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# Intersection Multiplicities in Commutative Algebra

Paul Roberts

In this paper we discuss some recent developments in Intersection Theory and their relation to problems in Commutative Algebra. After a short discussion on how the problem of defining intersection multiplicities originally gave rise to questions in Algebra, we describe some recent results, due mostly to W. Fulton, which enable one to do intersection theory in an algebraic setting, and we show how this makes it possible to give a new definition of intersection multiplicities which has several advantages over the old one. Finally, we show how this can then be used to answer some of the algebraic questions.

We consider the following situation: let $Y$ be a Noetherian scheme, let $X$ and $W$ be subschemes of $Y$, and let $p$ be an isolated point of $X \cap W$. A classical problem has been to give a good definition of the intersection multiplicity of $X$ and $W$ at $p$; we denote this as of yet undefined number by $m(X,W)$. This definition should satisy certain properties, and although there are many such properties which a good definition should satisfy, we mention here only three which are of particular interest in this paper:

1. (Additivity). If $X = X' + X''$, then $m(X,W) = m(X',W) + m(X'',W)$.

2. (Vanishing). If $\dim(X) + \dim(W) < \dim(Y)$, then $m(X,W) = 0$.

3. (Positivity). If $\dim(X) + \dim(W) = \dim(Y)$, then $m(X,W) > 0$.

Since we are basically interested in the local situation around p, we will assume throughout this paper that Y is Spec(A), where A is a commutative Noetherian local ring, and p is the closed point of Y. Then X and W will correspond to ideals of A. It should be mentioned at the outset that it is not in general possible to define m(X,W) for <u>all</u> subschemes X and W, and that the aim is to define it in as general a situation as possible.

We begin by recalling a simple case: let A be a two-dimensional Cohen-Macaulay ring, and suppose that X and W are defined by principal ideals (f) and (g). Then the ideal generated by f and g will be primary to the maximal ideal of A, and one defines

$$m(X,W) = \text{length}(A/(f,g)) = \text{length}(A/(f) \otimes A/(g)).$$

It is easy to verify the three properties above (and many others); in property 1, addition of subschemes is defined by multiplication of the generators of the corresponding principal ideals.

If A is an arbitrary local ring, and if X and W are defined by ideals I and J respectively, then the fact that p is an isolated point of the intersection means that the ideal I + J is again primary to the maximal ideal of A. The obvious generalization of the above is to define

$$m(X,W) = \mathrm{length}(A/(I+J)) = \mathrm{length}(A/I \otimes A/J).$$

However, this definition does not work; it is easy to see, for example, that it does not satisfy either of the first two properties. Serre [9] showed how to repair this by correcting it using higher Tors; he defines

$$m(X,W) = \sum_{i \geq 0} (-1)^i \mathrm{length}(\mathrm{Tor}_i(A/I, A/J)),$$

and, more generally, for finitely generated A-modules M and N for which $M \otimes_A N$ is a module of finite length, he defines

$$\chi(M,N) = \sum_{i \geq 0} (-1)^i \mathrm{length}(\mathrm{Tor}_i(M,N)).$$

For this definition to make sense, it is necessary that $\mathrm{Tor}_i(M,N) = 0$ for $i$ large; this will be true, for example, if either M or N has finite projective dimension. If A is regular (that is, if the maximal ideal of A is generated by n elements, where n is the dimension of A), then every module has finite projective dimension, and this gives a good definition of intersection multiplicities. This definition satisfies Property 1 where one defines addition of modules, modulo an appropriate equivalence relation, so that $M = M' + M''$ whenever there is a short exact sequence

$$0 \to M' \to M \to M'' \to 0.$$

In addition, Serre proved that if A is an equicharacteristic regular local ring, then properties 2 and 3 hold as well, and he conjectured that they hold for arbitrary regular local rings.

Shortly thereafter, the question arose whether the dimension properties 2 and 3 were not merely properties of regular local rings, but were, in fact, properties of modules of finite projective dimension over arbitrary local rings. In this setting the multiplicity $\chi(M,N)$ can be defined and is additive. We give three examples connected with this question.

Example 1. Properties 2 and 3 imply, at least implicitly, that the case in which $\dim(M) + \dim(N) > \dim(A)$ can never occur. There are, on the other hand, many examples where $M \otimes_A N$ has finite length but $\dim(M) + \dim(N) > \dim(A)$. For instance, let $A = k[[X,Y]]/(XY)$, $M = A/(X)$, and $N = A/(Y)$. However, it is still not known whether such an example exists where $M$ has finite projective dimension.

