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HEURISTIC METHODS FOR THE p-CENTER PROBLEM (*)

by Blas Pelegrin (1)

Abstract. — This paper deals with heuristic methods for the p-center problem in any metric space. Due to the complexity of this problem, a variety of heuristic algorithms have been proposed in different cases, usually in Network and Planar Location. Most of them are related to some class of method that can be used for the problem in any metric space. A review of these methods is presented and a new method is also proposed which is a 2-approximation heuristic for some particular cases of the problem.

Keywords : Location ; Cluster Analysis ; Heuristics.

1. INTRODUCTION

Let \( X \) be any metric space, with metric \( d(\cdot, \cdot) \), and \( M = \{ P_1, P_2, \ldots, P_m \} \) be any finite subset of \( X \), with positive weights \( w_i \) for each \( P_i \in M \). In a general framework, the \( p \)-center problem is defined as to find \( p \) points \( C_1, C_2, \ldots, C_p \) in \( X \) so that the maximal weighted distance between each point \( P_i \) and its closest point \( C_j, j = 1, 2, \ldots, p \), is minimized. The problem is formulated as:

\[
\text{(F1) } \begin{array}{c}
\text{Minimize } \sum_{C_1, \ldots, C_p \in X} \max_{1 \leq i \leq m} \min_{1 \leq j \leq p} d(P_i, C_j) \\
\end{array}
\]

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Alternatively, the problem can be seen as to find a partition 
\( \alpha = \{M_1, M_2, \ldots, M_p\} \) of the set \( M \) into \( p \) disjoint subsets, so that the 
maximum among the maximal weighted distances between the best point (1-
center) and the points in each subset is minimized. Then it is formulated as:

\[
(F2) \quad \text{Minimize } R_\alpha = \max \{ R(M_1), R(M_2), \ldots, R(M_p) \}
\]

where \( P(M, p) \) denotes the set of all partitions of \( M \) into \( p \) disjoint subsets 
and \( R(M_j) \) denotes the optimal value of the 1-center problem associated with 
the points in \( M_j, j=1, \ldots, p \). Then optimal centers, \( C_1, C_2, \ldots, C_p \), are 
optimal solutions to the 1-center problems associated with an optimal parti-
tion.

The problem arises typically in the field of Location Theory, where it has 
been studied mainly in two contexts: Network Location and Planar Location. 
The first, when \( X \) is a network, \( M \) is the set of nodes, and \( d \) is given by the 
shortest path (see [12 to 15, 18]). The second, when \( X \) is \( \mathbb{R}^2 \) and \( d \) is given 
by a norm function, usually the euclidean or the rectangular norm (see [1, 7, 
23, 29, 30]).

The problem can also be found in Cluster Analysis when the aim is to 
create a number \( p \) of groups in a set of objects \( M \), given by points in \( \mathbb{R}^n \), so 
that the maximal dissimilarity, measured by a metric \( d \), between each object 
\( P_i \) and the center \( C_j \) of its group is minimized (see [27]).

When the set of centers is constrained to be in a finite set, the problem is 
usually known as the \textit{p-center problem}. Otherwise, it is also known as the \textit{absolute p-center problem} in Network Location and as the \textit{continuous p-center} 
problem in Planar Location and Cluster Analysis. In all these cases, it has 
been proved to be \textit{NP}-hard, even to approximate the problem sufficiently 
closely (see [17, 19, 20]). Due to its complexity, a variety of heuristic algo-
rithms have been proposed in the literature in different cases, usually in 
Network and Planar Location (see [6, 9, 10, 12, 16, 23, 25]).

This paper deals with heuristic methods for the \( p \)-center problem in any 
metric space. Formulation (F2) is used and special attention is given to \( \delta \)-
approximate heuristics. Firstly, we review some methods, most proposed for 
particular cases of the problem, which can be used for any metric space. 
Secondly, we propose a new heuristic which is shown to be a 2-approximation 
algorithm in some particular cases. It is known that to produce partitions 
within \( \delta \) times the optimum is \textit{NP}-hard for \( \delta < 2 \) (see [16, 17]). Thus, in certain 
cases, the algorithm is “best possible”. Finally, some general considerations 
are mentioned.

