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A LINEAR ASSIGNMENT FORMULATION OF THE MULTIATTRIBUTE DECISION PROBLEM (*)

by Jean-Marie. BLIN (1)

Abstract. — *After proposing a general framework of analysis for multiattributed decision problems, this paper develops a linear assignment model to aggregate a set of individual ordinal evaluations of alternatives into a global ranking. A "best" aggregation scheme is defined as one which maximizes a linear function of individual agreement over alternative rankings. Due to the special features of this linear assignment problem, geometric formulation solutions are found. Both formulations are shown to be equivalent. Implications for aggregation theory and extensions of the model are briefly discussed.*

SECTION 1: STATEMENT OF THE PROBLEM

1.1. A large number of real-world decision problems cannot be properly assessed from a single viewpoint: a firm attempting to compare a set of alternative investment projects might want to rate them on the basis of (1) the net discounted profit expected from each investment (2) the payoff period and (3) the market share. An economist trying to assign a precise quantitative content to such expressions as the "rate of growth of the general price level" would want to compare prices of *a set* of commodities over several periods of time; similarly, in system analysis the question of how to take into account multiple criteria often arises; in the field of social choice theory the same problem is encountered and voting mechanisms are but one possible way of resolving it. To set the stage for our analysis it is convenient to adopt a few definitions to capture the sesential similarity between the various problems we have just mentioned (2).

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(2) See B. ROY, [16], for a general extensive discussion of the problem.

1.2. Basic definitions

A is a finite set of well-specified objects (e. g. investment projects, candidates in an election, etc.).

$$A = \{ a_1, a_2, \ldots, a_i, \ldots, a_m \}. \tag{1}$$

We are also given a finite class $\mathscr{S}$ of "criteria" (e. g. characteristics, features, voters):

$$\mathscr{S} = \{ S_1, S_2, \ldots, S_h, \ldots, S_l \} \tag{2}$$

Now each individual criterion $S_h \in \mathscr{S}$ is itselt a set endowed with a certain structure, algebraic, topological or both as the case may be.

For instance we could have

$$S_h = N \text{ (the set of natural numbers)}, \tag{3}$$

$$S_h = \{ 0, 1, 2, \ldots, n \}, \text{ the finite set of the first } n \text{ integers}, \tag{4}$$

$$S_h = \{ 0, 1 \} \text{ or } \{ \text{yes, no} \}, \tag{5}$$

$$S_h = \mathbf{R} \text{ or } \mathbf{R}_+. \tag{6}$$

More generally S_h could be a metric space or a topological space.

These criteria now give us a basis for representing the m objects of A. This "representation" process can be viewed as a set of l mappings φ_h:

$$\varphi_h : \quad A \to S_h \qquad (h = 1, 2, \ldots, l). \tag{7}$$

In general we would not expect these mappings to be identical (if they were, we would be faced with a single criterion decision problem and no aggregation would be necessary). Each object a_i is thus described by an l-dimensional image:

$$[\varphi_1 (a_i); \varphi_2 (a_i); \ldots ; \varphi_h (a_i); \ldots ; \varphi_l (a_i)]. \tag{8}$$

1.3. The Aggregation Problem

Informally we would like to "combine" the l-dimensional images of the m objects in a certain "best" way.

Letting the images set of A be denoted by $\Phi(A)$ $\left(\text{where } \Phi(A) \subset \prod_{h=1}^{l} S_h\right)$ the aggregation problem consists in finding a mapping σ that maps $\Phi(A)$ into a one-dimensional "aggregate space" 0:

$$\sigma : \quad \Phi(A) \rightarrow 0. \tag{9}$$

Now the nature of the aggregate space 0 will vary depending upon the problem at hand. For instance if all the criterion sets $S_h = \{1, 2, \ldots, n\}$ and the representation mappings Φ_h are permutations of S_h, we way want to require that

$$0 = \{1, 2, \ldots, n\} \tag{10}$$

and $\sigma \in \mathscr{S}$ (where $\mathscr{S}$ denotes the set of permutation operators, i. e. the group of permutations).

