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Inequalities in von Neumann algebras*

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Abstract Generalization of inequalities involving trace of matrices to von Neumann algebras not having traces in general is discussed.

§1. Introduction

There are some well-known useful inequalities involving the trace of matrices: Let $A^* = A$, $B^* = B$, $\rho \geqq 0$, $\sigma \geqq 0$ and x be finite matrices.

(i) Golden-Thompson inequality ([15], [22]):

$$\operatorname{tr}(e^A e^B) \geqq \operatorname{tr} e^{A+B}. \tag{1.1}$$

(ii) Peierls-Bogolubov inequality ([11], [18])

$$\operatorname{tr} e^{A+B} \geqq (\operatorname{tr} e^A)\exp\{\operatorname{tr}(e^A B)/\operatorname{tr} e^A\}. \tag{1.2}$$

(iii) Powers-Størmer inequality ([19]):

$$\| \rho - \sigma \|_{tr} \geqq \| \rho^{1/2} - \sigma^{1/2} \|^2_{H.S.} \tag{1.3}$$

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Here $\|x\|_{tr} \equiv tr\{(x^*x)^{1/2}\}$, $\|x\|_{H.S.} \equiv \{tr(x^*x)\}^{1/2}$.

(iv) Convexity of $\log tr\ e^A$ in A ([16]).

(v) Lieb concavity ([16]): $tr \exp(A+\log \rho)$ is convex in ρ.

(vi) Wigner-Yanase-Dyson-Lieb concavity ([16], [24]): Let $0 \leqq s$, $0 \leqq r$, $r+s \leqq 1$. Then $tr(x^*\sigma^s x \rho^r)$ is jointly concave in ρ and σ.

(vii) Properties of relative entropy ([17], [23]): The relative entropy

$$S(\sigma/\rho) = tr\bigl(\rho \log \rho\bigr) - tr\bigl(\rho \log \sigma\bigr) \qquad (1.4)$$

satisfies the following properties (in addition to being lower semicontinuous in ρ and σ):

(α) Positivity: $S(\sigma/\rho) \geqq 0$ ($S(\sigma/\rho)=0$ only if $\sigma=\rho$) if $tr\ \sigma = tr\ \rho$.

(β) Convexity: $S(\sigma/\rho)$ is jointly convex in ρ and σ.

(γ) Monotonicity: Let E_N denote the conditional expectation of matrices to a *-subalgebra N relative to the trace. Then

$$S(E_N\sigma/E_N\rho) \leqq S(\sigma/\rho) \qquad (1.5)$$

In this review, we describe how to rewrite these inequalities without using "trace" so that the resulting expressions are meaningful for a general von Neumann algebra and inequalities remains true. We also sketch proofs for rewritten inequalities (ii), (v), (vi) and (vii). The proofs of (i), (ii) and (iv) are given for a general von Neumann algebra in [3] and (iii) in [4]. Also see [20]. The proof of (vi) and (viii) for a general von Neumann algebra will appear in a forth coming paper ([7]). The proof of (vi), (vii) (α) and (β) has already been given in [9].

Just to give an indication of what are our general idea, consider (i), (ii), (iv) and (v). Let M be a * algebra of matrices to which A,B and ρ belong. Any linear functional φ on M, which is positive in the sense that $\varphi(x^*x) \geqq 0$ for all $x \in M$ can be expressed in terms of a density matrix $\rho_\varphi \in M$ as

$$\varphi(x) = \operatorname{tr}(\rho_\varphi x) \ , \quad x \in M. \tag{1.6}$$

If we consider the case where $\rho_\varphi = e^A$, then

$$\operatorname{tr} e^A e^B = \varphi(e^B), \tag{1.7}$$

$$\operatorname{tr} e^A = \varphi(1), \tag{1.8}$$

$$\operatorname{tr} e^A B = \varphi(B). \tag{1.9}$$

Hence, if we somehow manage to define a positive linear functional φ^B on M from given φ with $\rho_\varphi = e^A$ and from $B=B^* \in M$, so that

$$\varphi^B(x) = \operatorname{tr}(e^{A+B}x), \tag{1.10}$$

then (i) and (ii) can be rewritten as

$$\varphi(e^B) \geqq \varphi^B(1) \geqq \varphi(1)\exp\{\varphi(B)/\varphi(1)\}. \tag{1.11}$$

(iv) is the convexity of $\log \varphi^B(1)$ in B and (v) is the concavity of $\varphi^{\log\rho}(1)$ in ρ.

For general van Neumann algebra M, φ is taken to be normal faithful positive linear functional. Here "normal" refers to a continuity of $\varphi(x)$ in $x \in M$ relative to the σ-weak (or σ-strong) topology in M. Faithfulness refers to the property that $\varphi(x^*x) = 0$ occurs only if $x=0$. This property is equivalent to $\rho_\varphi > 0$ for the case of (1.6) and is automatically satisfied for $\rho_\varphi = e^A$. The only part which requires more sophiscated tool is the definition of φ^B —— a perturbed functional. The theory of modular operators [21] is used in an essential manner for this purpose.

