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MAXIMAL ORDERS IN AN AZUMAYA ALGEBRA OVER A VON NEUMANN REGULAR RING

By

BO STENSTRÖM

1. Introduction

The classical theory of maximal orders over a Dedekind domain R was generalized by Auslander and Goldman [1] to the case of a noetherian integrally closed domain R , and further by Fossum [10] to a Krull domain R . The methods used for these generalizations depend heavily on a reduction to the classical case by localization at the prime ideals of height 1 in R , and they are not practicable in the case of a more general ground-ring R . More recently, Kirkman and Kuzmanovich [14] have studied maximal orders over a hereditary ring R , using the Pierce representation of R as a sheaf of Dedekind domains to obtain a reduction to the classical case.

Our aim in this paper is to use the methods of [14] to study maximal orders over a commutative ring R whose total ring of fractions K is von Neumann regular. When Q is an Azumaya algebra over K , we shall define an R-*order* in Q to be full R-subalgebra A of Q such that every element of A is integral over R . Besides the development of the basic results of maximal orders, we shall obtain a characterization of Dedekind orders (cf. Robson [20]) as maximal orders over (generalized) Dedekind rings (Theorem 12.1).

Part I. General theory of maximal orders

2. Preliminaries

Let R be a commutative ring with total ring of fractions K, and let Σ be the set of non-zero-divisors of R. Throughout this paper we shall assume that K is von Neumann regular and that R is completely integrally closed in K, i.e. if $x \in K$ and there exists $s \in \Sigma$ such that $sx^i \in R$ for all $i \geq 0$, then $x \in R$. Since R is then integrally closed in K, every idempotent of K lies in R, so R is a p.p. ring, i.e. the principal ideals of R are projective modules [3].

The rings R and K thus have the same boolean algebra $\underline{B}$ of idempotents. Let $\underline{X}$ denote the boolean space of maximal ideals of $\underline{B}$. The stalk at $x \in \underline{X}$ of the Pierce sheaf associated to the ring R is $R_x = R/xR$, where xR is the ideal of R generated by the set x of idempotents. R_x is an indecomposable ring, i.e. its only idempotents are 0 and 1. More generally, the stalk at x for an R-module M is

$$M_x = R_x \otimes_R M \cong M/xM .$$

There is a canonical surjection $M \rightarrow M_x$, written as $m \mapsto m_x$. If $m_x = 0$ for some $x \in \underline{X}$, then $m_y = 0$ for all y in some closed-and-open neighborhood of x in $\underline{X}$, and $me = 0$ for some idempotent e of R. Furthermore, $\bigoplus_{x \in \underline{X}} R_x$ is faithfully flat as an R-module. (See [18] or [22] for details on the Pierce sheaf).

Since K is von Neumann regular, K_x is a field for each $x \in \underline{X}$. The ring R_x is an integral domain with K_x as its field of fractions.

We shall throughout the paper assume that Q <u>is an Azumaya algebra over</u> K. Then for each $x \in \underline{X}$ we have that Q_x is a central simple K_x-algebra [14]. In [14] it is shown how the reduced trace can be defined as a K-linear mapping $\mathrm{Trd}: Q \rightarrow K$. We shall need the following two results:

Lemma 2.1 The mapping $\psi : Q \to \mathrm{Hom}_K(Q,K)$ given by $\psi(a) = \mathrm{Trd}(a\,-)$ is a K-isomorphism.

Proof. See [14] (Lemma 2.3) for details. The essential point is that $\psi_x : Q_x \to \mathrm{Hom}_{K_x}(Q_x, K_x)$ is classically known to be an isomorphism for each $x \in \underline{X}$. $\square$

Lemma 2.2 If $a \in Q$ is integral over R , then $\mathrm{Trd}(a) \in R$.

Proof. It suffices to show this pointwise for each $x \in \underline{X}$. As is shown in [14], one is then reduced to the case when R_x is an integral domain, which is treated in [2]. $\square$

3. R-lattices

Let V be a finitely generated projective K-module. An R-submodule L of V is called an R-lattice in V if

1) L is full in V , i.e. $LK = V$;
2) L is contained in a finitely generated R-submodule of V .

Note that since K is R-flat, one has for every R-submodule L of V that

$$L \otimes_R K \cong LK \cong L[\Sigma^{-1}] ,$$

where $L[\Sigma^{-1}]$ denotes the module of fractions of L with respect to Σ .

Lemma 3.1 If L is an R-lattice in V and M is a full R-submodule of V , then $sL \subset M$ for some $s \in \Sigma$.

Proof. L is contained in an R-submodule of V generated by $x_1, \dots, x_n$. Since M is full, each x_i can be written as $x_i = \sum_j k_{ij} x_{ij}$ with $x_{ij} \in M$. Choose $s \in \Sigma$ such that all $sk_{ij} \in R$. Then $sL \subset M$. $\square$

Proposition 3.2 An R-submodule L of V is an R-lattice in V if and only if there exist finitely generated projective R-submodules P_1, P_2 of V such that $P_1 \subset L \subset P_2$ and $\mathrm{rank}_R P_1 = \mathrm{rank}_K V$.

