_n \cdot v &\geq v, \\ (s_k s_{k+1} \cdots s_n \cdot v)_k &\geq (A \cdot v)_k - v_k = (\alpha - 1)v_k \end{aligned}$$

for all k. Since s_k only changes the kth coordinate of v, we have

$$\begin{aligned} (s_{k+1} \cdots s_n \cdot v)_i &\geq (\alpha - 1)v_i, \ i > k, \\ (s_{k+1} \cdots s_n \cdot v)_i &= v_i, \qquad\quad i \leq k. \end{aligned}$$

Applying the same reasoning to $w^{-1} = s_n \cdots s_1$, we find

$$u = (s_{k+1} \cdots s_n \cdot v) + (s_{k-1} \cdots s_1 \cdot v)$$

satisfies $u_i \geq \alpha v_i$, $i \neq k$, and $u_k = 2v_k$. On the other hand, we have

$$((w + w^{-1}) \cdot v)_k = (s_k \cdot u)_k,$$

using again the fact that only s_k changes the kth coordinate. Therefore we have

$$\begin{aligned} ((w + w^{-1}) \cdot v)_k &= (\mathrm{A} \cdot u)_k - u_k \\ &\geq (\mathrm{A} \cdot (\alpha v))_k - 2v_k \quad = \quad (\alpha^2 - 2)v_k, \end{aligned}$$

establishing (5.3) and completing the proof. □

Corollary **5.5.** — *The bicolored Coxeter elements, if they exist, minimize $\lambda(w)$ among all Coxeter elements.*

Geometric interpretation. — Suppose (W, S) admits a two-coloring $\mathrm{S} = \mathrm{S}_1 \sqcup \mathrm{S}_2$ with corresponding Coxeter element $w = \sigma_1\sigma_2$. Then $\mathrm{F}_i = \bigcap_{s \in \mathrm{S}_i} \mathrm{F}(s)$ is a facet of F with a finite stabilizer in W; hence it meets K°. Let $[x, y] \subset \mathrm{X}$ be a line segment joining F_1 to F_2 in $(\mathbf{rad}(\mathrm{V}))^{\perp}$ and perpendicular to both. A loop γ that traces $[x, y]$ twice, once in each direction, gives a geodesic representing w; thus $\log \lambda(w) = 2\mathrm{L}([x, y])$.

In terms of the Hilbert metric on the quotient orbifold X, Theorem 4.1 implies:

Corollary **5.6.** — *The loop γ for a bicolored Coxeter element has minimal length among all loops that touch all the faces of* X.

Example. — Let $(\mathrm{W}, \mathrm{S}) = \langle a, b, c \ : \ a^2 = b^2 = c^2 = (ac)^2 = (ab)^3 \rangle$ be the $(2, 3, \infty)$ triangle group. Its Coxeter diagram is

$$a \,\text{——}\, b \overset{\infty}{\text{——}} c.$$

The Coxeter element $w = (ac)b$ is bicolored, and the corresponding segment $[x, y]$ joins the right angle $x = \mathrm{F}(a) \cap \mathrm{F}(c)$ of X to the opposite side $\mathrm{F}(b)$. The hyperbolic length of $[x, y]$ is given by the log of the golden mean, and therefore

$$\lambda(w) = \frac{3 + \sqrt{5}}{2} = 2.61803...$$

is the golden mean squared.

The corresponding tiling of $\mathbf{H}^2$ in the Poincaré model, and the geodesic stabilized by w (which has $[x, y]$ as a subsegment), are shown in Figure 4. As is well-known, W contains $\mathrm{PSL}_2(\mathbf{Z})$ as a subgroup of index two, and $w^2 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ when suitably normalized.

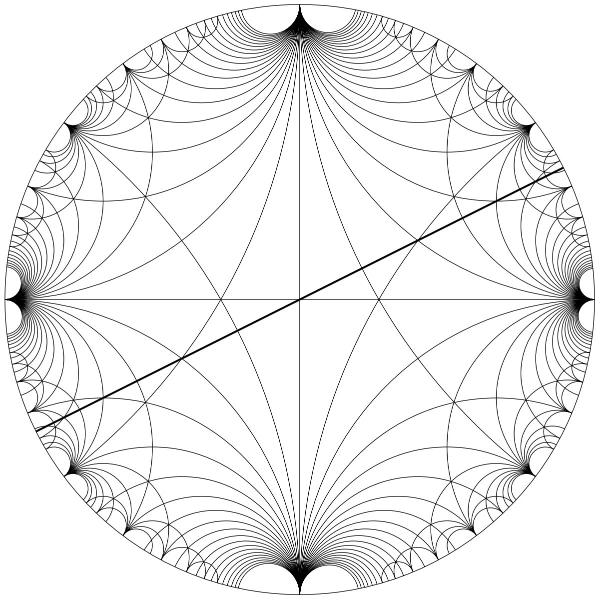


FIG. 4. — The geodesic stabilized by the Coxeter element for the $(2, 3, \infty)$ triangle group