§2. Modular operators

Let Ψ and Φ be cyclic and separating vector of a von Neumann algebra M on a Hilbert space $\mathfrak{H}$. (Ψ cyclic if $M\Psi$ is dense in $\mathfrak{H}$; separating if $x \in M$ and $x\Psi=0$ imply $x=0$ or equivalently $M'\Psi$ is dense.) Let $S_{\Phi,\Psi}$ be an antilinear operator defined on $M\Psi$ by

$$S_{\Phi,\Psi}\, x\Psi, = x^*\Phi, \quad x \in M. \tag{2.1}$$

Then $S_{\Phi,\Psi}$ has a closure $\bar{S}_{\Phi,\Psi}$, whose absolute square defines the relative modular operator:

$$\Delta_{\Phi,\Psi} = (S_{\Phi,\Psi})^* \bar{S}_{\Phi,\Psi} \, . \tag{2.2}$$

The special case $\Delta_{\Psi,\Psi}$ is denoted by Δ_Ψ and called the modular operator. For given Ψ, $\Delta_{\Phi,\Psi}$ depends only on the normal faithful positive linear functional

$$\varphi(x) = (\Phi, x\Phi), \quad x \in M \tag{2.3}$$

and not on its representative vector Φ.

One of the main ingredients of Tomita-Takesaki theory ([21], also see [12]) is that $x \in M$ implies

$$\sigma_t^{\varphi}(x) \equiv (\Delta_{\Phi,\Psi})^{it} x (\Delta_{\Phi,\Psi})^{-it} \in M \tag{2.4}$$

for all real t. σ_t^{φ} is a continuous one-parameter group of automorphisms of M, called <u>modular automorphisms</u>. σ_t^{φ} depends only on φ and not on Ψ nor on the choice of the representative vector Φ of φ.

The polar decomposition

$$S_{\Psi,\Psi} = J_{\Psi}(\Delta_{\Psi})^{1/2} \tag{2.5}$$

defines an antiunitary involution J_{Ψ}. (Namely $(J_{\Psi}f, J_{\Psi}g) = (g,\Psi)$, $(J_{\Psi})^2 = 1$.) The other main ingredient of Tomita-Takesaki theory is that $x \in M$ implies

$$j_{\Psi}(x) \equiv J_{\Psi} x J_{\Psi} \in M'. \tag{2.6}$$

The closure of the set of vectors $(\Delta_{\Psi})^{1/4} x \Psi$ where x runs over all positive elements of M is called <u>natural positive cone</u> and denoted by V_{Ψ} ([4], [8], [13]). It is a pointed closed convex cone, which is selfdual (i.e. $(f,g) \geq 0$ for all $g \in V_{\Psi}$ if and only if $f \in V_{\Psi}$). For any $\Phi \in V_{\Psi}$ and $x \in M$, $xj_{\Psi}(x)\Phi \in V_{\Psi}$ and the set of $xj_{\Psi}(x)\Psi$ for all $x \in M$ is dense in V_{Ψ}. Any vector $\Phi \in V_{\Psi}$ is cyclic if and only if it is separating. For such Φ in V_{Ψ}, $J_{\Phi} = J_{\Psi}$ and $V_{\Phi} = V_{\Psi}$ (the universality). For a general cyclic and separating Φ, there exists a unitary u' in

M' such that $V_\Phi = u'V_\Psi$, $J_\Phi = u'J_\Psi(u')^*$ and

$$S_{\Phi,\Psi} = u'J_\Psi(\Delta_{\Phi,\Psi})^{1/2}. \tag{2.7}$$

In our disscussion, we can use a fixed natural positive cone and hence we drop the suffix Ψ from J_Ψ, V_Ψ and j_Ψ in the following.

Any normal positive linear functional φ of M has a unique representative vector $\xi(\varphi)$ in V:

$$\varphi(x) = (\xi(\varphi), x\xi(\varphi)). \tag{2.8}$$

The mapping ξ is a concave monotone increasing (relative to the positive cones M^+ and V) homeomorphism, homogeneous of degree 1/2, satisfying

$$\|\xi(\varphi_1) + \xi(\varphi_2)\| \, \|\xi(\varphi_1) - \xi(\varphi_2)\| \geqq \|\varphi_1 - \varphi_2\| \geqq \|\xi(\varphi_1) - \xi(\varphi_2)\|^2. \tag{2.9}$$

For faithful φ of (2.3), $\xi(\varphi)$ is given by

$$\xi(\varphi) = (\Delta_{\Phi,\Psi})^{1/2}\Psi. \tag{2.10}$$

(For general φ with a support projection e, $\xi(\varphi)$ is obtained by the same formula in the subspace $ej(e)\mathfrak{H}$ with Ψ replaced by $ej(e)\Psi$ and with Δ defined relative to eMe.)

To understand all formulas above, we go back to the simple case of M being a matrix algebra and see what newly defined quantities look like.