Example 2. In the particularly simple case of a module of finite projective dimension where $M$ has a resolution of the form

$$0 \longrightarrow A \xrightarrow{a} A \longrightarrow M \longrightarrow 0$$

for some non-zero-divisor $a$ in $A$, the dimension properties follow from Krull's Principal Ideal Theorem. Several other cases can be reduced to this one; for instance, if $M$ has a resolution of the form

$$0 \longrightarrow A^n \xrightarrow{(a_{ij})} A^n \longrightarrow M \longrightarrow 0,$$

the multiplicity properties of $M$ can be reduced to those of $A/(\text{the determinant of } (a_{ij}))$.

<u>Example 3</u>. Another question concerning modules of finite projective dimension is the following: suppose $A$ is a Cohen-Macaulay ring and $M$ is a module of finite length and finite projective dimension. Then the only integer $i$ for which $\mathrm{Ext}^i(M,A)$ does not vanish is $i = n$, where $n = \dim(A)$, and $\mathrm{Ext}^n(M,A)$ is a module of finite length. It was asked whether one always had the equality $\mathrm{length}(\mathrm{Ext}^n(M,A)) = \mathrm{length}(M)$. This is true, for example, for regular (or Gorenstein) rings.

Several cases of the dimension conjectures were proven under the sole hypothesis that $M$ had finite projective dimension. For example, they were proven by Dutta [2] and Foxby [4] if $N$ has dimension 1, and by Peskine and Szpiro [7] if all modules are graded over a graded ring. The methods used in these proofs led to the conjecture that these theorems could be proven in general by constructing appropriate invariants of the modules under consideration (see Szpiro [10]). The idea was as follows: if $E_*$ is a free resolution of $M$, one constructs invariants $ch_i(E_*)$ for $i = 0,1,\dots$, and if $N$ is any finitely generated module, one constructs invariants $\tau_i(N)$, all satisfying the following properties:

(a). If $E_* \otimes N$ has homology of finite length, then

$$\chi(E_* \otimes N) \overset{\text{def.}}{=} \sum (-1)^i \mathrm{length}(H_i(E_* \otimes N)) = \sum_{i \geq 0} ch_i(E_*)\,\tau_i(N).$$

(b). If $i > \dim(N)$, then $\tau_i(N) = 0$.

(c). If $i < \mathrm{codim}(M)$, then $ch_i(E_*) = 0$, where $\mathrm{codim}(M) = \dim(A) - \dim(M)$.

If invariants satisfying all of this can be defined, then it is easy to prove Property 2 for $\chi(M,N)$ if $M$ has finite projective dimension. In fact, if $\dim(M) + \dim(N) < \dim(A)$, then, using properties (b) and (c), we see that all the terms in the sum $\sum_{i \geq 0} ch_i(E_*)\tau_i(N)$ vanish, so that $\chi(M,N) = 0$.

Several years ago, Baum, Fulton, and MacPherson [1] (see also Fulton [5], Chapter 18), constructed invariants $ch_i$ and $\tau_i$ as above for schemes of finite type over a field, and these satisfy property (b). More recently, Fulton ([5], Chapter 20) showed that the definitions can be extended to an arbitrary scheme of finite type over a regular scheme, and that property (a) also holds. However, shortly thereafter, Dutta, Hochster, and McLaughlin [3] produced an example which showed that the dimension properties are false if only $M$ is assumed to have finite projective dimension. In their example, $A = k[[X,Y,Z,W]]/(XY-ZW)$, where $k$ is an arbitrary field, $N = A/(X,Z)$, and $M$ is a module of finite projective dimension, of Krull dimension zero, and of length 15, with $\chi(M,N) = -1$.

In the remainder of the paper, we show that in spite of this apparent failure, these invariants can be used to answer a number of the questions discussed above. We first describe $ch_i(E_*)$ and $\tau_i(N)$ in a little more detail. We assume that $A$ is a homomorphic image of a regular local ring $R$.