Finite covers of Coxeter diagrams. — Here is another perspective on the bicolored eigenvalue. Let (W, S) be a Coxeter system with connected diagram D. Regarding D as a topological 1-complex with weights on its edges, consider the 2^d-fold covering space $D' \to D$ determined by the map

$$\pi_1(D) \to H_1(D, \mathbf{Z}/2).$$

All cycles in D′ have even length, so the associated Coxeter system (W′, S′) admits a bicolored Coxeter element $w' \in W'$. Clearly $\alpha(W, S) = \alpha(W', S')$, so we can alternatively define the bicolored eigenvalue of (W, S) by

$$\beta(W, S) = \lambda(w').$$

In other words, every Coxeter system admits a bicolored 'virtual' Coxeter element, whose leading eigenvalue is $\beta(W, S)$.

Indiscrete groups. — The proof of Theorem 5.1 uses only the fact that the adjacency matrix A_{st} is non-negative and symmetric, so it can easily be generalized beyond Coxeter groups. A corresponding result applies, for example, to the (possibly indiscrete) group generated by reflections through the sides of any simplex in hyperbolic space with interior dihedral angles $\leq \pi/2$.

6. Minimal hyperbolic diagrams

In this section we use the classification of minimal hyperbolic diagrams to prove a universal lower bound for eigenvalues in Coxeter groups.

Let (W, S) be a Coxeter system and let

$$\lambda(W, S) = \inf\{\lambda(w) \ : \ w \in W \text{ and } \lambda(w) > 1\}.$$

We set $\lambda(W, S) = 1$ if all elements of W have spectral radius one. Note that when (W, S) is hyperbolic, $\log \lambda(W)$ is the length of the shortest closed geodesic on the hyperbolic orbifold $\mathbf{H}^p/W$.

We will show:

Theorem **6.1.** — *Either* $\lambda(W, S) = 1$ *or* $\lambda(W, S) \geq \lambda_{Lehmer}$.

Recall $\lambda_{\text{Lehmer}} = 1.17628...$ is the largest real root of Lehmer's polynomial (1.1).

Minimal Coxeter elements. — We first show $\lambda(W, S)$ can be computed by examining a finite number of elements $w \in W$. Given a Coxeter system (W, S), let

$$\lambda_{\text{Cox}}(W, S) = \inf_{S_n} \lambda(s_{\sigma 1} s_{\sigma 2} \cdots s_{\sigma n}).$$

The infimum is realized by the *minimal Coxeter elements* in W.

Theorem **6.2.** — *For any Coxeter system with* $\lambda(W, S) > 1$*, we have*

$$\lambda(W, S) \quad = \quad \inf\{\lambda_{\text{Cox}}(W_I, I) \ : \ (W_I, I) \text{ is hyperbolic or higher-rank.}\}$$

Proof. — Any element $w \in W$ is conjugate to an essential element of (W_J, J) for some $J \subset S$. If $\lambda(w) > 1$ then (W_J, J) has a hyperbolic or higher-rank component (W_I, I) with the same minimal Coxeter eigenvalue as (W_J, J). By Theorem 4.1 we have

$$\lambda(w) \geq \lambda_{\text{Cox}}(W_J, J) = \lambda_{\text{Cox}}(W_I, I),$$

and the proof is completed by taking the infimum over all $w \in W$ with $\lambda(w) > 1$. □

Monotonicity. — Coxeter systems admit a natural partial order, defined by $(W, S) \leq (W', S')$ if there is an injective map $\iota : S \to S'$ such that $m(s, t) \leq m(\iota(s), \iota(t))$ for all $s, t \in S$. We write $(W, S) \cong (W', S')$ if ι extends to an isomorphism between W and W'; otherwise $(W, S) < (W', S')$. Since $m(s, t) \in \{1, 2, 3, ..., \infty\}$, this ordering satisfies the descending chain condition: any strictly decreasing sequence of Coxeter systems is finite.

The inequality on $m(s, t)$ is equivalent to the inequality

$$A_{st} \leq A'_{\iota(s)\iota(t)}$$

between adjacency matrices. Now the spectral radius of $A_{st} \geq 0$ increases as its entries do, since $\lambda(A) = \lim \|A^n\|^{1/n}$. The same is therefore true of the bicolored eigenvalue; we have:

Proposition **6.3.** — *If* $(W, S) \geq (W', S')$ *then* $\beta(W, S) \geq \beta(W', S')$.

Minimal hyperbolic diagrams. — A hyperbolic Coxeter system (W, S) is *minimal* if (W', S') has only spherical and affine components whenever $(W', S') < (W, S)$.

Proposition **6.4.** — *If* (W_0, S_0) *is hyperbolic or higher rank, then there is a minimal hyperbolic Coxeter system with* $(W, S) \leq (W_0, S_0)$.

Proof. — We will write the signature of a Coxeter system as $(p(W, S), q(W, S))$.

