

## PROBLEMS OF METHOD IN LEVI-CIVITA'S CONTRIBUTIONS TO HYDRODYNAMICS

PIETRO NASTASI & ROSSANA TAZZIOLI

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**ABSTRACT.** — Levi-Civita made important contributions to hydrodynamics: he solved D'Alembert's paradox, introduced the "wake hypothesis", deduced the general integral of any plane motion involving a wake, and gave a rigorous proof of the existence of the irrotational wave in a canal of finite depth. In this paper, we investigate Levi-Civita's results in this area, and connect them to the methods of the new theory of integral equations. Finally, we give some information on Levi-Civita's students. In our paper, we often use letters written by and addressed to Levi-Civita.

**RÉSUMÉ** (Problèmes de méthode dans les contributions de Levi-Civita à l'hydrodynamique)

Levi-Civita apporta des contributions remarquables à l'hydrodynamique; il a résolu le paradoxe de D'Alembert, introduit l'hypothèse du sillage, déduit l'intégrale générale d'un mouvement plan avec sillage et démontré de manière rigoureuse l'existence de l'onde irrotationnelle dans un canal de profondeur finie. Dans notre article, nous présentons les résultats de Levi-Civita dans cette discipline et en montrons le lien avec les méthodes de la nouvelle théorie des équations intégrales. Enfin, nous donnons quelques informations sur les étudiants de Levi-Civita. Dans notre article, nous employons souvent des lettres écrites par et adressées à Levi-Civita.

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## INTRODUCTION

In their well-known contribution to *Handbuch der Physik*, “The classical field theories”, Truesdell and Toupin [1960] give an overview of the history of hydrodynamics, without quoting either Levi-Civita’s paper on the “wake hypothesis” [Levi-Civita 1901] (see section 1) or his subsequent article in which this hypothesis leads to the general integral of any plane motion involving a wake, by means of an adequate conformal transformation [Levi-Civita 1907a]. However, they do often quote some of Levi-Civita’s students – mainly Umberto Cisotti<sup>1</sup> – who extended and deepened his results.

Other sources do reference Levi-Civita’s method [Levi-Civita 1907a]; for example, the classical works of Villat [1920; 1930] and Lamb [1932] on hydrodynamics, Gurevich’s treatise on the theory of jets [Gurevich 1966], and Weinstein’s paper on Levi-Civita’s contribution to the theories of jets and wakes [Weinstein 1975]. More recent expositions of the history of hydrodynamics do not quote Levi-Civita’s 1901 paper. There, he solved D’Alembert’s paradox – namely, if a solid body moves in a perfect fluid (originally motionless), then the resisting force acting on the body is always zero – using his wake hypothesis, and deduced the law for the resistance on a body due to the fluid. Levi-Civita assumes that a solid body moving in a fluid separates the fluid into two regions – one in front of the body and one behind it (the wake) – and that the separation surface is a discontinuity surface (see section 1 for details).

The wake hypothesis was well-known to Levi-Civita’s contemporaries, for instance Cisotti [1912a] and Villat [1918], who considered it very important for new and fruitful research. Today, it is Cisotti who tends to be referenced relative to D’Alembert’s paradox, since it was Cisotti who clarified and developed the ideas in Levi-Civita’s 1901 paper

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<sup>1</sup> Umberto Cisotti (1882-1946) was one of Levi-Civita’s students at the University of Padua, where he graduated in 1903. From 1907 to 1913 he was Levi-Civita’s assistant. He recalled his studies at the University of Padua in a letter to Levi-Civita on March 28, 1935: “The 29th of March 1903, in the morning, I was in your old study in via Altinate, in Padua, and you carefully looked over my dissertation. In the meantime, the door suddenly opened and your father, so distinctive and nice, with a cigar in his hand entered the room [...]” [Nastasi & Tazzioli 2003, p. 69-70]. Cisotti became full professor of rational mechanics at the Polytechnic of Milan. His main research fields concern plane hydrodynamics, which he (and his students) studied by means of complex variables.

[Levi-Civita 1901]. Anderson [1997, p. 252], however, provides a single (incorrect) reference to a nonexistent paper of Levi-Civita said to have been published in 1901 in the *Comptes rendus de l'Académie des sciences*. Anderson also quotes Levi-Civita in a free (and not very faithful) translation of lines from [Levi-Civita 1907a, p. 522].

Levi-Civita's method [Levi-Civita 1907a] is a fundamental contribution to hydrodynamics and provided the starting point for much research – research directly suggested by Levi-Civita to his students and research, as in the case of H. Villat and M. Brillouin, inspired by his papers. It is interesting to quote here what Olivier Darrigol recently wrote en passant on Levi-Civita on the basis of indirect reports (by Hadamard and Brillouin): “Jacques Hadamard [1903, pp. 355-361] gave a proof that surfaces of discontinuity cannot be formed in a perfect fluid as long as cavitation is excluded. This proof, however, does not exclude the growth of a pre-existing, tiny surface of discontinuity. Marcel Brillouin [1911] made this point, described the growth process, and extended the conformal methods of Kirchhoff, Rayleigh, and Levi-Civita to curved obstacles devoid of angular points.” [Darrigol 2002, p. 46, footnote 48]. As shown in sections 1 and 2 below, the two fundamental works by Levi-Civita [1901; 1907a] are based on the existence of this surface of discontinuity.

The other hydrodynamic subjects studied by Levi-Civita concern the theory of waves, where he restated intuitions and previous problems in terms of rigorous mathematical formalism. [Levi-Civita 1925] is his main work on the subject and concerns irrotational waves with finite amplitude. In particular, it deals with periodic waves that propagate without changing their shape. Levi-Civita deduced rigorous solutions instead of the (second-order) approximations obtained by Stokes and Lord Rayleigh. Their method did not lead naturally to further approximations and a fortiori did not prove the convergence of the approximation algorithm.

Lord Rayleigh, after a sequence of not quite satisfactory attempts, doubted the real existence of the phenomenon, that is, the rigorous solution of hydrodynamic equations corresponding to periodic and permanent waves (of Airy). However, towards the end of his life, he changed his mind because of new intuitions that made the existence of this wave type plausible from a physical point of view (as noted by Levi-Civita [1925, pp. 201-202]). Rayleigh was impressed by the results obtained

by Korteweg and de Vries, and especially by the celebrated Korteweg-de Vries equation<sup>2</sup>. In their paper, Korteweg and de Vries wrote: “[...] we find that, even when friction is neglected, long waves in a rectangular canal must necessarily change their form as they advance, becoming steeper in front and less steep behind. Yet since the investigations of De Boussinesq, Lord Rayleigh and St. Venant on the solitary wave, there has been some cause to doubt the truth of this assertion” [Korteweg & de Vries 1895, p. 422]. In one of his later works, Rayleigh [1917] studied the solitary wave again and made new calculations to the sixth order of approximation.

Levi-Civita developed and solved the problem with the utmost rigour, by means of a new approximating expression that he named the “stokian” (in honour of Stokes). In so doing, he then, in Lamb’s words, closed “an historic controversy” [Lamb 1932, p. 420] by representing both the exact outline of such waves and the mathematical equation linking height, length, transport, and velocity of propagation of waves by means of simple formulae. His students broached and solved many other problems on waves.

We plan to investigate Levi-Civita’s contribution to this area of fluid mechanics in detail in a subsequent work. In the present paper, we mainly consider Levi-Civita’s contribution to hydrodynamics in the first period of his scientific career, when he taught at the University of Padua (from 1892 to 1918). In the first part of our paper (sections 1, 2, 3), we consider the wake hypothesis, Levi-Civita’s method, and some further developments. In the second part (section 4), we concentrate on the role of Dini’s formula and – more generally – of the theory of integral equations in solving hydrodynamic problems concerning wake. Finally, we add some information on Levi-Civita’s school in the concluding remarks (section 5).

Dini’s formula – which connects the values of a function  $f$  on the circumference of a circle with the values of its normal derivative on the same circumference, if  $f$  is assumed to be harmonic in the circle – is an analytical relation which seems far from any hydrodynamic application. In reality, some questions of hydrodynamics (for example, certain

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<sup>2</sup> See [Blij 1978], [Darrigol 2003] on the Korteweg-de Vries equation and its history.

two-dimensional problems involving a wake) can be reduced to a particular conformal mapping by means of the theory of complex analysis and, in particular, using Dini's formula. A deeper analysis of Dini's formula has allowed us to draw attention to a new approach to the study of the partial differential equations of mathematical physics. This new study developed from the first decade of the 20-th century onward, and used the theory of integral equations. In fact, some students and correspondents of Levi-Civita, also influenced by his ideas, considered the theory of integral equations as the best approach to the study of the equations of mathematical physics. In this paper, we also aim to show that there were close connections between hydrodynamic problems and the emergence of the new methods of integral equations.

### 1. D'ALEMBERT'S PARADOX

In 1901, Levi-Civita published "Sulla resistenza dei mezzi fluidi", a paper that left an important mark on the history of hydrodynamics. It is part of a letter by Levi-Civita to Francesco Siacchi (1893-1907), who communicated it to the *Accademia dei Lincei*. By using his hypothesis concerning the wake, Levi-Civita was able to overcome all the theoretical difficulties connected with the so-called D'Alembert paradox: namely, if one assumes that in a perfect fluid a body produces a continuous motion, then – as a consequence of Bernoulli's theorem - the resistance on the body due to the fluid will be zero for any shape of the body. Many scientists pointed out that the underlying assumptions were illegitimate and probably responsible for D'Alembert's paradox: fluids were supposed to be ideal and without friction. But not everyone shared this idea.

