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On the Relation Between Yangians, Affine Hecke Algebras and Long-Range Interacting Models

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On the relation between Yangians, affine Hecke algebras and long-range interacting models.

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Plan :
1- Long-Range Interacting Models.
2- Yangians and Affine Hecke Algebras.
3- A Yangian deformation of the W-algebras.

1 Long-Range Interacting Models.

There is a large family of integrable long range interacting spin chains. In particular, a model introduced by Haldane and Shastry [2], see also [3], is a variant of the spin half Heisenberg chain, with exchange inversely proportional to the square distance between the spins. It possesses the remarkable properties that its spectrum is highly degenerate and additive, and that the elementary excitations are spin half objects obeying a half-fractional statistics intermediate between bosons and fermions. They are defined as follows. We consider a spin chain with N sites, labeled by integers $i, j, \cdots$ ranging from 1 to N. On each sites there is a spin variable σ_i which takes two values: $\sigma_i = \pm$. The hamiltonians, which are all su(2) invariant, are of the following form :

$$H = \sum_{i \neq j} h_{ij} \left(P_{ij} - 1 \right) \tag{1}$$

where P_{ij} is the operator which exchanges the spins at the sites i and j. For translation invariance $h_{ij} = h(i-j)$. Demanding the integrability of the model selects the functions h.

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The possible choices are :

$$h(x) = \begin{cases} \frac{\gamma^2}{(\sinh \gamma x)^2}, & \text{hyperbolic model } (\gamma \text{ real}) \\ \frac{(\pi/N)^2}{\left(\sin \frac{\pi x}{N}\right)^2}, & \text{trigonometric model} \\ \mathcal{P}(x), & \text{elliptic model.} \end{cases}$$

where $\mathcal{P}(x)$ is the Weierstrass function. When $\gamma \to \infty$, the hyperbolic model reduces to the Heisenberg spin chain: $h_{ij} = \delta_{i,j+1} + \delta_{j,i+1}$, and for $\gamma \to 0$, the interaction becomes the $1/x^2$ exchange. The hyperbolic model has not been completly solved for general γ, although a partial list of eigenstates is known. The elliptic model is even more intriguing since it interpolates between the Heisenberg spin chain of finite length and the trigonometric model [4].

The Haldane-Shastry spin chain is the trigonometric model. In the thermodynamical limit, $N \to \infty$, it reduces to the $1/x^2$ exchange model, but it also possesses remarkable properties at finite N. Notably, its hamiltonian commutes with an infinite dimensional algebra whose two first generators are [5] :

$$\vec{Q}_0 = \sum_i \vec{S}_i \tag{2}$$

$$\vec{Q}_1 = \sum_{i \neq j} cotg \left(\frac{\pi(i-j)}{N} \right) \vec{S}_i \times \vec{S}_j \tag{3}$$

with $\vec{S}_i$ the spin operators acting on the site i. The first generators are the usual su(2) generators. Together with the second ones, they form a representation of the su(2) Yangian, (which is a deformation of the su(2) current algebra, see section 3 for an introduction to the Yangians). This infinite dimensional symmetry is at the origin of the large degeneracy of the spectrum. The fact that the hamiltonian is Yangian invariant at finite N is particular to the Haldane-Shastry spin chain; in the Heisenberg spin chain, the Yangian symmetry only appears in the thermodynamical limit.

In order to grasp the rules describing the spectrum, we first construct few eigenstates. Clearly, the ferromagnetic vacuum $|\Omega\rangle = |++\cdots++\rangle$ is an eigenstate : its energy is zero. The eigenstates in the one-magnon sector are the plane waves :$|k\rangle = \sum_n \exp(i2\pi kn/N)\sigma_n^-|\Omega\rangle$, with pseudo-momentun k, $1 \leq k \leq (N-1)$: the one-magnon energy is $\epsilon(k) = \left(\frac{\pi}{N}\right)^2 k(k-N)$. In the two-magnon sectors, i.e. for states of the form $|\psi\rangle = \sum_{n,m} \psi_{n,m}\sigma_n^-\sigma_m^-|\Omega\rangle$, the eigenstates which are not degenerate with the zero or one-magnon eigenstates are labeled by two pseudo-momenta k_1, k_2, with $1 \leq k_1, k_2 \leq (N-1)$. They are given by :

$$\psi_{n,m}^{[k_1,k_2]} = (k_1 - k_2)\left(\omega^{nk_1+mk_2} + \omega^{mk_1+nk_2}\right) - \frac{\omega^n + \omega^m}{\omega^n - \omega^m}\left(\omega^{nk_1+mk_2} - \omega^{mk_1+nk_2}\right)$$

with $\omega = \exp(i2\pi/N)$. Note that these wave functions vanish if $k_1 = k_2$ but also if $|k_1 - k_2| = 1$. The energy of $|\psi^{[k_1,k_2]}\rangle$ is $E = \epsilon(k_1) + \epsilon(k_2)$.

From the two-magnon computation we learn two properties of the spectrum : (i) it is additive, e.g. the two-magnon energy is the sum of the one-magnon energies, but (ii) the pseudo-momenta satisfy a selection rule : they are neither equal nor they differ by a unit. These rules are the general rules, and the full spectrum can be described as follows [6].