Let the Hilbert space $\mathfrak{H}$ be M itself with inner product

$$\langle\eta(x),\ \eta(y)\rangle = \mathrm{tr}\ x^*y \tag{2.11}$$

where we have used the notation $\eta(x)$ for an element in $\mathfrak{h}$ to distinguish it from the operator $x \in M$, which is faithfully represented by the left multiplication:

$$\pi(x)\eta(y) \equiv \eta(xy). \tag{2.12}$$

The left multiplication

$$\pi'(x)\eta(y) \equiv \eta(yx) \tag{2.13}$$

defines operators $\pi'(x)$ which generates $\pi(M)'$. $\pi(M)$ which is isomorphic to M will take place of M in our general discussion.

Let ρ_ψ and ρ_φ be density matrices defined in (1.6). Let Ψ be $\eta(\rho_\psi{}^{1/2})$. Then for $x \in M$

$$\Delta_{\Phi,\Psi}\eta(x) = \eta(\rho_\varphi x \rho_\psi{}^{-1}), \tag{2.14}$$

$$J\eta(x) = \eta(x^*), \tag{2.15}$$

$$V = \eta(M^+), \tag{2.16}$$

$$\xi(\varphi) = \eta(\rho_\varphi{}^{1/2}), \tag{2.17}$$

$$\sigma_t^\varphi(\pi(x)) = \pi(\rho_\varphi x \rho_\varphi{}^{-1}). \tag{2.18}$$

It is now possible to rewrite inequalities (iii), (vi) and (vii) as follows. First note that

$$\|\xi(\varphi_1) - \xi(\varphi_2)\|^2 = \|\rho_{\varphi_1}^{1/2} - \rho_{\varphi_2}^{1/2}\|_{H.S.}^2 ,$$

$$\|\varphi_1 - \varphi_2\| = \sup_{\|x\| \leq 1} |\varphi_1(x) - \varphi_2(x)|$$

$$= \sup_{\|x\| \leq 1} |tr(\rho_{\varphi_1} - \rho_{\varphi_2})x| = \|\rho_{\varphi_1} - \rho_{\varphi_2}\|_{tr}.$$

Hence the second inequality of (2.9) is the generalization of the Powers-størmer inequality (iii).

Next note that

$$(\Delta_{\Phi,\Psi})^{s/2} x\Psi = \eta(\rho_{\varphi}^{s/2} x \rho_{\psi}^{(1-s)/2})$$

which implies

$$\|(\Delta_{\Phi,\Psi})^{s/2} x\Psi\|^2 = tr(x^* \rho_{\varphi}^{s} x \rho_{\psi}^{1-s}). \tag{2.19}$$

Hence the concavity of (2.19) generalizes the concavity in (vi) for $r + s = 1$. (The case $r + s \leq 1$ in (vi) follows from the case $r + s = 1$ and the operator concavity of $\rho \to \rho^p$ for $0 \leq p \leq 1$.)

Finally

$$S(\varphi/\psi) = -(\Psi, (\log\Delta_{\Phi,\Psi})\Psi) \tag{2.20}$$

coincides with (1.4) with $\sigma = \rho_{\varphi}$ and $\rho = \rho_{\psi}$. Hence the positivity for $\varphi(1) = \psi(1)$, convexity and monotonicity of (2.20) generalize (vii), where the conditional expectation E_N in (1.5) is to be replaced by the restriction of a functional to von Neumann subalgebra N of M, because of the following circumstances: $E_N(\rho)$ is defined as the unique element in N satisfying

$$\mathrm{tr}\, \rho x = \mathrm{tr}\, E_N(\rho) x$$

for all $x \in N$. For $\rho = \rho_\varphi$, it coincides with the definition of the density matrix for the functional

$$\varphi^N(x) = \mathrm{tr}\, \rho x = \varphi(x), \qquad x \in N,$$

whcih is the restriction of φ to N.

We note that the concavity and monotonicity of ξ correspond to the operator concavity and monotonicity of $\rho \to \rho^{1/2}$.

§3. Perturbation of functionals.

To generalize the perturbed functional φ^B given by (1.10) to a general von Neumann algebra M, we define a vector $\Phi(h) \in V$ for given $\Phi \in V$ and $h = h^* \in M$ so that

$$\varphi^h(x) = (\Phi(h), x\Phi(h)), \qquad x \in M \tag{3.1}$$

is the desired perturbed functional. The formula (2.14) and (1.10) suggest

$$\log \Delta_{\Phi(h),\Phi} - \log \Delta_\Phi = h \tag{3.2}$$

which implies, due to (2.10),

$$\Phi(h) = \exp\{(\log \Delta_\Phi + h)/2\}\Phi. \tag{3.3}$$

An alternative expression can be found by using the expansion

$$e^{(A+B)t}e^{-tA} = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \sigma^{\varphi}_{-it_n}(B) \cdots \sigma^{\varphi}_{-it_1}(B),$$

$$\sigma^{\varphi}_t(B) = e^{itA}Be^{-itA},$$

to the representative vector $(e^{(A+B)/2}e^{-A/2})e^{A/2}$, where $\varphi(x) = \mathrm{tr}(e^A x)$. The resulting expression, written in terms of the modular operator Δ_Φ of $\Phi = e^{A/2}$ is

$$\Phi(h) = \sum_{N=0}^{\infty} \int_0^{1/2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \Delta_\Phi^{t_n} h \Delta_\Phi^{t_{n-1}-t_n} h \cdots \Delta_\Phi^{t_1-t_2} h\Phi. \tag{3.4}$$

