Y. Guivarc'h
M. Keane

Transience of Random Walks on Nilpotent Groups

Publications des séminaires de mathématiques et informatique de Rennes, 1972, fascicule 2
« Probabilités », , p. 8-15
<http://www.numdam.org/item?id=PSMIR_1972___2_8_0>

TRANSIENCE OF RANDOM WALKS ON NILPOTENT GROUPS

by

Y. Guivarc'h and M. Keane

Let G be a locally compact topological group. For a given probability measure μ on G , the <u>random walk</u> R W (μ) generated by μ is the Markov process with state space G and transition probabilities

$$P\ (g,A) = \mu\ (g^{-1}A)$$

for $g \in G$ and $A \subseteq G$ measurable. Random walks on G fall into two classes :

- R W (μ) is <u>recurrent</u> if for any $g \in G$ and any open set V in G, the probability of entering V infinitely often, starting at g, is one.
- R W (μ) is <u>transient</u> if for any $g \in G$ and any relatively compact set W in G, the expected number of visits to W starting from g is finite (and thus the probability of entering W infinitely often is zero).

We call the group G <u>transient</u> if R W (μ) is transient for each probability μ whose support generates G topologically. Otherwise G is <u>recurrent</u>.

It is well known (see e.g. [SP]) that if G is abelian and compactly generated, then G is transient if and only if the dimension of G/K is greater than two, where K is the maximal compact subgroup of G. The following result is angeneralization of the abelian case.

Theorem.-

Let G be a locally compact and compactly generated nilpotent group with maximal compact subgroup K. Then G is recurrent if and only if G/K is isomorphic to one of the six following abelian groups :

$$\mathbb{R} \oplus \mathbb{R} \ , \ \mathbb{R} \oplus \mathbb{Z} \ , \ \mathbb{Z} \oplus \mathbb{Z} \ , \ \mathbb{R} \ , \ \mathbb{Z} \ , \ 0 \ .$$

Corollary (discrete case).-

Let G be finitely generated, torsion free and nilpotent. If G is recurrent, then G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z}$ or 0.

The details of the proof of the theorem will be published later ([G K] , [KGB]) . Here we would like to give a proof for the simplest nilpotent non - abelian group and for a particular random walk on the group. This example contains the basic ideas and will not entangle us in the technical details of the non - discrete case.

Thus we let N denote the group of matrices of the form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

with a, b, c $\in \mathbb{Z}$. Henceforth, we identify such a matrix with the point $(a,b,c) \in \mathbb{Z}^3$. Multiplication of matrices yields the following rule of composition, which we write additively because of the similarity with addition in $\mathbb{Z}^3$:

$$(a,b,c) \hat{+} (a',b',c') = (a+a', b+b', c+c'+ab')$$

Let μ be the probability measure assigning mass $\frac{1}{6}$ to each of the points

(*) $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$

In $\mathbb{Z}^3$ with the normal addition μ would generate a transient random walk. By comparing the random walk generated by μ on N with this abelian random walk, we shall arrive at the following result.

Theorem.-

R W (μ) is transient on N.

Proof.-

Suppose that we start walking randomly on N at time 0 at the point (x_o,y_o,z_o). This means that with equal probability we pick a point $(\alpha_1,\beta_1,\gamma_1)$ from the collection (*) and at time 1 move to

$$(x_1,y_1,z_1) = (x_o,y_o,z_o) \hat{+} (\alpha_1,\beta_1,\gamma_1).$$

Then we pick another point $(\alpha_2,\beta_2,\gamma_2)$ from (*) according to μ , independant of (x_o,y_o,z_o) and of the choice of $(\alpha_1,\beta_1,\gamma_1)$, and at time 2 we move to the point

$$(x_2,y_2,z_2) = (x_1,y_1,z_1) \hat{+} (\alpha_2,\beta_2,\gamma_2).$$

Continuing this procedure for n time units, we obtain an admissible n - path

$$(x_o,y_o,z_o) \ , \ (x_1,y_1,z_1) \ , \ \ldots \ , \ (x_n,y_n,z_n)$$

for the process R W (ω). Now if at a certain time we find ourselves at the point (x,y,z), then at the following time step, using (*) and the rule $\hat{+}$, we shall arrive at one of the " neighbors ".

$$(x \pm 1 \ , \ y \ , \ z)$$
$$(x \ , \ y \pm 1 \ , \ z \pm x)$$
$$(x \ , \ y \ , \ z \pm 1 \ )$$

of (x,y,z). Thus for our admissible n - path we have

$$x_i - x_{i-1} = a_i$$
$$y_i - y_{i-1} = b_i$$
$$z_i - z_{i-1} = c_i$$

with the following possibilities for (a_i,b_i,c_i) :

$$(\pm 1, 0, 0),(0,\pm 1, \pm x_{i-1}) \ , \ (0,0,\pm 1),$$

$1 \leq i \leq n$. Therefore, also

$$(a_i,b_i,c_i) = (\alpha_i,\beta_i,\gamma_i+\beta_i x_i)..$$

Now denote by P_n the probability starting at (0,0,0) to return to (0,0,0) at time n. We have

$$P_n = \frac{\# \text{ of admissible n-paths with } (x_o,y_o,z_o)=(x_n,y_n,z_n)=(0,0,0)}{B^n}$$

