RAIRO. RECHERCHE OPÉRATIONNELLE

J. RODRIGUES DIAS

Some approximate inspection policies for a system with imperfect inspections

RAIRO. Recherche opérationnelle, tome 24, nº 2 (1990), p. 191-199

<http://www.numdam.org/item?id=RO_1990__24_2_191_0>

Recherche opérationnelle/Operations Research
(vol. 24, n° 2, 1990, p. 191 à 199)

SOME APPROXIMATE INSPECTION POLICIES FOR A SYSTEM WITH IMPERFECT INSPECTIONS (*)

by J. Rodrigues Dias (1)

Abstract. — *Consider a system with an exponential lifetime distribution in which the bad working state is only known if it is inspected. Suppose that each inspection and each unit time of system bad working have fixed costs. If the inspections are not error-free, it was proved that the optimal inspection policy is not a periodic one.*

In this paper, we present for this model some approximate periodic inspection policies, with numerical results, proving that they are very good in practice. Particularly, we present and we analyse two simple solutions for the inspection period which are generalizations of previous approximations.

Keywords : Inspection policies; exponential lifetime; imperfect inspections; inspection period; approximations; costs comparison.

Résumé. — *Considérons un système avec un temps de vie exponentiel, dont l'état de mauvais fonctionnement est seulement connu s'il est examiné. Supposons aussi que chaque inspection et chaque unité de temps de mauvais fonctionnement ont des coûts constants. Si les inspections ne sont pas parfaites, on a prouvé que la politique d'inspection optimum n'est pas périodique.*

Dans cet article, nous présentons pour ce modèle quelques politiques approchées d'inspection périodique, avec des résultats numériques. On peut conclure que ces politiques sont très bonnes dans la pratique. En particulier nous présentons et nous analysons deux solutions très simples pour la période d'inspection, qui sont des généralisations d'autres approximations.

1. INTRODUCTION

Let us consider a system whose lifetime is a continuous random variable T with a reliability function $R(t)$ and a failure rate $h(t)$ given by

$$R(t)=1-F(t), \qquad t>0 \tag{1}$$

$$h(t)=f(t)/R(t), \qquad t>0 \tag{2}$$

(*) Received October 1989.
(1) University of Évora, Department of Mathematics, 7000 Évora, Portugal.

Recherche opérationnelle/Operations Research, 0399-0559/90/02 191 09/$ 2.90

where $f(t)$ and $F(t)$ are the density and the distribution functions of T. Let $E(T)$ be the expected value of T.

Also, let us suppose that the system failure is only known if the system is inspected.

Generally, a problem that we can consider is to obtain the inspection times t_k, with $t_0=0$ and $k=1, 2, 3, \ldots$, so that the expected total cost $E(C)$ is minimum during a cycle, with

$$E(C)=C_1 E(N)+C_2 E(D) \tag{3}$$

where: (*a*) $E(N)$ is the expected number of inspections during a cycle; (*b*) $E(D)$ is the expected time between the system failure and its detection; (*c*) C_1 is the cost of each inspection; (*d*) C_2 is the cost of each unit time of not detected bad working of the system. We consider that a cycle begins when the system is new and ends when the failure is detected.

This problem can find applications in quality control, medicine, nuclear energy, defense, etc.

Many models have been studied in the literature. In particular, we can refer, by its importance, a model analysed by Barlow and Proschan (1965). In Pierskalla and Voelker (1976) and in Sherif (1982) we can find a lot of different models and references. More recently, in Rodrigues Dias (1987 *a*) and in Kaio and Osaki (1988) other models and approximations can be found.

In this paper we are going to consider the next particular model: (*a*) the system has only two working states: a good one (concerning its lifetime) and a bad one (after a random failure); (*b*) the inspections are instantaneous and don't interfere in the system state; (*c*) there is a probability w smaller (or equal) than one ($w \leqq 1$) of detecting the system failure in a next inspection (so, the inspections are imperfect); (*d*) the system lifetime T has an exponential distribution, with $R(t)$ and $h(t)$ given by:

$$R(t)=\exp(-t/\alpha), \qquad t>0 \tag{4}$$

$$h(t)=1/\alpha. \tag{5}$$

This model was presented and studied by Sengupta (1982), proving that the optimal inspection policy is a not periodic one. This fact is surprising, if we have in mind that we are considering a constant failure rate.

In previous papers, we have already analysed this model and we have obtained interesting results [Rodrigues Dias (1985), (1988 *b*) and (1989)]. In particular, we could say that the minimum total expected cost strongly depends on the value of the probability w [on the contrary, this was not the conclusion of Sengupta (1982)]. Also, we have proved [Rodrigues Dias (1987 *a*), (1988 *b*)], in terms of costs, an equivalence between this model and another one concerning periodic and perfect inspections, but with a delay between the instant of each inspection and the instant in which the system state is known. From this equivalence, it was possible to conclude some results.