Let $A_*Y = A_0Y \oplus A_1Y \oplus \ldots$ be the (rational) Chow group of $Y$; $A_kY$ is the free $Q$-module on integral subschemes of $Y$ of dimension $k$ (or, equivalently, on prime ideals $P$ of $A$ with

$\dim(A/P) = k$) modulo rational equivalence. Rational equivalence can be defined in this situation as follows: if $Q$ is a prime ideal with $\dim(A/Q) = k + 1$, and if $x$ is a non-zero element of $A/Q$, then the cycle

$$\sum_{\substack{P \supseteq Q \\ \dim(A/P)=k}} \text{length}(A/(Q,x))_P \,[A/P]$$

is defined to be rationally equivalent to zero; rational equivalence is the equivalence relation generated by this relation. The class of an integral subscheme $V$ corresponding to a prime ideal $P$ will be denoted $[V]$ or $[A/P]$. If $N$ is a finitely generated $A$-module, there is a cycle class $[N]$ in $A_*Y$ defined as follows:

$$[N] = \sum_{\substack{P \text{ minimal} \\ \text{in } \operatorname{Supp}(N)}} \text{length}(N_P)\,[A/P].$$

Now let $E_*$ be a bounded complex of finitely generated free modules, and let $X$ be the support of $E_*$; that is, we have $X = \{P \mid (E_*)_P \text{ is not exact}\}$. The <u>local Chern character</u> $ch(E_*) = ch_0(E_*) + ch_1(E_*) + \ldots$ is defined as an intersection operator on $A_*Y$ by means of the "graph construction"; we give here a very brief description of this construction, details of which can be found in Fulton [5], Chapter 18. The basic idea is to take the graphs of the maps $d_i : E_i \to E_{i-1}$ of the complex $E_*$, which are free submodules of $E_i \oplus E_{i-1}$ of rank $e_i = \text{rank}(E_i)$ and thus define sections of the Grassmannians $\text{Grass}_{e_i}(E_i \oplus E_{i-1})$. One then deforms these sections to cycles lying over the support of $E_*$ and operates on these with the Chern characters of the canonical bundles of the Grassmannians. The resulting cycle class can be pushed down to $X$ and is defined to be the result of applying the operator $ch(E_*)$ to the cycle $[Y]$. If $V$ is an integral subscheme of $Y$, one defines the action of $ch(E_*)$ on $[V]$ by restricting the complex $E_*$ to $V$ and proceeding as before; the result is then an element of $A_*(V \cap X)$. In this way operators $ch_i(E_*)$ are defined for each integer $i$, and $ch_i(E_*)$ gives, for every subscheme $Z$ of $Y$ and for every integer $k$, an "intersection" map from $A_k(Z)$ to $A_{k-i}(Z \cap X)$. The invariants $\tau_i(N)$ are defined using the local Chern character as follows: let $A$ be a homomorphic image of the regular local ring $R$ of dimension $m$, and let $G_*$ be a free resolution of the module $N$ over $R$. Then

$$\tau_i(N) = ch_{m-i}(G_*)([R]).$$

We note that the cycle class $[R]$ is in $A_m(\mathrm{Spec}(R))$, and, since $\mathrm{Supp}(G_*) = \mathrm{Supp}(N)$, operating on this class by $ch_{m-i}(G_*)$ gives an element of $A_i(\mathrm{Supp}(N))$. Since $\mathrm{Supp}(N) \subseteq Y$, this can be pushed forward to give an element of $A_iY$, which we will also denote $\tau_i(N)$. We now have two elements of $A_*Y$ canonically associated to the module $N$, namely $[N]$ as defined above and $\tau(N) = \sum \tau_i(N)$. These are in general not the same, but one always has

$$\tau(N) = [N] + (\text{terms of dimension} < \dim(N)).$$

We now state several properties of these invariants.

(a). (The local Riemann-Roch formula). If $E_* \otimes N$ has homology of finite length, then

$$\chi(E_* \otimes N) = \sum ch_i(E_*)(\tau_i(N)).$$

In this formula, we take $\tau_i(N)$ in $A_*(\mathrm{Supp}(N))$. The condition that $E_* \otimes N$ has homology of finite length implies that $(\mathrm{Supp}(E_*) \cap (\mathrm{Supp}(N))$ $=$ p (the closed point of Y), so $ch_i(E_*)(\tau_i(N))$ is an element of $A_0(\mathrm{Supp}(N) \cap \mathrm{Supp}(E_*)) = A_0(p) \cong \mathbb{Q}$, and it can be identified with a number. A more precise statement of this formula would be:

$$(\chi(E_* \otimes N))[p] = \sum ch_i(E_*)(\tau_i(N)).$$

(b). If $i > \dim(N)$, then $\tau_i(N) = 0$.