Helmholtz [1868] assumed that the region between the wake and the region outside of it was a discontinuity surface formed at any sharp angle of the walls along which the fluid moved<sup>3</sup>. Kirchhoff [1869] and Rayleigh [1876] developed the dead-water theory of resistance according to which the body in motion drags behind it an infinite liquid column that moves with it. Therefore, there are two different regions in the fluid – the wake and the region outside of it – which are divided by a (vortical) surface of

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<sup>3</sup> On the prehistory of discontinuity surfaces, see [Darrigol 2002].

discontinuity for the velocity, since between fluid and wake a discontinuous change of velocity occurs. Therefore, one of the assumptions necessary for the validity of D'Alembert's paradox is abandoned. The length of the wake is supposed to be infinite, otherwise D'Alembert's paradox is valid again, as Marcel Brillouin [1911] had proven rigorously. Brillouin's proof is valid in two dimensions and was extended to three dimensions by Duhem [1914].

Levi-Civita wrote in his paper: "It seems to me that Newton's law on incompressible fluids (the resistance is proportional to the square of the velocity  $v$ ) can be theoretically deduced, without stepping outside of pure hydrodynamics"<sup>4</sup>. His paper is devoted precisely to proving such a claim. Levi-Civita remarked that by experience a body moving "fairly rapidly" in a fluid drags after it a fluid column, which creates a surface of discontinuity. More specifically, Levi-Civita assumed the following [Levi-Civita 1901, pp. 130-131]:

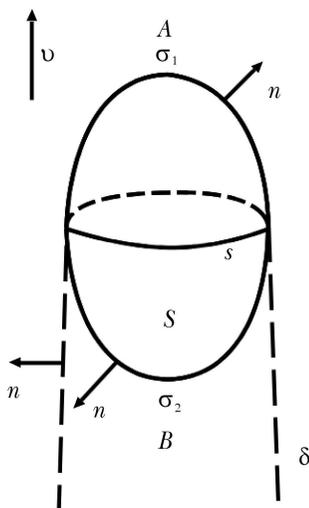


Figure 1: redrawn from [Levi-Civita 1901, p. 131]

"1) The motion of the fluid produced by the body  $S$  [stationary with respect to  $S$ ] has a surface of discontinuity  $\delta$  behind the body; such a surface extends for an infinite distance behind the body and starts from a certain curve  $s$  on the boundary  $\sigma$  of  $S$ .

<sup>4</sup> [Levi-Civita 1901, pp. 130]. The translations from Italian and French into English are ours.

2) The particles of the fluid of the region  $B$  behind the body behave as if they were rigidly connected to  $S$ .

3) The motion of the fluid in the region  $A$  [in front of the body] is irrotational, and satisfies the usual conditions at infinity" [See Figure 1].

Therefore, there are two different regions in the fluid – inside and outside the wake – separated by a surface of discontinuity. By expressing his three hypotheses in analytical terms and by appealing to physical evidence, Levi-Civita deduced a formula for the resistance in accordance with Newton's law.

As Levi-Civita himself subsequently wrote, "the resistance is thus fundamentally distinct from both viscosity and friction, which are to be considered as secondary dissipative phenomena. One should be able to explain resistance without appealing to such phenomena, which are to be considered only in concrete applications by calculating – or at least by approximately estimating – the corresponding corrective terms" [Levi-Civita 1907a, p. 520]. Thanks to his wake hypothesis, which is according to Villat "the only acceptable interpretation of the natural phenomena" [Villat 1918, p. 47], Levi-Civita was then able to overcome D'Alembert's paradox and to express the resistance by a law that accorded with the physical data.

The relevance of Levi-Civita's paper is corroborated in letters from Jacques Hadamard (1865-1963). Writing from Paris on April 19, 1902 Hadamard noted that

"I have received with great pleasure your work on electricity, 'L'influenza d'un schermo, etc.' and thank you heartily. It arrived at the moment when I was going to write to you about another one of your notes, which I received in 1901 and which interested me highly, the one entitled 'Sulla resistenza dei mezzi fluidi'.

I am indeed now editing the courses I gave in 1898–99 and 1899–1900<sup>5</sup>, which were based exactly on the necessity (absolutely general necessity, not depending merely on d'Alembert's paradox) to take into account discontinuities in the motion of gases. On this occasion, if I have the time for it, I intend to take up again the study of the problems to which you draw attention in your work. However, there is one case in which I cannot agree with you: that of liquids. It seems to me that the true theory of the phenomenon cannot be found (for liquids) in discontinuities of the kind you introduce. Indeed,

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<sup>5</sup> From these courses originated his *Leçons sur la propagation des ondes et les équations de l'hydrodynamique* [Hadamard 1903].

these discontinuities should propagate being affected at different moments by different molecules (unless your surface  $\delta n'$  has the shape of a cylinder with generators parallel to the motion), which is impossible in the case of liquids.

On the contrary, I believe like you that in the case of gases one must introduce the discontinuities. In the case of gases, I should be very grateful if you could provide me with some bibliographic information. The case of liquids is dealt with everywhere or almost, but in that of gases there must be more than the very sparse ones I know about. In this case, could you inform me about them? You would do me a great favour”<sup>6</sup>.

Unfortunately, we do not have Levi-Civita’s answer, but we report what Hadamard wrote in a subsequent, undated letter:

“Dear Sir,

You are perfectly right: the objection to your hypothesis about the existence of a discontinuity in the case of liquids is a simple inadvertence on my part. Apparently, there is no reason that a discontinuous motion of the kind you consider should not be possible.

At most I would be tempted not to consider it (once we suppose it established) a full solution to the question, because it would remain to know how it was created and in particular – a difficulty that does not exist for gases – why the continuous motion, which is compatible with all the conditions of the problem, is not actually brought about. Above a certain value for the velocity one understands quite well that this is so. But up to that, that is, for sufficiently small velocities?

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<sup>6</sup> [Nastasi & Tazzioli 2003, pp. 151-152]: “J’ai reçu avec grand plaisir votre travail d’Electricité ‘l’Influenza d’un schermo’, etc. et vous en remercie vivement. Il m’est arrivé au moment où j’allais vous écrire relativement à une autre de vos notes, reçue en 1901 et qui m’a tant intéressé, celle qui est intitulée ‘Sulla resistenza dei mezzi fluidi’.

Je suis en effet en train de rédiger mes cours professés en 1898-99 et 1899-1900, lesquels ont précisément pour base la nécessité (absolument générale, et non point relative au seul paradoxe de d’Alembert) de faire intervenir les discontinuités dans le mouvement des gaz. A cette occasion, je compte, si j’en ai le loisir, reprendre l’étude des problèmes sur lesquels vous appelez l’attention dans votre travail. Seulement, il est un cas dans lequel je ne puis être d’accord avec vous : c’est celui des liquides. La véritable théorie du phénomène ne me paraît pas pouvoir être cherchée (pour les liquides) dans les discontinuités de l’espèce que vous introduisez. Car ces discontinuités devraient se propager, affectées, à des moments différents, des molécules différentes (à moins que votre surface  $\delta n'$  ait la forme d’un cylindre à génératrices parallèles au mouvement) ce qui est impossible dans le cas des liquides.

Au contraire, dans le cas des gaz, je crois avec vous qu’il y a lieu d’introduire les discontinuités. Sur ce cas de gaz, je vous serais bien reconnaissant si vous pouviez me fournir quelques indications bibliographiques. Le cas des liquides est traité un peu partout, mais sur celui des gaz il doit y avoir d’autres travaux que les très rares que je connais. Pourriez vous, dans ce cas, me les indiquer? Vous me rendriez grand service”.

It is true that, for those, the matter perhaps presents itself in conformity with d'Alembert's paradox. I do not know whether experiments have yielded data on this point.

I therefore ask you please not to give any importance to my previous letter and to believe in my very friendly feelings"<sup>7</sup>.

Again, unfortunately, we do not have Levi-Civita's response, but in another letter (on February 19, 1903), Hadamard asked Levi-Civita some questions, the answers to which form the subject of Levi-Civita's well-known 1907 paper [Levi-Civita 1907a] (see § 2). Hadamard wrote:

"Dear Sir,

I always intended to write to you about the question which occupies us, and your letter renews my regret not to have done it.

As you may have seen, in my note I have taken care to exclude objections like those which you raise. These, evidently, are those to which one should have recourse in order to eliminate the difficulty. Only, then, can we no longer, at least given the present state of science, base ourselves on mathematical reasoning? It becomes a little bit, as you say very well, a matter of faith or, if you prefer, of intuition.

The phenomena presented by fluids and which rational hydrodynamics fails to explain, or at least the most important of them, seem to be:

1° friction. Friction intervenes in the formation of vortices. I do not believe it necessary to appeal to it in order to explain slidings, since friction is known to destroy these slidings when they exist.

2° and above all – the mixing of layers, local turbulence or *swirls*, by which originally separate molecules meet, and others originally in contact separate: all of this disregarding sliding. For my part, I have no doubt that this phenomenon plays an important role in [the case in point].

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<sup>7</sup> [Nastasi & Tazzioli 2003, pp. 152-153]: "Cher Monsieur,

Vous avez parfaitement raison : la critique faite à votre hypothèse sur l'existence d'une discontinuité dans le cas des liquides est une simple inadvertance de ma part. Il n'y a, semble-t-il, aucune raison qu'un mouvement discontinu de la nature de ceux que vous considérez ne soit pas possible.

Tout au plus serais-je tenté de ne pas la considérer (en le supposant une fois obtenu) comme résolvant entièrement la question, parce qu'il resterait à savoir comment il naîtrait et surtout – difficulté qui n'existe pas pour les gaz – pourquoi le mouvement continu, qui est compatible avec toutes les conditions du problème ne se produit pas en réalité. A partir d'une certaine valeur de la vitesse, on comprend très bien qu'il en est ainsi. Mais jusque là, je veux dire pour les vitesses suffisamment petites ?