Now, the purpose of this paper is to present approximate periodic inspection policies for this model, which are very simple in practice, proving with numerical results that they are very good. In special, we present and we analyse two simple solutions for the inspection period which are generalizations of previous approximations.

2. THE OPTIMAL INSPECTION POLICY

Sengupta (1982) proved that for the model we are here considering the optimal inspection times t_k are given by

$$t_k = T_0 + k P, \qquad k = 1, 2, 3, \ldots \tag{6}$$

having presented the following pair of equations (with other symbols) to obtain T_0 and P:

$$C_2 - [(C_1 + C_2 P)/\alpha] \exp[-(T_0 + P)/\alpha]/[1 - \exp(-P/\alpha)] = 0 \tag{7}$$

$$(1 - w)/w - \exp(-P/\alpha)/[1 - \exp(-P/\alpha)] + C_2/[(C_1 + C_2 P)/\alpha] = 0. \tag{8}$$

In this model we can obtain for $E(N)$ and $E(D)$ the expressions:

$$E(N) = \exp[-(T_0 + P)/\alpha]/[1 - \exp(-P/\alpha)] + 1/w \tag{9}$$

$$E(D) = P E(N) + T_0 - E(T). \tag{10}$$

Minimizing $E(C)$ given by (3), with $E(N)$ and $E(D)$ given by (9) and (10), we have obtained [Rodrigues Dias (1987 *a*)] the pair of simpler equations to obtain T_0 and P:

$$\exp(T_0/\alpha) = [(r + P)/\alpha]/[\exp(P/\alpha) - 1], \qquad r = C_1/C_2 \tag{11}$$

$$\exp(T_0/\alpha) = w \exp(-P/\alpha)/[\exp(-P/\alpha) + w - 1]. \tag{12}$$

These equations, which have the same first member, are quite interesting because in the first of them we have only the coefficient r between the costs

C_1 and C_2 (T_0 and P don't depend on the individual values of C_1 and C_2) and in the second one we have only the probability w.

If we replace in (11) the first member by the second member of (12), we can calculate P. So, the value of T_0 can be calculated using the expression:

$$T_0 = \alpha \log \{[(r+P)/\alpha]/[\exp(P/\alpha)-1]\}. \quad (13)$$

It is easy to prove that when $w=1$, so we obtain $T_0=0$. In this case we have periodic inspections, with inspection period P calculated from the equation:

$$\exp(P/\alpha) - P/\alpha - 1 - r/\alpha = 0. \quad (14)$$

3. APPROXIMATE PERIODIC INSPECTION POLICIES

As the periodic inspections are very simple in practical applications, in this point we are going to present three approximate periodic inspection policies for the model we are analysing.

In the first one, we present the optimal periodic inspection policy considering that the inspections are imperfect. In the other two policies, we obtain approximate solutions obtained from the first one.

In particular, we present two simple expressions for the inspection period, which are generalizations of previous ones.

We calculate for these three periodic policies the expected total costs and we compare them with the minimum total expected cost of the optimal not periodic inspection policy, concluding that the approximations are very good. We present results for different values of r and w.

3.1. The optimal periodic inspection policy

When we are minimizing the total expected cost $E(C)$, given by (3), with $E(N)$ and $E(D)$ given by (9) and (10), if we consider $T_0=0$ (periodic inspections), we can obtain the following equation from which it is possible to calculate the optimal inspection period P, concerning this optimal periodic inspection policy (for imperfect inspections) :

$$-r/\alpha + [1 - P/\alpha - \exp(-P/\alpha)] + [\exp(P/\alpha) + \exp(-P/\alpha) - 2]/w = 0. \quad (15)$$

If we do $w=1$, we obtain, naturally, the equation (14).

We are going to represent by $E(C)_0$ the minimum total expected cost concerning the optimal not periodic inspection policy and by $E(C)_1$ the minimum total expected cost of this periodic inspection policy.

To compare $E(C)_1$ with $E(C)_0$ for different values of r and w, we present in Table I the relative errors, in percentage, given by S_1:

$$S_1 = [E(C)_1 - E(C)_0]/E(C)_0 . 100\,\%. \qquad (16)$$

TABLE I

Relative errors (S_1), in percentage, concerning the optimal periodic inspection policy.

		$r = C_1/C_2$				
		.05	.1	.2	.4	.8
w	1.0	0	0	0	0	0
	.9	.12	.16	.20	.25	.29
	.8	.45	.58	.75	.92	1.08
	.7	.95	1.25	1.59	1.95	2.28
	.6	1.63	2.14	2.73	3.33	3.86
	.5	2.51	3.29	4.17	5.07	5.82

From these values, we can point out the next conclusions:

(*a*) When $w = 1$, the relative errors are, obviously, equal to zero. In fact, in this case, the optimal inspection policy is a periodic one.