To each eigenstate multiplet is associated a set of pseudo-momenta $\{k_p\}$ which are non-consecutive integers ranging from 1 to $(N-1)$. The energy of an eigenstate $|\{k_p\}\rangle$ with pseudo-momenta $\{k_p\}$ is:

$$H|\{k_p\}\rangle = \left(\sum_p \epsilon(k_p)\right)|\{k_p\}\rangle \qquad with \quad \epsilon(k) = \left(\frac{\pi}{N}\right)^2 k(k-N) \tag{4}$$

Furthermore, the degeneracy of the multiplet with pseudo-momenta $\{k_p\}$ is described by its su(2) representation content as follows. Encode the pseudo-momenta in a sequence of $(N-1)$ labels 0 or 1 in which the 1's indicate the positions of the pseudo-momenta; add two 0's at both extremities of the sequence which now has length $(N+1)$. Since the pseudo-momenta are neither equal nor consecutive, two labels 1 cannot be adjacent. The sequence corresponding to the ferromagnetic vacuum is a line of 0, those of the one-magnon states have N label 0 and only one label 1, and so on. A sequence can be decomposed into the product of elementary motifs, which are series of $(Q+1)$ consecutives 0's. The multiplicity of the spectrum is recovered if to each elementray motif of length $(Q+1)$ we associate a spin $Q/2$ representation of su(2). The representation content of the full sequence is then given by the tensor product of its elementary motifs.

The magnons are the excitations over the ferromagnetic vacuum; the excitations over the antiferromagnetic vacuum are conveniently described in terms of spinons. For N even, the antiferromagnetic vacuum corresponds to the alternating sequence of symbols $010101\cdots010$. The excitations are obtained by flipping and moving the symbols 0 and 1. Let us give the sequences corresponding to the first few excitations over the antiferromagnetic vacuum, (for concreteness we choose $N=10$) :

$$\begin{cases} 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ , & \text{antiferromagnetic vacuum (o)} \\ 0\ 1\ 0\ 1\ 0\ 1\ 0_x 0_x 0\ 1\ 0\ , & \text{a two-spinon excitations (2a)} \\ 0\ 1\ 0_x 0\ 1\ 0_x 0\ 1\ 0\ 1\ 0\ , & \text{a two-spinon excitations (2b), etc...} \end{cases}$$

We have inserted a x between any two consecutive labels 0. These crosses represent the spinon excitations, their number is the spinon number. Note that there is no one-spinon excitation for N even. By convention, we will say that consecutive crosses not separated by any label 1 correspond to spinons in the same orbital, while crosses separated by labels 1 correspond to spinons in different orbitals. From the rules described above, it follows that the degeneracy of the excitations (2a) and (2b) are different : it is three in the case (2a) and four in the case (2b). These degeneracy are recovered by giving a su(2) spin half to the spinons and by assuming that spinons in the same orbital are in a fully symmetric states. Hence, in the case (2a), there are two spinons in the same orbital and therefore they form a spin one representation of su(2), and in the case (2b), the two spinons are in two different orbitals and therefore they form a su(2) representation isomorphic to the tensor product of two spin half representations of su(2). The fact that the spinons are spin half excitations can also be seen by looking at the excitations of a spin chain of length N with N odd.

This description of the states generalizes to the full spectrum. We can classify the sequences by their number M of pseudo-momenta. The spinon number N_{sp} of a sequence is then defined by $M = \frac{N-N_{sp}}{2}$. Since M is an integer, $(N-N_{sp})$ is always even : this means that the spinons are always created by pairs. A sequence of pseudo-momenta $\{k_p; p=1,\cdots,M\}$, in the N_{sp} spinon sector, can be decomposed into $(M+1)$ elementary motifs where, as before an elementary motif is a series of consecutive 0. We call the elementary motifs the accessible orbitals to the spinons. At fixed N_{sp}, there are $N_{orb} = (1+\frac{N-N_{sp}}{2})$ orbitals. Hence, a sequence of pseudo-momenta $\{k_p\}$ corresponds to the filling of the N_{orb} orbitals with respective spinon occupation numbers $n_p = (k_{p+1} - k_p - 2)$, with $k_0 = -1$ and $k_{M+1} = N + 1$ by convention. Since an elementary motif of length $(Q + 1)$ corresponds to a spin $Q/2$ representation of su(2), the full degeneracy of the sequences is then recovered by assuming that the spinons are spin half objects which behave as bosons in each orbitals.

The spinons are not bosons but "semions": they obey a half fractional statistics. This follows from the fact that the number of available orbitals varies with the total occupation number [6]. Indeed, at spinon number N_{sp}, the number of orbitals is $N_{orb} = (1 + \frac{N-N_{sp}}{2})$. Therefore, we have the statistical interaction :

$$g_{sp} = -\frac{\partial N_{orb}}{\partial N_{sp}} = 1/2$$

In the following section, we will describe how the fractional statistics of the spinons is encoded in the Yangian representation theory.

The spinon description of spectrum is very similar to the description of the excitations of the Heisenberg spin chain given by Faddeev and Takhtajan [7].

Note that the model is gapless. Its low energy properties belong to the same universality class as the Heisenberg model. The low energy, low temperature, behavior is described by the level one su(2) WZW conformal field theory. The spinon formulation of the Haldane-Shastry spin chain provides a new quasi-particle description of the states in the WZW model [8].

2 Yangians and Affine Hecke Algebras.

In this section we review few of the new results on integrable models and on the Yangian representation theory which emerged from the study of the long-range interacting models. But we first need to recall standard result concerning the algebraic Bethe ansatz, cf e.g. [9].

2.1 Algebraic Bethe ansatz and Yangians.

We introduce the basic notion of the algebraic Bethe ansatz, using the quantum Heisenberg chain as an example. We consider a chain of length N: on each site there is a spin variable σ_j. We denote by S_j^{ab}, $a, b = 1,\ 2$, the spin operators satisfying the su(2) commutation relations :

$$\left[\ S_j^{ab}\ ,\ S_k^{cd}\ \right] = \delta_{jk}\Big(\ \delta^{cb}\ S_j^{ad} - \delta^{ad}\ S_j^{cb}\ \Big) \tag{5}$$

The Heisenberg hamiltonian is :

$$H = \sum_{k=1}^{N} \sum_{ab} S_k^{ab} S_{k+1}^{ba} = \sum_{k=1}^{N} (P_{k,k+1} - 1) \tag{6}$$

Here, we have assumed periodic boundary conditions. As is well known, in order to preserve the integrability the spin operators S_k^{ab} should act on the spin half representation of su(2). So, the spin variables take only two values, $\sigma_j = \pm$, and the operator S_j^{ab} which acts only the j^{th} spin is represented by the canonical matrix $|a\rangle\langle b|$.