Il est vrai que, pour celles-ci, la chose se présente peut être conformément au paradoxe de d'Alembert. Je ne sais si l'expérience a fourni des données sur ce point.

Veuillez donc je vous prie, n'attacher aucune importance à ma lettre précédente et croire à mes sentiments très amicaux".

Does it lead to the creation of a fluid pump like the one you consider? It is possible. But after all I do not find this very certain<sup>a</sup>, and since, on the other hand, the creation of swirls would probably be sufficient *on its own* to eliminate the paradox, the most urgent to explain would perhaps be how they are created. All of it is very complicated.

Please accept, dear Sir, the assurance of my best feelings

J. Hadamard

<sup>a</sup> However, M. Marey tells me that, in his experiments, he finds a fluid pump, with exchange between this pump and the surrounding fluid due to fluid curls. This, as you know, agrees fairly well with your hypothesis

To some of his students, Levi-Civita suggested hydrodynamic problems related to the wake hypothesis and to D'Alembert's paradox. One of them – Cisotti – extended the validity of D'Alembert's paradox to a body with any shape [Cisotti 1905], to any fluid (perfect and not subjected to external forces) [Cisotti 1906], and to the case where the fluid moves in a cylindrical pipe [Cisotti 1909]. The wake hypothesis, as well as Levi-Civita's method (see § 2) and further developments due to many authors, were eventually organized systematically in Cisotti's book *Idromeccanica*

<sup>8</sup> [Nastasi & Tazzioli 2003, pp. 153-154]: "Cher Monsieur,

J'avais toujours l'intention de vous écrire à propos de la question qui nous occupe, et votre lettre vient renouveler le remords de ne pas l'avoir fait.

Comme vous avez pu le voir, j'ai eu soin, dans ma note, de réserver les objections du genre de celles que vous soulevez. Ce sont bien elles, évidemment, auxquelles il faut recourir pour lever la difficulté. Seulement, alors, nous ne pouvons plus, au moins dans l'état actuel de la Science, nous appuyer sur le raisonnement mathématique? Cela devient un peu, comme vous le dites fort bien, une affaire de foi ou, si vous le voulez, d'intuitions.

Les phénomènes présentés par les fluides et qui échappent à l'Hydrodynamique rationnelle, ou, du moins, les plus importants d'entre eux, paraissent être :

1° le frottement. Celui-ci intervient dans la production de tourbillons. Je ne crois pas qu'il y ait lieu de l'invoquer pour expliquer la production de glissements, puisqu'on sait que le frottement détruit ces glissements quand ils existent.

2° et surtout, – le mélange des couches, le tourbillonnement local ou remous, par lequel des molécules primitivement séparées se rejoignent et d'autres primitivement en contact se séparent : tout cela abstraction faite du glissement. Je ne doute pas, quant à moi, que ce phénomène ne joue un rôle important en [l'espèce].

Aboutit-il à la production d'une pompe fluide comme celle que vous considérez? C'est possible. Mais cela ne me semble pas, en somme, très certain<sup>a</sup>, et comme, d'autre part, la production de remous suffirait probablement à elle seule, à lever le paradoxe, le plus pressé serait peut être d'expliquer comment ils se produisent. C'est toute chose compliquée.

Agrérez, cher Monsieur, l'assurance de mes meilleurs sentiments

J. Hadamard

<sup>a</sup> M. Marey me dit toutefois que, dans ses expériences, il constate une pompe fluide, avec échange entre elle et le fluide ambiant par l'intermédiaire de volutes liquides. Ceci, comme vous savez, est assez d'accord avec votre Hypothèse".

*piana* [Cisotti 1921], which is a valuable compendium of the state of knowledge on fluid mechanics in the first decades of the 20-th century.

## 2. LEVI-CIVITA'S METHOD

Levi-Civita's paper, "*Scie e leggi di resistenza*" ("Wakes and laws of resistance"), published in the *Rendiconti del Circolo matematico di Palermo* [Levi-Civita 1907a], is a classic in the history of hydrodynamics and greatly stimulated analytical studies of hydraulic problems. His deductions are limited to two dimensions and to incompressible fluids, and are based on his wake hypothesis [Levi-Civita 1901]. Levi-Civita found the general integral of a plane motion with wake – if the shape of the body moving in the fluid is polygonal or curvilinear – based solely on the conditions of continuity and differentiability except at an angular point  $O$  called the bow.

Levi-Civita's method is founded on complex analysis and is contained in nuce in works by Kirchhoff [1869] and Rayleigh [1876] and developed by N. Joukowski [1890]. However, their procedure is not very rigorous and can be applied to certain particular cases only – such as the case of polygonal boundaries. "I am very pleased", Levi-Civita wrote to Villat in a letter on June 19, 1911, "that the perfection of the Helmholtz-Kirchhoff method which I have indicated has given rise to many scientific studies"<sup>9</sup>.

In fact, Levi-Civita extended Kirchhoff's method to a tunnel with curved walls and uniform current, and established the resistance law for any boundary (also curvilinear), "by studying the problem *ab initio* and deeply analyzing its mathematical character" [Levi-Civita 1907a, p. 521]. More in detail, let  $C$  be the body,  $B$  the wake moving with it – that is, the wake is fixed with respect to the axes  $x$ ,  $y$  – and  $A$  the region in front of  $C$  where the motion of the fluid is stationary and irrotational (see Figure 2). Therefore, in region  $A$  the velocity potential  $\varphi(x, y)$  of the fluid is such that

$$(1) \quad d\varphi = u dx + v dy.$$

<sup>9</sup> "Je me réjouis infiniment que le perfectionnement de la méthode de Helmholtz-Kirchhoff signalé par moi, ait donné l'essor à bien des recherches savantes." Levi-Civita's letters to Villat are held in Dossier Villat (Archives de l'Académie des sciences, Paris) and published in [Nastasi & Tazzioli 2003, pp. 371-410].

Since the fluid is incompressible ( $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ ), the function  $\varphi$  is harmonic, and it is possible to define a function  $\psi$  (the stream function) associated with  $\varphi$  and such that:

$$(2) \quad d\psi = u - v dx + u dy.$$

If one considers the complex variable  $z = x + iy$  over the plane of motion and posits

$$(3) \quad w = u - iv; \quad f = \varphi + i\psi,$$

then from (2) and (3),  $f$  is an analytic function of  $z$  in the region  $D$  satisfying

$$\frac{df}{dz} = w = e^{-i\Omega} = e^{-(\theta + i\tau)}.$$

In each point the velocity has the value  $V = e^\tau$  and makes the angle  $\tau$  with the  $x$ -axis. If the function  $\Omega$  of  $f$  is known, then the equation

$$dz = w = e^{i\Omega} df$$

allows for the determination of the motion of the fluid.

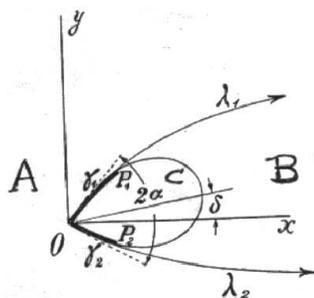


Figure 2: [Pistoiesi 1932, p. 342]

From known results of complex analysis, the domain  $D$  of the plane  $f$  can be mapped conformally onto a region  $D'$  of a new auxiliary plane  $t$ . “The success of the method”, wrote Villat, “depends on the choice of  $D'$ , which should be made in such a way that the determination of the function  $\Omega(t)$  is possible on it. T. Levi-Civita has made a great step forward, by making this representation on the area of a semicircle of

the plane  $t$ , in such a way that the edges of the wake correspond to the horizontal diameter that delimits the semicircle”<sup>10</sup>.

In order to deduce such a conformal mapping (“representation”), Levi-Civita considered the equation:

$$f = a^2 \left[ \cos s_0 - \frac{1}{2} \left( t + \frac{1}{t} \right) \right]^2,$$

where  $a$  is a constant, and  $s_0$  is a fixed angle between 0 and  $p$  such that  $e^{is_0}$  corresponds to the point  $O$  in Figure 2. This choice allows Levi-Civita to obtain, by assuming  $O$  to be the unique singular point, the more general expression of  $\Omega(t)$ :

$$\Omega(t) = \delta + \frac{2\alpha i}{\pi} \log \frac{i(t - e^{is_0})}{1 - te^{is_0}} + c_0 + c_1 t + c_2 t^2 + \dots + c_n t^n + \dots$$

( $\alpha, \delta$  are such that the tangents to the lines  $\lambda_1, \lambda_2$  in the point  $O$  make the angles  $\delta = \pm\alpha$  with the  $x$ -axis (see Figure 2)) under the conditions:

$$c_0 = \frac{2\alpha}{\pi} \left( s_0 - \frac{\pi}{2} \right)$$

$$c_0 + c_1 \cos s_0 + c_2 \cos 2s_0 + \dots + c_n \cos ns_0 + \dots = 0$$

All the elements of the motion can be easily found. Finally, by using complex analysis, and in particular Cauchy’s theorem, Levi-Civita was able to express the resistance of components ( $P_x, P_y$ ) due to the motion of the body in a very simple way:

$$P_x + iP_y = \frac{\pi a^2}{4} \Omega^2(0) + i \frac{i\pi a^2}{2} \left[ 2 \cos s_0 \Omega'(0) - \frac{1}{2} \Omega''(0) \right].$$

In the paper, no actual cases are dealt with, and it is generally difficult to find an explicit mathematical relation between the function  $\Omega(t)$  and the shape of the body, which is supposed to be known. At any rate, in the case when  $\Omega = 0$ , it is possible to deduce the motion of an angular contour via published results<sup>11</sup>. Any shape (and the resistance which

<sup>10</sup> [Villat 1918, pp. 48-49]: “Le succès de la méthode dépend du choix de  $D'$ , qu'on doit faire de telle manière que la détermination de la fonction  $\Omega(t)$  y soit possible. T. Levi-Civita a introduit un grand progrès, en imaginant de faire cette représentation sur l'aire d'un demi-cercle du plan  $t$ , de manière que les bords du sillage correspondent au diamètre horizontal qui délimite le demi-cercle”.