Proof. Suppose L is an R-lattice. Since K is regular, we may write $V = \oplus Ku_i$, where each Ku_i is isomorphic toa principal ideal of K , i.e. Ku_i is isomorphic to Ke_i for some idempotent $e_i \in R$. Since L is full, we may assume that $u_i \in L$. Then $P_1 = \oplus Ru_i$ is a finitely generated projective R-module in L and of same rank as V . By Lemma 3.1 there exists $s \in \Sigma$ such that $sL \subset P_1$, and then $L \subset s^{-1}P_1 = P_2$.

The converse is clear, for if P_1 is a finitely generated projective R-module of same rank as V , then P_1 is full in V . ▯

Remark Similar arguments show that if M is an R-lattice in V , then an R-submodule L of V is an R-lattice if and only if $rM \subset L \subset s^{-1}M$ for some $r, s \in \Sigma$.

4. R-orders

An R-subalgebra A of the Azumaya K-algebra Q is an R-order in Q if A is full in Q and every $a \in A$ is integral over R .

Lemma 4.1 If A is an R-order in Q , then A is a central R-algebra.

Proof. If $a \in \text{cen}(A)$, then $a \in \text{cen}(AK) = \text{cen}(Q) = K$. Since a is integral over R , and R is integrally closed in K , it follows that $a \in R$. ▯

The ring Q may thus be described as the ring $A[\Sigma^{-1}]$ of central fractions of A . Of course Q is also the total left and right ring of fractions of A , since every non-zero-divisor is invertible in an Azumaya algebra.

Proposition 4.2 There exists an R-order in Q .

Proof. As in the proof of Prop. 3.2 we may write $Q = \oplus Ku_i$, with

$u_1 = 1$. Then $u_i u_j = \sum_k a_{ijk} u_k$ for some $a_{ijk} \in K$. Let $s \in \Sigma$ with all $s a_{ijk} \in R$. Put $v_1 = 1$, $v_i = s u_i$ for $i \neq 1$. Then $Rv_1 + \sum Rv_i$ is a full R-algebra, and it is an R-order since it is a finitely generated R-module . ▯

Proposition 4.3 An R-subalgebra A of Q is an R-order in Q if and only if A_x is an R_x-order in Q_x for each $x \in \underline{X}$.

Proof. A is full in Q if and only if A_x is full in Q_x for each $x \in \underline{X}$, since $\bigoplus_x R_x$ is faithfully flat. If an element $a \in A$ is integral over R , then of course $a_x \in A_x$ is integral over R_x at each $x \in \underline{X}$. Suppose on the other hand that A_x is an R_x-order for all $x \in \underline{X}$. For each $a \in A$ and $x \in \underline{X}$ there is then an equation of integral dependence for a holding at all y in a neighborhood of x . Because of the compactness of $\underline{X}$ one can multiply together finitely many of these equations to get an equation of integral dependence for a holding at all $y \in \underline{X}$, i.e. holding globally for a . ▯

Theorem 4.4 An R-subalgebra A of Q is an R-order in Q if and only if A is an R-lattice.

Proof. Suppose A is an R-order in Q . Write $Q = \bigoplus Ku_i$ with $Ku_i = Ke_i$ for idempotents $e_i \in R$, and with $u_i \in A$. Define $g_i : Q \to K$ as $g_i(u_i) = e_i$, $g_i(u_j) = 0$ for $i \neq j$. By Lemma 2.1 there exist $v_i \in Q$ such that $g_i(a) = \mathrm{Trd}(v_i a)$ for all $a \in Q$. Since the g_i's generate the K-module $\mathrm{Hom}_K(Q,K)$, the v_i's generate Q over K . Similarly $e_i g_i = g_i$ implies $e_i v_i = v_i$. For each $a \in A$ we write $a = \sum k_j v_j$ with $k_j \in K$. Then

$$\mathrm{Trd}(au_i) = \mathrm{Trd}(\sum_j k_j v_j u_i) = \sum_j k_j g_j(u_i) = k_i e_i ,$$

so $k_i e_i \in R$ by Lemma 2.2. Then

$$a = \sum k_i v_i = \sum k_i e_i v_i \in \sum Rv_i ,$$

and hence A is contained in the finitely generated R-module $\sum Rv_i$

Suppose conversely that the R-algebra A is an R-lattice in Q. By Prop. 4.3 it suffices to show that A_x is an R_x-order for each $x \in \underline{X}$. We may therefore assume that R is an integral domain with field of fractions K. Let B be any R-order in Q (it exists by Prop. 4.2). By Lemma 3.1 there exists $s \in \Sigma$ such that $sA \subset B$. One may now proceed by arguing as in the proof of Prop. 1.2 of [7], and one obtains that a is integral over R. $\square$

Remarks. 1. By Schelter [21] (p. 253) there exists a noetherian R-order over a Krull domain R, such that A is not a finitely generated R-module.