Clearly a vey large number of mappings σ could be chosen. To discriminate among them, some "goodness of fit" criterion is needed. Intuitively we would like the aggregate representation mapping σ to respect as much as possible the individual mappings φ_h. The question then revolves around the choice of an objective function that will evaluate the goodness of fit between the "extensive" image $\Phi(A)$ of the set A and its aggregate image $\sigma[\Phi(A)] \subset 0$. Once such an objective function has been chosen, the problem is then to search for a class of aggregation mappings σ that meet this optimality requirement. Clearly the answer to the first question, i. e. the choice of a goodness of fit index, is partly dependent upon the choice of a structure for the criterion sets S_h. In the next sections we will illustrate this approach by using a simple linear form as our objective function to be maximized. The metric interpretation of this solution concept will also be discussed.

SECTION 2: AGGREGATING A SET OF (l) COMPLETE ORDERINGS OF m OBJECTS

2.1. Introduction and Background

We shall now assume that the individual representation mappings φ_h of the m objects $a_i \in A$ are permutation operators, i. e.

$$S_h = \{1, 2, \ldots, i, \ldots, m\}, \tag{11}$$

$$\varphi_h \in \mathscr{S}_m \qquad \forall\, h = 1, 2, \ldots, l, \tag{12}$$

where $\mathscr{S}_m$ denotes the group of permutation of the first m integers.

As we know each operator φ_h can be represented by a *permutation matrix* **P**, i. e. an $m \times m$ nonnegative matrix each row and column of which has only one entry equal to 1 (and the others are 0). And finally we want to find an aggregate mapping $\sigma \in \mathscr{S}_m$ to represent the individuals mappings φ_h.

2.2. Maximizing Agreement Among the Various Ranking Criteria

DEFINITION 1: An *agreement matrix* Π is a square $(m \times m)$ nonnegative matrix whose entries π_{ij} represent the number of individual orderings where the ith alternative (of the reference order) is placed in the jth position:

$$\pi_{ij} = k \quad \Leftrightarrow \quad \exists\, K \subset \{1, 2, \ldots, l\} \ni |K| = k \tag{13}$$

and $\varphi_h(a_i) = a_j$ iff $h \in K$, $\forall\, h \in \{1, 2, \ldots, l\}$. It is clear that we may agree to assign unequal weights w_h to each individual criterion S_h; in this case this amounts to assuming that the hth individual ordering is replicated w_h times.

DEFINITION 2: The agreement index I_A is a real-valued linear mapping such that:

$$I_A = \sum_{i,j} \pi_{ij} P_{ij}, \tag{14}$$

where the p_{ij}'s are the entries of an $(m \times m)$ permutation matrix P, representing the hth ordering.

The *first formulation* of the aggregation problem in this framework is then: Find $P^* \in \mathscr{S}_m$ such that

$$\sum_{i,j} \pi_{ij} P_{ij} \leqq \sum_{i,j} \pi_{ij} P^*_{ij} \tag{15}$$

for all P matrices of the mth order, Of course, the first solution method we can think of is simply to enumerate the $m!$ permutation matrices P and choose that matrix P^* which maximizes I_A. Clearly, this is computationally inefficient and even infeasible as m becomes large. An alternative formulation of the problem is now proposed, which will greatly reduce this computational burden.

The *second formulation* of our problem consists in allowing fictitious stochastic orderings. More specifically we want to find a bistochastic solution matrix $[b_{ij}]$ (1) which

$$\text{Max} \sum_{i,j} \pi_{ij} b_{ij} \tag{16}$$

(1) A bistochastic matrix B is a nonnegative $(m \times m)$ matrix whose coefficients satisfy the following properties:

(i) $\forall\, i, \forall\, j \; b_{ij} \geqq 0 \; (i, j = 1, 2, \ldots, m)$;

(ii) $\forall\, i \; \sum_{j=1}^{m} b_{ij} = 1$;

(iii) $\forall\, j \; \sum_{i=1}^{m} b_{ij} = 1$.

subject to

$$\left.\begin{aligned} &\sum_{j=1}^{m} b_{ij} = 1, \qquad (17)\\ &\sum_{i=1}^{m} b_{ij} = 1, \qquad (18)\\ &b_{ij} \geqq 0, \qquad (19) \end{aligned}\right\} \quad i, j = 1, 2, \ldots, m.$$

We can readily recognize the b_{ij}'s of this formulation as the entries of an $(m \times m)$ bistochastic matrix B (1). This new problem is, of course, a simple linear programming problem.