Consider the set of all Coxeter systems with $(W_\alpha, S_\alpha) \leq (W_0, S_0)$ and $q(W_\alpha, S_\alpha) \geq 1$. By the descending chain condition, this set has at least one minimal element (W, S). The minimal system (W, S) must be irreducible — otherwise one of its hyperbolic or higher-rank components would be strictly smaller. By minimality, $q(W', S') = 0$ if $(W', S') < (W, S)$, and thus any strictly smaller system has only spherical and affine components.

To see (W, S) is hyperbolic, pick $s \in S$ and let $I = S - \{s\}$; then $(W_I, I) < (W, S)$ so $q(W_I, I) = 0$. Adding s back in increases the signature by at most 1, so $q(W, S) = 1$.

Therefore (W, S) is a minimal hyperbolic Coxeter system. □

By Theorem 6.2 we have:

Proposition **6.5.** — *If* (W, S) *is a minimal hyperbolic Coxeter system, then* $\lambda(W, S) = \lambda_{\mathrm{Cox}}(W, S)$.

Theorem **6.6.** — *Up to isomorphism, there are 38 minimal hyperbolic Coxeter systems. Their diagrams are shown in Table 5.*

Proof. — Since the affine and spherical diagrams are known, the enumeration of minimal hyperbolic Coxeter systems is a straightforward combinatorial problem, albeit with many cases.

As an alternative argument, we note that if (W, S) is a minimal hyperbolic Coxeter system, then (W, S) is irreducible and $\mathrm{rad}(V) = (0)$. (Indeed, $\mathrm{rad}(V) \neq (0)$ implies $\mathbf{R}^I + \mathrm{rad}(V) = \mathbf{R}^S$ for some proper subset $I \subset S$, and then $(W_I, I) < (W, S)$ is still hyperbolic, contradicting minimality.)

Moreover, the condition that all proper parabolic subgroups of W are affine or spherical implies that the vertices of the simplex $\mathbf{P}F \subset \mathbf{P}V^*$ lie inside or on the boundary of hyperbolic space $\mathbf{H}^{n-1}$. Therefore $\mathbf{H}^{n-1}/W$ has finite volume.

The hyperbolic Coxeter systems of finite covolume with trivial radical are known and appear, for example, in [Hum, §6.8].[1] There are 72 such Coxeter systems with $|S| \geq 4$. For $|S| \leq 3$ there are infinitely many, namely the (p, q, r) triangle groups with $1/p + 1/q + 1/r < 1$. However among these, only the $(3, 3, 4)$, $(2, 4, 5)$ and $(2, 3, 7)$ groups are minimal. Thus we obtain a list of 75 Coxeter systems containing all the minimal ones. Removing the non-minimal elements from this list of 75, we are left with the 38 diagrams shown in Table 5. □

Guide to Table 5. — The first column in Table 5 gives the notation for the Coxeter system (W, S); the second, its diagram.

The third column gives the approximate value of $\lambda(W, S) = \lambda_{\text{Cox}}(W, S)$. Note that $\lambda(W, S) = \beta(W, S)$ for bipartite diagrams, so it is easily computed from the adjacency matrix. For the 5 diagrams which cannot be two-colored, $\beta(W, S)$ is shown in parentheses.

The last column gives the characteristic polynomial $p(x) = \det(xI - w)$ of a minimal Coxeter element. By Proposition 6.5, $\lambda(W, S)$ is a zero of $p(x)$.

Our notation for Coxeter systems is based in part on the standard notation A_n, B_n, D_n, E_n spherical diagrams. To each of these spherical diagrams one can adjoin an extending node to obtain an affine diagram. Attaching one more *hyperbolic node* to the extending node by a single edge, we obtain the *hyperbolic diagrams* Ah_{n+2}, Bh_{n+2}, Dh_{n+2} and Eh_{n+2}. Note that $\lambda_{\text{Lehmer}} = \lambda(Eh_{10})$.

The notation L_{4343} indicates a linear graph with four edges, whose weights are 4, 3, 4, and 3. Similarly K_{343} indicates a linear graph with edge weights 3, 4 and 3, but with an additional edge of weight 3 attached to the penultimate node. We denote by Q_n a loop of n edges, one of which is doubled, and by X_5 and X_6 a pair of star-shaped diagrams in no particular series.

Proof of Theorem 6.1. — Suppose $\lambda(W, S) > 1$. By the preceding results and the bicolored bound (Theorem 5.1), there is a hyperbolic or higher-rank subsystem (W_I, I), $I \subset S$, and a minimal hyperbolic diagram $(W', S') \leq (W_I, I)$ such that

$$\lambda(W, S) \;=\; \lambda_{\text{Cox}}(W_I, I) \;\geq\; \beta(W_I, I) \;\geq\; \beta(W', S').$$

Inspection of Table 5 shows $\beta(W', S') \geq \lambda_{\text{Lehmer}}$ for all minimal hyperbolic Coxeter systems, completing the proof. □

[1] In the first printing of this book, X_5 is missing a weight on one of its edges.

