MICHAL MISIUREWICZ

A Short Proof of the Variational Principle for a $\mathbb{Z}_+^N$ Action on a Compact Space

Publications des séminaires de mathématiques et informatique de Rennes, 1975, fascicule S4
« International Conference on Dynamical Systems in Mathematical Physics », , p. 1-10
<http://www.numdam.org/item?id=PSMIR_1975___S4_A18_0>

A SHORT PROOF OF THE VARIATIONAL PRINCIPLE

FOR A $\mathbb{Z}_+^N$ ACTION ON A COMPACT SPACE

by

Michał Misiurewicz

0. Introduction. Ruelle in [12] introduced the notion of pressure for an action of the group $\mathbb{Z}^N$ on a compact metric space. It is a generalization of the notion of topological entropy. The variational principle (proved in [12] under some strong conditions) is a generalization of the Dinaburg's theorem ([5,8,7]) on a connection between the topological and measure entropies. A general proof of the variational principle was given by Walters in [13] (see also Denker [3]) for an action of $\mathbb{Z}_+$ and by Elsanousi in [6] for an action of $\mathbb{Z}_+^N$.

The first part of the proof given below (an inequality $h_\mu(T)+\mu f \leqslant P(T,f)$) is a natural generalization of the proof from [10]. The second part ($\sup_\mu (h_\mu(T)+\mu f) \geqslant P(T,f)$) is quite new (although the idea is close to the Ruelle's one).

1. Notations.

$\mathbb{Z}_+$ denotes the set of all non-negative integers. Let us fix a positive integer N.

$G = \mathbb{Z}_+^N$ is a commutative semigroup with respect to addition. For $n \in G$ we denote by n_i the i-th coordinate of all T^{*n} , $n \in G$).

$\mathcal{W}$ is the set of all neighbourhoods of the diagonal in $X \times X$, directed by the inclusion. It is a uniform structure (uniformity) for X (see [9]).

2. Definitions of pressure and entropy.

We may in the natural way extend Ruelle's definition of pressure of $\mathbb{Z}^N$ - action ([12]).

Let $n \in G$, $\delta \in \mathcal{W}$, $f \in C(X)$. We define succesively:

$$\Lambda(n) = \{m \in G : m_i < n_i \quad \text{for} \quad i = 1,\dots,N\} \quad ,$$

$$\lambda(n) = \operatorname{Card} \Lambda(n) = n_1 \cdot \ldots \cdot n_N \quad ,$$

$$\delta_n = \bigcap_{k \in \Lambda(n)} (T^k \times T^k)^{-1} \delta \quad ,$$

$$f_n = \sum_{k \in \Lambda(n)} T^{*k} f$$

Of course $\delta_n \in \mathcal{W}$, $f_n \in C(X)$.

A finite subset e of X is called:

(n,δ)-separated, if for any $x,y \in e$, $x \neq y$, we have $(x,y) \notin \delta_n$,

(n,δ)-spanning, if for any $x \in X$ there exists $y \in e$ such that $(x,y) \in \delta_n$.

We denote $p(f,e) = \log \sum_{x \in e} \exp f(x)$. We define further

$$(1) \qquad P_{n,\delta}(T,f) = \sup \{ p(f_n,e) : e \text{ is } (n,\delta)\text{-separated} \}$$

$$(2) \qquad P_\delta(T,f) = \limsup_{n \in G} \frac{1}{\lambda(n)} P_{n,\delta}(T,f)$$

Of course, $P_\delta(T,f) \geq P_\varepsilon(T,f)$ for $\delta \subset \varepsilon$. Therefore it is possible to define the pressure

$$(3) \qquad P(T,f) = \lim_{\delta \in \mathcal{W}} P_\delta(T,f) = \sup_{\delta \in \mathcal{W}} P_\delta(T,f)$$

In the sequel we shall use the measure entropy function of n ($i = 1,\ldots,N$). For $n,m \in G$ let $nm = (n_1m_1,\ldots,n_Nm_N)$. The relation $\geqslant$ ($n \geqslant m$ iff $n_i \geqslant m_i$ for $i = 1,\ldots,N$) directs G .

X is a non-empty compact Hausdorff space.

$C(X)$ is the space of all continuous real functions on X with the norm $\|f\| = \sup_{x \in X} |f(x)|$.