The algebraic Bethe ansatz goes in few steps.

•) The first step consists in constructing the local monodromy matrices $T_j(u)$. These matrices are 2×2 matrices whose elements $T_j^{ab}(u)$ are operators. The matrices $T_j(u)$ are defined by :

$$T_j^{ab}(u) = u\delta^{ab} + \lambda S_j^{ab} \tag{7}$$

where u is a complex number, called the spectral parameter, and λ a coupling constant. Note that the matrix $T_j(u)$ only acts on the j^{th} spin. The important point is that we can compute the commutation relations between its matrix elements. These relations can be gathered into the famous relations of the algebraic Bethe anstaz, see e.g. [9]:

$$R(u-v)(T(u)\otimes 1)(1\otimes T(v)) = (1\otimes T(v))(T(u)\otimes 1)R(u-v) \tag{8}$$

where $R(u)$ is Yang's solution of the Yang-Baxter equation, $R(u) = u - \lambda\ P$, with P the exchange operator $P(x\otimes y) = y\otimes x$.

••) The second step consists in constructing the complete monodromy matrix, which we denote by $T(u)$. It is obtained by taking the ordered product of the local monodromy matrices. Namely,

$$T^{ab}(u) \ = \ \sum_{a_2\cdots a_N} T_1^{aa_2}(u)\ T_2^{a_2a_3}(u)\ \cdots\ T_N^{a_Nb}(u) \tag{9}$$

It admits an $(\frac{1}{u})$-expansion:

$$u^{-N}\ T^{ab}(u) = \delta^{ab} + \frac{\lambda}{u}\left(\sum_k S_k^{ab}\right) + \frac{\lambda^2}{u^2}\left(\sum_{j<k}\sum_d S_j^{ad}S_k^{db}\right) + \cdots$$

The crucial point is the fact that the complete monodromy matrix (9) satisfy the relations (8) if the local monodromy matrices do. These relations are equivalent to the following quadratic commutation relations :

$$(u-v)\left[\ T^{ab}(u)\ ,\ T^{cd}(v)\ \right] = \lambda\left(T^{cb}(u)T^{ad}(v) - T^{cb}(v)T^{ad}(u)\right) \tag{10}$$

An important consequence of the relations (8) is that the transfer matrix $\mathcal{T}(u)$, which is the trace of the monodromy matrix, $\mathcal{T}(u) = tr(T(u)) = T^{11}(u) + T^{22}(u)$, is a generating function of commuting hamiltonians :

$$[\ \mathcal{T}(u)\ ,\ \mathcal{T}(v)\ ] = 0$$

The Heisenberg hamiltonian is recovered by expanding the logarithm of the trace to first order: $H \propto \partial_u \log \mathcal{T}(u)\Big|_{u=0}$.

Another generating function of commuting quantities is given by the quantum determinant $det_qT(u)$. It is defined by :

$$det_qT(u) = T^{22}(u-\lambda)T^{11}(u) - T^{21}(u-\lambda)T^{12}(u) \tag{11}$$

It commutes with all the matrix elements of the monodromy matrix : $\left[det_qT(u), T^{ab}(v)\right] = 0$.

The quadratic algebra (10) is called a su(2) Yangian [10]. More precisely, consider a T-matrix satisfying the commutation relations (8) or (10), and normalized to have a quantum determinant equal to one: $det_q T(u) = 1$. Assume that the T-matrix possesses a $(\frac{1}{u})$-expansion as follows :

$$T^{ab}(\lambda) = \delta^{ab} + \lambda \sum_{n=0}^{\infty} u^{-n-1} t^{ab}_{(n)} \tag{12}$$

Then, the su(2) Yangian is the associative algebra generated by the elements $t^{ab}_{(n)}$. For these elements, the relations (10) are equivalent to :

$$\begin{aligned} \left[t^{ab}_{(0)}, t^{cd}_{(m)}\right] &= \delta^{cb} t^{ad}_{(m)} - \delta^{ad} t^{cb}_{(m)} \\ \left[t^{ab}_{(n+1)}, t^{cd}_{(m)}\right] - \left[t^{ab}_{(n)}, t^{cd}_{(m+1)}\right] &= \lambda \left(t^{cb}_{(m)} t^{ad}_{(n)} - t^{cb}_{(n)} t^{ad}_{(m)} \right) \end{aligned} \tag{13}$$

Note that with the quantum determinant constraint, the $(\frac{1}{u})$-expansion of the monodromy matrix can be reconstructed from its two first components $t^{ab}_{(0)}$ and $t^{ab}_{(1)}$. The relations (13) clearly shows the Yangians are deformation of loop algebras.

• • •) The next step consists in diagonalizing the transfer matrix. The algebraic Bethe ansatz provides a way to perform this diagonalization inside a finite dimensional irreducible representation of the su(2) Yangian. Similarly as for the unitary representations of su(2), any finite dimensional irreducible Yangian representation is specified by an highest weight vector $|\Omega\rangle$. It is characterized by the following equations :

$$T(u)|\Omega\rangle = \begin{pmatrix} f_1(u) & 0 \\ \star & f_2(u) \end{pmatrix} |\Omega\rangle \tag{14}$$

where $f_1(u)$ and $f_2(u)$ are C-number functions, not operators. The product of these functions is related to the quantum determinant by : $det_q T(u) = f_2(u-\lambda) f_1(u)$. Due to the fact that the quantum determinant commutes with the T-matrix, only the ratio $f_1(u)/f_2(u)$ encodes the data of the representation. Moreover, the Yangian representation with highest weight vector $|\Omega\rangle$ is finite dimensional if and only if this ratio satisfies [10] :

$$\frac{f_1(u)}{f_2(u)} = \frac{P(u+\lambda)}{P(u)} \tag{15}$$

for some polynomial $P(u)$. These polynomials are called Drinfel'd polynomials. The condition (15) is the analogue of the fact that finite dimensional su(2) representations correspond to half integer spins.