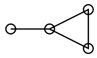
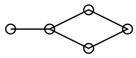
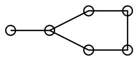
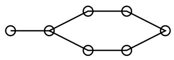
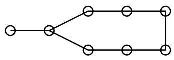
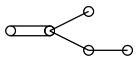
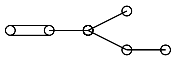
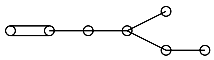
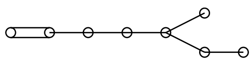
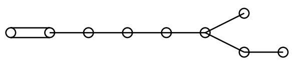
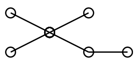
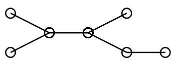
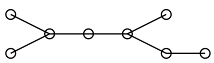
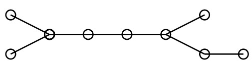
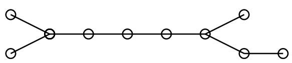
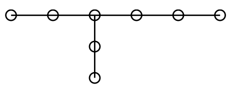
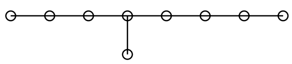
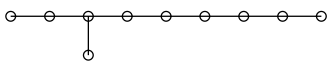
Coxeter system		$\lambda(W, S)$	$\det(xI - w)$
Ah_4		2.36921 (2.26844)	$1 - x - 3x^2 - x^3 + x^4$
Ah_5		2.08102	$(1 + x)(1 - x - 2x^2 - x^3 + x^4)$
Ah_6		1.98779 (1.96355)	$1 - 2x^2 - 3x^3 - 2x^4 + x^6$
Ah_7		1.88320	$(1 + x)(1 + x + x^2)(1 - 2x + x^2 - 2x^3 + x^4)$
Ah_8		1.83488 (1.82515)	$1 - x^2 - 2x^3 - 3x^4 - 2x^5 - x^6 + x^8$
Bh_5		1.72208	$(1 + x)(1 - x - x^2 - x^3 + x^4)$
Bh_6		1.58235	$1 - x^2 - 2x^3 - x^4 + x^6$
Bh_7		1.50614	$(1 + x)(1 - x - x^3 - x^5 + x^6)$
Bh_8		1.45799	$1 - x^2 - x^3 - x^5 - x^6 + x^8$
Bh_9		1.42501	$(1 + x)(1 - x - x^3 + x^4 - x^5 - x^7 + x^8)$
Dh_6		1.72208	$(1 + x)^2(1 - x - x^2 - x^3 + x^4)$
Dh_7		1.58235	$(1 + x)(1 - x^2 - 2x^3 - x^4 + x^6)$
Dh_8		1.50614	$(1 + x)^2(1 - x - x^3 - x^5 + x^6)$
Dh_9		1.45799	$(1 + x)(1 - x^2 - x^3 - x^5 - x^6 + x^8)$
Dh_{10}		1.42501	$(1 + x)^2(1 - x - x^3 + x^4 - x^5 - x^7 + x^8)$
Eh_8		1.40127	$(1 + x + x^2)(1 - x^2 - x^3 - x^4 + x^6)$
Eh_9		1.28064	$(1 + x)(1 - x^3 - x^4 - x^5 + x^8)$
Eh_{10}		1.17628	$1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}$

TABLE 5. — The 38 minimal hyperbolic Coxeter diagrams (Continued on next page)

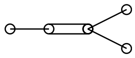
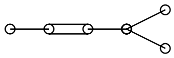
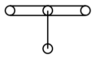
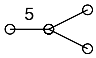
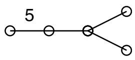
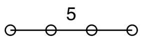
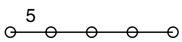
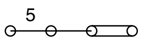
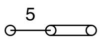
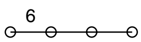
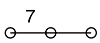
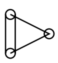
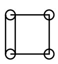
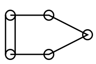
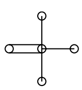
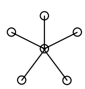
Coxeter system		$\lambda(W, S)$	$\det(xI - w)$
K_{343}		2.08102	$(1+x)(1-x-2x^2-x^3+x^4)$
K_{3433}		1.88320	$(1+x)^2(1-2x+x^2-2x^3+x^4)$
K_{44}		2.61803	$(1+x)^2(1-3x+x^2)$
K_{53}		2.15372	$(1+x)^2(2-3x-\sqrt{5}x+2x^2)$
K_{533}		1.91650	$(1+x)(2-x-\sqrt{5}x-x^3-\sqrt{5}x^3+2x^4)$
L_{33433}		1.58235	$1-x^2-2x^3-x^4+x^6$
L_{34333}		1.40127	$1-x^2-x^3-x^4+x^6$
L_{353}		1.84960	$2+x-\sqrt{5}x-2\sqrt{5}x^2+x^3-\sqrt{5}x^3+2x^4$
L_{4343}		1.88320	$(1+x)(1-2x+x^2-2x^3+x^4)$
L_{443}		2.08102	$1-x-2x^2-x^3+x^4$
L_{5333}		1.36000	$(1+x)(2-x-\sqrt{5}x+2x^2-x^3-\sqrt{5}x^3+2x^4)$
L_{534}		1.91650	$2-x-\sqrt{5}x-x^3-\sqrt{5}x^3+2x^4$
L_{54}		2.15372	$(1+x)(2-3x-\sqrt{5}x+2x^2)$
L_{633}		1.72208	$1-x-x^2-x^3+x^4$
L_{73}		1.63557	$(1+x)(1+x+x^2-4x\cos^2\pi/7)$
Q_3		3.09066 (2.89005)	$(1+x)(1-2x-\sqrt{2}x+x^2)$
Q_4		2.57747	$1-x-x^2-2\sqrt{2}x^2-x^3+x^4$
Q_5		2.43750 (2.3963)	$(1+x)(1-2x+x^2-\sqrt{2}x^2-2x^3+x^4)$
X_5		2.61803	$(1+x)^3(1-3x+x^2)$
X_6		2.61803	$(1+x)^4(1-3x+x^2)$