The crucial point of this formulation, however, is the fact that any solution to this second problem will necessarily be a solution to the first one. The proof of this result is obvious: (i) it is a well known fact of linear programming theory that if there exists an optimal solution, there will always be at least one solution at a vertex of the polyhedral feasible region; and (ii) this vertex is nothing else but a permutation matrix P, according to the Birkhoff-von Neumann theorem (2). Hence, solving problems (16-19) will give us all the solution(s) to problem (15) as we had claimed. This second approach, however, eliminates the computational limitation described before (3).

2.3. A Minimal Distance Algorithm

Another approach to the aggregation problem in the context of l individual complete strict orderings on A, is now presented (4).

The basic idea here is to exploit the geometrical properties of the set $\mathscr{S}_m$ and β as described by the Birkhoff-von Neumann theorem. In order to do that we must first prove a simple result on agreement matries Π.

LEMMA 1: *Let* Π *be an* $(m \times m)$ agreement matrix. *Then the following relations always hold:*

$$\left.\begin{aligned} &\sum_{j=1}^{m} \pi_{ij} = l, \qquad \forall i = 1, 2, \ldots, m, \qquad (21)\\ &\sum_{i=1}^{m} \pi_{ij} = l, \qquad \forall j = 1, 2, \ldots, m, \qquad (22) \end{aligned}\right\}$$

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(1) The idea for this formulation was first proposed by T. C. Koopmans and M. Beckmann in the context of a location problem [11].

(2) THEOREM (Birhkoff-von Neumann): *The set* β *of bistochastric matrices of order m forms a convex polyhedron in* $\mathbf{R}^{m^2}$, *whose vertex set is identical with the set* $\mathscr{S}_m$ *of permutation matrices.*

(3) The approach we propose in this paper can be compared with that of Jacquet-Lagrèze [11]. In both cases the aggregation problem is stated as a discrete programming problem. The main difference consists in the form of the objective function (linear, here, vs, quadratic).

(4) To extend it to weak orderings, it suffices to enter $1/t$ in the Π matrix whenever an element is tied with t others.

Intuitively, this result is obvious if we realize that each row of II defines a (different) partition of the set of criteria $\{ 1, 2, \ldots, l \}$.

Proof: It suffices to prove (20) for any row i since the labeling of the alternatives (the reference order in the set A) is entirely arbitrary.

(i) By contradiction, let us suppose first that $\sum_{j=1}^{m} \Pi_{ij} > l$.

Then $\exists j$ and $k \in \{ 1, 2, \ldots, m \}$ and $h \in \{ 1, 2, \ldots, l \}$ such that $\varphi_h(a_i) = \{ a_j\, a_k \}$ with $j \neq k$ contrary to our assumption that the φ_h are *strict* orderings.

(ii) Now suppose $\sum_{j=1}^{m} \Pi_{ij} < l$.

Then $\exists\, i \in \{ 1, 2, \ldots, m \}$ and $h \in \{ 1, 2, \ldots, l \}$ such that $\varphi_h(a_i) = \emptyset$ contrary to our *completeness* assumption for the φ_h mappings. Hence we must have $\sum_{j=1}^{m} \Pi_{ij} = l$. The proof of (21) for any column j is exactly parallel to that for (20).

Q. E. D.

We now define a normalized agreement matrix Π^{norm}.

DEFINITION 3: A *normalized agreement matrix* Π^{norm} is an agreement matrix Π the rows and columns of which have all been divided by $l : \Pi^{\text{norm}} = (1/l)\,\Pi$.

An immediate corollary to Lemma 1 can now be stated.

COROLLARY 1: *Any normalized* $(m \times m)$ *agreement matrix* Π^{norm} *is a bistochastic matrix* β *of the same order.*

Proof: This follows directly from Lemma 1.

We can now make use of the geometrical characterization of the sets $\mathcal{S}_m$ and β afforded by the Birkoff-von Neumann theorem. We known from Corollary 1 that any agreement matrix Π—obtained from the individual strict orderings $\varphi_h \in \mathcal{S}_m$ as explained above—can be transformed through a simple normalization operation into an element $B \in \beta$, the convex polyhedron of all bistochastic matrices of the mth order. In a sense one can view this normalized agreement matrix as defining a complete stochastic (agregate) ordering on the set of alternatives A.