(*b*) When w decreases and r increases, the relative errors increase. However, for the wide ranges of values of w and r that we have considered, these relative errors are always smaller than 6% (and almost smaller than 2%).

(*c*) The relative errors depend more significantly on the values of the probability w than the values of r.

(*d*) Considering values of w not very small, which can be found in practical applications (corresponding to a not strongly imperfect inspection), the relative errors are very small (smaller than 1%).

Finally, we can say that this optimal periodic inspection policy is a very good approximation (a nearly optimal one) of the optimal not periodic inspection policy, presented by Sengupta (1982). In Rodrigues Dias (1985) we have presented other results and conclusions.

3.2. An approximate periodic inspection policy

We have seen that the optimal periodic inspection policy is a nearly optimal one concerning the Sengupta (1982) model. However, to obtain the period P we have to solve the equation (15).

So, a practical question we can pose is to obtain a simple expression to calculate approximately the period P. Nakagawa and Yasui (1979), considering a result obtained by Schneeweiss (1976), had obtained the next approximation, when the inspections are perfect:

$$P \approx [2\,r\,E(T)]^{1/2}. \tag{17}$$

We have analysed this result in terms of costs [Rodrigues Dias (1983)], having concluded that it was much better than their authors had referred (in terms of the relative errors of the inspection period).

When the lifetime distribution is exponential [with $E(T)=\alpha$], the result (17) can be obtained from (14) if we consider:

$$\exp(P/\alpha) \approx 1 + P/\alpha + (P/\alpha)^2/2. \tag{18}$$

In a quite similar way [Rodrigues Dias (1985)], we have obtained from (15) the next approximate expression P^* for the period P:

$$P^* = [2\,r\,E(T)]^{1/2}\,.\,[w/(2-w)]^{1/2}. \tag{19}$$

It can be seen easily that when $w=1$ (perfect inspections) we have the result (17). So, it can be said that this new result (19) is a generalization of that one obtained by Nakagawa and Yasui (1979) for the case of imperfect inspections, beeing $[w/(2-w)]^{1/2}$ the generalization factor.

It is now important to analyse how good this new approximation is in terms of costs. To do that, we are going to calculate the relative errors S_2 of the total expected cost $E(C)_2$ concerning this approximate periodic inspection policy, with period P^*:

$$S_2 = [E(C)_2 - E(C)_0]/E(C)_0\,.\,100\%. \tag{20}$$

The results obtained are in Table II.

The conclusions are similar to those of the previous section 3.1. The relative errors are, naturally, greater than those concerning the optimal periodic policy. However, the differences are irrelevants. So, we can say that this approximate periodic inspection policy is a nearly optimal periodic inspection policy.

Finally, we can say that the inspection period P^* concerning a periodic inspection policy is a nearly optimal solution for the Sengupta (1982) model.

TABLE II

Relative errors (S_2), in percentage, concerning an approximate periodic inspection policy.

		$r=C_1/C_2$				
		.05	.1	.2	.4	.8
w	1.0	.13	.26	.49	.92	1.69
	.9	.19	.30	.49	.81	1.35
	.8	.49	.67	.91	1.25	1.73
	.7	.97	1.29	1.69	2.14	2.66
	.6	1.65	2.17	2.8	3.4	4.08
	.5	2.5	3.3	4.2	5.1	5.9

3.3. Another approximate periodic inspection policy

In Rodrigues Dias (1987 *b*) we have presented a new approximation for the inspection period, when the inspections are perfect and when the failure rate is constant (exponential distribution):

$$P \approx [2\,r\,E(T)]^{1/2}/[1+.234\sqrt{r'}], \qquad r'=r/E(T). \tag{21}$$

This result can be interpreted as a generalization of that one of Nakagawa and Yasui (1979). It was seen that this approximate solution is a nearly optimal one. In Rodrigues Dias (1988 *a*) we have analysed it for other types of failure rates.

Now, the idea is to include the factor $1/[1+.234\sqrt{r'}]$ in the expression of P^* given by (19) to obtain a new approximation P^{**}:

$$P^{**}=[2\,r\,E(T)]^{1/2}.[w/(2-w)]^{1/2}/[1+.234\sqrt{r'}]. \tag{22}$$

If $E(C)_3$ is the expected total cost concerning this new approximation, we can calculate the relative errors S_3, in percentage:

$$S_3=[E(C)_3-E(C)_0]/E(C)_0\,.\,100\%. \tag{23}$$

The results obtained are in the Table III.

We can point out the next conclusions:

(*a*) For $w=1$ the relative errors are so small that they can be considered, for practical purposes, equal to zero, as it was shown in Rodrigues Dias (1987 *b*). In fact, they are, for the values of r considered here, smaller than 0.000 1%.