2. Kirkman and Kuzmanovich [14] show that if R is hereditary, then every R-order in Q is finitely generated as an R-module, but that this no longer holds if R is only semihereditary.

5. The left and right orders of a lattice

Lemma 5.1 *If I is a full R-submodule of Q, then* $I \cap \Sigma \neq \emptyset$.

Proof. We have $1 = \sum x_i k_i$ with $x_i \in I$, $k_i \in K$. Choose $s \in \Sigma$ with all $sk_i \in R$. Then $s = \sum x_i sk_i \in I$. $\square$

For the converse we have:

Lemma 5.2 *If A is an R-order in Q and I is a left A-submodule of Q such that* $I \cap \Sigma \neq \emptyset$, *then I is full in Q.*

Proof. Suppose $s \in I \cap \Sigma$. If $q \in Q$, then $q = \sum a_i k_i$ with $a_i \in A$, $k_i \in K$. But then $q = \sum a_i k_i = \sum a_i s \cdot s^{-1} k_i \in IK$. Hence I is full. $\square$

Let A be an R-order in Q. A left A-submodule I of Q, such that I also is an R-lattice, is called a *left A-lattice*. Similarly right A-lattices and (two-sided) A-B-lattices are defined.

If I and J are R-submodules of Q, put

$$I{}^{\cdot}\!.J = \{q \in Q \mid qJ \subset I\}, \quad I.{}^{\cdot}J = \{q \in Q \mid Jq \subset I\}.$$

Lemma 5.3 *If* I *and* J *are* R-*lattices, then also* $I{}^{\cdot}\!.J$ *and* $I.{}^{\cdot}J$ *are* R-*lattices.*

Proof. I contains elements $x_1,\dots,x_n$ which generate Q over K, and $J \subset Rq_1 + \dots + Rq_m$. We may write $x_i q_j = \sum_k c_{ijk} x_k$ with $c_{ijk} \in K$. Choose $s \in \Sigma$ with all $sc_{ijk} \in R$. Then $sx_i q_j \in I$, so $sx_i \in I{}^{\cdot}\!.J$ for $i = 1,\dots,n$, and it follows that $I{}^{\cdot}\!.J$ is full.

If $t \in J \cap \Sigma$ (Lemma 5.1), then $(I{}^{\cdot}\!.J)t \subset I$, so $I{}^{\cdot}\!.J \subset t^{-1}I$, which is contained in a finitely generated R-submodule of Q. Hence $I\ J$ is an R-lattice. $\square$

For each R-lattice I we define the *left*, resp. *right*, *order of* I as

$$o_l(I) = \{q \mid qI \subset I\}, \quad o_r(I) = \{q \mid Iq \subset I\},$$

which by Lemma 5.3 and Theorem 4.4 are R-orders. We also put

$$I^{-1} = \{q \mid IqI \subset I\} = o_l(I).{}^{\cdot}I = o_r(I){}^{\cdot}\!.I,$$

which by Lemma 5.3 also is an R-lattice. Note that while I is an $o_l(I)$-$o_r(I)$-lattice, I^{-1} is an $o_r(I)$-$o_l(I)$-lattice. In the usual way one shows:

Proposition 5.4 *Let* Λ *be an* R-*order in* Q. *If* I *and* J *are left* Λ-*submodules of* Q *and* J *is full, then*

$$I.{}^{\cdot}J \cong \mathrm{Hom}_\Lambda(J,I).$$

In particular one obtains for every R-lattice I in Q that

$$\mathrm{Hom}_{o_l(I)}(I,I) \cong o_r(I),$$

$$\mathrm{Hom}_{o_l(I)}(I,o_l(I)) \cong I^{-1}.$$

6. Maximal orders

An R-order A in Q is maximal if there is no R-order B in Q such that $A \subsetneq B$. It is immediate from the definition of orders, and Zorn's lemma, that every R-order in Q is contained in a maximal R-order.

Proposition 6.1 An R-order A in Q is maximal if and only if A_x is a maximal R_x-order in Q_x for each $x \in \underline{X}$.

Proof. Suppose each A_x is a maximal R_x-order. If B is an R-order containing A, then $A_x = B_x$ for all $x \in \underline{X}$ by Lemma 4.3, and the faithfulness of $\bigoplus_x R_x$ implies that A = B. Hence A is a maximal R-order.

Suppose on the other hand that A is a maximal R-order, and consider any $x \in \underline{X}$. Suppose $A_x \subset C$ for some R_x-order C. Put $B = \varphi^{-1}[C]$ under the mapping $\varphi : Q \to Q_x$. So B is an R-algebra containing A. Let $b \in B$. Then $b_x \in C$ is integral over R_x, so $e(b^n + r_{n-1}b^{n-1} + \ldots + r_0) = 0$ for some idempotent e of R, and hence eb is integral over R. It follows that all elements of $A + eB = (1-e)A \oplus eB$ are integral over R, and hence A + eB is an R-order. The maximality of A implies B = A and thus C = $= A_x$, so also A_x is maximal. ∎

Proposition 6.2 The following properties of an R-order A in Q are equivalent:

(a) A is a maximal R-order.