All the states in an irreducible Yangian representation are obtained by iterative actions of $T^{21}(u)$ on $|\Omega\rangle$:

$$|\Psi\rangle = T^{21}(u_1)\, T^{21}(u_2) \cdots T^{21}(u_M)|\Omega\rangle \tag{16}$$

The Bethe states, which are eigenstates of the transfer matrix, are of this form, but for particular values of the parameters u_p. The relations determining these u_p's are called the Bethe ansatz equations. They can be summarized as follows. Let us define a polynomial $Q(u)$ of degree M whose roots are the u_p's :

$$Q(u) = \prod_{p=1}^{M} (u - u_p) \tag{17}$$

The state (16) is then an eigenstate of the transfer matrix $\mathcal{T}(u)$ if the roots u_p of $Q(u)$ are such that this polynomial is solution of the following difference equation :

$$t(u)Q(u) = f_1(u)Q(u-\lambda) + f_2(u)Q(u+\lambda) \tag{18}$$

where $t(u)$, a polynomial of degree N, is the eigenvalue of the transfer matrix on the Bethe state (16). Notice that eq. (18) at the same time gives the equations determining the Bethe roots u_p and the eigenvalue $t(u)$. Eq.(18) was introduced by Baxter in its solution of the 8-vertex model [11].

Following an idea due to Sklyanin [12], the Bethe eigenstates can then be rewritten in terms of the polynomial $Q(u)$. Since the operator $T^{21}(u)$ is a polynomial of degree $(N-1)$, let us assume that we can factorized it as follows,

$$T^{21}(u) = \lambda \, S^- \, . \prod_{k=1}^{N-1} (u - x_k) \tag{19}$$

where the x_k are operators and $S^- = \sum_j S_j^{21}$. It follows from the relations (10) that the x_k are commuting operators. The Bethe eigenstates (16) are then given by

$$|\Psi\rangle = \left(S^-\right)^M \; Q(x_1)\; Q(x_2) \; \cdots \; Q(x_{N-1})|\Omega\rangle \tag{20}$$

The eqs.(18,20) reflect the separation of the variables, since the eigenstates are determined from the solutions of one equation for a function of one variable only.

Finalizing the solution of the models consists in analyzing the Bethe ansatz equations and their solutions. This can analytically be done explicitely only in the thermodynamical limit.

2.2 Quantization of the spectral parameter and Dunkl operators.

The long-range interacting models cannot be solved using the algebraic Bethe ansatz. This follows from the fact the hamiltonian commutes with the T-matrix, and therefore non-degenerate eigenstates cannot be obtained by iterative action of the lowering operators $T^{21}(u)$. Nevertheless, the tools of the algebraic Bethe ansatz are useful for constructing integrable long range interacting models and for deciphering the symmetries of these models.

To illustrate this fact, we now consider su(2) generalizations of the Calogero-Sutherland models [1]. These models describe M particles interacting by long range forces. Their positions are parameterized by complex numbers z_i, $i = 1, \cdots, M$, and each particle carries a spin $\sigma = \pm$. The Hamiltonian is :

$$H_D = \sum_{j=1}^{M} (z_j \partial_{z_j})^2 - \sum_{i \neq j} \lambda (P_{ij} + \lambda) \frac{z_i z_j}{(z_i - z_j)^2} \tag{21}$$

where λ is a coupling constant and P_{ij} exchanges the spins of the particles i and j. Notice we recover the Haldane-Shastry spin chain in the static limit $\lambda = \infty$.

The construction of these models relies on the definition a monodromy matrix in which the spectral parameter has been quantized. More precisely, let us consider the monodromy matrix (9) but in which the spectral parameters have been shifted to $(u - \widehat{D}_i)$, where the $\widehat{D}_i$

are operators, commuting among themselves and with the spin operators. More precisely, we define a $\widehat{T}$-matrix by [13] :

$$\widehat{T}^{ab}(u) = \sum_{a_2\cdots a_N} \widehat{T}_1^{aa_2}(u)\, \widehat{T}_2^{a_2a_3}(u) \,\cdots\, \widehat{T}_N^{a_Nb}(u) \tag{22}$$

with

$$T_i^{ab}(u) = \frac{(u-\widehat{D}_i)\delta^{ab} + \lambda S_i^{ab}}{u-\widehat{D}_i} \tag{23}$$

The operators $\widehat{D}_i$ we consider are defined as follows [13] :

$$\widehat{D}_i = z_i\partial_{z_i} + \lambda\sum_{j>i}\theta_{ij}K_{ij} - \lambda\sum_{j<i}\theta_{ji}K_{ij} \tag{24}$$

where $\theta_{ij} = \frac{z_i}{z_i-z_j}$ and K_{ij} the operators which exchange the particles at positions z_i and z_j : $K_{ij}z_j = z_iK_{ij}$. They obey the defining relations of a degenerate affine Hecke algebra :

$$\begin{aligned} \left[\,\widehat{D}_i\,,\,\widehat{D}_j\,\right] &= 0 \\ \left[\,K_{i,i+1}\,,\widehat{D}_k\,\right] &= 0 \quad if\ k\neq i,i+1 \\ K_{i,i+1}\widehat{D}_i - \widehat{D}_{i+1}K_{i,i+1} &= -\lambda \end{aligned} \tag{25}$$

In the mathematics literature, the role of the affine Hecke algebra in this context was revealed by Cherednik [14]. In the physics literature, operators similar but different to the $\widehat{D}_i$ were introduced by Polykronakos [15]. Notice that these relations imply that :

$$\left[K_{ij},\widehat{\Delta}_M(u)\right] = 0, \quad with \quad \widehat{\Delta}_M(u) = \prod_{i=1}^{M}(u-\widehat{D}_i) \tag{26}$$

I.e. $\widehat{\Delta}_M(u)$ is symmetric by permutation of the particles. This property follows from $\left[K_{ii+1},(u-\widehat{D}_i)(u-\widehat{D}_{i+1})\right] = 0$, which is valid for all i.