<sup>11</sup> In fact, M. Rethy in 1879 and D. Bobileff in 1881 had already found the resistance law of a bow made by two segments having equal length and touching in their extreme points. For details, see [Lamb 1924, p. 96].

follows from it) corresponds to a particular expression of  $\Omega$ . Even if he did not completely solve the general problem, Levi-Civita did succeed in introducing more mathematical rigour in an important field of hydrodynamics.

In fact, many of Levi-Civita's works were devoted to making hydrodynamics more rigorous. For example, his paper on the wake hypothesis [Levi-Civita 1901] eliminated d'Alembert's paradox, which had threatened the foundations of hydrodynamics. Another paper, published by Levi-Civita in 1905, proceeded along the same line [Levi-Civita 1905], aiming to solve mathematically some fundamental problems on the contraction of liquid veins. Many "experimental studies" had been developed on the subject, which were "exhaustive" from the point of view of hydraulics; however, a proper "theoretical" treatment of the phenomenon was missing. "Even the theorem of existence is missing", Levi-Civita [1905, p. 459] remarked, "and the quantitative solutions cannot be found, not even with approximate procedures." It is true, Levi-Civita pointed out in a footnote, that Stokes and Helmholtz had worked on two-dimensional motions and had elaborated a "complete" theory of the "fluid veils" going out from "sufficiently long" fissures, but their results did not necessarily contain a realistic approximation for outflows with closed boundary. In this note, Levi-Civita rigorously established that the coefficient of contraction can be made less than  $\frac{1}{2}$ , if the orifice is provided with a suitable (divergent) nozzle. Cisotti also devoted a paper [Cisotti 1908] to such a problem, where he used the method and results obtained by Levi-Civita [1905; 1907a] to deduce the general integral of irrotational fluids passing through an orifice.

Levi-Civita published another note in the *Comptes rendus* of the Paris Academy of Sciences (1913) on the *foundations* of hydrodynamics, in which he proved Torricelli's theorem in a more general case. The velocity of the flow of a heavy liquid through a (small) orifice is expressed, via Torricelli's theorem, as

$$(4) \quad v^2 = 2gh,$$

$h$  being the level of the orifice under the free surface of the liquid [Levi-Civita 1913, p. 385].

However, Levi-Civita remarked, the motion is assumed to be stationary in all classical proofs. "To my knowledge, nobody has shown that (4)

should also be valid for the beginning of the motion, that is, at the moment where the flow begins by the sudden opening of an orifice in the wall of a recipient containing liquid at rest" [Levi-Civita 1913, p. 385]. Levi-Civita was able to prove that (4) rigorously defined the initial velocity of the fluid for any elements  $d\Omega$ , where  $\Omega$  represents the size of the orifice. Robert Mazet (1903-1991), who studied with Painlevé and Villat in Paris, was then a Rockefeller student in Rome, in 1926 and 1927, and took up a similar subject in his dissertation [Mazet 1929], which was suggested (and supervised) by Levi-Civita<sup>12</sup>.

### 3. BRILLOUIN AND VILLAT

"The method so brilliantly developed by Levi-Civita [Levi-Civita 1907a] stimulated other authors to formulate many problems in hydrodynamics, always leading to specific results with accuracy, clarity, and elegance" [Cisotti 1912a, p. 493]. By following these researches, many students of his *school* devoted themselves to the systematic treatment of liquid jets and, in particular, to the treatment of outflow problems, motion with wake in a channel, the derivation of channels, the bifurcation of liquid veins, the motion of currents moving between a rigid wall and a free water surface and the confluence of two (or more) jets. The first part of Cisotti's book [Cisotti 1921] deals with all these subjects.

Brillouin and Villat, in particular, improved Levi-Civita's method by making it easier to apply in actual cases. In his classic paper, Marcel Brillouin (1854-1948) required that the arbitrary function  $\Omega$  satisfy certain physical conditions – which were expressed analytically – in order to solve the hydrodynamic problem. Levi-Civita's method was indeed too general; in fact, it also applied to cases which were not possible from the physical point of view. In order to avoid any paradox (negative pressure in the fluid), Brillouin assumed the stream of the fluid to be faster than the solid body moving in the fluid.

Henri Villat (1879-1972) extended such researches in many papers, mainly published in the *Comptes rendus* between 1910 and 1913 [Villat 1911c;d; 1912b]. He often used the following procedure: conformally

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<sup>12</sup> On Levi-Civita's influence on Mazet's dissertation, see the letters by Mazet to Levi-Civita in [Nastasi & Tazzioli 2003, pp. 195-203].

map the region  $D$  of the function  $f$  (the notations are from [Levi-Civita 1907a], see § 2) over a new region  $D'$ , where  $D'$  is a circular semi-annulus whose semi-circumferences represent the two lines of discontinuity of the wake [Villat 1911b; 1912a]. In this way, Villat extended in his thesis [Villat 1911a] Levi-Civita's method and established the general integral of a permanent plane motion in a fluid bounded by an infinite wall, and where a given body is plunged. Villat's formula is expressed in terms of elliptic functions, and becomes Levi-Civita's formula when the distance between obstacle and wall tends to infinity. Villat also introduced a new function, which gives  $\Omega$ , and has a "close and evident" connection with the shape of the obstacle plunged in the fluid [Villat 1918, p. 50]. In this way, Villat completed Levi-Civita's theory.

Villat [1913a] again employed a similar method – to map the given region over a circular crown conformally – to solve a problem of existence and uniqueness concerning the equations of hydrodynamics. "Is this solution unique?", Villat [1913a, p. 442] asked of Levi-Civita's solution, which gives the motion of a solid body plunged in an ideal fluid [Levi-Civita 1907a]. "In other words, do the equations for the permanent motion of an irrotational fluid in the presence of a given obstacle possess a well-determined solution, for which the velocities are continuous almost everywhere (except perhaps for certain surfaces of discontinuity), and for which the pressure is continuous and everywhere positive?"<sup>13</sup>

Villat [1914; 1920] proved that hydrodynamic equations can lead to more than one solution if, for example, the shape of the obstacle is made by two concurrent segments with concavity towards the current (see Figures 3a, 3b). In such a case, there are two different solutions: in the first, the current laps the obstacle, while in the second, the current goes away from the obstacle and then reaches it again, leaving a region of dead fluid between its concave boundary and a line of discontinuity  $\lambda$  ( $O$  is the dead point). Both solutions are possible, and neither is better a priori than the other from the physical point of view.

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<sup>13</sup> [Villat 1913a, p. 442-443]: "En d'autres termes, les équations du mouvement permanent d'un fluide irrotationnel en présence d'un obstacle donné, ont-elles une solution bien déterminée, pour laquelle les vitesses soient continues presque partout (exception faite peut-être pour quelques surfaces de discontinuité) et pour laquelle la pression soit continue et partout positive?"

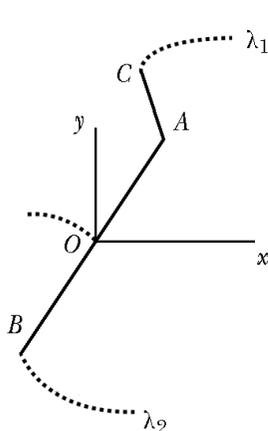
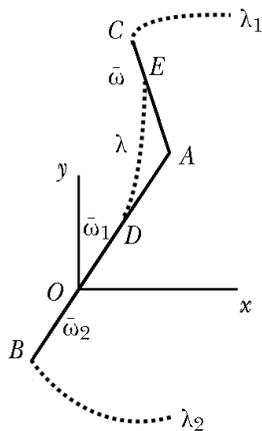


Figure 3a [Villat 1920, p. 94]


 Figure 3b (*Ibidem*)

René Thiry – one of Villat's students – developed the research concerning the uniqueness of the solution in his thesis, and published his main results in the *Annales de l'École normale supérieure* [Thiry 1921]. He proved that under certain conditions the solutions are infinite and constitute a continuous succession between the two solutions obtained by Villat.

#### 4. DINI'S FORMULA AND FUNCTIONAL ANALYSIS

As we have seen, complex analysis is an important tool in hydrodynamics and, in particular, in Levi-Civita's method. In this context, a functional relation due to Ulisse Dini (1845-1918) played an important role [Dini 1871b]; it connected the values of a function  $f$  on the circumference of a circle with the values of its normal derivative on the same circumference, if  $f$  is assumed to be harmonic in the circle.

Levi-Civita [1911a; 1911b] communicated two notes to the *Accademia dei Lincei*, in which he studied Dini's formula, if the function  $f$  is conformally mapped from the circle to any region  $S$  (finite and regular). He proved that  $f$  (expressed in the new variables) is also harmonic on  $S$  and that Dini's formula becomes a functional relation – valid on the mapped circumference – between  $f$  and its new normal derivative.

In these papers, Levi-Civita studied (the extension of) Dini's formula when the region  $S$  is assumed to be more general (for example, if  $S$  is unbounded). It is very interesting to consider a conformal mapping of

the circle to any rectilinear infinite strip, which “makes a brilliant analytical artifice developed by Lord Rayleigh – in order to study the nature of a solitary wave – fully rigorous”<sup>14</sup> [Levi-Civita 1911a, p. 164]. Levi-Civita himself proved this claim in a third note communicated to the *Accademia dei Lincei* [Levi-Civita 1911c].