(b) $o_l(I) = A$ for every left A-lattice I, and $o_r(J) = A$ for every right A-lattice J.

(c) $o_l(I) = o_r(I) = A$ for every A-A-lattice I.

(d) If J is an A-A-lattice and there exists $s \in \Sigma$ such that $sJ^n \subset A$ for all $n \geq 1$, then $J \subset A$.

Proof. (a) $\Rightarrow$ (b) is clear since $o_l(I)$ and $o_r(J)$ are R-orders containing Λ, while (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d): If $sJ^n \subset \Lambda$ for all $n \geq 1$, put $J' = \sum_{n\geq 1} J^n$. Then also J' is an Λ-Λ-lattice, and we have $J \subset o_l(J') = \Lambda$.

(d) $\Rightarrow$ (a): Suppose $\Lambda \subset B$, where B is an R-order in Q. Then B is an Λ-Λ-lattice by Theorem 4.4, and by Lemma 3.1 there exists $s \in \Sigma$ such that $sB \subset \Lambda$. Since B is a ring, condition (d) therefore gives $B \subset \Lambda$. □

We give two examples of maximal orders:

Example 1 If Λ is an Azumaya algebra over R, then Λ is a maximal R-order in the Azumaya K-algebra $\Lambda \otimes_R K$.

Proof: See e.g. [14], Prop. 1.8. □

Example 2 If Λ is a maximal R-order in Q, then $M_n(\Lambda)$ is a maximal R-order in $M_n(Q)$.

Proof (cf. [19], p. 110). Suppose B is an R-order in $M_n(Q)$ with $M_n(\Lambda) \subset B$. Let C be the set of elements $q \in Q$ such that there exists a matrix $M = (m_{ij})$ in B with some entry $m_{ij} = q$. In that case also the matrix $E_{1i}ME_{j1} = qE_{11}$ belongs to B, where E_{ij} denote the matrix units. Hence $C = \{q \mid qE_{11} \in B\}$, and therefore C is an R-order in Q with $\Lambda \subset C$. Hence $\Lambda = C$, and it follows that $B = M_n(\Lambda)$. □

Note that both these examples imply that $M_n(R)$ is a maximal R-order in $M_n(K)$.

7. The groupoid of divisorial lattices

We shall briefly indicate how the usual foundations for a multiplicative ideal theory can be developed in this general context. An R-lattice I is normal if $o_l(I)$ and $o_r(I)$ are maximal R-orders. In that case also I^{-1} is normal, with $o_l(I^{-1}) = o_r(I)$

and $o_r(I^{-1}) = o_l(I)$. A normal R-lattice I is <u>divisorial</u> if $I = (I^{-1})^{-1}$. The operation $I \mapsto (I^{-1})^{-1}$ is a closure operation on normal R-lattices. Every normal R-lattice I is contained in a smallest divisorial R-lattice, namely $(I^{-1})^{-1}$. For any maximal R-orders A and B in Q we let $\underline{N}(A,B)$ denote the set of R-lattices I with $o_l(I) = A$ and $o_r(I) = B$. If $I \in \underline{N}(A,B)$ and $J \in \underline{N}(B,C)$, then $IJ \in \underline{N}(A,C)$. With this "proper multiplication", i.e. with IJ defined defined when $o_r(I) = o_l(J)$, the set $\underline{N}$ of all normal R-lattices becomes an abstract category.

If $I, J \in \underline{N}(A,B)$, we put $I \prec J$ when $I^{-1} \subset J^{-1}$, and we call I and J <u>Artin equivalent</u> if $I^{-1} = J^{-1}$. The preordering $\prec$ is compatible with proper multiplication in $\underline{N}$, and

$$\underline{D} = \underline{N}/\text{Artin equivalence}$$

becomes an ordered category under the relation $\leq$ induced from $\prec$. The image of $I \in \underline{N}$ in $\underline{D}$ will be denoted by $[I]$. Each equivalence class contains precisely one divisorial ~~xxxxxx~~ R-lattice. Actually $\underline{D}$ is a groupoid, where the inverse of $[I]$ is $[I^{-1}]$.

For each maximal R-order A we put

$$\underline{D}(A) = \{[I] \mid I \in \underline{N}(A,A)\} ,$$

which is a subgroup ("vertex group") of the groupoid $\underline{D}$. As usual one concludes (by a theorem of Iwasawa) that the group $\underline{D}(A)$ is commutative ([4], p. 317). If A and B are maximal R-orders, then $\underline{D}(A)$ and $\underline{D}(B)$ are isomorphic groups; the isomorphism is given by $[J] \mapsto [I^{-1}JI]$ for any $I \in \underline{N}(A,B)$, e.g. $I = A \cdot B$, and it is independent of the choice of I since the vertex groups are commutative.

We note:

<u>Proposition 7.1</u> <u>Every maximal proper divisorial ideal of a maximal R-order</u> A <u>is a minimal full prime ideal of</u> A .