This is clear, since $\tau_i(N)$ is in $A_i(\mathrm{Supp} N)$, which is zero for $i > \dim(N)$.

(c). (Multiplicativity). If $E_*$ and $F_*$ are bounded complexes of free modules with supports X and W respectively, then for each integer k we have

$$ch_k(E_* \otimes F_*) = \sum_{i+j=k} ch_i(E_*)ch_j(F_*).$$

The multiplication here is composition of operators. This can be stated more concisely:

$$ch(E_* \otimes F_*) = ch(E_*)ch(F_*).$$

(d). (Commutativity). For all i and j, we have

$$ch_i(E_*)ch_j(F_*) = ch_j(F_*)ch_i(E_*).$$

Before returning to the questions raised above concerning modules of finite projective dimension, we would like to introduce another possible definition of intersection multiplicity. Let $\alpha$ be an element of $A_*X$ for $X$ a closed subscheme of $Y$. We say that $\alpha$ has finite projective dimension if there is a bounded complex of free modules $E_*$ with support contained in $X$ such that $\alpha = ch(E_*)([Y])$. If $\beta = ch(F_*)([Y])$ in $A_*W$ is another cycle class of finite projective dimension, and if $X \cap W = p$, we define the intersection multiplicity of $\alpha$ and $\beta$ by the formula

$$(m(\alpha,\beta))[p] = ch(E_*)(\beta) \quad \text{in } A_0(p).$$

From the multiplicativity of local Chern characters, we have

$$ch(E_*)(\beta) = ch(E_*)ch(F_*)([Y]) = ch(E_* \otimes F_*)([Y]) = ch(F_*)(\alpha),$$

and this definition is in fact symmetric. The same kind of argument shows that it does not depend on the choice of the complex $E_*$ with $ch(E_*)([Y]) = \alpha$, since if $E'_*$ were another one, we would have

$$ch(E_*)(\beta) = ch(E_*)ch(F_*)([Y]) = ch(F_*)ch(E_*)([Y]) =$$

$$= ch(F_*)(\alpha) = ch(F_*)ch(E'_*)([Y]) = ch(E'_*)(\beta).$$

This definition is clearly additive, and we now show that it satisfies the vanishing property (Property 2). For simplicity, we assume that all components of $Y$ have dimension $n$; otherwise the statement must be modified slightly.

Theorem. Let $\alpha$ and $\beta$ be cycle classes of finite projective dimension in $A_*X$ and $A_*W$ respectively. Assume that $X \cap W = p$ and that $\dim(X) + \dim(W) < \dim(Y) = n$. Then $m(\alpha,\beta) = 0$.

Proof. We have

$$(m(\alpha,\beta))\, p = ch(E_*)(\beta) = ch(E_* \otimes F_*)([Y]).$$

Since all components of Y have dimension n, this simplifies to become

$$(m(\alpha,\beta))\ p \;=\; ch_n(E_*\otimes F_*)([Y]) \;=\; \sum_{i+j=n} ch_i(E_*)ch_j(F_*)([Y]).$$

Suppose that $j < \operatorname{codim}(W)$ $( = \dim(Y) - \dim(W))$. Then $ch_j(F_*)([Y])$ is an element of $A_{n-j}(W)$, and, since the dimension of W is less that $n - j$, $A_{n-j}(W) = 0$. Hence the term $ch_i(E_*)ch_j(F_*)([Y])$ is equal to zero. Similarly, if $i < \operatorname{codim}(X)$, using the commutativity of the local Chern characters, we have that $ch_i(E_*)ch_j(F_*)([Y]) = 0$ in this case as well. The hypothesis that $\dim(X) + \dim(W) < \dim(Y)$ implies that one of these two cases must hold for every i and j with $i + j = n$, so we deduce that $m(\alpha,\beta) = 0$.