As some of his letters to Volterra show, Levi-Civita thought that it would have been very fruitful to connect Rayleigh’s theory (on the solitary wave) with the theory of integral equations. Writing to Volterra on January 31, 1911, he announced that

“I think I will have the pleasure to come to Rome on March 4 and to communicate some remarks about the solitary wave in *Seminario Matematico*; of course, my lecture will consist of a friendly exchange of ideas. There should indeed exist a bridge between a certain resolutive artifice of Lord Rayleigh and the modern theory of integral equations; it would be fruitful to find it. I have been unable to find it up to now and am afraid I might not succeed; however, I can point out an elegant question that may not be as difficult as it seems to my colleagues after all”<sup>15</sup>.

A month later on February 27, Levi-Civita proudly wrote to Volterra

“that next Saturday I will be in Rome; then I will be very pleased to come to the meeting of *Seminario Matematico* and to communicate some remarks on some expressions of the functional equation, which gives the profile of the waves in a canal. I finally succeeded in realizing the connection between a genial (and at first glance rash) artifice, which allowed Lord Rayleigh to deduce actual results, and rigorous methods, which did not let any gleam of light leak out. I do not despair of deducing the complete solution of the problem”<sup>16</sup>.

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<sup>14</sup> A solitary wave is one consisting of a unique swelling (of height not necessarily small with respect to the depth of fluid). It was studied experimentally by J. Scott Russell (1844) and theoretically (as well as by approximations) by J. Boussinesq (1874) and Lord Rayleigh (1876).

<sup>15</sup> [Nastasi & Tazzioli 2000, p. 115]: “Penso tuttavia di potermi in ogni caso procurare il piacere di venire a Roma pel 4 Marzo e di presentare al Seminario Matematico alcune osservazioni sull’onda solitaria: si intende, sotto forma di amichevole scambio di idee. Fra un certo artificio risolutivo di Lord Rayleigh e la teoria moderna delle equazioni integrali dovrebbe pur esistere un ponte di passaggio, e il trovarlo sarebbe indubbiamente remunerativo. Io non vi sono riuscito e temo ormai di non riuscirci più, ma potrò sempre segnalare ai colleghi una questione elegante e, in definitiva, forse meno difficile che non sembri”.

<sup>16</sup> [Nastasi & Tazzioli 2000, p. 115]: “Mi faccio dovere di confermarle che Sabato prossimo mi troverò a Roma; mi sarà quindi ben gradito d’intervenire alla seduta del Seminario Matematico e di presentarvi alcune osservazioni sui diversi aspetti che può assumere l’equazione funzionale, la quale definisce il profilo delle onde di canale. Sono arrivato finalmente a riconoscere il collegamento fra un geniale (e a prima vista temerario) artificio, con cui Lord Rayleigh è giunto a risultati concreti, e i metodi rigorosi, che non lasciano invece trapelare alcun spiraglio di luce. Non disperso di poterne ricavare la soluzione completa del problema”.

Villat [1911b; 1912a] deduced a relation similar to Dini's formula for solving the Dirichlet problem in an annulus by employing elliptic functions. Villat [Villat 1911a] used his formula, which in a circle reduces to Dini's formula", to find the motion of a given body in a fluid.

In 1871, Ulisse Dini (1845-1918) had already established a formula for the complex function in the region between two concentric circumferences, if the values of its real part on the boundary were known [Dini 1871a]. Villat deduced his relation from this formula in the *Rendiconti del Circolo matematico di Palermo* in 1912. In a memoir published more than thirty years after his initial research, Dini [1913, p. 323] commented: "The route followed by Villat is certainly very clever and remarkable", and added: "In October 1870, I solved the same problem simply and naturally, but in a different way (which very easily reduces to the formula for elliptic functions given by Villat)". Dini continued: "I then also deduced the solution if the region is between two homofocal ellipses". Dini's result was expressed by a power series, and many pages of long calculations were necessary to show that his relation led to Villat's formula.

Villat's method in the case of an annulus was very similar to the procedure he had already applied to a circle [Villat 1911f]; this method was, in Villat's words, "very simple and capable of being extended to the resolution of a very large number of much more general problems"<sup>17</sup> Villat reduced the problem of finding 'a harmonic function in a circle, which has given continuous values on the circumference' to finding "a holomorphic function  $\Omega(z) = P(x, y) + iQ(x, y)$  ( $z = x + iy$ ), where  $P(x, y)$  is the harmonic function which solves the original problem'. For this purpose, he divided the circumference into a collection of small arcs, and on each of these he imposed a constant value for  $P$ , setting it equal to zero on the rest of the circumference. Using appropriate series developments, Villat found a series of functions in which all the terms represented the solution on one portion of the circle. A limit process provided the solution to the initial problem:

$$\Omega(z) = \frac{1}{2\pi} \int_0^{2\pi} \Psi(\theta) \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\theta$$

<sup>17</sup> [Villat 1911f, p. 443]: "fort simple et susceptible d'être étendue à la résolution d'un très grand nombre de problèmes beaucoup plus généraux".

( $\Psi$  is the given function on the circumference) whose real part is the Poisson integral, which is the classical solution of the problem.

Then Villat studied the particular case in which the given function  $\Psi$  is symmetric on the circumference:  $\Psi(2\pi - \theta) = \Psi(\theta)$ . “This case is particularly interesting, for it is present in a large number of problems from mathematical physics”<sup>18</sup>. In fact, he showed “by an immediate

calculation” that the previous relation became formula (103) of his thesis (see [Villat 1911a]), namely,

$$\Omega(z) = \int_0^\pi \Psi(\theta) \frac{1 - z^2}{1 - 2z \cos \theta + z^2} d\theta$$

Villat deduced this formula using both Levi-Civita’s method and a relation already established by Cisotti [1908, § 12] for the case in which the given values on the circumference are constant over arcs of it<sup>19</sup>.

Tommaso Boggio<sup>20</sup> [1911] also used Cisotti’s formula to find the general integral of a fluid moving in a curved and smooth channel. A more detailed study of Dini’s formula was developed by Boggio himself in a subsequent paper, which opens with the following words: “In some questions of hydrodynamics, and in particular in questions concerning discontinuous two-dimensional motions, it is necessary to know the complex function in a circle whose real part has given values on the circumference” [Boggio 1912, p. 22]. Boggio was also proud of having shown that Villat’s formula (103) can be deduced simply and directly from the Poisson integral. Boggio [Boggio 1912, p. 33] wrote in a footnote: “Such a formula was established by Villat in his Memoir (p. 269). He deduced it from another formula due to Cisotti [...]; he thought that it was unlikely (his Memoir, p. 285) to be obtained from the Poisson integral directly, while I showed that it is easily possible”. The same formula could also be deduced from certain equations due to Somigliana [1888], which connected the

<sup>18</sup> [Villat 1911f, p. 454-455]: “Ce cas est particulièrement intéressant, car il se présente dans un grand nombre de problèmes de Physique mathématique”.

<sup>19</sup> On Cisotti’s formula, see [Weinstein 1975, pp. 274-276].

<sup>20</sup> Tommaso Boggio (1877-1963) was full professor of rational mechanics at the University of Turin. His works concern potential theory, harmonic and biharmonic functions, integral equations, the theory of elasticity, and hydrodynamics.

real and imaginary parts of a complex function on the circumference of a circle, where such a function is regular.

Boggio expressed his doubts about Villat's paper again in a letter to Levi-Civita on August 9, 1911. He wrote<sup>21</sup>:

"Yesterday I read the second part of Villat's memoir ([Villat 1911a]), but I was really somewhat disappointed because it seemed to me – by reading the Preface – that the Author found the solution for any given body a priori; but it is not the case, since – substantially – he only found the expression of  $\omega(\zeta)$  as a function, when the values of its real part are given on the circumference, while you obtained  $\omega(\zeta)$  as a power series [see above for the notation].

In his memoir (p. 285), Villat writes that it is unlikely that  $\omega(\zeta)$  can be deduced directly from the Poisson integral, which solves the Dirichlet problem in a circle (and for real functions). I examined the question, and I found  $\omega(\zeta)$  from the Poisson integral in a very simple way. In such a way, Villat's checks of the calculation – which are interesting from an analytical point of view, but not from a mechanical point of view – are all unnecessary. My method is also useful for finding the complex function, whose real part  $u$  satisfies the condition  $cu - \frac{du}{dx}$  given function ( $c$  positive constant) on the circumference. If the real part of the function is that function given by Dini's formula which you used in your latest notes communicated to the Accademia dei Lincei ([Levi-Civita 1911a;b]). I hope to send you this short work soon.

In your card, you kindly suggested that I establish the connection of my formula – which expresses  $\omega(\zeta)$  in terms of the values of  $\frac{d\theta}{d\alpha}$  on the circumference – with Dini's formula, or better with the corollaries that you developed for a strip. I did not well understand what I should do, and therefore I will be grateful if you would suggest what I have to do in more detail"<sup>22</sup>.

<sup>21</sup> See U. Lucia, "Corrispondenza con Tommaso Boggio", in [Nastasi & Tazzioli 2003, pp. 427-551, pp. 509-510].

<sup>22</sup> [Nastasi & Tazzioli 2003, p. 510]: "Ho letto ieri la 2<sup>a</sup> parte della memoria di Vilat; ma, a dir vero, ho provato una piccola delusione, perché leggendo la prefazione pareva che l'A. fosse riuscito a trovare la soluzione per un profilo assegnato a priori, mentre invece ciò non è perché, in sostanza, egli ha solo trovato sotto forma finita l'espressione di  $\omega(\zeta)$ , dati i valori della sua parte reale sulla circonferenza, mentre Lei aveva dato la  $\omega(\zeta)$  come serie di potenze.