Proof. (Cf. [8], Th. 1.6). Let P be a maximal divisorial ideal of A. Suppose I, J are ideals $\supsetneq P$ with $IJ \subset P$. We must have $I^{-1} = A$, for $(I^{-1})^{-1}$ is a divisorial ideal properly containing P. Likewise we have $J^{-1} = A$. For each $q \in P^{-1}$ we have $qIJ \subset qP \subset A$, so $qI \subset J^{-1} = A$ and $q \in I^{-1} = A$. Hence $P^{-1} \subset$ $\subset A$, which is impossible. This shows that P is prime.

Suppose now Q is a full prime ideal with $Q \subsetneq P$. Then $QP^{-1} \subset$ $\subset PP^{-1} \subset A$. But we also have $QP^{-1} \cdot P \subset Q$, and since Q is prime, this gives $QP^{-1} \subset Q$. So $P^{-1} \subset o_r(Q) = A$, which is impossible. □

8. Prime ideals

Since the Azumaya algebra Q is a PI-ring (it satisfies all polynomial identities holding in some matrix ring over a splitting algebra for Q), also every R-order is a PI-ring. Therefore there are available several results on the lifting of prime ideals. For the convenience of the reader we reproduce them here (see [5], [12], [13] for proofs):

Proposition 8.1 Let A be an R-order in Q. Then:

(i) For every prime ideal $\underline{p}$ of R there exists a prime ideal P of A such that $P \cap R = \underline{p}$.

(ii) If $\underline{p} \subset \underline{q}$ are prime ideals of R and P is a prime ideal of A with $P \cap R = \underline{p}$, then there exists a prime ideal Q of A with $P \subset Q$ and $Q \cap R = \underline{q}$.

(iii) There cannot exist prime ideals $P_1 \subsetneq P_2$ in A with $P_1 \cap R = P_2 \cap R$.

It follows in particular that if $\underline{m}$ is a maximal ideal of R and P is a prime ideal of A with $P \cap R = \underline{m}$, then P is a maximal ideal of A. Similarly it follows that if P is a maximal ideal of A, then $P \cap R$ is a maximal ideal of R.

9. Invertible lattices

An R-lattice I in Q is called <u>invertible</u> if $II^{-1} = o_l(I)$ and $I^{-1}I = o_r(I)$. In that case there is a Morita context derived from the obvious mappings

$$I \otimes_{o_r(I)} I^{-1} \longrightarrow o_l(I) \ , \quad I^{-1} \otimes_{o_l(I)} I \longrightarrow o_r(I) \ .$$

Hence an invertible R-lattice I is a finitely generated projective generator for both left $o_l(I)$-modules and right $o_r(I)$-modules, and the rings $o_l(I)$ and $o_r(I)$ are Morita equivalent. In particular one has as usual:

<u>Lemma 9.1</u> <u>Let</u> I <u>be an</u> R-<u>lattice in</u> Q. <u>Then</u> $I^{-1}I = o_r(I)$ <u>if and only if</u> I <u>is projective as a left</u> $o_l(I)$-<u>module; in that case</u> I <u>is also a finitely generated left</u> $o_l(I)$-<u>module.</u>

If I is an invertible R-lattice, then I^{-1} is invertible with $o_l(I^{-1}) = o_r(I)$ and $o_r(I^{-1}) = o_l(I)$. If I and J are invertible R-lattices with $o_r(I) = o_l(J)$, then IJ is invertible with $o_l(IJ) = o_l(I)$, $o_r(IJ) = o_r(J)$. Hence the invertible R-lattices form a groupoid under proper multiplication.

Let A be an R-order in Q. An R-lattice I is called A-<u>invertible</u> if it is invertible and $o_l(I) = o_r(I) = A$. The A-invertible lattices form a multiplicative group $\underline{I}(A)$. If A is a maximal R-order, then $\underline{I}(A)$ is a subgroup of $\underline{D}(A)$ since every invertible lattice is divisorial.

The group $\underline{I}(A)$ may be compared with the Picard group $Pic_R(A)$ of isomorphism classes over R of invertible A-A-bimodules. There is the usual exact sequence of groups

$$1 \longrightarrow R^* \longrightarrow K^* \xrightarrow{\varphi} \underline{I}(A) \xrightarrow{\psi} Pic_R(A) \xrightarrow{\tau} Pic_K(Q) \ ,$$

where R^* and K^* are the subgroups of invertible elements of R resp. K, and $\varphi(x) = Ax$, $\psi(I) = [I]$, $\tau([M]) = [M \otimes_R K]$.

But $\mathrm{Pic}_K(Q) = \mathrm{Pic}(K)$ since Q is an Azumaya K-algebra, and $\mathrm{Pic}(K) = 0$ since K is von Neumann regular (Marot [17]). Hence:

Proposition 9.2 The sequence

$$1 \to R^* \to K^* \to \underline{I}(A) \to \mathrm{Pic}_R(A) \to 0$$

is exact.