Since the operators $\widehat{D}_i$ commute, the $\widehat{T}$-matrix (22) satisfies the RTT relation (8). However, the positions and the spin variables are totally uncoupled since the operators $\widehat{D}_i$ commute with the spin operators. In order to couple them, we define a projection π which consists in replacing the permutation K_{ij} by the permutation P_{ij} after it has been moved to the right of an expression. One can view this projection as the result of acting on wave functions totally symmetric under simultaneous permutations of the positions and of the spins. In more mathematical words, this procedure consists in quotienting the algebra generated by the permutations K_{ij} and P_{ij} by the left ideal generated by $(K_{ij} - P_{ij})$. We use it to eliminate the permutations of the particles by replacing them with those of the spins.

The transfer matrix $T(u)$ defined by

$$T(u) = \pi(\widehat{T}(u)) \tag{27}$$

will then satisfy the Yang-Baxter equation if we can replace the projection of the product $(1\otimes\widehat{T}(v))(\widehat{T}(u)\otimes 1)$ by the product of the projections. Since, $\widehat{\Delta}_M(u)$ is symmetric under permutation, it is equivalent to check this property for $\widehat{T}'(u) = \widehat{\Delta}_M(u)\widehat{T}(u)$. For this to be true, $\widehat{T}(u)$ applied on a totally symmetric wave function must still be a totally symmetric wave function. Equivalently, we must have:

$$\pi\left(K_{ij}\widehat{T}(u)\right) = P_{ij}\pi\left(\widehat{T}(u)\right) \tag{28}$$

Since the permutation groups are generated by the permutations $K_{i,i+1}$ and $P_{i,i+1}$, eq.(28) is equivalent to : $\pi\left(K_{i,i+1}\widehat{T}_i(u)\widehat{T}_{i+1}(u)\right) = P_{i,i+1}\pi\left(\widehat{T}_i(u)\widehat{T}_{i+1}(u)\right)$, with $\widehat{T}_i(u)$ defined in (23). This is garanteed if the commutation relations of the degenerate Hecke algebra (25) are satisfied. Thus, the relations (25) are the necessary relations for this T-matrix to satisfy the RTT-relation.

An alternative presentation of this T-matrix was obtained in ref.[13] :

$$T^{ab}(u) \;=\; \delta^{ab} + \lambda \sum_{i,j=1}^{M} S_i^{ab}\left(\frac{1}{u-L}\right)_{ij} \tag{29}$$

where L is the matrix defined by : $L_{ij} = \delta_{ij}z_j\partial_{z_j} + (1-\delta_{ij})\lambda\theta_{ij}P_{ij}$, with $\theta_{ij} = z_i/(z_i - z_j)$. In eq.(29), the projection π has been explicitely done.

The immediate consequences of this construction are the following. Since the T-matrix (27) satisfies the relation (8) it defines a representation of the su(2) Yangian. As explained in the previous section, the relation (8) implies that $\mathcal{T}(u) = tr(T(u))$ is a generating function of commuting hamiltonian. However, $\mathcal{T}(u)$ is not Yangian invariant since it does not commute with T itself. A clever choice consists in choising the quantum determinant $det_qT(u)$ as the generating function of commuting hamiltonians. It is the projection of the quantum determinant of $\widehat{T}(u)$:

$$det_qT(u) = \pi\left(\frac{\widehat{\Delta}_M(u+\lambda)}{\widehat{\Delta}_M(u)}\right) \tag{30}$$

where $\widehat{\Delta}_M(u)$ is defined in eq.(26). The hamiltonian (21) is the u^{-2}-term in (30). It is therefore Yangian invariant. The quantum determinant (30) has been diagonalized in ref.[13] by directly diagonalizing the operators $\widehat{D}_i$.

2.3 Application to the Haldane-Shastry spin chain.

We now explain how the previous construction can be used to derive the fractional selection rules satisfied by the eigenstates of the Haldane-Shastry spin chain.

As mentioned in section 2, the Haldane-Shastry spin chain is Yangian invariant. Therefore, there exists a T-matrix commuting with the hamiltonian (1) and satisfying the relations (8). It was constructed in [13]. It is the limit $u, \lambda \to \infty$ with $x = u/\lambda$ fixed, of the T-matrix (29). Its expression is :

$$T^{ab}(x) = \delta^{ab} + \sum_{i,j=1}^{N} S_i^{ab}\left(\frac{1}{x-L'}\right)_{ij} \tag{31}$$

with $L'_{ij} = (1-\delta_{ij})\theta_{ij}P_{ij}$, $\theta_{ij} = z_i/z_{ij}$ with $z_{ij} = z_i - z_j$, and S_i^{ab} is the canonical matrix $|a\rangle\langle b|$ acting on the i^{th} spin only. For any values of the complex numbers z_j, the transfer matrix (31) form a representation of the exchange algebra (10) with u changed into x and λ normalized to one. The trigonometric spin chain corresponds to $z_j = \omega^j$ with ω a primitive

N^{th} root of the unity. For these values of z_j, the transfer matrix (31) commutes with the hamiltonian (1).