$\mathfrak{M}(X)$ is the space of all positive Borel regular normed measures on X . It can be identified with the ~~xxxx~~ space of all positive linear functionals on $C(X)$ having norm 1 (therefore we shall write μf instead of $\int_X f\, d\mu$ for $f \in C(X)$). We consider the weak-$*$ topology on $\mathfrak{M}(X)$ (then $\mathfrak{M}(X)$ is compact).

T ($n \mapsto T^n$) is an action of G on X , i.e. a homomorphism of G into the semigroup of all continuous transformations of X into itself (i.e. $T^n : X \longrightarrow X$, $T^{n+m} = T^n \circ T^m$).

$T^{*n} : C(X) \longrightarrow C(X)$ (for $n \in G$) is the operator induced by T^n (i.e. $T^{*n}f = f \circ T^n$).

$T^{**n} : \mathfrak{M}(X) \longrightarrow \mathfrak{M}(X)$ (for $n \in G$) is a restriction of the operator induced by T^{*n} (i.e. $T^{**n}\mu = \mu \circ T^{*n}$) to $\mathfrak{M}(X)$. It is easy to check that indeed $T^{**n}(\mathfrak{M}(X)) \subset \mathfrak{M}(X)$ and that T^{**n} is continuous.

Of course T^* and T^{**} are actions of G on $C(X)$ and $\mathfrak{M}(X)$, respectively.

$\mathfrak{M}(X,T)$ is a space of all T-invariant measures (i.e. these elements of $\mathfrak{M}(X)$ which are fixed points

$H_\mu(\cdot)$ (for the definition and properties see e.g. [11]).

For $\mu \in \mathcal{M}(X,T)$ the entropy $h_\mu(T)$ may be defined in the same way as in [2] for the action of $\mathbb{Z}^N$. For a Borel finite partition A of the space X we define

$A^n = \bigvee_{k \in \Lambda(n)} (T^k)^{-1}A$ for $n \in G$,

$h_\mu(T,A) = \lim_{n \in G} \frac{1}{\lambda(n)} H_\mu(A^n)$ (it is easy to show that the limit exists, cf. [2]). Finally,

$h_\mu(T) = \sup \{ h_\mu(T,A) : A$ - Borel finite partition$\}$.

3. The variational principle.

We shall prove the following variational principle:

$P(T,f) = \sup_{\mu \in \mathcal{M}(X,T)} (h_\mu(T) + \mu f)$. The proof consists of two parts.

Part I. $h_\mu(T) + \mu f \leq P(T,f)$ for $\mu \in \mathcal{M}(X,T)$.

Proof. Let $\mu \in \mathcal{M}(X,T)$. Let us fix $\xi > 0$ and a Borel finite partition A . Take $m \in G$ such that

(4) $\log 2 \leq \xi \cdot \lambda(m)$

Let A^m consist of the sets $a_1, \ldots, a_s$. For any of them there exists a compact set $b_i \subset a_i$ such that $\mu(a_i \setminus b_i) \leq \frac{\xi}{s \log s}$. Let $b_0 = X \setminus \bigcup_{i=1}^{s} b_i$. For the partition $B = \{b_0, b_1, \ldots, b_s\}$ we have

(5) $H_\mu(A^m | B) \leq \mu(b_0) \cdot \log s \leq \xi$

We take $\varepsilon = (X \times X) \setminus \bigcup_{\substack{i,j=1 \\ i \neq j}}^{s} (b_i \times b_j) \in \mathcal{W}$ and next $\delta \in \mathcal{W}$ such that $\delta \circ \delta \subset \varepsilon$ (i.e. if $(x,y),(y,z) \in \delta$, then $(x,z) \in \varepsilon$) and $|f(x) - f(y)| \leq \xi$ if $(x,y) \in \delta$.

Let us fix $n \in G$. There exists a maximal (nm,δ)-separated (i.e. being also (nm,δ)-spanning) set $e \subset X$.

Denote by $C := \bigvee_{k \in \Lambda(n)} (T^{km})^{-1}B$. Further, denote $\alpha(b) = \sup_{x \in b} f_{nm}(x)$ for $b \in C$, $\beta = \sum_{b \in C} \exp \alpha(b)$.

We have $\int_b f_{nm}\, d\mu \leq \alpha(b) \cdot \mu(b)$, therefore

$$H_\mu(C) + \mu f_{nm} \leq \sum_{b \in C} \mu(b) \ (\alpha(b) - \log \mu(b)) =$$

$$= \beta \cdot \sum_{b \in C} \frac{\exp \alpha(b)}{\beta} \cdot \eta\left(\frac{\mu(b)}{\exp \alpha(b)}\right)$$, where $\eta(x) = -x \log x$.