Il Vilat nella sua memoria (pag. 285) dice di ritenere poco verosimile l'aver  $\omega(\zeta)$  partendo direttamente dall'integrale di Poisson che risolve il problema di Dirichlet per il cerchio (e per funzioni reali). Orbene, avendo esaminato la questione, ho trovato un modo semplicissimo per trovare la  $\omega(\zeta)$  partendo dall'integrale di Poisson. In tal modo si ha pure il vantaggio di fare a meno di tutte quelle verifiche che fa il Villat, le quali se sono interessanti dal punto di vista dell'analisi, hanno scarso interesse meccanico. Questo mio metodo serve anche per trovare la funzione di variabile complessa, la cui parte reale  $u$  soddisfa sulla circonferenza alla condizione  $cu - \frac{du}{dx} =$  funzione data (con  $c$  costante positiva). Per  $c = 0$  la parte reale della funzione non è altro che quella tale funzione, data dalla formula del Dini, da Lei adoperata nei Suoi ultimi lavori dei Lincei. Spero di poterle mandare presto questo breve scritto.

Boggio [1912] did write the paper (discussed above) on the subject. In this connection, he wrote to Levi-Civita on December 9, 1911:

“I received the proofs of the paper on complex functions ([Boggio 1912]), which I started last summer and completed later on. Since I communicated to you what I was doing at that time, and since you suggested some useful things, I think it is right to send you the proofs, so that you can see if it is better to make some changes”<sup>23</sup>.

In the same letter, Boggio made some remarks on a paper by E.E. Levi<sup>24</sup> [Levi 1908], in which Levi deduced one or more complex functions, the real and imaginary parts of which satisfy certain conditions on a closed and analytical boundary:

“I recently read a paper by E.E. Levi on Riemann’s problem in the “*Nachrichten*” of Göttingen ([Levi 1908]), where it is said that Hilbert solved Riemann’s and other similar problems by means of integral equations.

I noticed that one of these problems – to find the complex function in a region, if the linear relation between its real and imaginary parts is known on the boundary – can be solved using Dirichlet’s problem. I know that in order to solve Dirichlet’s problem one uses integral equations; anyway, this remark does not seem completely useless to me”<sup>25</sup>.

The paper quoted by Boggio had been suggested to Levi by Levi-Civita himself, as Levi remarked in a footnote of his paper. The content of the

Nella sua cartolina, Lei mi ha gentilmente fatto presente l’opportunità di mettere in relazione la mia formula che esprime  $\omega(\zeta)$  in termini dei valori di  $\frac{d\omega}{d\sigma}$  sulla circonferanza, colla formula del Dini, o meglio con quei corollari che Lei ha sviluppato per la striscia. Non ho veramente capito bene ciò che dovrei fare, e Le sarei perciò grato se mi volesse suggerire, con maggiori dettagli, quanto debba fare”.

<sup>23</sup> [Nastasi & Tazzioli 2003, p. 510]: “Ho ricevuto le bozze di quel lavoro sulle funzioni di variabile complessa, che iniziai nella scorsa estate e che completai di poi. Poiché ebbi allora occasione di comunicarle quanto stavo facendo, e Lei mi suggerì varie utili cose al riguardo, così credo non inopportuno inviarle le bozze, affinché veda se fosse il caso di modificare qualche cosa”.

<sup>24</sup> Eugenio Elia Levi (1883-1917) died in the First World War when he was a full professor of analysis at the University of Genoa for many years. His untimely death did not prevent him from being considered as one of the best Italian analysts of the twentieth century; in particular, his research on ‘totally elliptic’ partial differential equations became a classic.

<sup>25</sup> [Nastasi & Tazzioli 2003, p. 511]: “Avendo riletto recentemente un articolo di E.E. Levi delle ‘*Nachrichten*’ di Göttinga sul problema di Riemann, ho visto che vi è detto che Hilbert ha risolto colle equazioni integrali il problema di Riemann ed altri analoghi.

Ora, ho osservato che uno di essi, e cioè quello di trovare la funzione di variabile complessa per un’area, quando sul contorno si conosce la relazione lineare fra la parte reale e immaginaria, si può risolvere col problema di Dirichlet. Capisco che per risolvere il problema di Dirichlet si ricorre alle equaz. integrali, tuttavia mi pare che la mia osservazione sia non del tutto inutile”.

paper indeed coincided with that of a letter – written to Levi-Civita in 1908<sup>26</sup> – aimed at overcoming “a small difficulty which one encounters in M. Hilbert’s memoir (\*)<sup>27</sup>, and [at showing] how one may protect the result from this objection: I hope to be forgiven for having presented these very simple considerations, given the fundamental importance of M. Hilbert’s results” [Levi 1908, p. 1].

In fact, Hilbert had noticed that in the integral equations the singular part of some functions disappears completely. Hilbert’s conclusion, Levi pointed out, “does not seem completely exact to me”. However, Levi proved that “fortunately” the mistake did not influence the final result. Villat also thought that Dini’s formula was important; indeed, he wrote to Levi-Civita on June 24, 1911:

“Meanwhile, I permit myself to send you some very short notes which have appeared recently in the *Comptes-Rendus de l’Académie des Sciences*. Perhaps the one that concerns Dirichlet’s problem in a circular ring ([Villat 1911b]) will be of particular interest to you: there I have determined yet another function of a complex variable from the values of its real part on the border; which is another encounter, very flattering for me, with the idea that is expressed in your own work “Sur la Transformation d’une équation fonctionnelle de Dini ([Levi-Civita 1911a;b]). In this connection, I wish to thank you for your kindly sending [your reprint], which I have read with the keenest interest, raised even more by the just-mentioned analogy”<sup>28</sup>.

In the same letter, Villat also considered the following hydrodynamic problem, which was subsequently the subject of a note published in the *Comptes rendus*:

“I have just taken up a problem which may interest you in particular, namely: ‘to see how the motion of an indeterminate fluid changes around a given obstacle when one modifies the intrusion of this obstacle on the currents *without*

<sup>26</sup> Levi’s letter to Levi-Civita is published in [Nastasi & Tazzioli 2004a, p. 95-98].

<sup>27</sup> Levi refers to the third of Hilbert’s communications (see below) published in 1904 ([Hilbert 1904]). His footnote (\*) reads: “This difficulty was pointed out to me by M. Levi-Civita”.

<sup>28</sup> [Nastasi & Tazzioli 2003, p. 374]: “En attendant, je me permets de vous envoyer quelques notes très brèves parues récemment aux *Comptes-Rendus de l’Académie des Sciences*. Peut-être celle qui est relative au problème de Dirichlet dans une couronne circulaire ([Villat 1911b]), vous intéressera t’elle particulièrement : j’y ai déterminé encore une fonction de variable complexe par les valeurs de sa partie réelle sur la frontière ; ce qui est une nouvelle rencontre, très flatteuse pour moi, avec l’idée qui se trouve dans votre propre travail ‘Sur la Transformation d’une équation fonctionnelle de Dini’ ([Levi-Civita 1911a;b]). A ce propos, je tiens à vous remercier de votre gracieux envoi, que j’ai lu avec le plus vif intérêt, augmenté encore par suite de cette analogie susdite”.

*changing its shape*'. Admittedly, the change of orientation entails a change of the point on the wall where the current divides. For certain cases I have been able ultimately to reduce the full solution of the problem to the resolution of an integral equation very close in form to that of Fredholm, namely:

$$\varphi(x) + \int_0^a F(x, y)[\varphi(x) - \varphi(y)] ds = \psi(x),$$

where  $\varphi(x)$  is the unknown function;  $F(x, y)$  and  $\psi(x)$  are given;  $x$  and  $s$  fall between 0 and  $a$ ; but  $F(x, s)$  is infinite as  $\frac{1}{s-x}$  for  $s = x$ ; in consequence the equation cannot be reduced to that of Fredholm. I am thwarted by the resolution of this equation, which however it would be a pity not to integrate. Allow me to say – knowing how masterly you have dealt with integral equations – that a suggestion from you would be infinitely precious to me; do you think the equation can be solved?"<sup>29</sup>.

Levi-Civita answered two days later, on June 26:

"The question to which you draw my attention is all the more interesting as you have been able to express it in a very elegantly reduced form. But I fear that some essential difficulty may still be hidden there. That is at least the impression I get from a very analogous case where my efforts have failed. Substantially I deal (using your notation) with the following integral equation:

$$\varphi(x) + \int_0^a \log|x-s|\varphi'(s) ds = \psi(x) \quad (\varphi' = \frac{d\varphi}{ds})$$

You see clearly that the whole difficulty comes from the singularity of the kernel, exactly as in your case. If, instead of  $\log$ , the kernel was a function  $F(x, s)$ , finite and integrable at the same time as  $\frac{\partial F}{\partial s}$ , a simple integration by parts would be enough to reduce it to the typical Fredholm case"<sup>30</sup>.

<sup>29</sup> [Villat 1911e]: "Je suis en ce moment occupé d'un problème que vous intéressera peut être particulièrement, et alors voici l'énoncé: 'Voir comment se modifie le mouvement d'un fluide indéfini autour d'un obstacle donné, lorsqu'on modifie l'intrusion de cet obstacle sur les courants, sans changer sa forme'. Bien entendu, le changement d'orientation entraîne le changement du point de la paroi, où la courant se divise. J'ai pu, dans certain cas, ramener en dernière analyse la solution complète de la question, à la résolution d'une équation intégrale d'une forme très voisine de celle de Fredholm, à savoir:

$$\varphi(x) + \int_0^a F(x, y)[\varphi(x) - \varphi(y)] ds = \psi(x),$$

où  $\varphi(x)$  est la fonction inconnue;  $F(x, y)$  et  $\psi(x)$  sont donnés;  $x$  et  $s$  sont compris entre 0 et  $a$ ; mais  $\frac{1}{s-x}$  est infini comme pour  $s = x$ ; de sorte que l'équation ne se ramène pas à celle de Fredholm. Je suis pour le normal arrêté par la résolution de cette équation, qu'il serait cependant dommage de ne pas intégrer. Permettez moi de vous dire, – sachant avec quelle maîtrise vous vous êtes occupé des équations intégrales – qu'une indication de votre part me serait infiniment précieuse; l'équation vous paraît-elle résoluble?"