In this context an aggregation process could thus be considered as a mapping σ from the interior of β onto the set $\mathcal{S}_m$, i. e. the set of vertices of the convex polyhedron β, by the Birkhoff-von Neumann theorem. Such a σ mapping would clearly not be bijective since $\mathcal{S}_m$, the set of vertices of β (the permutation operators φ) is finite, whereas it is very easy to show that the set β has the power of the continuum.

The solution concept we shall now propose could be thought of as a *vertex projection method:* given some normalized agreement matrix Π^{norm}, we can search for the vertex $\hat{P}$ which is "closest" to B under some metric in $\mathbf{R}^{m^2} \supset \beta$. Here the goodness of fit criterion for determining the "best" aggregate ordering in the set $\mathscr{S}_m$ can be of a metric nature because of the properties of β and $\mathscr{S}_m$.

Under this solution concept, the aggregation problem can thus be stated:

$$\underset{p \in \mathscr{S}m}{\text{Min}}\ d(P, \Pi^{\text{norm}}) \tag{22}$$

$$\hat{P},\ \Pi^{\text{norm}} \in \mathbf{R}^{m^2};\ \Pi^{\text{norm}} \in \beta;\ \mathscr{S}_m \subset \beta \subset \mathbf{R}^{m^2}$$

Let us now choose as our metric d:

$$d(P, \Pi^{\text{norm}}) = \sum_{i,j} |\, p_{ij} - \pi_{ij}^{\text{norm}} \,|. \tag{23}$$

The maximal agreement approach and the minimal distance approach lead to the same solutions(s). This is proved in the following lemma.

LEMMA 2: *The maximal agreement problem* (equation 15) *and the minimal distance problem* (equation 23) *always lead to the same solution(s).*

Proof: As we recall the maximal agreement problem consisted in maximizing the agreement index I_A over the set $\mathscr{S}_m$ of all permutation matrices

$$\underset{[p_{ij}] \in \mathscr{S}_m}{\text{Max}}\ I_A = \sum_{i,j} \pi_{ij}\, p_{ij}, \tag{24}$$

where $[\pi_{ij}]$ is the agreement matrix defined in 2.2 (Def. 1). Clearly, any optimal solution $[p_{ij}^*]$ to (24) is also an optimal solution to:

$$\underset{[p_{ij}] \in \mathscr{S}_m}{\text{Max}}\ I'_A = \sum_{i,j} \pi_{ij}^{\text{norm}}\, p_{ij}. \tag{25}$$

Thus we simply need to show the equivalence between the "normalized" maximum problem (equation (25) and the minimal distance problem which is written

$$\underset{[p_{ij}] \in \mathscr{S}_m}{\text{Min}}\ d(P, \pi^{\text{norm}}) = \sum_{i,j} |\, p_{ij} - \pi_{ij}^{\text{norm}} \,|. \tag{23}$$

Let us denote $\{ i, j \}^*$ the set of row and column indices which maximize (25) above, i. e. :

$$\{ i, j \}^* = \{ (i, j) \,|\, p_{ij} = 1 \text{ and } \sum_{i,j} \pi_{ij}^{\text{norm}}\, p_{ij} \text{ is maximum} \}.$$

(The set $\{ i, j \}^*$ has m elements by definition.)

For this maximal solution $\{ i, j \}^*$, equation (23) becomes

$$\begin{aligned} &\sum_{(i,j) \in \{i,j\}^*} |\, 1 - \pi_{ij}^{\text{norm}} \,| + \sum_{(i,j) \notin \{i,j\}^*} \pi_{ij}^{\text{norm}} \\ &= m - \sum_{(i,j) \in \{i,j\}^*} \pi_{ij}^{\text{norm}} + \sum_{(i,j) \notin \{i,j\}^*} \pi_{ij}^{\text{norm}} \qquad (\text{as } \pi_{ij}^{\text{norm}} \in [0, 1]). \end{aligned} \tag{26}$$

