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A Non-Central Functional Limit Theorem for Quadratic Forms in Martingale Difference Sequences

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Abstract

For quadratic forms with nulls on the diagonal the partial-sum process is both a martingale and a stochastic integral. Using corresponding tools we derive a result on the convergence to the double Wiener-Itô integral.

Let \( \{A^n = (a^n_{i,j})\}_{n \in \mathbb{N}} \) be a sequence of infinite matrices with nulls on the diagonals:

\[
a^n_{i,i} = 0, \quad i = 1, 2, \ldots, n \in \mathbb{N}.
\]  

Let \( \{X_j\}_{j \in \mathbb{N}} \) be a martingale difference sequence with respect to some filtration \( \{\mathcal{F}_j\}_{j \in \mathbb{N} \cup \{0\}} \) such that

\[
E(X^2_j | \mathcal{F}_{j-1}) = 1, \quad j = 1, 2, \ldots.
\]  

Then for each \( n \in \mathbb{N} \) the process

\[
S_{n,k} = \sum_{1 \leq i,j \leq k} a^n_{i,i} X_i X_j = \sum_{j \leq k} \left( \sum_{i<j} (a^n_{i,i} + a^n_{i,j}) X_i \right) X_j \quad k = 1, 2, \ldots,
\]

is both a square integrable martingale and a stochastic integral with respect to \( \{\mathcal{F}_j\} \).

Using the martingale structure, a Functional Central Limit Theorem for suitably scaled \( \{S_{n,k}\}_{k \in \mathbb{N}} \) was proved in [JaMe90]. In the present paper we state a non-central functional limit theorem based on limit theorems for stochastic integrals given in [JMP89]. The limit is identified with a double Wiener-Itô stochastic integral (see e.g. [Ito51] or [Maj81]). In some sense it is not surprising: such limits arise, for example, in limit theory for quadratic forms in stationary gaussian sequences exhibiting long-range dependence (see [Ros79], also [SuHo86]).

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For each $n \in \mathbb{N}$, let $A_n^\text{sym}$ be the symmetrization of $A^n$, i.e. the matrix with entries $(a^n_{ij} + a^n_{ji})/2$. For $A^n$, define its representation in $L^2([0, T]^2)$ by the formula

$$A_n^\text{sym}(u, v) = a^n_{[nu],[nv]} \quad \text{for} \quad T \geq u, v \geq 0.$$  

(4)

Finally, let

$$X^n(t) = n^{-1/2} \sum_{1 \leq j \leq \lfloor nt \rfloor} X_j$$

(5)

$$Y^n(t) = n^{-1} \sum_{1 \leq i, j \leq \lfloor nt \rfloor} a^n_{ij}X_iX_j.$$  

(6)

**Theorem** Suppose $A^n$'s satisfy (1) and $\{X_j\}$ is a martingale difference sequence such that (2) holds.

If $A_n^\text{sym}$ converges in $L^2([0, T]^2)$ to some function $A$, then

$$Y^n \xrightarrow{\mathcal{D}} Y,$$

(7)

where

$$Y(t) = \int \int A(u, v)I_{[0,t]^2}(u, v) \, dW_u \, dW_v.$$  

(8)

is the classical Wiener-Ito integral.

**Proof.** For each $\varepsilon > 0$ there are continuous functions $\phi_l, \psi_l$, $l = 1, 2, \ldots, m$ and numbers $\alpha_1, \ldots, \alpha_m$ such that

$$\Phi(s, t) = \sum_{l=1}^{m} \alpha_l \cdot \frac{\phi_l(s)\psi_l(t) + \psi_l(s)\phi_l(t)}{2}$$

satisfies

$$\|A - \Phi\|_2 < \varepsilon.$$  

(9)

Hence for $n \geq n_1$, $\|A_n^\text{sym} - \Phi\|_2 < 2\varepsilon$ and by continuity of $\Phi \|A_n^\text{sym} - \Phi_n\|_2 < 3\varepsilon$ for $n \geq n_2$, where $\Phi_n(s, t) = \Phi([ns]/n, [nt]/n)$.

If

$$Z^n(t) = n^{-1} \sum_{1 \leq i, j \leq \lfloor nt \rfloor} \Phi(i/n, j/n)X_iX_j,$$

(10)

then for $n \geq n_2$

$$E \sup_{1 \leq t \leq T} |Y^n(t) - Z^n(t)|^2 \leq 4E|Y^n(T) - Z^n(T)|^2 \leq 4\|A_n^\text{sym} - \Phi_n\|_2^2 \leq 4 \cdot 9\varepsilon^2.$$  

(11)

Let

$$Z(t) = \int \int \Phi(u, v)I_{[0,t]^2}(u, v) \, dW_u \, dW_v.$$  

(12)

By the isometry property for Wiener-Ito integrals

$$E \sup_{1 \leq t \leq T} |Y(t) - Z(t)|^2 \leq 4E|Y(T) - Z(T)|^2 = 4\|A - \Phi\|_2^2 \leq 4\varepsilon^2.$$  