<sup>30</sup> [Nastasi & Tazzioli 2003, p. 375]: "La question sur laquelle vous attirez mon attention est d'autant plus piquante que vous êtes réussi à lui donner une forme analytique très élégamment réduite. Mais je crains qu'il s'y cache encore quelque difficulté essentielle. C'est du

“It is curious”, Villat wrote on July 2, 1911, “that the integral equations, the solution of which I search, meets thus with an equation which you have studied yourself: more precisely, an integration by parts, of  $\log|x-s|F(x,y)\varphi'(s)$ , taking the precaution to write  $\varphi(s) - \varphi(x)$  as the primitive of  $\varphi'(s)$ , reduces your equation to a particular case of mine”<sup>31</sup>.

Finally, Villat wrote to Levi-Civita on October 15 [Nastasi & Tazzioli 2003, pp. 376-377] that he had solved the problem. Levi-Civita’s equation is indeed solved by Villat’s formula (1):

$$\varphi(x) = \frac{1}{\pi^2} \int_0^a \frac{\psi(s) - \psi(x)}{s-x} ds + \frac{\psi(x)}{1 + \pi^2} \left[ 1 + \log \frac{a-x}{x} \right].$$

where  $\psi(x)$  is supposed to satisfy the conditions

$$\int_0^a \frac{\psi(x)}{x} dx = \int_0^a \frac{\psi(x)}{x-a} dx = 0$$

And Villat added:

“I do not know in what connection you have encountered equation (1). Mightn’t my partial result be applied to it?

If it interests you, I shall send you the detailed demonstration: do you think it worth publishing? In that case, I should be extremely honoured if you would allow me to say which question you encountered that made equation (1) interesting”<sup>32</sup>.

Levi-Civita’s answer was immediate. He wrote to Villat on October 18:

“This is how I have come to the functional equation

moins l’impression que je tire d’un cas assez analogue où mes efforts ont échoués. J’en ai affaire substantiellement (en employant vos notations) à l’équation intégrale que voici :

$$\varphi(x) + \int_0^a \log|x-s|\varphi'(s) ds = \psi(x) \quad (\varphi' = \frac{d\varphi}{ds})$$

Vous voyez bien que toute la difficulté provient de la singularité du noyau, justement comme dans votre cas. Si, au lieu du  $\log$ , on avait pour noyau une fonction  $F(x,s)$ , finie et intégrable en même temps que  $\frac{\partial F}{\partial s}$ , une simple intégration par parties suffirait à ramener au cas typique de Fredholm”.

<sup>31</sup> [Nastasi & Tazzioli 2003, p. 376]: “Il est curieux que l’équation intégrale dont je recherche la solution, se rencontre ainsi avec une équation que vous avez étudiée vous-même : plus précisément, une intégration par partie, de  $\log|x-s|F(x,y)\varphi'(s)$ , en prenant la précaution d’écrire  $\varphi(s) - \varphi(x)$  comme primitive de  $\varphi'(s)$ , ramène votre équation à être un cas particulier de la mienne”.

<sup>32</sup> [Nastasi & Tazzioli 2003, p. 377]: “Je ne sais à propos de quelle question vous avez rencontré l’équation (1). Est-ce que par hasard mon résultat partiel pourrait lui être appliqué ? Si cela vous intéresse, je vous communiquerai la démonstration détaillée : pensez-vous que cela vaille la peine d’être publié ? Auquel cas, je serai extrêmement honoré si vous m’autorisiez à dire quelle est la question rencontrée par vous, qui donne à l’équation (1) son intérêt”.

$$(1) \quad \varphi(x) + \int_0^a \log|x-s|\varphi'(s) ds = \psi(x).$$

You know well that the case of a function  $F(x, s)$ , having a logarithmic singularity for  $s = x$ , arises when one has to determine a function of a complex variable (holomorphic within a plane area) according to a linear relation, at the contour, between the real part and the coefficient of  $\sqrt{-1}$ . It is exactly such a question (for a specific form of  $F$ ) which presents itself in the first approximation for the solitary wave. In order to grasp well the nature of the difficulty, I began with the simplest type, that is, equation (1), but I did not succeed in understanding it clearly. So much greater is the pleasure with which I follow and shall follow your investigations. As far as the solitary wave is concerned (which I intend to take up in the near future), I have realized that it is better to approach the problem from a different point of view; besides, your partial result would longer apply to it. Independently of that, your partial result seems to me so simple and interesting that it is well worth making it known”<sup>33</sup>.

Villat’s contribution was published in the *Comptes rendus* [Villat 1911e], while Levi-Civita’s results on the solitary wave – whose connections with the theory of integral equations had been communicated to the *Accademia dei Lincei* [Levi-Civita 1911a;b;c] (see above) – were obtained from “another point of view” in a note communicated to the *Accademia dei Lincei* some months later [Levi-Civita 1912].

We would like explicitly to note the importance of functional analysis in the study of the equations of mathematical physics. From 1903 onward, Fredholm, Hilbert, and E. Schmidt published a series of papers on the theory of integral equations, in which a new method of solving equations

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<sup>33</sup> [Nastasi & Tazzioli 2003, p. 378]: “Voici comment je suis parvenu à l’équation fonctionnelle

$$(1) \quad \varphi(x) + \int_0^a \log|x-s|\varphi'(s) ds = \psi(x).$$

Vous savez bien que le cas général d’un rayon  $F(x, s)$ , ayant une singularité logarithmique pour  $s = x$ , se rencontre lorsqu’il s’agit de déterminer une fonction de variable complexe (holomorphe à l’intérieur d’une aire plane) d’après une relation linéaire, au contour, entre la partie réelle et le coefficient de  $\sqrt{-1}$ . C’est justement une telle question (pour une forme bien déterminée de  $F$ ) qui se présente dans la première approximation de l’onde solitaire. Pour me rendre bien compte de la nature de la difficulté, j’avais commencé à étudier le type le plus simple, c’est-à-dire l’équation (1), mais je ne suis pas réussi à y voir clair. D’autant plus grand est le plaisir avec lequel je suis et je suivrai vos recherches. Quant à l’onde solitaire (que je compte reprendre prochainement), j’ai reconnu qu’il va mieux aborder le problème d’après un autre point de vue; d’ailleurs votre résultat partiel ne lui serait plus applicable. Indépendamment de cela, votre résultat partiel me paraît si simple et piquant qu’il vaut bien la peine de le faire connaître”.

of mathematical physics was developed. When Italian mathematicians became aware of the new theory of integral equations, they changed their approach completely by resorting to Fredholm's theory. Their new point of view was shared by the majority of mathematicians, who at that time began to study the equations of mathematical physics by means of the theory of integral equations. Many Italian mathematicians – such as Boggio, Lauricella<sup>34</sup>, Marcolongo<sup>35</sup>, and Orlando<sup>36</sup> – abandoned the method of Green's functions<sup>37</sup> and adopted the new theory after reading the papers on integral equations published mainly by Hilbert, Fredholm, and Picard. In fact, as Boggio [1907, p. 248] remarked, “all static problems of Mathematical Physics can be reduced to solving integral equations”. Sometimes, known solutions were found again using the new and fruitful method of integral equations. This is the case of Boggio, who proudly showed that both methods (Green's function and Fredholm's theory) “determine the strain of an elastic plane plate, if forces act on its boundary in the plane of the plate, and the strain of the boundary is known” [Boggio 1907, p. 250]. Boggio's letters to Levi-Civita show how Levi-Civita encouraged him to study integral equations, by even suggesting to him the Italian translations of some technical words (“*nucleo*”, “*autovalore*”, and “*autofunzione*”). In fact, in his letter of November 24, 1906 Boggio wrote to Levi-Civita:

“The reading of the clear and interesting paper of Picard: *Sur quelques applications de l'équation fonctionnelle*, published in the latest volume of the *Rendiconti di Palermo* [Picard 1906], persuaded me to study deeply the theory of integral equations; and then I studied Fredholm's *Memoir* in the *Acta Math.* [Fredholm 1903] and Hilbert's papers in the *Nachrichten* of Göttingen (see [Hilbert 1912]).

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<sup>34</sup> Giuseppe Lauricella (1867-1913) was full professor of analysis at the University of Catania, but he gave contributions to mathematical physics as well. Some of his results in these fields are remarkable and still mentioned.

<sup>35</sup> Roberto Marcolongo (1862-1943) was full professor of rational mechanics at the Universities of Messina and Naples. He wrote excellent treatises on many subjects of mathematical physics and also researched on history of mathematics – his historical studies on Leonardo da Vinci and on the three body problem are to be mentioned.

<sup>36</sup> Luciano Orlando (1887-1915) died in the First World War as officer of artillery. His main works concern the theory of elasticity and the theory of integral equations, where he recognised the importance of the so-called “Goursat's kernels”.

<sup>37</sup> On Green's method and its connections with the history of integral equations see [Nastasi & Tazzioli 2004a].

In addition, I read Kneser's note published in the latest number of Palermo [Kneser 1906]; but some obscure passages remain. [...]

I translate with '*perno*' [now called 'nucleo'] the meaning of Hilbert's *Kern*; it seems to me that the idea of *perno* is close enough to the idea which Hilbert intended to represent with *Kern*"<sup>38</sup>.

Then Boggio discussed some passages in Kneser's paper and in Hilbert's first communication; in particular, he drew his attention to Hilbert's theory: If the kernel ("*perno*") is symmetric, then the eigenvalues ("*valori eccezionali*") cannot be complex numbers.