In the representation (31), the quantum determinant is a pure number for any values of the z_j's given by :

$$det_q\, T(x) \;=\; 1+\sum_{i,j=1}^{N}\Big(\frac{1}{x-\Theta}\Big)_{ij} \;=\; \frac{\Delta_N(x+1)}{\Delta_N(x)} \tag{32}$$

with $\Delta_N(x)$ the characteristic polynomial of the $N\times N$ matrix Θ with entries θ_{ij}: $\Delta_N(x) = det(x-\Theta)$. For the Haldane-Shastry spin chain $z_j = \omega^j$ and we have :

$$\Delta_N(x) = \prod_{j=1}^{N}(x+\frac{N+1}{2}-j) \tag{33}$$

Since the monodromy matrix (31) commutes with the hamiltonian, the long-range interacting spin chain cannot be solved using the algebraic Bethe ansatz. A way to solve it consists first in decomposing the spin chain Hilbert space into irreducible sub-representation of the Yangian, and then in computing the energy in each of these irreducible blocks. For the values $z_j = \omega^j$, the representation (31) is completely reducible. Each irreducible sub-representation possesses a unique highest weight vector $|\Lambda\rangle$ which is annihilated by $T^{12}(x)$ and which is an eigenvector of the diagonal components of the transfer matrix, as in eq.(14). In ref.[13], it was shown that the corresponding eigenvalues of $T^{11}(x)$ and $T^{22}(x)$ can be expressed in terms of two polynomials $P(x)$ and $Q(x)$:

$$T(x)|\Lambda\rangle = \frac{Q(x+1)}{Q(x)}\begin{pmatrix}\frac{P(x+1)}{P(x)} & 0\\ \star & 1\end{pmatrix}|\Lambda\rangle \tag{34}$$

These polynomials characterize the irreducible sub-representations. The polynomials $Q(x)$ and $P(x)$ are not independent, since the quantum determinant (31) take the same value in any of the irreducible blocks. They should satisfy :

$$\Delta_N(x) = P(x)\; Q(x)Q(x-1). \tag{35}$$

Therefore, the roots of $P(x)$ and $Q(x)$ are among those of $\Delta_N(x)$. This implies that $Q(x)$ factorized as :

$$Q(x) = \prod_{p=1}^{M}(x+\frac{N+1}{2}-k_p) \tag{36}$$

where the $\{k_p\}$ are integers bewteen 1 and $(N-1)$. The equation (35) then admits solutions if and only if the roots of $Q(x)$ are not adjacent, or equivalently, if and only if the integers $\{k_p\}$ are neither equal nor adjacent. These integers will be identified with the rapidities labeling the eigenmultiplets of the spin chain.

This provides a purely algebraic way to recover the rapidity selection rule. It also shows that the fractional statistics of the spinon excitations is an echo of the Yangian symmetry.

3 A Yangian deformation of the W-algebras.

We begin this section by introducing different representations of the $su(n)$ loop algebra and of the $su(n)$ Yangian. We then show how they can be gathered into a representation of a bigger algebra, which we called a "Yangian W_∞ algebra" [17].

3.1 A set of operators.

For our purpose we define the following two operators:

$$J_0^{ab} = \sum_{j=1}^{N} E_j^{ab}, \tag{37}$$

$$J_1^{ab} = \sum_j E_j^{ab}\frac{\partial}{\partial x_j} - \lambda\sum_{j\neq k}(E_jE_k)^{ab}\frac{1}{x_j - x_k}. \tag{38}$$

where E_j^{ab} are the canonical $n \times n$ matrices acting on the j^{th} space. We have used the conventional notations, $(E_jE_k)^{ab} = \sum_{c=1}^{n} E_j^{ac}E_k^{cb}$. The generators J_0^{ab} and J_1^{ab} satisfy the following relations:

$$[J_0^{ab}, J_0^{cd}] = \delta^{bc}J_0^{ad} - \delta^{da}J_0^{cb}, \tag{39}$$

$$[J_0^{ab}, J_1^{cd}] = \delta^{bc}J_1^{ad} - \delta^{da}J_1^{cb}, \tag{40}$$

$$[J_0^{ab}, [J_1^{cd}, J_1^{ef}]] - [J_1^{ab}, [J_0^{cd}, J_1^{ef}]] = 0. \tag{41}$$

The third equation is known as the Serre relation for the loop algebra. These relations imply that the higher generators $J_{n>1}^{ab}$, which are defined recursively using the generator J_1^{ab}, form a representation of su(n) loop algebra,

$$[J_n^{ab}, J_m^{cd}] = \delta^{bc}J_{n+m}^{ad} - \delta^{da}J_{n+m}^{cb}. \tag{42}$$

Consider now the following set of generators as [13, 5],

$$Q_0^{ab} = J_0^{ab}, \tag{43}$$

$$Q_1^{ab} = \sum_j E_j^{ab}\Big(x_j\frac{\partial}{\partial x_j} + \frac{1}{2}\Big) - \frac{\lambda}{2}\sum_{j\neq k}(E_jE_k)^{ab}\frac{x_j + x_k}{x_j - x_k}. \tag{44}$$

We have the following commutation relations:

$$[J_0^{ab}, Q_1^{cd}] = \delta^{bc}Q_1^{ad} - \delta^{da}Q_1^{cb}, \tag{45}$$

$$\begin{aligned}&[J_0^{ab}, [Q_1^{cd}, Q_1^{ef}]] - [Q_1^{ab}, [J_0^{cd}, Q_1^{ef}]]\\ &= \frac{\lambda^2}{4}\Big([J_0^{ab}, [(J_0J_0)^{cd}, (J_0J_0)^{ef}]] - [(J_0J_0)^{ab}, [J_0^{cd}, (J_0J_0)^{ef}]]\Big).\end{aligned} \tag{46}$$

These relations together with equation (39) are the defining relations of the Yangian Y(su(n)). The second equation (46) is called the "deformed" Serre relation. It reduces to the Serre relation (41) for the loop algebra when $\lambda \to 0$. In this sense, the Yangian can be viewed as a "deformed" loop algebra. The relations (39) and (45-46) show that the generators Q_0^{ab} and Q_1^{ab} form a representation of the Yangian algebra Y(su(n)).