(13)
Hence it is enough to prove that
\[ Z^n \xrightarrow{p} Z \] (14)
on the space \( D([0,T]) \). But
\[
Z_n(t) = 2n^{-1} \sum_{1 \leq i < j \leq [nt]} \Phi(i/n, j/n)X_i X_j
\]
\[
= \sum_{l=1}^{m} \alpha_l \left( \sum_{1 \leq i < j \leq [nt]} \phi_l(i/n)\psi_l(j/n) \frac{X_i X_j}{\sqrt{n}/\sqrt{n}} + \sum_{1 \leq i < j \leq [nt]} \psi_l(i/n)\phi_l(j/n) \frac{X_i X_j}{\sqrt{n}/\sqrt{n}} \right)
\]
\[
= \sum_{l=1}^{m} \alpha_l \left( \int_0^t \psi_l(v) \left( \int_0^v \phi_l(u) dX_u^n \right) dX_v^n + \int_0^t \phi_l(v) \left( \int_0^v \psi_l(u) dX_u^n \right) dX_v^n \right)
\]
\[
\xrightarrow{p} \sum_{l=1}^{m} \alpha_l \left( \int_0^t \psi_l(v) \left( \int_0^v \phi_l(u) dW_u \right) dW_v + \int_0^t \phi_l(v) \left( \int_0^v \psi_l(u) dW_u \right) dW_v \right)
\]
\[
= Z(t),
\]
where the convergence in distribution holds by [JMP89, Theorem 2.6] \( \Box \)

**Example 1.** Let \( f : [0,1]^2 \to \mathbb{R}^1 \). Define quadratic forms
\[
a_{i,j} = f(i/n, j/n).
\]
If \( g_n(u,v) = f([nu],[nv]) \to \mathbb{L}^2 \) \( f(u,v) \), then
\[
n^{-1} \sum_{1 \leq i,j \leq [nt]} a_{i,j} X_i X_j \xrightarrow{p} \int_0^1 \int_0^1 \bar{f}(u,v) dW_u dW_v,
\]
where \( \bar{f} \) is the symmetrization of \( f \).

**Example 2.** Let
\[
c_0 = 0, c_1 = b_1, c_2 = b_2, \ldots, c_d = b_d, c_{d+1} = b_1, c_{d+2} = b_2, \ldots
\]
Define a matrix \( A \) by
\[
a_{i,j} = c_{|i-j|}.
\]
In this case \( A^n \) does not converge, so we cannot apply directly our theorem.
Take \( n = N \cdot d \) and define a rearranging of coordinates:

\[
\begin{align*}
&\quad e_1 \mapsto e_1 \\
&\quad e_2 \mapsto e_{N+1} \\
&\quad e_3 \mapsto e_{2N+1} \\
&\quad \vdots \\
&\quad e_d \mapsto e_{(d-1)N+1} \\
&\quad e_{d+1} \mapsto e_{2} \\
&\quad e_{d+2} \mapsto e_{N+2} \\
&\quad \vdots
\end{align*}
\]

(The general formula is of the form \( e_{k+d+i} \mapsto e_{(i-1)N+k+1} \) if \( 0 \leq k \leq N-1, 1 \leq i \leq d \).)

Under this rearranging, the distribution of \( X^n \) remains unchanged, while \( \mathbb{A}^n \) transforms to the form \( D^n = (d^n_{p,q}) \), where for \( p = (i-1)N + k + 1 \) and \( q = (j-1)N + l + 1 \)

\[
d^n_{p,q} = \begin{cases} 
0 & \text{if } j = i, l = k \\
b_d & \text{if } j = i, l \neq k \\
b_{|j-i|} & \text{if } j > i, l \geq k \text{ or } j < i, l \leq k \\
b_{d-|j-i|} & \text{if } j > i, l < k \text{ or } j < i, l > k.
\end{cases}
\]

It is easy to see that now \( \overrightarrow{D^n} \to D \) in \( L^2([0,1]^2) \), where

\[
D(u,v) = \begin{cases} 
0 & \text{if } u = v \\
b_d & \text{if } [d \cdot u] = [d \cdot v], u \neq v \\
b_{[d \cdot u] - [d \cdot v]} & \text{if } [d \cdot u] > [d \cdot v] \text{ and } \{d \cdot u\} \geq \{d \cdot v\} \\
& \text{or } [d \cdot u] < [d \cdot v] \text{ and } \{d \cdot u\} \leq \{d \cdot v\} \\
b_{d-[d \cdot u] - [d \cdot v]} & \text{if } [d \cdot u] > [d \cdot v] \text{ and } \{d \cdot u\} < \{d \cdot v\} \\
& \text{or } [d \cdot u] < [d \cdot v] \text{ and } \{d \cdot u\} > \{d \cdot v\}
\end{cases}
\]

Here \([x]\) is the integer part of \( x \) and \( \{x\} = x - [x] \).

The convergence is easily extendible from subsequence \( N \cdot d \) to the whole \( \mathbb{N} \). But we do not have the functional convergence, since

\[
n^{-1} \sum_{1 \leq i,j \leq [nt]} a^n_{i,j} X_i X_j
\]

and

\[
n^{-1} \sum_{1 \leq i,j \leq [nt]} d^n_{i,j} X_i X_j
\]

have nothing common except for \( t = 1 \), where they coincide.
References