We adopt (3.4) as the definition of $\Phi(h)$ and (3.1) as the definition of φ^h for a general von Neumann algebra M. The absolute convergence of (3.4), uniform over $h \in (M)_k$ (the ball of radius k in M), follows from the following Lemma ([2], Theorem 3.1):

<u>Lemma 1</u> (1) A cyclic and separating vector Φ is in the domain of the operator

$$Q(z) \equiv \Delta_\Phi^{z_1} Q_1 \Delta_\Phi^{z_2} Q_2 \cdots \Delta_\Phi^{z_n} Q_n \tag{3.5}$$

for any integer n, any $Q_j \in M$ $(j=1,\dots,n)$ and any complex number z_j $(j=1,\dots,n)$ in the tube domain

$$\bar{I}_n^{1/2} \equiv \{z=(z_1,\ldots,z_n);\ \mathrm{Re}\ z_1 \geq 0,\ldots,\mathrm{Re}\ z_n \geq 0,$$
$$1/2 \geq \mathrm{Re}(z_1+\ldots z_n)\}. \qquad (3.6)$$

(2) The vector-valued function $Q(z)\Phi$ of $z = (z_1,\ldots,z_n)$ is strongly continuous on $\bar{I}_n^{1/2}$, holomorphic in the interior $I_n^{1/2}$ of $\bar{I}_n^{1/2}$ and uniformly bounded by $\|\Phi\|\ \|Q_1\|\cdots\|Q_n\|$.

(3) Let $(M)_k^{*st}$ be the ball of radius k in M, equipped with *-strong operator topology. The vector $Q(z)\Phi$ is strongly continuous as a function of

$$(Q_1\ldots Q_n) \in (M)_k^{*st} \times \ldots \times (M)_k^{*st} ,$$

the continuity being uniform in $z_1\ldots z_n$ over any compact subset of the tube $\bar{I}_n^{1/2}$. ($k>0$ is arbitrary.)

(For the proof of (3), see Remark at the end of the section.)

The perturbed vector $\Phi(h)$ is automatically a cyclic and separating vector in the same natural cone as Φ and satisfies (3.2), (3.3) and the following properties ([2]):

$$\Phi(h_1) = \Phi(h_2) \text{ if and only if } h_1 = h_2. \qquad (3.7)$$

$$[\Phi(h_1)](h_2) = \Phi(h_1+h_2). \qquad (3.8)$$

$$[\Phi(h)](-h) = \Phi. \qquad (3.9)$$

$$[\Phi(\lambda\mathbf{1})] = e^{\lambda/2}\Phi. \qquad (3.10)$$

$$\log \Delta_{\Phi(h)} = \log \Delta_\Phi + h - j(h). \qquad (3.11)$$

$$\sigma_t^{\varphi^h}(x) = u_t \sigma_t^{\varphi}(x) u_t^* , \qquad (3.12)$$

$$u_t \equiv (\Delta_{\Phi(h),\Phi})^{it}\Delta_\Phi^{-it}$$
$$= \sum_{n=0}^{\infty}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \sigma_{t_n}^{\varphi}(h)\ldots\sigma_{t_1}^{\varphi}(h). \qquad (3.13)$$

$$(d/dt)\{\sigma_t^{\varphi^h}(x) - \sigma_t^{\varphi}(x)\}_{t=0} = i[h,x]. \qquad (3.14)$$

$$(d/dt)u_t = u_t\sigma_t^{\varphi}(h). \qquad (3.15)$$

From Lemma 1(3) and the uniform bound of Lemma 1(2), it follows that $\Phi(h)$ is strongly continuous as a function of $h \in (M)_k$.

For our application, it is important to find an analytic continuation in h. For example, the vector $\Phi(h)$ can be defined for arbitrary $h \in M$ by (3.4). It is then seen from the uniform bound of Lemma 1(2) that $\Phi(h(z))$ is holomorphic in z if $h(z)$ is holomorphic in z. The following Lemma ([2], Theorem 3.2) yields such result for $\varphi^h(1)$:

<u>Lemma 2</u> (1) For any $Q_j \in M$ $(j=1,\ldots,n+1)$, the following formula defines a single-valued function $f(z)$ for $z \in \bar{I}_n^1$ (defined by (3.6) in which 1/2 is replaced by 1):

$$f_{n+1}(z) = (\Delta_\Phi^{\bar{z}_{j2}} Q_{j+1}^* \Delta_\Phi^{\bar{z}_{j+1}} \ldots \Delta_\Phi^{\bar{z}_n} Q_{n+1}^* \Phi\ ,$$
$$\Delta_\Phi^{z_{j1}} Q_j \Delta_\Phi^{z_{j-1}} \ldots \Delta_\Phi^{z_1} Q_1 \Phi), \qquad (3.16)$$

where

$$z = (z_1,\ldots,z_n) \in \bar{I}_n^1\ , \qquad z_j = z_{j1}+z_{j2},$$

$$\mathrm{Re}(z_1+\ldots+z_{j-1}+z_{j1}) \leqq 1/2,$$

$$\mathrm{Re}(z_{j2}+z_{j+1}+\ldots+z_n) \leqq 1/2.$$

(2) The function $f_{n+1}(z)$ so defined is continuous on $\bar{I}_n^1$, holomorphic in the interior I_n^1 of $\bar{I}_n^1$, and uniformly bounded on $\bar{I}_n^1$ by $\|\Phi\| \|Q_1\| \ldots \|Q_{n+1}\|$.

(3) The values of $f_{n+1}(z)$ at distinguished boundaries of $\bar{I}_n^1$ are given by

$$f_{n+1}(it_1-it_2,\ldots,it_n-it_{n+1}) = \varphi(\sigma^{\varphi}_{t_{n+1}}(Q_{n+1})\ldots\sigma^{\varphi}_{t_1}(Q_1)), \quad (3.17)$$

$$f_{n+1}(it_1-it_2,\ldots,it_j-it_{j+1}+1,\ldots,it_n-it_{n+1})$$
$$= \varphi(\sigma^{\varphi}_{t_j}(Q_j)\ldots\sigma^{\varphi}_{t_1}(Q_1)\sigma^{\varphi}_{t_{n+1}}(Q_{n+1})\ldots\sigma^{\varphi}_{t_{j+1}}(Q_{j+1})), \quad (3.18)$$

where $t_1,\ldots,t_{n+1}$ are real and $j=1,\ldots,n$.

(4) $f_{n+1}(z)$ is a continuous function of

$$(Q_1,\ldots,Q_{n+1}) \in (M)_k^{st} \times\ldots\times (M)_k^{st} ,$$

the continuity being uniform in z over any compact subset of $\bar{I}_n^1$. ($k>0$ is arbitrary.) Here $(M)_k$ is equipped with strong operator topology. (For Bergman-Weil formula, see [1], Corollary 3.4 and Remark 3.5.)

<u>Remark</u> (1) Lemma 2(4) can be proved as follows: To make dependence on $Q = (Q_1,\ldots,Q_{n+1})$ explicit, we write

$$F(z;Q) = e^{(z_1^2+\dots+z_n^2)} f_{n+1}(z) \tag{3.19}$$

where the Gaussian factor is introduced to make F uniformly vanishing for infinite z in $\bar{I}^1_{n+1}$. It is enough to show that for any $\varepsilon>0$,

$$|F(z;Q') - F(z;Q)| < \varepsilon$$

for Q' in a suitable strong neighbourhood of Q within $(M)^{st}_k \times\dots\times(M)^{st}_k$, the neighbourhood being independent of z as long as z is in any given compact subset of $\bar{I}^1_{n+1}$. Due to the analyticity in z and vanishing at infinite z, $|F(z;Q') - F(z;Q)|$ is bounded by the supremum of its values on distinguished boundaries, which consists of the following $n+1$ planes:

$$B_0 = \{z \; ; \; \mathrm{Re}\, z = 0\} \; , \tag{3.20}$$

$$B_j = \{z \; ; \; \mathrm{Re}\, z_j = 1 \text{ and } \mathrm{Re}\, z_\ell = 0 \text{ for } \ell \neq j\} \; , \tag{3.21}$$

where $j=1,\dots,n$. Since $F(z;h)$ tends to 0 as $z\to\infty$ from within $\bar{I}^1_{n+1}$, uniformly in $h \in (M)^{st}_k\times\dots\times(M)^{st}_k$, it is enough to see that the supremum of $|F(z;Q') - F(z;Q)|$ over z in some compact subset of a distinguished boundary is bounded by a given ε. For this it is enough to see that $F(z;Q)$ is a continuous function of $(z,Q)\in B_j\times(M)_k\times\dots\times(M_k)$ for $j=0,\dots,n$. The function $f(z;Q)$ is given by Lemma 2(3), which can be rewritten as the expectation value in Φ of a product of some of operators $Q_1,\dots,Q_{n+1},\Delta_\Phi^{i(t_{n+1}-t_1)},\dots,\Delta_\Phi^{i(t_n-t_{n+1})},\Delta_\Phi^{i(t_{n+1}-t_1)}$ in a certain order. Since a product of operators is simultaneously strongly continuous as long as operators are in a uniformly bounded set, and since Δ_Φ^{is} is strongly continuous in real variable s (with norm 1), we have the desired continuity of $f(z;Q)$ in (z,Q) with z on distinguished boundaries.