We now wish to apply this to the questions concerning intersection multiplicities for modules. By the local Riemann-Roch formula, we have, if M and N are modules of finite projective dimension with resolutions $E_*$ and $F_*$ respectively,

$$\chi(M,N) \;=\; \chi(E_*\otimes F_*) \;=\; ch(E_*\otimes F_*)(\tau(Y)),$$

where $\tau(Y) = \tau$(the A-module A). From the Theorem just proven, we can deduce the vanishing property for $\chi(M,N)$ whenever we have $ch(E_*\otimes F_*)(\tau(Y)) = ch(E_*\otimes F_*)([Y])$; for example, this will hold when $\tau(Y) = [Y]$. This can be shown to be true, for instance, when A is a complete intersection, which includes the case when A is regular conjectured by Serre. This case of the vanishing property has also been proven by K-theoretic methods by Gillet and Soulé [6]. One can show that if Y is an isolated singularity, or, more generally, if the singular locus of Y has dimension less than or equal to one, then the hypothesis that $\dim(X) + \dim(W) < \dim(Y)$ implies that $ch(E_* \otimes F_*) = 0$, so that one can prove the vanishing property in this case also (see Roberts [8]). On the other hand, the positivity is not yet known, even in the case in which A is regular.

The above discussion shows that to answer questions on intersection multiplicities for modules, it is important to know something about $\tau(Y)$, and in particular, to know when it holds that $\tau(Y) = [Y]$. In fact, the crucial question is whether there exist bounded complexes $E_*$ of free modules with homology of finite length with

$$ch(E_*)(\ \tau(Y) - [Y]\ ) \neq 0.$$

As mentioned above, such a complex cannot exist over a complete intersection, and there is at present no known example over a Gorenstein ring. We show, however, that the example of Dutta, Hochster, and McLaughlin referred to above makes it possible to construct a complex of this type over a Cohen-Maxaulay ring.

Let $B = k[[X,Y,Z,W]]/(XY-ZW)$, where k is a field of characteristic $\neq 2$. Let I be the ideal $(X,Z)$. Let A be the integral closure of $B[\sqrt{XW}]$ in its quotient field; it is quite easy to show that an element $a + b\sqrt{XW}$, with a and b in the quotient field of B, is integral over B (and thus over $B[\sqrt{XW}]$) if and only if $a \in B$ and $b = b'/X$, with $b' \in I$. Thus $A \cong B \oplus I$ as a B-module. From the short exact sequence

$$0 \longrightarrow I \longrightarrow B \longrightarrow B/I \longrightarrow C$$

and the fact that $B/I \cong k[[Y,W]]$ we deduce that $\operatorname{depth}(I) = 3$, and A is Cohen-Macaulay. Let $E_*$ be a free resolution of the B-module of length 15 in the example, and let $F_* = E_* \otimes_B A$. Since B is a complete intersection, we have $\tau(B) = [B] = \tau_3(B)$. By the local Riemann-Roch formula, we have

$$ch_2(E_*)(\tau_2(B/I)) = -1.$$

Using the additivity of $\tau$, we can conclude that

$$\tau_2(A) = \tau_2(I) = -\tau_2(B/I),$$

and, in fact, that

$$ch_2(F_*)(\tau_2(A)) = 1.$$

We conclude with an application to the question raised in Example 3. Let A be the ring just constructed, and let $F_*$ be $E_* \otimes_B A$ as above. We remark that the local Chern characters have the following property: if $\check{F}_* = \mathrm{Hom}(F_*, A)$ is the dual complex, then

$$ch_i(\check{F}_*) = (-1)^i ch_i(F_*) \quad \text{for each integer } i.$$

Taking into account the shift in degree, the statement that M and $\mathrm{Ext}^n(M,A)$ have the same length in Example 3 can be reformulated as follows:

$$\chi(\check{F}_*) = (-1)^n \chi(F_*),$$

where $F_*$ is a resolution of M. In terms of local Chern characters, we have

$$\chi(\check{F}_*) = \sum ch_i(\check{F}_*)(\tau_i(Y)) = \sum (-1)^i ch_i(F_*)(\tau_i(Y)),$$

whereas $$\chi(F_*) = \sum ch_i(F_*)(\tau_i(Y)).$$

Thus if $\tau_k(Y) = 0$ whenever $n - k$ is odd, the formula holds, while in the example under consideration, where $ch_2(F_*)(\tau_2(Y)) = 1$

and $n = 3$, it does not. In fact, computing using the above formulas, one finds that $\text{length}(M) = 31$, while $\text{length}(\text{Ext}^n(M,A)) = 29$.

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