TABLE 5. — (Continued)

7. Small Salem numbers

In this section we conclude by detailing some connections between the simplest Coxeter systems and small Salem and Pisot numbers.

Salem and Pisot numbers. — An algebraic integer $\lambda > 1$ is a *Pisot number* if its conjugates (other than λ itself) satisfy $|\lambda'| < 1$. Similarly, an algebraic integer $\lambda > 1$ is a *Salem number* if its conjugates satisfy $|\lambda'| \leq 1$ and include $1/\lambda$. (We allow quadratic Salem numbers.)

It is known that the Pisot numbers form a closed subset $P \subset \mathbf{R}$, homeomorphic to the ordinal ω^ω, and that every Pisot number is a limit of Salem numbers (see e.g. [Sa]). The smallest Pisot number, $\lambda_{\text{Pisot}} \approx 1.324717$, is a root of $x^3 = x + 1$, while the smallest accumulation point in P is the golden mean,

$$\lambda_{\text{Golden}} = \frac{1+\sqrt{5}}{2} \approx 1.61803,$$

a root of $x^2 = x + 1$. All Pisot numbers $\lambda < \lambda_{\text{Golden}} + \epsilon$ are known [DP].

The Salem numbers are less well-understood. It is conjectured that $\lambda_{\text{Lehmer}} \approx 1.17628$, a root of the 10th degree polynomial discovered by Lehmer and given in (1.1), is the smallest Salem number [Leh], [GH]. The catalog of 39 Salem numbers given in [B1] includes all Salem numbers $\lambda < 1.3$ of degree ≤ 20 over $\mathbf{Q}$ [B3]; it will be sufficient for the applications below. At present there are 47 known Salem numbers $\lambda < 1.3$, and the list of such is known to be complete through degree 40; see [B2], [Mos] and [FGR].

Salem numbers from Coxeter groups. — A Coxeter system (W, S) is *crystallographic* if W preserves a lattice $V(\mathbf{Z}) \subset V$.

A Coxeter system is crystallographic iff every cycle in its diagram contains an even number of edges with weight 4 and an even number with weight 6, and no edge weights other than 3, 4, 6 and ∞ occur in the diagram [Hum, §5.13].

Proposition **7.1.** — *Let* (W, S) *be a hyperbolic crystallographic Coxeter system, and suppose* $w \in W$ *satisfies* $\lambda(w) > 1$. *Then* $\lambda(w)$ *is a Salem number of degree at most* $|S|$ *over* $\mathbf{Q}$.

Proof. — Since w acts by an automorphism of $V(\mathbf{Z}) \cong \mathbf{Z}^{|S|}$, $\lambda = \lambda(w)$ is an algebraic integer of degree at most $|S|$. Since V is hyperbolic, w has exactly two eigenvalues outside the unit circle, namely $\lambda^{\pm 1}$. All the other conjugates λ' of λ also occur as eigenvalues of w, so they satisfy $|\lambda'| \leq 1$. Finally $1/\lambda$ must be a conjugate of λ, because the product of all conjugates of λ is an integer dividing $\det(w) = \pm 1$. □

Corollary **7.2.** — *If* (W, S) *is hyperbolic, crystallographic, and bipartite, then* $\beta(W, S)$ *is a Salem number.*

Note. — The Coxeter system Ah_{2n} is hyperbolic, crystallographic but not bipartite, and in fact $\beta(Ah_{2n})$ fails to be a Salem number for $n \geq 5$ (it has 2 conjugates outside the unit circle).