(2) Lemma 1 (3) can be proved as follows: Let

$$\Phi(z;Q) = e^{z_1^2+\ldots+z_n^2} Q(z)\Phi. \tag{3.22}$$

We have to show that

$$\|\Phi(z;Q') - \Phi(z;Q)\| = \sup_{\|\Psi\|=1} |(\Psi,\Phi(z;Q') - \Phi(z;Q))| < \varepsilon$$

for $Q' = (Q_1'\ldots Q_n')$ in a suitable strong neighbourhood of $Q = (Q_1\ldots Q_n)$ within $(M)_k^{*st}\times\ldots\times(M)_k^{*st}$, the neighbourhood being independent of z as long as z is in a given compact subset of $\bar{I}_{n+1}^1$. As above, the problem is reduced to the strong continuity of $\Phi(z;Q)$ in (z,Q) for z in the distinguished boundaries of $\bar{I}_n^{1/2}$ and Q in $(M)_k^{*st}\times\ldots\times(M)_k^{*st}$. This follows again from the strong continuity of product of operators in a uniformly bounded set applied to the following expressions for real $s = (s_1\ldots s_n)$:

$$\Phi(is_1\ldots is_n;Q) = \Delta_\Phi^{is_n}Q_n\ldots\Delta_\Phi^{is_1}Q_1\Phi,$$

$$\Phi(is_1\ldots is_j+1/2\ldots is_n;Q) = \Delta_\Phi^{is_n}Q_n\ldots\Delta_\Phi^{is_{j+1}}Q_{j+1}\Delta_\Phi^{i(s_1+\ldots+s_j)}$$
$$Q_1^*\Delta_\Phi^{-is_1}Q_2^*\Delta_\Phi^{-is_2}\ldots\Delta_\Phi^{-is_{j-1}}Q_j^*\Phi.$$

(3) In the proof of Theorem 3.2 of [2], a factor $e^{-(z_1^2+\ldots+z_n^2)}$ is missing from the definition of $F^{\beta}(z)$ on page 173. With this factor, it is enough to prove the simultaneous continuity of $F^{\beta}(x-i\lambda^{(j)})$ in Q's and x's for each j, which follows again from the strong continuity of product on bounded set.

§4. Proof of Lieb convexity

We use the method of Epstein ([14]), for which we need an analytic continuation of $\varphi^h(1)$ in h, given by the following formula:

$$f(Q,\varphi) \equiv \varphi(1)+\varphi(Q)+\sum_{n=2}^{\infty}\int_0^1 dt_1\ldots\int_0^{t_{n-1}} dt_n f_n(t_1-t_2,\ldots,t_{n-1}-t_n). \tag{4.1}$$

By Lemma 2(2), the expression (4.1) is convergent and defines a holomorphic function of Q in the sense that $f(Q(z),\varphi)$ is holomorphic in z whenever Q(z) is holomorphic in z. It is also strongly continuous as long as Q is in a bounded set. If $Q = h = h^*$, then

$$f(h,\varphi) = \varphi^h(1), \tag{4.2}$$

which can be proved as follows.

It is enough to prove (4.2) for a dense set of h and hence we assume that $\sigma_t^{\varphi}(h)$ is an entire function of t. In this case the following formula holds for real z and $H = \log \Delta_{\Phi}$:

$$e^{iz(H+h)}e^{-izH} = \sum_{n=0}^{\infty}(iz)^n\int_0^1 dt_1\ldots\int_0^{t_{n-1}} dt_n\sigma^{\varphi}_{zt_n}(h)\ldots\sigma^{\varphi}_{zt_1}(h). \tag{4.3}$$

See, for example, [6] Theorem 14.) Due to $H\Phi = 0$, we have

$$e^{iz(H+h)}\Phi = \sum_{n=0}^{\infty}(iz)^n\int_0^1 dt_1\ldots\int_0^{t_{n-1}} dt_n\sigma^{\varphi}_{zt_n}(h)\ldots\sigma^{\varphi}_{zt_1}(h)\Phi, \tag{4.4}$$

at first for real z. Since

$$(e^{-i\bar{z}(H+h)}\Psi,\ \Phi)$$

for any entire vector Ψ of $H+h$ (which is selfadjoint) and the inner product of Ψ with the right hand side of (4.4) are both an entire function of z and coincides for real t, they are equal. It follows that Φ is in the domain of $e^{iz(H+h)}$ and (4.4) holds for all z. For $z = -i/2$, (4.4) gives $\Phi(h)$ (the right handside gives (3.4) and the left hand side gives (3.3)). Hence

$$\varphi^h(1) = (\Phi,\ e^{H+h}\Phi)$$

$$= \varphi(1) + \varphi(h) + \sum_{n=z}^{\infty}\int_0^1 dt_1\ldots\int_0^{t_{n-1}}dt_n(\Phi,\sigma^{\varphi}_{-it_n}(h)\ldots\sigma^{\varphi}_{-it_1}(h)\Phi). \tag{4.5}$$

The desired result (4.1) follows (4.5) due to the formula

$$(\Phi,\sigma^{\varphi}_{t_n}(h)\ldots\sigma^{\varphi}_{t_1}(h)\Phi) = f_n(it_1-it_2,\ldots,it_n-it_{n-1}), \tag{4.6}$$

which obviously holds for real t and hence by analytic continuation for all t where f_n is defined. This concludes the proof of (4.2).