Part II. Maximal orders over Krull rings

10. Krull rings

The results on multiplicative ideal theory in § 7 may be applied to the case when the K-algebra Q is equal to K. One then obtains a generalization of the classical theory of divisors (as developed in [6], Chap. 7). In particular this leads to a study of Krull subrings of the von Neumann regular ring K; a study which has been undertaken by J. Marot [16], [17] (cf. also G.M. Bergman [3]). Since Marots work is not easily available, we shall in this section recapitulate relevant parts of it.

Let R be a completely integrally closed subring of the von Neumann regular ring K. We shall always assume $R \neq K$. An R-submodule $\underline{a}$ of K is full if and only if $\underline{a} \cap \Sigma \neq \emptyset$.

Lemma 10.1 If $x \in R$ and $s \in \Sigma$, then there exists $y \in R$ such that $x + ys \in \Sigma$.

Proof. There is an idempotent e such that $x = ex$ and $e = xu$ for some $u \in K$. We assert that $x + (1-e)s \in \Sigma$. For suppose $zx + z(1-e)s = 0$ for some $z \in R$. Then $ezx = 0$, so $zx = 0$. But $s \in \Sigma$ then implies $z(1-e) = 0$ and $z = ze = zxu = 0$. ∎

Lemma 10.2 Every full R-submodule of K is generated by non-zero-divisors.

Proof. Let $\underline{a}$ be an R-submodule of K with $s \in \underline{a} \cap \Sigma$. To find non-zero-divisor generators for $\underline{a}$, it suffices to do so for $Rs + Rx$ for each $x \in \underline{a}$, and this is easily done by Lemma 10.1. □

An R-submodule $\underline{a}$ of K is an R-lattice (also called a fractional R-ideal) if and only if there exist $s, t \in \Sigma$ with $s \in \underline{a}$ and $t\underline{a} \subset R$. A fractional R-ideal $\underline{a}$ is called divisorial if $\underline{a} = R:(R:\underline{a})$, where $\underline{b}:\underline{a}$ in general denotes the set $\{x \in K \mid x\underline{a} \subset \underline{b}\}$.

Lemma 10.3 $R:(R:\underline{a})$ is equal to the intersection $\tilde{\underline{a}}$ of all principal fractional ideals containing $\underline{a}$.

Proof. Let $x \in K$. Then $x \in R:(R:\underline{a})$ if and only if $xy \in R$ for every non-zero-divisor $y \in R:\underline{a}$ (by Lemma 10.2). Thus $x \in R:(R:\underline{a})$ if and only if $x \in Ry^{-1}$ for every y such that $\underline{a} \subset Ry^{-1}$, i.e. if and only if $x \in \tilde{\underline{a}}$. □

Two fractional ideals $\underline{a}$ and $\underline{b}$ are Artin equivalent if and only if $\tilde{\underline{a}} = \tilde{\underline{b}}$; the equivalence class of $\underline{a}$ is called the divisor of $\underline{a}$ and is denoted div $\underline{a}$. The divisors form an ordered abelian group $\underline{D}(R)$, which is denoted additively so that

$$\text{div } \underline{a}\,\underline{b} = \text{div } \underline{a} + \text{div } \underline{b} .$$

One has div $\underline{a} \leqslant$ div $\underline{b}$ if and only if $\tilde{\underline{a}} \supset \tilde{\underline{b}}$.

A discrete valuation on K is a mapping $\nu : K \to \underline{Z} \cup \{\infty\}$ such that

$\nu(xy) = \nu(x) + \nu(y)$,

$\nu(x+y) \geqslant \inf\{\nu(x), \nu(y)\}$,

$\nu(1) = 0$, $\nu(0) = \infty$,

$\nu(x) = 1$ for some non-zero-divisor $x \in K$.

The ring $V = \{x \in K \mid \nu(x) \geqslant 0\}$ is the (discrete) valuation ring of ν, and $\underline{p} = \{x \in K \mid \nu(x) \geqslant 1\}$ is a full prime ideal of V.

Clearly K is the total ring of fractions of V, and V is completely integrally closed in K. All full ideals of V are principal and of the form Vp^n $(n \geq 0)$ for a certain $p \in V$, and Vp is the unique full prime ideal of V.

More generally, a subring V of K, with K as its total ring of fractions, is a valuation ring in K if the full ideals of V are totally ordered under inclusion. As in the classical case one shows (cf. [6], Chap. 6, § 4):

Lemma 10.4 Let V be a valuation ring in K. Then any over-ring of V in K is a valuation ring, and the over-rings of V in K are totally ordered under inclusion.

R is a Krull ring iff there is a family $(\nu_i)_{i \in I}$ of discrete valuations on K such that

K 1) R is the intersection of the valuation rings of the ν_i;

K 2) For every $s \in \Sigma$, $\nu_i(s) = 0$ except for finitely many i.

Proposition 10.5 The following properties of the ring R are equivalent:

(a) R is a Krull ring.

(b) R satisfies ACC on divisorial ideals.

(c) R_x is a Krull domain for each $x \in \underline{X}$, and for each $s \in \Sigma$, s_x is invertible in R_x for all but finitely many x.