Since Coxeter elements minimize $\lambda(w)$, they provide a geometric source of small Salem numbers. For example, from Table 5 one can verify:

Proposition **7.3.** — *The smallest Salem numbers of degrees 6, 8 and 10 coincide with the eigenvalues of Coxeter elements for* Eh_8, Eh_9 *and* Eh_{10}. *In particular,* $\beta(Eh_{10}) = \lambda_{Lehmer}$.

Note these 3 diagrams are the hyperbolic versions of the exceptional spherical diagrams E_6, E_7 and E_8.

Pisot numbers as limits. — A sequence of Coxeter systems can give a geometric form to a sequence of Salem numbers converging to a Pisot number. To give examples of this phenomenon, let $Y_{a,b,c}$ denote the Coxeter system whose diagram is a tree with 3 branches of lengths a, b and c, joined at a single node. For example, $Eh_8 = Y_{3,3,4}$, $Eh_9 = Y_{2,4,5}$ and $Eh_{10} = Y_{2,3,7}$.

Theorem **7.4.** — *As* $n \to \infty$, *we have*

$$\begin{aligned} \beta(Ah_n) &\to \lambda_{Golden} && \textit{from above,} \\ \beta(Bh_n) &\to \lambda_{Pisot} && \textit{from above,} \\ \beta(Dh_n) &\to \lambda_{Pisot} && \textit{from above, and} \\ \beta(Y_{2,3,n}) &\to \lambda_{Pisot} && \textit{from below.} \end{aligned}$$

The values of β *above, excluding the subsequence* $\beta(Ah_{2n})$, *are all Salem numbers.*

Proof. — The sequences of Coxeter systems above are all hyperbolic, crystallographic and (excluding Ah_{2n}) bipartite, so $\beta(W_n, S_n)$ ranges through Salem numbers. The limiting behavior of the $\beta(W_n, S_n)$ is calculated in [Hof] for the case of Ah_n; the other cases are similar. □

Infinite diagrams. — We remark that the diagrams for Bh_n, Dh_n and $Y_{2,3,n}$ all converge to the infinite diagram $Y_{2,3,\infty}$ if we use the triple-point as a basepoint. Similarly, Ah_n converges to $Y_{2,\infty,\infty}$. Suitably interpreted, we have $\beta(Y_{2,3,\infty}) = \lambda_{\mathrm{Pisot}}$ and $\beta(Y_{2,\infty,\infty}) = \lambda_{\mathrm{Golden}}$. See [MRS] for more on Pisot numbers and infinite graphs.

Proposition **7.5.** — *If an irreducible Coxeter system satisfies* $1 < \beta(W, S) \leq \lambda_{Golden}$ *then its diagram is a tree.*

Proof. — If the diagram is not a tree then $(W, S) \geq Ah_n$ or $(W, S) \geq Q_n$ for some n. In the first case we have $\beta(W, S) \geq \beta(Ah_n) > \lambda_{\mathrm{Golden}}$. In the second case we have $\beta(W, S) \geq \beta(Q_n)$, and one can check that $\beta(Q_n) > 2$ for all n. □

Small Coxeter systems. — Using Theorem 7.4 we can enumerate the Coxeter systems (W, S) that are sufficiently close to spherical, in the sense that $\beta(W, S)$ is sufficiently close to 1.

Theorem **7.6.** — *The only irreducible Coxeter systems with*

$$1 < \beta(W, S) < \lambda_{Pisot}$$

are $Y_{2,4,5}$ *and* $Y_{2,3,n}$, $n \geq 7$.

Proof. — Suppose $1 < \beta(W, S) < \lambda_{Pisot}$. By Proposition 7.5, the diagram D of (W, S) is a tree.

We claim D has at least one vertex of degree 3 or more. Indeed, there exists a minimal hyperbolic Coxeter system with $(W', S') \leq (W, S)$ and hence $\beta(W', S') < \lambda_{Pisot}$. Referring to Table 5, we find

$$\begin{aligned}(W, S) &\geq Eh_9 = Y_{2,4,5} \quad \text{or}\\ (W, S) &\geq Eh_{10} = Y_{2,3,7}.\end{aligned}$$

In particular, D contains a copy of the $Y_{2,3,5}$ diagram, possibly with higher weights.

Next we claim all the edges of D have weight 3. Indeed, an edge of weight 4 or more implies $(W, S) \geq Bh_n$ for some n, which is impossible because $\beta(W, S) < \lambda_{Pisot}$. In fact the tree D consists of 3 branches joined at a single node; otherwise $(W, S) \geq Dh_n$ for some n, which is impossible because $\lambda(Dh_n) > \lambda_{Pisot}$.