We now apply Lemma 3 of [14] to the function $\rho \to f(\log\rho, \varphi)$ defined on

$$D = \bigcup \{A;\ \mathrm{Re}\ e^{-i\theta} A \geqq \varepsilon\} \tag{4.7}$$

where the union is over real $\varepsilon > 0$ and $\theta \in [-\pi/2, \pi/2]$, and $\mathrm{Re}\ C$ denotes $(C+C^*)/2$. The convexity of $\Phi(\log \rho) = f(\log \rho, \ )$ in $\rho \in M^+$ follows from the following conditions to be satisfied by f:

(i) f is holomorphic in $\rho \in D$.

(ii) If $\mathrm{Im}\ \rho > 0$ and $\rho \in D$, then $\mathrm{Im}\ f(\log \rho, \varphi) \geqq 0$. If $\mathrm{Im}\ \rho < 0$ and $\rho \in D$, then $f(\log \rho, \varphi) \leqq 0$. Here $\mathrm{Im}\ \rho$ denotes $(\rho - \rho^*)/(2i)$.

(iii) For every real r and $\rho \in D$,

$$f(\log (r\rho), \varphi) = r^s f(\log \rho, \varphi) \tag{4.8}$$

where $0 < s \leqq 1$.

Since $\rho \to \log \rho$ is holomorphic in the domain (4.7) ([14]), (i) is satisfied. Since $\varphi^{h+c\mathbf{1}}(1) = e^c \varphi^h(1)$, the corresponding equation holds for its analytic continuation and hence (4.8) holds with $s = 1$.

To prove (ii), we introduce

$$h_\beta \equiv \int \sigma_t^{\varphi}(\log \rho) e^{-t^2/\beta} dt/(2\pi\beta)^{1/2}. \tag{4.9}$$

We can verify (ii) if we show that $\mathrm{Im}\, f(h_\beta, \varphi) \geqq 0$ if $\mathrm{Im}\,\rho > 0$, $\rho \in D$ and $f(h_\beta, \varphi) \leqq 0$ if $\mathrm{Im}\,\rho > 0$, $\rho \in D$, because $\lim_{\beta \to +0} h_\beta = \log \rho$ and $f(Q, \varphi)$ is continuous in Q.

Let E_λ for $\lambda \in [0,1]$ be the spectral projection of Δ_Φ for the spectral set $[\lambda, 1/\lambda]$. Then $E_\lambda H$ is bounded and $\lim_{\lambda \to 0} E_\lambda = 1$. By Remark 4 of [14], $0 < \mathrm{Im} \log \rho < \pi$ if $\mathrm{Im}\,\rho > 0$. This implies $0 < \mathrm{Im}\, h_\beta < \pi$ if $\mathrm{Im}\,\rho > 0$. By Remark 2 of [14], $0 <$ $\mathrm{Im}\; \mathrm{Sp}\, h_\beta < \pi$ where Sp denotes the spectrum. Hence $\mathrm{Im}\, \mathrm{Sp}(e^{HE_\lambda + h_\beta}) \geqq 0$ and

$$\mathrm{Im}\, (\Phi,\, e^{HE_\lambda + h_\beta}\Phi) \geqq 0$$

whenever $\mathrm{Im}\,\rho > 0$. We now prove

$$\lim_{\lambda \to 0} (\Phi,\, e^{HE_\lambda + h_\beta}\Phi) = f(\log \rho, \varphi), \tag{4.10}$$

which will complete the proof of Lieb convexity for a general von Neumann algebra.

By the formula (4.3) with H replaced by HE_λ and iz by 1, we obtain by using $e^{-HE_\lambda}\Phi = \Phi$

$$(\Phi,\, e^{HE_\lambda + h_\beta}\Phi) = \sum_{n=0}^{\infty} \int_0^1 dt_1 \cdots \int_0^{t_{n-1}} dt_n\, g(t_1 \ldots t_n), \tag{4.11}$$

$$g(t_1 \ldots t_n) = (\Phi,\, h_\beta e^{(t_{n-1} - t_n)HE_\lambda} \cdots e^{(t_1 - t_2)HE_\lambda} h_\beta \Phi). \tag{4.12}$$