Now we note that $\sum_{i,j} \pi_{ij}^{\text{norm}} = m$. Hence

$$\sum_{(i,j) \notin \{i,j\}^*} \pi_{ij}^{\text{norm}} = m - \sum_{(i,j) \in \{i,j\}^*} \pi_{ij}^{\text{norm}}. \tag{27}$$

In view of equation (27), we can write (26) as

$$\begin{aligned} m - \sum_{(i,j) \in \{i,j\}^*} \pi_{ij}^{\text{norm}} + m - \sum_{(i,j) \in \{i,j\}^*} \pi_{ij}^{\text{norm}} \\ = 2m - 2 \sum_{(i,j) \in \{i,j\}^*} \pi_{ij}^{\text{norm}}. \end{aligned}$$

To minimize this last equation for some $\{i, j\}$ permutation is equivalent to maximizing

$$-2m + 2 \sum_{(i,j) \in \{i,j^*\}} \pi_{ij}^{\text{norm}},$$

which has the same solution as the original maximization problem (25) since they only differ by a constant.

Q. E. D.

Furthermore, we note that the value of the objective function I_A for the normalized maximal agreement problem ranges over the closed interval $[1, m]$:

— if all orderings are identical we obtain:

$$\pi_{ij}^{\text{norm}} = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ 0 & 0 & . & . & . & 1 \end{bmatrix}$$

and $I'_A = m$;

— if all orderings are uniformly distributed, the agreement matrix becomes:

$$\pi_{ij}^{\text{norm}} = \begin{bmatrix} \frac{1}{m} & . & . & . & \frac{1}{m} \\ . & . & . & . & . \\ . & . & \frac{1}{m} & . & . \\ . & . & . & . & . \\ \frac{1}{m} & . & . & . & \frac{1}{m} \end{bmatrix}.$$

In this case the problem has multiple optima expressing the fact that any ordering is as good as any other as a representation of the distribution of individual orderings. Then π_{ij}^{norm} is the center of gravity of the β polyhedron and our aggregation method simply states that all elements of $\mathscr{S}_m$ (all vertices of β) are equally "accurate" in representing π_{ij}^{norm} (1).

Based in these upper and lower bounds for I'_A, we can define an overall similarity index (S) between all orderings as follows:

$$S = \frac{\text{Actual value of } I'_A \text{ at the optimum} - \text{Minimum}}{\text{Maximum} - \text{Minimum}},$$

or

$$S = \frac{\text{Actual} - 1}{m - 1}.$$

This index ranges over [0, 1] with

$S = 0$ when $I'_A = \text{Minimum} = 1$ and $S = 1$ when $I'_A = \text{Maximum} = m$.

Finally one may note that the solution to the "minimal" distance problem would be identical if we picked the Euclidean metric.

LEMMA 3 (2): *Let δ represent the Euclidean metric and d the "city-block" metric (as defined in 23 above).*

$\hat{P}$ is a solution to

$$\underset{P \in \mathscr{S}_m}{\text{Min}}\ \delta(P, \Pi^{\text{norm}}) = \Big[\sum_{i,j} (p_{ij} - \pi_{ij})^2\Big]^{1/2},$$

(1) It is easily shown that if the individuals orderings lead to such a point Π^{norm} at equal distance from all vertices of $\mathscr{S}_m$, we have a case of the "paradox of voting" — i. e. the fact that an intransitive group ordering results from pairwise majority voting aggregation of individual transitive orderings. For instance with three orderings:

$$\varphi_1(A) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \varphi_2(A) = \begin{bmatrix} b \\ c \\ a \end{bmatrix}, \quad \varphi_3(A) = \begin{bmatrix} c \\ a \\ b \end{bmatrix}$$

and we were to decide on an aggregate ordering by majority voting over each pair of alternatives we would obtain the following *intransitive* order: (a, b, c, a). In such a case, however, the normalized agreement matrix π^{norm} is given by equation (32) above; and our approach clearly indicates that any one of the three individual orderings is "optimal". The occurrence of any intransitivity is only a poor indicator of such an indeterminacy. A possible unique solution could then be reached through a completely randomized choice.