Thus $(W, S) = Y_{a,b,c}$ for some $a \leq b \leq c$. We have $(W, S) = Y_{2,4,5}$ if $(W, S) \geq Y_{2,4,5}$, since otherwise we would have

$$\beta(W, S) \;\geq\; \min\,(\beta(Y_{3,4,5}), \beta(Y_{2,5,5}), \beta(Y_{2,4,6})) \;>\; 1.36 > \lambda_{Pisot}.$$

Similarly, $(W, S) = Y_{2,3,n}$, $n \geq 7$, if $(W, S) \geq Y_{2,3,7}$, since otherwise we would have

$$\beta(W, S) \;\geq\; \min\,(\beta(Y_{3,3,7}), \beta(Y_{2,4,7})) \;\geq\; 1.40 > \lambda_{Pisot}.$$

To see these Coxeter systems qualify, just note that $\beta(Y_{2,3,n}) < \lambda_{Pisot}$ for all n by Theorem 7.4, and $\beta(Y_{2,4,5}) < \lambda_{Pisot}$ by Table 5. □

Corollary **7.7.** — *We have* $\lambda_{Pisot} = \inf\{\beta(W, S) \,:\, (W, S)$ *has higher rank*$\}$.

Proof. — Since $Y_{2,4,5}$ and $Y_{2,3,n}$, $n \geq 7$ are hyperbolic Coxeter systems, we have $\beta(W, S) \geq \lambda_{Pisot}$ if (W, S) has higher rank. To show this bound is best possible, let $Y_{2,3,n} \vee Y_{2,3,n}$ be the diagram obtained from two copies of $Y_{2,3,n}$ by identifying the nodes at the ends of the branches of length n. Let (W_n, S_n) be the associated Coxeter system (the 'double' of $Y_{2,3,n}$). Then it is straightforward to check that $\beta(W_n, S_n) \to \lambda_{Pisot}$ and $\mathrm{sig}(W_n, S_n) = (p_n, 2)$ for all $n \gg 0$. Thus λ_{Pisot} is a limit of $\beta(W, S)$ for higher-rank Coxeter systems. □

λ	Salem polynomial	Coxeter data	
1.17628	$1+x-x^3-x^4-x^5-x^6-x^7+x^9+x^{10}$	$\beta(Y_{2,3,7})$	
1.21639	$1-x^4-x^5-x^6+x^{10}$	$Y_{2,3,7}$	
1.23039	$1-x^3-x^5-x^7+x^{10}$	$\beta(Y_{2,3,8})$	
1.26123	$1-x^2-x^5-x^8+x^{10}$	$\beta(Y_{2,3,9})$,	$Y_{2,3,7}$
1.28064	$1-x^3-x^4-x^5+x^8$	$\beta(Y_{2,3,10})$,	$\beta(Y_{2,4,5})$
1.29349	$1-x^2-x^3+x^5-x^7-x^8+x^{10}$	$\beta(Y_{2,3,11})$,	$Y_{2,3,7}$

TABLE 6. — The 6 Salem numbers < 1.3 that can arise as $\lambda(w)$

Corollary **7.8.** — *Let* (W, S) *be a Coxeter system, and suppose* $w \in W$ *satisfies*

$$1 < \lambda(w) < \lambda_{Pisot}.$$

Then $\lambda(w)$ *is a Salem number.*

Proof. — We may assume (W, S) is irreducible and w is essential; then $1 < \beta(W, S) \leq \lambda(w) < \lambda_{\text{Pisot}}$, so (W, S) is either $Y_{2,4,5}$ or $Y_{2,3,n}$, $n \geq 7$. All these Coxeter systems are hyperbolic and crystallographic, so $\lambda(w)$ is a Salem number. □

Realizing small Salem numbers. — As remarked above, there are 47 known Salem numbers < 1.3. By the preceding Corollary, $\lambda(w)$ is also a Salem number whenever $1 < \lambda(w) < 1.3 < \lambda_{\text{Pisot}}$. Using the catalog of small Salem numbers, we can identify which ones occur.

Theorem **7.9.** — *Let* (W, S) *be a Coxeter system, and suppose* $1 < \lambda(w) < 1.3$, $w \in W$. *Then* $\lambda(w)$ *coincides with one of the 6 Salem numbers given in Table 6, and these all arise.*

Guide to Table 6. — The first column in Table 6 gives the approximate value of the Salem number λ; the second, the irreducible *Salem polynomial* $S(x)$ it satisfies; and the third, one or two ways in which λ arises in Coxeter groups as $\lambda(w)$. For example, $\lambda \approx 1.26123$ arises as $\lambda(w) = \beta(Y_{2,3,9})$ for any Coxeter element w in $Y_{2,3,9}$, and as $\lambda(w')$ for a suitable (non-Coxeter) element w' in $Y_{2,3,7}$.

Automorphisms of lattices. — To aid in the realization of Salem numbers via Coxeter groups, we quote a result from [GM].

Let $O(II_{p,1})$ denote the orthogonal group of the unique even, unimodular lattice of signature $(p, 1)$, and let $O^+(II_{p,1})$ be the subgroup of index two preserving one sheet of the hyperboloid $v \cdot v = -1$. It is known that $O^+(II_{9,1})$ is isomorphic with the Coxeter group $Y_{2,3,7}$ in its geometric representation [Vin2], [CS, Ch. 27].

A Salem polynomial is *unramified* if $|S(-1)S(1)| = 1$.