3.2 The Yangian Deformed W_∞ Algebra.

To combine the loop algebra J_n^{ab} and the Yangian algebra Q_n^{ab}, we introduce another set of generators K_n^{ab} :

$$K_n^{ab} = \sum_{j=1}^{N} E_j^{ab}x_j^n. \tag{47}$$

It is easy to see that the generators K_n^{ab} represent the su(n) loop algebra,

$$[K_n^{ab}, K_m^{cd}] = \delta^{bc} K_{n+m}^{ad} - \delta^{da} K_{n+m}^{cb}, \tag{48}$$

All the K_n^{ab} can be defined recursively from the two lowest generators,

$$K_0^{ab} = J_0^{ab}, \tag{49}$$
$$K_1^{ab} = \sum_j E_j^{ab} x_j. \tag{50}$$

By construction, they satisfy the relations (39-41) with J_n^{ab} replaced by K_n^{ab}.

Consider now the algebra generated by the elements $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$. The Yangian current Q_1^{ab} appears from an inter-relation formula between these operators;

$$[J_1^{ab}, K_1^{cd}] + [K_1^{ab}, J_1^{cd}] = 2\big(\delta^{bc} Q_1^{ad} - \delta^{da} Q_1^{cb}\big). \tag{51}$$

Besides the relation (46) for Q_1^{ab}, we also have the following Serre-like relations,

$$[J_0^{ab}, [J_1^{cd} + Q_1^{cd}, J_1^{ef} + Q_1^{ef}]] - [J_1^{ab} + Q_1^{ab}, [J_0^{cd}, J_1^{ef} + Q_1^{ef}]] = 0, \tag{52}$$
$$[J_0^{ab}, [K_1^{cd} + Q_1^{cd}, K_1^{ef} + Q_1^{ef}]] - [K_1^{ab} + Q_1^{ab}, [J_0^{cd}, K_1^{ef} + Q_1^{ef}]] = 0. \tag{53}$$

The relations (39-41), (45-46) and (51-53) possess an interesting interpretation: consider the generators $Q_1^{ab}(x,y)$ defined by

$$Q_1^{ab}(x,y) \equiv Q_1^{ab} + x\, J_1^{ab} + y\, K_1^{ab}, \tag{54}$$

for any complex numbers x and y. Then, all the previously written Serre relations can be summarized into the following compact equations :

$$[J_0^{ab}, Q_1^{cd}(x,y)] = \delta^{bc} Q_1^{ad}(x,y) - \delta^{da} Q_1^{cb}(x,y), \tag{55}$$

$$\begin{aligned}&[J_0^{ab}, [Q_1^{cd}(x,y), Q_1^{ef}(x,y)]] - [Q_1^{ab}(x,y), [J_0^{cd}, Q_1^{ef}(x,y)]] \\ &= \frac{\lambda^2}{4}\Big([J_0^{ab}, [(J_0J_0)^{cd}, (J_0J_0)^{ef}]] - [(J_0J_0)^{ab}, [J_0^{cd}, (J_0J_0)^{ef}]]\Big).\end{aligned} \tag{56}$$

In other words, the commutation relations between the generators J_n^{ab} and K_n^{ab} of the two loop subalgebras are such that the generators $Q_1^{ab}(x,y)$ form a representation of the Yangian for any x and y. We thus have an infinite number of Yangian subalgebras constructed from $Q_1^{ab}(x,y)$, but they all have λ as deformation parameter.

In the limit $\lambda \to 0$, the generators J_n^{ab} reduce to $J_n^{ab} = \sum_j E_j^{ab}(\partial_{x_j})^n$. Together with the operators K_n^{ab}, they generate a W_∞-algebra with elements,

$$Q_n^{(s)ab} = \sum_{j=1}^{N} E_j^{ab} x_j^{s-1} (\partial_{x_j})^{n+s-1}, \tag{57}$$

which satisfy the commutation relations,

$$\begin{aligned}[Q_n^{(s)ab}, Q_m^{(s')cd}] &= \delta^{bc} \cdot \sum_{k=0}^{n+s-1} \frac{(n+s-1)!(s'-1)!}{k!(n+s-k-1)!(s'-k-1)!} Q_{n+m}^{(s+s'-1-k)ad} \\ &\quad -\delta^{da} \cdot \sum_{k=0}^{m+s'-1} \frac{(m+s'-1)!(s-1)!}{k!(m+s'-k-1)!(s-k-1)!} Q_{n+m}^{(s+s'-1-k)cb}.\end{aligned} \tag{58}$$

As a consequence, this algebra is generated by the elements $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$. Moreover, it is easy to see that this W_∞-algebra possesses an infinite number of su(n) loop subalgebras.

For $\lambda \neq 0$, our algebra is naturally called a "Yangian deformed W_∞-algebra", and denoted $YW_\infty(su(n))$. The algebra includes the loop algebra, the Virasoro algebra [16], and the Yangian as subalgebras.

3.3 An infinite number of Yangian subalgebras.

We now analyze a little more the structure of the algebra. Let us first identify another Yangian subalgebra. Define another set of operators $\tilde{Q}_1^{ab}(h,\omega)$ by

$$\tilde{Q}_1^{ab}(h,\omega) = h^2 J_2^{ab} - \omega^2 K_2^{ab}, \tag{59}$$

where h and ω are arbitrary complex numbers. By direct computation, we see that the operators $\tilde{Q}_1^{ab}(h,\omega)$ constitute a representation of the Yangian since they satisfy the following relations:

$$[J_0^{ab}, \tilde{Q}_1^{cd}(h,\omega)] = \delta^{bc}\,\tilde{Q}_1^{ad}(h,\omega) - \delta^{da}\,\tilde{Q}_1^{cb}(h,\omega), \tag{60}$$

$$\begin{aligned} &[J_0^{ab}, [\tilde{Q}_1^{cd}(h,\omega), \tilde{Q}_1^{ef}(h,\omega)]] - [\tilde{Q}_1^{ab}(h,\omega), [J_0^{cd}, \tilde{Q}_1^{ef}(h,\omega)]] \\ &\quad = (\lambda h\omega)^2 \Big([J_0^{ab}, [(J_0J_0)^{cd}, (J_0J_0)^{ef}]] - [(J_0J_0)^{ab}, [J_0^{cd}, (J_0J_0)^{ef}]]\Big). \end{aligned} \tag{61}$$

Notice that the deformation parameter is now $2\lambda h\omega$.