(2) I am indebted to Pierre Batteau for pointing out this result to me.

if and only if it is a solution to

$$\operatorname*{Min}_{P \in \mathscr{S}_m} d(P, \Pi) = \sum_{i,j} | p_{ij} - \pi_{ij} |.$$

Proof: Expand $\delta^2 (P, \Pi) = \sum_{i,j} (p_{ij} - \pi_{ij})^2$:

$$\begin{aligned}\delta^2 (P, \Pi) &= P_{11}^2 + \pi_{11}^2 - 2 P_{11} \pi_{11} + \ldots + P_{mm}^2 + \pi_{mm}^2 - 2 P_{mm} \pi_{mm} \\ &= \| P \|^2 + \| \Pi \|^2 - 2 P^T \Pi,\end{aligned}$$

where P^T denotes the transpose of the permutation matrix P written as an m^2-dimensional vector (and similarly for Π).

Note that $\| P \|^2 = K$ (a constant), $\forall P \in \mathscr{S}_m$.

So to minimize $\delta^2 (P, \Pi)$ is equivalent to maximizing the scalar product $P^T \Pi$ which is precisely the maximal agreement problem (1).

Q. E. D.

Let us now examine another view of our aggregation model.

2.3. Aggregation as a statistical estimation problem

Given the properties of the normalized agreement matrix π_{ij}^{norm} namely its being a bi-stochastic matrix, we can interpret each row (or column) of π_{ij}^{norm} as a probability distribution on the set of slots (or alternatives). Once we have obtained π_{ij}^{norm} the aggregation problem could be stated as a statistical estimation problem: which permutation matrix P_{ij} "best fits" this π_{ij}^{norm} matrix ?

If we select the least squares estimation criterion (2), the problem reads

$$\operatorname*{Min}_{[p_{ij}] \in \mathscr{S}_m} \sum_{i,j} (p_{ij} - \pi_{ij}^{\text{norm}})^2. \tag{27}$$

The following corollary follows directly from lemmas 2 and 3.

COROLLARY: *The least squares criterion yields the same solution as the maximal agreement and minimal distance criteria.*

This result casts a rather different light on the nature of our original solution concept and provides a link between the aggregation problem and the statistical inference problem. It appears that the analogy between the two problems could be explored further (3).

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(1) Actually, this equivalence holds not only for this maximal agreement problem, but whenever $p_{ij} \in \{ 0, 1 \}$.

(2) Here, the least squares solution is constrained to be a permutation matrix.

(3) See, for instance, [5].

APPENDIX: ILLUSTRATION [1]

Let $S = \{ a, b, c, d, e \}$:

$$l = 6,$$

$$S_1 = \begin{bmatrix} a \\ d \\ c \\ b \\ e \end{bmatrix}; \quad S_2 = \begin{bmatrix} a \\ c \\ d \\ b \\ e \end{bmatrix}; \quad S_3 = \begin{bmatrix} c \\ a \\ d \\ b \\ e \end{bmatrix};$$

$$S_4 = \begin{bmatrix} b \\ c \\ a \\ d \\ e \end{bmatrix}; \quad S_5 = \begin{bmatrix} e \\ b \\ a \\ d \\ c \end{bmatrix}; \quad S_6 = \begin{bmatrix} b \\ c \\ a \\ e \\ d \end{bmatrix},$$

$$\pi_{ij} = \begin{bmatrix} 2 & 1 & 3 & 0 & 0 \\ 2 & 1 & 0 & 3 & 0 \\ 1 & 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 4 \end{bmatrix},$$

$$\pi_{ij}^{\text{norm}} = \frac{1}{6} [\pi_{ij}].$$

The optimal orderings computed by the linear assignment procedure are

$$\begin{bmatrix} b \\ c \\ a \\ d \\ e \end{bmatrix} \text{ and } \begin{bmatrix} a \\ c \\ d \\ b \\ e \end{bmatrix},$$

for which the objective function takes on the value 14.

Or, if π_{ij}^{norm} is used

$$I'_A = \frac{2}{6} + \frac{3}{6} + \frac{3}{6} + \frac{2}{6} + \frac{4}{6} = \frac{14}{6}$$
$$= 2.33.$$

The value of the similarity index is:

$$S = \frac{2.33 - 1}{4} = .5825.$$

[1] We are indebted to an anonymous referee for suggesting this example.

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