We replace each exponential in (4.12) by the formula

$$e^{sHE_\lambda} = \{\Delta_\Phi^s E_\lambda + (1 - E_\lambda)\}$$

and obtain 2^{n-1} terms of the following type

$$(\Phi,\ h_\beta e_{n-1}\sigma^{\varphi}_{-is_{n-1}}(h_\beta)\dots e_1\sigma^{\varphi}_{-is_1}(h_\beta)\Phi), \tag{4.13}$$

where

$$e_j = \varepsilon_j E_\lambda + (1-\varepsilon_j)(1-E_\lambda),$$

$$s_j = \sum_{\ell=j}^{n-1} \varepsilon_\ell (t_\ell - t_{\ell+1}),$$

and ε_j is either 0 or 1. By the continuity of the product of uniformly bounded operators, (4.13) is continuous in $(\lambda, s_1, \dots, s_{n-1})$ and hence tends to zero as $\lambda \to 0$, except that the term with all $\varepsilon_j = 1$ tends to

$$(\Phi,\ h_\beta\sigma^{\varphi}_{-i(t_{n-1}-t_n)}(h_\beta)\dots\sigma^{\varphi}_{-i(t_1-t_n)}(h_\beta)\Phi)$$

$$= (\Phi,\ \sigma^{\varphi}_{-it_n}(h_\beta)\dots\sigma^{\varphi}_{-it_1}(h_\beta)\Phi)$$

where all convergence is uniform in $(t_1\dots t_n)$ within the compact region of integration in (4.11). (4.13) is also bounded by

$$2^{n-1}\{\sup_{0\leq s\leq 1}\|\sigma^{\varphi}_{-is}(h_\beta)\|\}^n\|\Phi\|^2$$

independent of $(\lambda, t_1, \dots, t_n)$. Hence the series (4.11) is absolutely convergent uniformly in λ and we obtain (4.10) from the convergence of (4.13).

§5. Relative Entropy

Let E_λ be the spectral projection of $\Delta_{\Phi,\Psi}$. Then the definition (2.20) is

$$S(\varphi/\psi) = -\int_0^\infty \log \lambda \ d(\Psi, E_\lambda \Psi). \tag{5.1}$$

By a numerical inequality

$$\log \lambda \leqq \lambda - 1, \tag{5.2}$$

we have

$$\begin{aligned} S(\varphi/\psi) &\geqq \int_0^\infty (1-\lambda) d(\Psi, E_\lambda \Psi) \\ &= \|\Psi\|^2 - \|(\Delta_{\Phi,\Psi})^{1/2}\Psi\|^2 \\ &= \psi(1) - \varphi(1). \end{aligned} \tag{5.3}$$

Hence we have the positivity

$$S(\varphi/\psi) \geqq 0 \tag{5.4}$$

if $\varphi(1) = \psi(1)$. Since the equality in (5.2) holds only if $\lambda = 1$, the eqality in the inequality of (5.3) holds if the measure $d(\Psi, E_\lambda \Psi)$ is concentrated at $\lambda = 1$, i.e.

$$\Phi = (\Delta_{\Phi,\Psi})^{1/2}\Psi = \Psi.$$

Hence if $\varphi(1) = \psi(1)$, then

$$S(\varphi/\psi) = 0$$

holds if and only if $\varphi = \psi$. (Strict positivity.)

We now consider perturbed functional φ^{h-c1} where $h = h^* \in M$ and the number c is chosen to be

$$c = \log(\varphi^h(1)/\varphi(1)) \tag{5.5}$$

so that $\varphi^{h-c1}(1) = \varphi(1)$. By (3.2) and $\Delta_\Phi \Phi = \Phi$, we have

$$\begin{aligned} S(\varphi^{h-c1}/\varphi) &= -\varphi(h-c1) \\ &= \varphi(1)c - \varphi(h). \end{aligned} \tag{5.6}$$

The positivity and (5.5) imply

$$\varphi(h) \leq \varphi(1) \log(\varphi^h(1)/\varphi(1)), \tag{5.7}$$

which is the Peierls-Bogolubov inequality (the second inequality of (1.11)).

The WYDL concavity has been generalized ([7],[9]) to the joint concavity of $\|(\Delta_{\Phi,\Psi})^{p/2}x\Psi\|^2$ in faithful normal positive functionals φ and ψ for $0 \leq p \leq 1$. This implies the concavity of

$$\begin{aligned} S_p(\varphi/\psi) &\equiv \int_0^\infty \lambda^p d(\Psi, E_\lambda \Psi) \\ &= \|(\Delta_{\Phi,\Psi})^{p/2}\Psi\|^2 \end{aligned} \tag{5.8}$$

and hence the convexity of

$$S(\varphi/\psi) = \lim_{p\to 0} p^{-1}\{\psi(1)-s_p(\varphi/\psi)\} \tag{5.9}$$

jointly in φ and ψ.

This convexity can by used to prove the monotonicity

$$S(\varphi/\psi) \geqq S(E_N\varphi/E_N\psi) \tag{5.10}$$

where E_N denotes the restriction of functionals to N and the proof has been found so far ([7]) for a general M and for a von Neumann subalgebra N of M belonging to one of the following cases:

(1) $M = N \otimes N_1$ for $N_1 = M \cap N'$.

(2) $N = A' \cap M$ for a finite dimensional abelian von Neumann subalgebra A of M.

(3) N is an approximate finite von Neumann algebra. This includes any finite dimensional N, which is the case needed in applications ([5], [10]).

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