(*b*) For values of w between 1 and .8, the relative errors are smaller than those obtained using P^* (almost smaller than 1%).

TABLE III

Relative errors (S_3), in percentage, concerning another approximate periodic inspection policy.

		$r=C_1/C_2$				
		.05	.1	.2	.4	.8
w	1.0	.00	.00	.00	.00	.00
	.9	.13	.17	.22	.28	.35
	.8	.47	.62	.81	1.03	1.26
	.7	.99	1.32	1.72	2.17	2.62
	.6	1.69	2.25	2.91	3.65	4.36
	.5	2.59	3.42	4.41	5.48	6.48

(*c*) For $w < .7$ the relative errors are greater than those obtained using P^*.

As a main conclusion, we can say that this new approximate solution P^{**} for the inspection period is also a nearly optimal one and it is a generalization of (17) and (19) for the case of imperfect inspections.

4. CONCLUSIONS

In this paper we analyse a model considered by Sengupta (1982). We present simpler expressions to calculate the optimal not periodic inspection policy. Also, we refer that it is possible to obtain interesting results about this model [Rodrigues Dias (1985), (1988 *b*) and (1989)].

However, the fundamental purpose of this paper is to present and analyse three approximate periodic inspection policies for this model with imperfect inspections. We must note that, in practical applications, periodic inspection policies are simpler than not periodic ones.

The main conclusion is that the relative errors concerning all the three periodic inspection policies are very small, specially if the system inspection is not strongly imperfect. In this case, the relative errors are smaller than 1%. So, we can say that these approximate periodic inspection policies are very good for practical purposes. In this way, they can be considered nearly optimal ones.

On other hand, we present two approximate solutions for the inspection period which can be easily calculated and which are generalizations (for the case of imperfect inspections) of a previous approximation obtained by Nakagawa and Yasui (1979). We prove, with numerical results, that these two approximate solutions are very good in practice, since the corresponding relative errors are very small (in many practical cases they are smaller than 1%).

REFERENCES

R. E. Barlow and F. Proschan, *Mathematical Theory of Reliability*, John Wiley, New York, 1965.

N. Kaio and S. Osaki, *Inspection Policies: Comparisons and Modifications*, Recherche Opérationnelle/Operations Research, Vol. 22, No. 4, 1988, pp. 387-400.

T. Nakagawa and K. Yasui, *Approximate Calculation of Inspection Policy With Weibull Failure Times*, I.E.E.E. Trans. on Reliability, Vol. R-28, No. 5, 1979, pp. 403-404.

W. P. Pierskalla and J. A. Voelker, *A Survey of Maintenance Models: The Control and Surveillance of Deteriorating Systems*, Naval Research Logistics Quarterly, Vol. 23, No. 3, 1976, pp. 353-388.

J. Rodrigues Dias, *Influence de la Période d'Inspection sur les Coûts dans l'Inspection Périodique de Systèmes*, Revue de Statistique Appliquée, Vol. XXXI, No. 4, 1983, pp. 5-15.

J. Rodrigues Dias, *A Nearly Optimal Periodic Policy With Imperfect Inspection and an Exponential System Failure*, Proceedings of the VII Conference of Mathematicians of Latin Expression (G.M.E.L.), Vol. II, 1985, pp. 229-232, University of Coimbra.

J. Rodrigues Dias, *Systems Inspection Policies*, Ph. D. Thesis, University of Évora, 1987*a*.

J. Rodrigues Dias, *A Solution for the Inspection Period of Systems*, XII Portuguese-Spanish Conference on Mathematics, University of Minho, 1987*b*.

J. Rodrigues Dias, *A New Approximation for the Inspection Period of Systems With Different Failure Rates*, European Journal of Operational Research, Vol. 45, No. 2-3, 1990, pp. 219-223.

J. Rodrigues Dias, *Equivalence Between Two Inspection Models of Systems*, XIII Spanish-Portuguese Conference on Mathematics, University of Valladolid, 1988*b*.

J. Rodrigues Dias, *Some Results Concerning Two Equivalent Inspection Models of Systems*, XIV Spanish-Portuguese Conference on Mathematics, University of La Laguna, 1989.

W. G. Schneeweiss, *On the Mean Duration of Hidden Faults in Periodically Checked Systems*, I.E.E.E. Trans. on Reliability, Vol. R-25, No. 5, 1976, pp. 346-348.

B. Sengupta, *An Exponential Riddle*, Journal of Applied Probability, Vol. 19, 1982, pp. 737-740.

Y. S. Sherif, *Reliability Analysis: Optimal Inspection and Maintenance Schedules of Failing Systems*, Microelectronic Reliability, Vol. 22, No. 1, 1982, pp. 59-115.