Proof. [3], Prop. 6.2. □

Let R be a Krull ring. The group $\underline{D}(R)$ is the free abelian group on the set of minimal divisors > 0, called the prime divisors. The prime divisors correspond to the maximal proper divisorial ideals in R. For each $x \in K$ we can write

$$\operatorname{div} Rx = \sum \nu_P(x)\, P ,$$

with summation over the set of prime divisors P; here

$\nu_{\underline{p}}$ are discrete valuations satisfying K 1-2, and are called the essential valuations of R.

For each full prime ideal $\underline{p}$ of R we let $R_{\underline{p}}$ denote the ring of fractions $S^{-1}R$ with $S = \Sigma \cap (R \setminus \underline{p})$.

The following three lemmas deal with a Krull ring R, and they are proved essentially as in the classical case ([6], Chap. 7, § 1).

Lemma 10.6 Let ν_i $(i \in I)$ be the essential valuations of R, and let R_i be the valuation ring of ν_i. If S is a multiplicatively closed set in Σ, then $S^{-1}R = \bigcap_{j \in J} R_j$, where $J = \{ i \in I \mid \nu_i(s) = 0$ for all $s \in S \}$, and $S^{-1}R$ is a Krull ring.

Lemma 10.7 Let $\underline{p}$ be the divisorial ideal corresponding to a prime divisor P of R. Then $\underline{p}$ is a minimal full prime ideal of R, and $R_{\underline{p}}$ is the valuation ring of ν_P.

Lemma 10.8 A full ideal $\underline{p}$ is a maximal proper divisorial ideal of R if and only if $\underline{p}$ is a minimal full prime ideal of R. There is thus a bijective correspondence between essential valuations on R and minimal full prime ideals of R.

We shall write $\underline{P}$ for the set of minimal full prime ideals of R.

Proposition 10.9 The following properties of the ring R are equivalent:

(a) Every full ideal of R is projective.

(b) R is a Krull ring where every full prime ideal is maximal.

(c) R is a semihereditary Krull ring.

(d) R_x is a Dedekind domain for each $x \in \underline{X}$, and for each $s \in \Sigma$, s_x is invertible in R_x for all but finitely many x.

Proof. (a) $\Leftrightarrow$ (d): [3], Cor. 4.5.

(c) $\Leftrightarrow$ (d): Prop. 10.4 and [3], Th. 4.1.

(c) $\Rightarrow$ (d) is clear.

(c) $\Rightarrow$ (b): Let $\underline{m}$ be a full maximal ideal of R, and consider the over-ring $R_{\overline{\underline{m}}}$ of R. Since R is semihereditary, $R_{\overline{\underline{m}}}$ is a flat R-module ([9], Th. 5), and as in [15], Prop. 4 one shows that $R_{\overline{\underline{m}}}$ is a valuation ring in K. But $R_{\overline{\underline{m}}}$ is the intersection of a family $(R_j)_J$ of valuation rings of essential valuations of R (Lemma 10.6). From Lemma 10.4 follows that $R_{\overline{\underline{m}}} = R_j$ for some j J, and it follows that $\underline{m}$ must be a minimal full prime ideal. $\square$

A ring satisfying the conditions of Prop. 10.9 is called a <u>Dedekind ring</u> (in K).

<u>Proposition 10.10</u> <u>If</u> K <u>is hereditary, then every Dedekind ring</u> R <u>in</u> K <u>is hereditary.</u>

<u>Proof</u>. Let $\underline{a}$ be an ideal in R. We can write $\underline{a}K = \bigoplus_I Ke_i$, where $(e_i)_I$ is a family of orthogonal idempotents. If $a \in \underline{a}$, then $a = \sum k_i e_i$ with $k_i \in K$ and almost all $k_i = 0$. Since $k_i e_i = ae_i \in Re_i \cap \underline{a} = \underline{a}_i$, it follows that $\underline{a} = \bigoplus_I \underline{a}_i$.

Since $e_i \in \underline{a}K$, we see that $\underline{a}$ contains an element $s_i e_i$ with $s_i \in \Sigma$, for each $i \in I$. Let $x \in \underline{a}_i$. By Lemma 10.1 there exists $y \in R$ such that $z = x + ys_i \in \Sigma$. Then $x = xe_i = ze_i - rs_i e_i \in$ $\in RS_i e_i$, where $S_i = \{ t \in \Sigma \mid te_i \in \underline{a}_i \}$, and so $\underline{a}_i = RS_i e_i$. Since RS_i is a full ideal of R, it is projective, and so is then also $\underline{a}_i$. $\square$

11. <u>Krull orders</u>

<u>Lemma 11.1</u> <u>Let</u> R <u>be a Krull ring and</u> A <u>an</u> R-<u>order in</u> Q. <u>If</u> a <u>is a non-zero-divisor in</u> A, <u>then</u> a_x <u>is invertible in</u> A_x <u>for all but finitely many</u> x.

<u>Proof</u>. One may write $a^{-1} = bs^{-1}$ with $b \in A$ and $s \in \Sigma$. Since s_x is invertible in R_x for all but finitely many x (Prop. 10.5), it follows that $a_x^{-1} \in A_x$ for all but finitely many x. $\square$

Theorem 11.2 Let A be a maximal R-order in Q. The following conditions are equivalent:

(a) A satisfies ACC on divisorial ideals.