Our problem is to show that the expected number of visits to (0,0,0) , given by

$$\sum_{n=0}^{\infty} P_n$$

is finite. Obviously, the conditions on our n-path for returning to (0,0,0) are

$$\sum_{i=1}^{n} \alpha_i = 0$$

$$\sum_{i=1}^{n} \beta_i = 0$$

$$\sum_{i=1}^{n} \gamma_i = - \sum_{i=1}^{n} \beta_i x_{i-1} = - \sum_{i=1}^{n} \beta_i \left(\sum_{j=1}^{i-1} \alpha_j \right).$$

Now, let π_n denote the probability corresponding to P_n for the abelian random walk on $\mathbb{Z}^3$ with the same measure μ . An admissibl n-path for this walk is again given by choosing independently $(\alpha_1,\beta_1,\gamma_1)$,... , $(\alpha_n,\beta_n,\gamma_n)$ from (*), and the conditions for such a path to return to (0,0,0) at time n are

$$\sum_{i=1}^{n} \alpha_i = 0$$

$$\sum_{i=1}^{n} \beta_i = 0$$

$$\sum_{i=1}^{n} \gamma_i = 0.$$

To compare P_n and π_n, fix a set $I \subseteq \{1,\dots,n\}$ and for $i \in I$ choose $(\alpha_i,\beta_i,\gamma_i)$ with $\gamma_i = 0$ and such that

$$\sum_{i \in I} \alpha_i = \sum_{i \in I} \beta_i = 0 .$$

Suppose for the moment that the number of indices left in $J = \{1,\dots,n\} \setminus I$ is even, say $|J| = 2k$. Then the number of ways to choose the remaining $(\alpha_j,\beta_j,\gamma_j)$, $j \in J$, with $(\alpha_j,\beta_j,\gamma_j) = (0,0,\pm 1)$, such that the abelian random walk will return to (0,0,0) is given by

$$\binom{2k}{k},$$

while the number of such choices yielding a return of the non - abelian walk to (0,0,0) is either

$$0 \quad (\text{if } c = -\sum_{i=1}^{n} \beta_i \, (\sum_{j=1}^{i-1} \alpha_j) \text{ is odd})$$

or

$$\binom{2k}{\frac{1}{2}c} \qquad (\text{if } c \text{ is even}).$$

If $|J| = 2k+1$ is odd, then likewise the abelian walk returns to (0,0,1) for

$$\binom{2k+1}{k}$$

choices, while the non - abelian walk returns to (0,0,0) in less than

$$\binom{2k+1}{[\frac{c}{2}]}$$

cases, c as above. Noting that the binomial coefficients for the abelian case are maximal, and varying the choice of I and $(\alpha_i,\beta_i,\gamma_i)$, $i \in I$, we see that the number of admissible non -

abelian n-paths from (0,0,0) to (0,0,0) is less than or equal to the number of admissible abelian n-paths from (0,0,0) to (0,0,0) or (0,0,1). Thus if π'_n denotes the probability of landing at (0,0,1) at time n, we have $P_n \leq \pi_n + \pi'_n$

$$\sum_{n=0}^{\infty} P_n \leq \sum_{n=0}^{\infty} (\pi_n + \pi'_n) < \infty ,$$

since the abelian walk is known to be transient. This concludes the proof of the theorem.

Using only the above ideas and a bit of harmonic analysis on the circle group, it is easy to generalize the above result to any probability measure on N (note that it is not necessary to use the same measure on $\mathbb{Z}^3$ for comparison, as any probability measure on $\mathbb{Z}^3$ generating $\mathbb{Z}^3$ will yield a transient walk). Since any finitely generated torsion free nilpotent group contains a copy of N, we have the same result for such groups by considering the induced walk on the copy of N, supposing recurrence. Thus we can prove the corollary announced earlier. This procedure is impossible in the continuous case and the methods become more complicated.

Using similar techniques ([KGB]), we obtain also a renewal theorem for transient nilpotent groups G · if

$$\mathcal{U} = \sum_{n \geq 0} \mu^{*n}$$

is the potential kernel of R W (μ) , then

$$\lim_{g \to \infty} \mathcal{U} (g . V) = 0$$

for any V relatively compact in G.

Consider now the group S of all matrices of the form

$$(a,b) = \begin{bmatrix} 1 & 0 \\ a & 2^b \end{bmatrix}$$

with $a \in \mathbb{R}$ and $b \in \mathbb{Z}$. The multiplication is given by

$$(a,b) \hat{+} (a',b') = (a+2^b a', b+b')$$

S is a solvable group, and we conjecture that S is recurrent. More precisely, let μ be the measure giving mass $\frac{1}{4}$ to each of the points $(\pm 1, 0)$, $(0, \pm 1)$. It is not hard to see that recurrence is present in each component separately. Is R W (μ) recurrent ? We note that Azencott has constructed transient random walks with symmetric probabilities μ on the group

$$S' = \{ (a,b) \mid a,b \in \mathbb{R} \}.$$

REFERENCES

[SP] Stone and Port , Acta Math. 122, 19-144 (1969)

[GK] Guivarc'h et Keane, CR 272, 1462-3 (1971)

[KGB] Séminaire KGB Rennes, to appear

Y. GUIVARC'H
M. KEANE
Laboratoire de Probabilités
Université de RENNES
Equipe associée du CNRS n° 250
B.P. 25 A
35 - RENNES -