This subalgebra is actually a simple example of a more general structure. As we now explain, in the Yangian deformed W_∞-algebra generated by $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$, there exists an infinite number of "slices" in which a Yangian subalgebra can be constructed.

To prove it, we need to introduce the Dunkl operators D_i for the Calogero model [15].

$$D_i = \frac{\partial}{\partial x_i} - \lambda \sum_{j:j\neq i} \frac{1}{x_i - x_j} K_{ij}. \tag{62}$$

where K_{ij} is the operator permuting the coordinates x_i and x_j: $x_i\,K_{ij} = K_{ij}\,x_j$. We have the commutation relations:

$$\begin{aligned} D_i\,K_{ij} &= K_{ij}\,D_j, &(63)\\ [D_i, D_j] &= [x_i, x_j] = 0, &(64)\\ [D_i, x_j] &= \delta_{ij}\,(1 + \lambda \sum_{l:l\neq i} K_{il}) - (1-\delta_{ij})\,\lambda\,K_{ij}. &(65) \end{aligned}$$

Introduce now the operators Δ_i defined by:

$$\Delta_i = (hD_i + \omega x_i + y)\,(h'D_i + \omega' x_i + y'). \tag{66}$$

They depend on the c-numbers h, ω, y and h', ω', y'. They satisfy

$$[\Delta_i, \Delta_j] = \lambda\,(h\omega' - h'\omega)\,(\Delta_i - \Delta_j)\,K_{ij}. \tag{67}$$

This relation allows us to construct a representation of the Yangian algebra. Following Ref. [13], we introduce a monodromy matrix $T(u)$ by

$$T^{ab}(u) = \delta^{ab} + \lambda\,(h\omega' - h'\omega) \sum_i \pi\Big(\frac{1}{u - \Delta_i}\Big)\,E_i^{ab}, \tag{68}$$

where π is the projection consisting in replacing K_{ij} by P_{ij} once the permutation K_{ij} has been moved to the right of the expression. As in the previous section, the matrix $T^{ab}(u)$ satisfies,

$$[T^{ab}(u), T^{cd}(v)] = \frac{\lambda(h\omega' - h'\omega)}{u-v}\Big(T^{cb}(u)\, T^{ad}(v) - T^{cb}(v)\, T^{ad}(u)\Big) \tag{69}$$

This is another presentation of the Yangian. Therefore, the matrix (68) forms a representation of the Yangian. As usual, the quantum determinant of $T(u)$ defines a generating function of commuting operators which all commute with the matrix $T(v)$ itself.

We thus have identified an infinite number of Yangian subalgebra in the deformed W_∞-algebra. They are parametrized by the complex number h, ω, y and h', ω', y'. Notice that their deformation parameters are $\lambda(h'\omega - h\omega')$. The previously discussed loop and Yangian subalgebras can be recovered as particular cases of this construction.

We essentially worked with a specific class of representations of the algebra. But the algebra can be defined abstractly as the associative algebra generated by the elements $\{J_0^{ab}, J_1^{ab}, K_1^{ab}\}$ with the appropriate Serre relations. So it is important to decipher the statements which are representation dependent from those which are true in the algebra. Also we did not discuss the Hopf algebra structure, if any, of our algebra.

References

[1] F. Calogero, J. Math. Phys. 10 (1969) 2191, 2197;
B. Sutherland, J. Math. Phys. 12 (1971) 251.

[2] F.D.M. Haldane, Phys. Rev. Lett. 60 (1988) 635 ;
B.S. Shastry, Phys. Rev. Lett. 60 (1988) 639.

[3] See, e.g., F.D.M. Haldane, in the proceedings of the 16th Taniguchi Symposium, eds. N. Kawakimi and A. Okiji, Springer Verlag 1994.

[4] V.I. Inozemtsev, J. Stat. Phys. 59 (1990) 1143.

[5] F.D.M. Haldane, Z. Ha, J. Talstra, D. Bernard and V. Pasquier, Phys. Rev. Lett. 69 (1992) 2021.

[6] F.D.M. Haldane, Phys. Rev. Lett. 67 (1991) 937.

[7] L.D.Faddeev abd L. Takhtajan, Phys. Lett. 85A (1981) 375.

[8] D.Bernard, D.Serban and V.Pasquier, preprint hep-th/9404050
P. Bouwknegt, A. Ludwig and K. Schoutens, preprint hep-th/9406020.

[9] L.D.Faddeev, *Integrable models in 1+1 dimensional quantum field theory*, Les Houches, Elsevier Science Publishers (1994)

[10] V.G. Drinfel'd, Funct.Anal.Appl. 20 (1988) 56.

[11] R.J. Baxter, *Exactly solved models in statistical mechanic*, New York Academic, (1982).

[12] E.K. Sklyanin, in "Integrable and superintegrable systems", 1990, ed. B. Kupershmidt, World Scientific.

[13] D.Bernard, M.Gaudin,F.D.M. Haldane and V.Pasquier, J.Phys.A 26 (1993) 5219.

[14] I.Cherednik, Invent. Math. 106 (1991) 411.

[15] A. Polykronakos, Phys. Rev. Lett. 69 (1992) 703.

[16] K. Hikami and M. Wadati, J. Phys. Soc. Jpn 62 (1993) 4203.

[17] D. Bernard, K. Hikami and M. Wadati, in "New developments of integrable systems and long-ranged interaction models", ed. M-L Ge and Y-S Wu, 1995, World Scientific.