(b) $\underline{D}(A)$ is a free abelian group with the set of maximal proper divisorial ideals as basis.

(c) R is a Krull ring.

A maximal R-order A satisfying these conditions is called a Krull order.

Proof. (a) $\Leftrightarrow$ (b) is standard.

(a) $\Rightarrow$ (c): Let $\underline{a}$ be divisorial ideal in R, and put $I = ((A\underline{a})^{-1})^{-1}$. Then I is a divisorial ideal in A, and it suffices to show that $I \cap R = \underline{a}$, because then ACC for divisorial ideals in R will follow, and we can apply Prop. 10.4. Now

$$(I \cap R) \cdot (R:\underline{a}) \subset I \cdot (A\underline{a})^{-1} \cap K \subset A \cap K = R .$$

Hence $I \cap R \subset R:(R:\underline{a}) = \underline{a}$, so $I \cap R = \underline{a}$. (Cf. [7], Lemma 1.3).

(c) $\Rightarrow$ (a): From Lemma 6.1 follows that A_x is a maximal order over the Krull domain R_x, for each $x \in \underline{X}$. If I is a divisorial ideal of A, then $I_x = A_x$ for all but finitely many x, by Lemma 11.1. Since each A_x satisfies ACC on divisorial ideals ([2], p. 151), it follows that also A does so. □

Let R be a Krull ring. An R-lattice in Q is said to be $\underline{P}$-divisorial if $I = \bigcap_{\underline{P}} I_{\underline{p}}$. Similarly to ([2], p. 154) one has:

Proposition 11.3 Let R be a Krull ring, and let A be an R-order in Q. Then A is a maximal R-order if and only if A is $\underline{P}$-divisorial and $A_{\underline{p}}$ is a maximal $R_{\underline{p}}$-order for each $\underline{p} \in \underline{P}$.

12. Dedekind orders

Theorem 12.1 The following properties are equivalent for a maximal R-order A in Q :

(a) Every full ideal of A is invertible.

(b) Every full ideal of A is a projective left A-module.

(c) Every A-A-lattice is invertible.

(d) The A-A-lattices form under multiplication a free abelian group with the set of full maximal ideals as basis.

(e) A satisfies ACC on full ideals, and every full prime ideal of A is a maximal ideal.

(f) Every full left ideal of A is a finitely generated projective left A-module.

(g) R is a Dedekind ring.

A maximal R-order A satisfying these conditions is called a Dedekind order.

Proof. (a) $\Rightarrow$ (c) is clear since for every A-A-lattice I there exists $s \in \Sigma$ such that sI is a full ideal in A .

(c) $\Rightarrow$ (d): The A-A-lattices now form the group $\underline{D}(A)$, since every A-A-lattice is divisorial, and this group is free abelian on the set of maximal divisorial ideals.

(d) $\Rightarrow$ (e): Clearly A satisfies ACC on full ideals. Since every full ideal is a product of maximal ideals, a full prime ideal must be maximal.

(e) $\Rightarrow$ (g): R is a Krull ring by Theorem 11.2, and every full prime ideal of R is maximal by Prop. 8.1, so R is Dedekind by Prop. 10.9.

(g) $\Rightarrow$ (f): Each R_x , $x \in \underline{X}$, is a Dedekind domain by Prop. 10.9, and A_x is therefore a hereditary R_x-order (Prop. 6.1 and [1], Th. 2.9). Every full left ideal of A is finitely generated projective by the argument used in the proof of Lemma 3.3 of [14].

(f) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (a): Let I be a full ideal of A. Then $I^{-1}I = A$ by Lemma 9.1. This also gives

$$(II^{-1})^{-1}I = (II^{-1})^{-1}II^{-1}I \subset I ,$$

and hence $(II^{-1})^{-1} \subset o_l(I) = A$. But $II^{-1} \subset A$ then implies $II^{-1} = A$. $\square$

<u>Proposition 12.2</u> <u>Let</u> A <u>be a Dedekind</u> R-<u>order. If</u> I <u>is a left</u> A-<u>lattice, then</u> $o_r(I)$ <u>is a Dedekind</u> R-<u>order, and</u> I <u>is inverible</u>.

<u>Proof</u>. Put $J = II^{-1}$, which is a full ideal in A. Hence J is invertible, and $JJ^{-1} = A$, i.e. $II^{-1}J^{-1} = A$. It follows that $I^{-1}J^{-1} \subset I^{-1}$, so $J^{-1} \subset o_r(I^{-1}) = A$. Therefore $J = A$, and I is invertible. Also $o_r(I)$ is a Dedekind R-order, since it is Morita equivalent to A. $\square$

<u>Remark 1</u>. If R is hereditary ring, then every Dedekind R-order is a left and right hereditary ring by [14].

<u>Remark 2</u>. One may ask whether every Dedekind R-order is finitely generated as an R-module.

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