The function η is concave, therefore

(6) $$H_\mu(C) + \mu f_{nm} \leq \beta \cdot \eta\left(\sum_{b \in C} \frac{\exp \alpha(b)}{\beta} \cdot \frac{\mu(b)}{\exp \alpha(b)}\right) = \log \beta$$

e is a (nm,δ)-spanning set, hence for every $b \in C$ there exists a point $z(b) \in e$ such that

$\alpha(b) = \sup \{ f_{nm}(x) : x \in b , (x,z(b)) \in \delta_{nm} \}$. But if $(x,z(b)) \in \delta_{nm}$, then for $k \in \Lambda(nm)$ $(T^k x, T^k z(b)) \in \delta$, thus, in view of the definition of δ ,

$|(T^{*k} f)(x) - (T^{*k} f)(z(b))| \leq \xi$. Hence

(7) $$f_{nm}(z(b)) \geq \alpha(b) - \xi \cdot \lambda(nm)$$

From the definitions of δ and ε we obtain for $y \in e$, $k \in \Lambda(n)$ $\mathrm{Card} \{ a \in B : \exists_{x \in a} (T^{km}x, T^{km}y) \in \delta \} \leq 2$,

thus for $y \in e$ $\mathrm{Card} \{ b \in C : \exists_{x \in b} (x,y) \in \delta_{nm} \} \leq 2^{\lambda(n)}$,

because $\delta_{nm} \subset \bigcap_{k \in \Lambda(n)} (T^{km} \times T^{km})^{-1} \delta$. Hence

(8) $$\mathrm{Card} \{ b \in C : z(b) = y \} \leq 2^{\lambda(n)}$$

Hence, from (7) and (8) we get:

$$2^{\lambda(n)} \cdot \sum_{y \in e} \exp f_{nm}(y) \geqslant \sum_{b \in C} \exp \alpha(b) \cdot \exp(-\xi \cdot \lambda(nm)) \quad , \text{ thus}$$

$$(9) \qquad \lambda(n) \cdot \log 2 + p(f_{nm}, e) \geqslant \log \beta - \xi \cdot \lambda(nm)$$

But $\mu f_{nm} = \lambda(nm) \cdot \mu f$, so from this, from (1),(4),(6) and (9) we obtain (notice that $\lambda(nm) = \lambda(n) \cdot \lambda(m)$) :

$$(10) \qquad \frac{1}{\lambda(nm)} H_\mu(C) + \mu f \leqslant \frac{1}{\lambda(nm)} P_{nm,\delta}(T,f) + 2\xi$$

In view of (5), for $k \in \Lambda(n)$ we have

$H_\mu((T^{km})^{-1}A^m | (T^{km})^{-1}B) \leqslant \xi$, therefore

$H_\mu(A^{nm}|C) = H_\mu(\bigvee_{k \in \Lambda(n)} (T^{km})^{-1}A^m | \bigvee_{k \in \Lambda(n)} (T^{km})^{-1}B) \leqslant \xi \cdot \lambda(n)$.

Hence $H_\mu(A^{nm}) \leqslant H_\mu(C) + H_\mu(A^{nm}|C) \leqslant H_\mu(C) + \xi \cdot \lambda(n)$,

thus we obtain from (10): $\frac{1}{\lambda(nm)} H_\mu(A^{nm}) + \mu f \leqslant$

$\leqslant \frac{1}{\mu(nm)} P_{nm,\delta}(T,f) + 3\xi$. Taking lim sup with respect to n we get $h_\mu(T,A) + \mu f \leqslant P_\delta(T,f) + 3\xi \leqslant P(T,f) + 3\xi$.

But ξ and A were arbitrary , hence $h_\mu(T) + \mu f \leqslant P(T,f)$. $\square$

<u>Part II.</u> $\sup_{\mu \in \mathcal{M}(X,T)} (h_\mu(T) + \mu f) \geqslant P(T,f)$.

<u>Proof.</u> Let us fix $\delta \in \mathcal{W}$. For every $n \in G$, we choose such an (n,δ)-separated set e_n that

$$(11) \qquad p(f_n, e_n) \geqslant P_{n,\delta}(T,f) - 1$$

Let us define a measure σ_n , concentrated on e_n , by a formula $\sigma_n(\{y\}) = \exp(f_n(y) - p(f_n, e_n))$ for $y \in e_n$.