Theorem **7.10.** — *Let* $S(x)$ *be an unramified Salem polynomial of degree* $8n + 2$. *Then* $S(x) = \det(xI - g)$ *for some* $g \in O^+(II_{8n+1,1})$.

Proof of Theorem 7.9. — Let (W, S) be a Coxeter system with $1 < \lambda(w) < 1.3$, $w \in W$. As remarked above, $\lambda = \lambda(w)$ is a Salem number. We may assume w is essential; then $1 < \beta(W, S) < \lambda$. Since

$$\beta(Y_{2,3,n}) \geq \beta(Y_{2,3,12}) \approx 1.30227$$

for $n \geq 12$, Theorem 7.6 implies (W, S) is isomorphic to $Y_{2,4,5}$ or to $Y_{2,3,n}$, $7 \leq n \leq 11$.

Let d be the degree of λ over $\mathbf{Q}$. Since λ is a Salem number, d is even; and we have $d \leq |S|$ by Proposition 7.1.

Suppose (W, S) is isomorphic to $Y_{2,4,5}$. Then the condition $d \leq |S| = 9$ leaves only one possibility for λ, namely the degree 8 Salem number given in Table 6. In fact, in the catalog of Salem numbers in the range [1, 1.3] given in [B1] (known to be complete through degree 20), every other number has degree 10 or more.

Now suppose (W, S) is isomorphic to $Y_{2,3,n}$, $7 \leq n \leq 11$, and $d > 8$. Then $|S| = n + 3$, so $d = 10$, 12 or 14. If $d = 12$ then we have $\lambda \in [\beta(Y_{2,3,9}), 1.3]$, and if $d = 14$ then $\lambda \in [\beta(Y_{2,3,11}), 1.3]$. Referring to the catalog again, we find there are no Salem numbers of the required degrees in these ranges. Thus $d = 10$. There are 5 Salem numbers of degree 10 in the range [1, 1.3], and these complete the list of 6 numbers given in Table 6.

To conclude, we check that all 6 Salem numbers arise via Coxeter groups. Five of them can be recognized as the Coxeter eigenvalues $\beta(Y_{2,3,n})$, $7 \leq n \leq 11$. (The degree 8 number also arises as $\beta(Y_{2,4,5})$.) Four of the degree 10 numbers in Table 6 are unramified; by Theorem 7.10, these arise as $\lambda(g)$ for $g \in O^+(II_{9,1})$, and hence as $\lambda(w)$ for w in $Y_{2,3,7}$. All 6 numbers in the table are covered by at least once by these constructions, completing the proof. □

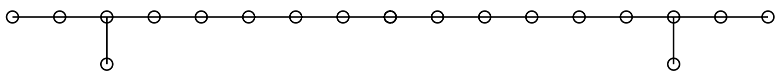


FIG. 7. — The Coxeter diagram for $W \subset O^+(II_{17,1})$

The second smallest Salem number. — After Lehmer's number, the second smallest known Salem number is $\lambda \approx 1.188368$, with unramified minimal polynomial

$$S(x) = 1 - x + x^2 - x^3 - x^6 + x^7 - x^8 + x^9 - x^{10} + x^{11} - x^{12} - x^{15} + x^{16} - x^{17} + x^{18}.$$

It is known that reflections in the roots of $II_{17,1}$ generate a Coxeter subgroup W of index two in $O^+(II_{17,1})$ [Vin2], [CS, Ch. 27]; its diagram is shown in Figure 7. Combining Theorems 7.9 and Theorem 7.10 we obtain:

Corollary **7.11.** — *The Salem number* $\lambda \approx 1.188368$ *arises as* $\lambda(g)$ *for* $g \in O^+(II_{17,1})$, *but not as* $\lambda(w)$ *for any* w *in the Coxeter subgroup* $W \subset O^+(II_{17,1})$.

In fact one can take $g = g_1 g_2$, where g_1 comes from the order 2 symmetry of the Coxeter diagram of W, and g_2 is the bicolored Coxeter element of a $Y_{2,3,7}$ subdiagram.

Graph theory. — We remark that the study of Coxeter systems via the values of $\beta(W, S)$ contains, as a special case, the study of graphs G via the leading eigenvalues $\alpha(G)$ of their adjacency matrices.

For example, Shearer has shown the values of $\alpha(G)$ (even when restricted to trees) are dense in the interval $[\sqrt{2+\sqrt{5}}, \infty)$ [Sh]. It follows that the values of $\beta(W, S)$ are dense in $[\lambda_{\text{Golden}}, \infty)$. On the other hand, graphs with $\alpha(G) < \sqrt{2+\sqrt{5}}$ have been classified, and it seems likely that a similar classification can be completed for Coxeter systems with $\beta(W, S) < \lambda_{\text{Golden}}$.

A survey of work on the leading eigenvalues of graphs can be found in [CR].

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Manuscrit reçu le 9 octobre 2001.