In a letter on December 3, 1906, Boggio thanked Levi-Civita for suggesting that he write to Schmidt to obtain a copy of his famous dissertation on the theory of integral equations supervised by Hilbert [Schmidt 1905].

Of course, many Italian analysts were aware of the papers by Fredholm and Hilbert and of their relevance, as the abovementioned 1908 letter by E.E. Levi to Levi-Civita shows. One year before, Levi [Levi 1907] remarked that the theory of linear integral equations is useful for solving classical problems in the theory of (ordinary and partial) differential equations. Therefore, it is not by chance that Levi-Civita pointed out to Levi "a difficulty" concerning a certain reduction to an integral equation due to Hilbert in the third of his "*communications*"<sup>39</sup>, namely, a difficulty concerning the solution of Riemann's problem (and analogous ones) on one or more complex functions, whose real and imaginary parts satisfy some conditions on a closed boundary. Levi-Civita's "difficulty" was rapidly resolved by Levi; the solution is one of his better contributions to analysis, as many mathematicians agreed.<sup>40</sup>

<sup>38</sup> [Nastasi & Tazzioli 2003, p. 472-477]: "La lettura della chiara ed interessante Nota di Picard: Sur 'quelques applications de l'équation fonctionnelle', comparsa nell'ultimo fascicolo del Rendiconti di Palermo, mi ha invogliato a studiare un po' a fondo la teoria delle equazioni integrali, ed ho appunto perciò studiato la Memoria di Fredholm negli Acta Math. e quelle di Hilbert delle Nachtrichten di Gottinga.

Ho inoltre letto la nota di Kneser contenuta nell'ultimo fascicolo di Palermo, ma mi è rimasto qualche punto oscuro. [...]

Traduco con *perno* il senso di *Kern* di Hilbert; mi pare che l'idea di *perno* si avvicini abbastanza bene all'idea che Hilbert ha voluto rappresentare con *Kern*".

<sup>39</sup> Hilbert's notes are collected in [Hilbert 1912].

<sup>40</sup> Interesting comments on Levi's contributions are to be found in [Dieudonné 1981, p. 68, pp. 255-258, p. 266, p. 271].

## 5. CONCLUDING REMARKS

As noted, Levi-Civita's wake hypothesis [Levi-Civita 1901] strongly influenced the contemporary research on hydrodynamics; the hypothesis also allowed him to solve the D'Alembert paradox and to develop his analytic method [Levi-Civita 1907a], which became the starting point for many results deduced by Villat and Brillouin as well as some of Levi-Civita's students.

But Levi-Civita made other important contributions to hydrodynamics. In fact, in 1907 he published another influential paper [Levi-Civita 1907b], which concerns progressive permanent waves in a canal with horizontal bed. Here, "progressive" means that the motion appears stationary to an observer moving along with the apparent translation of the fluid.

This work differed fundamentally from earlier research, as it considered motions in the vertical plane, in which gravity intervenes as a fundamental force that cannot be ignored. In this context, Levi-Civita once more used results and theorems from the theory of functions of a complex variable, in a procedure which allowed him to reduce the problem to the solution of a mixed (namely, differential and difference) equation related to a single holomorphic function. Cisotti extended some of Levi-Civita's results to more general canals [Cisotti 1911] and to the case of non-stationary flows [Cisotti 1919]. The functional differential equation obtained by Levi-Civita was solved perturbationally at different orders of approximations by Cisotti and Crudeli in a series of articles ([Cisotti 1918; 1920; Crudeli 1919; 1923]). Recently, by starting from Levi-Civita's functional differential equation, Decio Levi [1994] re-obtained the Korteweg-de Vries equation in the shallow-water, small perturbations approximation. At first, he obtained a complex Korteweg-de Vries equation in the complex domain, and then deduced the usual one, by requiring that the vertical velocity be small. Levi wrote in conclusion that

"Many interesting problems can now be easily attacked and will be the subject of future work. Among them, let me just mention the possibility of getting better approximating equations. Moreover, we can extend the Levi-Civita approach to the case of a one-dimensional channel with variable bottom, a two-dimensional homogeneous fluid in a channel or in the open sea, a stratified fluid, etc." [Levi 1994, p. 709].

In a paper published in 1912, Levi-Civita had gone on to study permanent waves in a canal with horizontal bed under certain physical conditions [Levi-Civita 1912]. In particular, he had proved the so-called generalized Stokes-Rayleigh's theorem (the transport of fluid mass increases without limits with time), and deduced a new formula for the kinetic energy of waves.

If one puts an obstacle in the canal (for instance, if a vertical bar is welded to the bed of the canal), then the motion of the fluid will change. In particular, the surface of the fluid will present an intumescence over the obstacle. Cisotti [1912b] found the height of the intumescence, which is connected with the height of the bar, and cast the problem in the more general case where a series of identical obstacles are put at the same mutual distance on the bed of the canal.

Levi-Civita systematically treated the theory of canal waves, making use of the physical concept of a *wave-motion phenomenon*, during lectures given at the University of Barcelona [Levi-Civita 1922]. But a fundamental problem remained unsolved – the determination of periodic and irrotational permanent waves. In 1802, F. von Gerstner (1756-1823) had given the solution for periodic and permanent rotational waves moving in canals of infinite depth. Afterwards, Airy, Stokes, and Rayleigh had approached the case of irrotational waves by applying Gerstner's results and limiting the analysis to first approximations<sup>41</sup>. As we have already pointed out, in a classical memoir published in the *Mathematische Annalen*, Levi-Civita proved the existence of the irrotational wave in a canal of infinite depth [Levi-Civita 1925]. Many of his students worked on this subject and extended his method to more general cases. Dirk Struik (1894-2000) generalized Levi-Civita's result to canals of finite depth, and Marie-Louise Dubreil-Jacotin (1905-1972) proved the existence of infinite rotational waves, including the irrotational wave, the existence of which had already been determined with Levi-Civita's [1925] and Gerstner's wave as particular cases.

If water is dumped into the beginning of a canal, then a wave will swell – it is called a solitary wave (see note 12). Many students of Levi-Civita studied this wave phenomenon; among them, Cisotti, Struik,

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<sup>41</sup> On the history of wave-theory see [Darrigol 2003].

Dubreil-Jacotin, Alexander Weinstein (1897-1979), and Luigi Sante da Rios<sup>42</sup> (1881-1965). The latter also successfully studied another question, which Levi-Civita had considered very important: the asymptotic form of the Newtonian potential for slender tubes. In the 1930s, Levi-Civita published an important work on this subject [Levi-Civita 1932], which was intimately connected to hydrodynamics. There, he analyzed Saturnian rings and vortices by means of results due to da Rios on vortex filament dynamics [Da Rios 1906; 1910].

As noted above, we plan to analyze the contributions of Levi-Civita and his students to waves and their related applications in a subsequent paper. Here, we underline Levi-Civita's role as a leader of the mathematical school in Padua (until 1918) and then in Rome. He attracted students from numerous countries, whom he encouraged and followed with competence and kindness. Evidence of this side of Levi-Civita's personality is found in his private correspondence published in [Nastasi & Tazzioli 2004b].

We have discussed Cisotti and da Rios, but Giuseppe Picciati (1868-1908) also merits mention. Picciati was a student of Volterra in Pisa, where he graduated in physics in 1890 and then in mathematics in 1895. From 1901 onward, he published important papers on electrodynamics and electromagnetism under the supervision of Levi-Civita. These rapidly qualified him for university teaching at the University of Padua. Levi-Civita [1908, p. 366] wrote: "Picciati put his lasting mark – in a very short time – on another area, rational hydrodynamics." Picciati made contributions to problems on motion in a viscous fluid [Picciati 1907b;c;d] – "which only the unique genius of Stokes had dared to face" [Levi-Civita 1908, p. 366]. Picciati [1907b; 1907c], in fact, studied the rectilinear motion (uniform or not) of a sphere in an incompressible and viscous fluid; he deduced the motion and the general integral of the resistance acting on the sphere, if the motion is "slow". In order to find these results, he reduced the original problem to integrating the equation of heat

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<sup>42</sup> Luigi Sante da Rios (1881-1965) was one of Levi-Civita's students at the University of Padua, where he graduated with a dissertation on hydrodynamics in 1906. Even if da Rios obtained some results which are nowadays appreciated, his university career was unlucky; da Rios indeed qualified for university teaching in rational mechanics but taught in high schools all his life.

propagation for given boundary conditions. In fact, a general method due to Volterra [1904] for integrating differential equations of parabolic type, together with the suitable Green function, had already allowed Picciati [1907a, p. 750] “to represent the [general] integral [of heat propagation] with certain boundary conditions in a simple and fruitful way”. Picciati [1907d] used the same method if an unbounded cylinder – as opposed to the sphere – moved in a viscous fluid.

Many young mathematicians referred to Levi-Civita’s works, even if they were not his immediate students. They also asked questions or consulted him about their researches, for instance, relative to hydrodynamics, Tommaso Boggio, Bruto Caldonazzo (1886-1960), Gustavo Colonnetti (1886-1968), Carlo Ferrari (1903-1996), Bruno Finzi (1899-1974), Modesto Panetti (1875-1957), Enrico Pistolesi (1889-1968), and, of course, Henri Villat. In addition, many foreign students were attracted by his international reputation and came to Rome to study. For example, Dubreil-Jacotin, Mazet, Struik, and Weinstein are well-known and earned international reputations.

Finally, we wish to emphasize Levi-Civita’s role – documented in the letters cited in section 4 – in the development of the theory of integral equations in Italy. We also underscore the effectiveness of new methods elaborated by Fredholm, Hilbert, and E. Schmidt applied to mathematical physics, in particular to hydrodynamics. The letters exchanged by Levi-Civita with Villat and Volterra attest to the fruitfulness of applying Fredholm’s theory in hydrodynamic.

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