We have $\sum_{y \in e_n} \sigma_n(\{y\}) = 1$, therefore $\sigma_n \in \mathcal{M}(X)$.

Let $\mu_n = \frac{1}{\lambda(n)} \sum_{k \in \Lambda(n)} T^{**k} \sigma_n$.

For some sequence $(n_i)_{i=1}^{\infty}$ cofinal with G we have

(12) $\lim_{i\to\infty} \frac{1}{\lambda(n_i)} P_{n_i,\delta}(T,f) = P_\delta(T,f)$

We choose some cluster point of the sequence $(\mu_{n_i})_{i=1}^{\infty}$ and denote it by μ. Of course, μ is also a cluster point of the net $(\mu_n)_{n\in G}$. For $g \in C(X)$ and $k \in G$ fixed, the function $\Phi: \mathfrak{M}(X) \longrightarrow \mathbb{R}$, given by the formula $\Phi\nu = \nu g - \nu(T^{*k}g)$, is continuous, therefore $\Phi\mu$ is a cluster point of the net $(\Phi\mu_n)_{n\in G}$. We have

$|\Phi\mu_n| \leqslant \frac{1}{\lambda(n)} \cdot 2 \cdot (\lambda(n) - \lambda(n-k)) \cdot \|g\|$ (for $n \geqslant k$),

because $\mathrm{Card}\,(\Lambda(n) \setminus (k+\Lambda(n))) = \mathrm{Card}\,((k+\Lambda(n)) \setminus \Lambda(n)) = \lambda(n) - \lambda(n-k)$. But $\lim_n \frac{\lambda(n-k)}{\lambda(n)} = 1$, thus $\Phi\mu = 0$. Hence $\mu g = (T^{**k}\mu)g$. But g and k were arbitrary, therefore $\mu \in \mathfrak{M}(X,T)$.

There exists a Borel finite partition A of X such that $a \times a \subset \delta$ for $a \in A$. Then for $a \in A^n$ $a \times a \subset \delta_n$, therefore $\mathrm{Card}\,(e_n \cap a) \leqslant 1$. Hence

$H_{\sigma_n}(A^n) + \sigma_n f_n = \sum_{y\in e_n} \sigma_n(\{y\})(f_n(y) - \log \sigma_n(\{y\})) = p(f_n,e_n)$.

Let us fix $m,n \in G$, $n \geqslant 2m$. For given $j \in \Lambda(m)$ let $s(j) = (E(\frac{n_1-j_1}{m_1}),\dots,E(\frac{n_N-j_N}{m_N}))$. We have:

$A^n = \bigvee_{r\in\Lambda(s(j))} (T^{rm+j})^{-1}A^m \vee \bigvee_{k\in\Xi} (T^k)^{-1}A$, where

$\Xi = \Lambda(n) \setminus (j+\Lambda(ms(j)))$. But $\mathrm{Card}\,\Xi = \lambda(n) - \lambda(ms(j)) \leqslant \lambda(n) - \lambda(n-2m)$, thus $p(f_n,e_n) = H_{\sigma_n}(A^n) + \sigma_n f_n \leqslant$

$\leqslant \sum_{r\in\Lambda(s(j))} H_{\sigma_n}((T^{rm+j})^{-1}A^m) + \sigma_n f_n + (\lambda(n) - \lambda(n-2m)) \log \mathrm{Card}\, A.$

Summing the inequalities obtained for $j \in \Lambda(m)$ we get (notice that for $k \in \Lambda(n)$ there exists a unique j (m) and a unique $r \in \Lambda(s(j))$ such that $k = rm + j$) :

(13) $$\sum_{k \in \Lambda(n)} H_{\sigma_n}((T^k)^{-1}A^m) + \lambda(m) \cdot \sigma_n f_n \geqslant$$

$$\geqslant \lambda(m) \cdot (p(f_n, e_n) - (\lambda(n) - \lambda(n-2m)) \cdot \log \mathrm{Card}\, A)$$

We have also

(14) $$\sigma_n f_n = \sigma_n \Big(\sum_{k \in \Lambda(n)} T^{*k} f \Big) = \Big(\sum_{k \in \Lambda(n)} T^{**k} \sigma_n \Big) f = \lambda(n) \cdot \mu_n f$$

From the definition of entropy it follows that

(15) $$H_{\sigma_n}((T^k)^{-1}A^m) = H_{T^{**k}\sigma_n}(A^m) .$$

From the definition of entropy and from the concavity of the function $-x \log x$ it follows that

(16) $$H_{\mu_n}(A^m) \geqslant \frac{1}{\lambda(n)} \sum_{k \in \Lambda(n)} H_{T^{**k}\sigma_n}(A^m)$$

The formulas (11) and (13) - (16) give us

(17) $$\frac{1}{\lambda(m)} H_{\mu_n}(A^m) + \mu_n f \geqslant \frac{1}{\lambda(n)} P_{n,\delta}(T,f) - \frac{1}{\lambda(n)} \cdot$$

$$\cdot ((\lambda(n) - \lambda(n-2m)) \log \mathrm{Card}\, A + 1)$$

The partition A can be chosen in such a way that the boundaries of elements of A have measure μ zero. (see [1] , Chapt. IV , §5 , exerc. 13 d ; see also [10]). Then A^n has the same property. But for a set a with the boundary of measure μ zero, the function $\mathfrak{M}(X) \longrightarrow \mathbb{R}$, given by $\nu \longmapsto \nu(a)$, is continuous in the point μ .

Hence the function $\mathfrak{M}(X) \longrightarrow \mathbb{R}$, given by $\nu \longmapsto H_\nu(A^m)$ is also continuous at the point μ , therefore in view of (12), (17) and the definition of μ , we have

$$\frac{1}{\lambda(m)} H_\mu(A^m) + \mu f \geqslant P_\delta(T,f) \, .$$

Taking the limit with respect to m and using the inequality $h_\mu(T,A) \leqslant h_\mu(T)$, we obtain $h_\mu(T) + \mu f \geqslant P_\delta(T,f)$. But δ was arbitrary, hence $\sup_{\mu \in \mathfrak{M}(X,T)} (h_\mu(T) + \mu f) \geqslant P(T,f)$. □

4. Remark .

If $h_\mu(T) + \mu f = P(T,f)$, then μ is called an equilibrium state for (T,f) (measure with maximal entropy in the case of $f = 0$). The above construction shows that if $P_\delta(T,f) = P(T,f)$ for some $\delta \in \mathscr{W}$, then there exists an equilibrium state for (T,f) . In the case of $N = 1$, $f = 0$, this can be reformulated as follows:

If there exists an open cover A such that $h(T,A) = h(T)$, then there exists a measure with maximal entropy.

This is a particular case of the theorem of Denker ([4]), but obtained without assuming X finite dimensional.

References .

[1] N. Bourbaki - Éléments de Mathématique, Livre VI (Intégration) Chap. IV - Hermann & C^{ie} , Paris, 1952

[2] J. P. Conze - Entropie d'un groupe abélien de transformations - Z. Wahrscheinlichkeitstheorie verw. Geb., 25 (1973), 11 - 30

[3] M. Denker - Remarques sur la pression pour les transformations continues - C. R. Acad. Sci. Paris, 279 (1974), 967 - 970

[4] M. Denker - Measures with maximal entropy - preprint

[5] E. I. Dinaburg - Сьязь между различными энтропийными характеристиками динамических систем - Изв. А. Н. СССР, 35 (1971), 324 - 366

[6] S. A. Elsanousi - A variational principle for the pressure of a continuous $\mathbb{Z}^2$-action on a compact metric space - preprint

[7] T. N. T. Goodman - Relating topological entropy and measure entropy - Bull. Lond. Math. Soc., 3 (1971), 176-180

[8] L. W. Goodwyn - Topological entropy bounds measure-theoretic entropy - Proc. Amer. Math. Soc., 23 (1969), 679 - 688

[9] J. L. Kelley - General Topology - Van Nostrand, Princeton, 1955

[10] M. Misiurewicz - Topological conditional entropy - to appear in Studia Math , 55

[11] V. A. Rohlin - Лекции по энтропийной теории преобразований с инвариантной мерой - УМН, 22 (1967), 3 - 56 (Lectures on the entropy theory of measure preserving transformations - Russian Math. Surveys, 22 (1967), 1-52)

[12] D. Ruelle - Statistical mechanics on a compact set with $\mathbb{Z}^\nu$ action satysfying expansiveness and specification - Trans. Amer. Math. Soc., 185 (1973), 237 - 252

[13] P. Walters - A variational principle for the pressure of continuous transformations - preprint

M. Misiurewicz
Mathematical Institute, Univ. of Warsaw
PKiN IX p.
00-901 Warszawa
Poland