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2. REVIEW OF HEURISTIC METHODS

Many heuristic algorithms, proposed for some particular cases of the \( p \)-center problem, are related to some class of methods that can be used for the problem in the above general framework. In the following, we review heuristic methods of two types that can be extensively used.

**Type 1: heuristics based on the 1-center problem**

Most of heuristic proposed for other location-allocation problems, for instance the alternate location-allocation method and the exchange method (see [3, 4, 26, 28]), can be modified to be used for the \( p \)-center problem. Each of them iterates in the set of partitions \( P(M, p) \) and contains a single center subroutine (SCS). This modified type of methods for the \( p \)-center problem is as follows: Start with any partition \( \alpha = \{ M_1, M_2, \ldots, M_p \} \). Use SCS to evaluate \( R(M_j) \) and to obtain the center \( C_j \) associated with each subset \( M_j \). Once \( R_\alpha \) is evaluated, either a new partition \( \alpha' \) with \( R_\alpha < R_\alpha \) is generated or the procedure is finished.

These methods can be used for the problem in any metric space whenever a SCS is given. If each center is constrained to be in \( M \), which often happens in Cluster Analysis, the SCS is given by enumeration. In other cases, some SCS have been proposed for absolute and continuous centers under different measures of distance (see [2, 8, 11, 18, 21, 24]).

**Type 2: \( \delta \)-approximation heuristics**

Other proposed heuristics generate output partitions without using any SCS. The most interesting are those that generate \( \delta \)-approximate solutions, i.e., some partition \( \alpha \) such that \( R_\alpha \leq \delta R^* \) where \( R^* \) denotes the optimal value of (F2). To our knowledge, only one of these heuristics has been proposed for any metric space. It is as follows:

*Heuristic H1*

1. Choose \( C_1 = P_k \) such that \( w_k = \max \{ w_i : i = 1, \ldots, m \} \). Set \( j = 1 \) and \( d(P_i) = w_i d(P_i, C_1) \) for each \( P_i \in M \).
2. While \( j < p \) do:

\[
C_{j+1} = P_i \quad \text{such that} \quad d(P_i) = \max \{ d(P_j) : i = 1, \ldots, m \}
\]

\[
d(P_0) = \min \{ d(P_i), w_i d(P_i, C_{j+1}) \} \quad \text{for each} \quad P_i \in M.
\]
3. Determine $\alpha = \{M_1, M_2, \ldots, M_p\}$ where

$$M_j = \{P_i : w_i d(P_i, C_j) \leq w_i d(P_i, C_k), k = 1, \ldots, m\}$$

for $j = 1, \ldots, m$. If a point $P_i$ belongs to more than one set $M_j$ assign it arbitrarily to one set.

H1 was given by Dyer and Frieze (9) for points in any metric space, and guarantees $\delta_0 = \min\{3, 1 + \beta\}$, where $\beta$ is the maximal ratio between the weights of points in $M$.

However, when a finite set $L$ of possible values for $R^*$ is known, as happens for $X = M$ and for some continuous problems, the following heuristic can also be used:

**Heuristic H2**

1. Arrange all the values in $L$ into an increasing sequence $R_1 < R_2 < \ldots < R_q$.

2. Find the least value $R \in \{R_1, R_2, \ldots, R_q\}$ for which the following subroutine yields an output $S$ with $|S| \leq p$

   RANGE:
   
   - Set $S = \emptyset$ and make unlabelled all points in $M$.
   - While there is any unlabelled point in $M$ do:
     
     Choose an unlabelled point $P_t$ of maximum weight, put $S = S \cup P_t$, and label $P_t$ and every unlabelled point $P_i$ such that $w_i d(P_i, P_t) \leq 2R$.

3. Augment $S$ arbitrarily to a set of $p$ points. Denote $C_1, C_2, \ldots, C_p$ these points.

4. Determine $\alpha = \{M_1, M_2, \ldots, M_p\}$ where

$$M_j = \{P_i : w_i d(P_i, C_j) \leq w_i d(P_i, C_k), k = 1, \ldots, m\}$$

for $j = 1, \ldots, m$. If a point $P_i$ belongs to more than one set $M_j$ assign it arbitrarily to one set.

H2 was given by Plesnik (25) for points in a network and guarantees $\delta = 2$ for both the $p$-center and the absolute $p$-center problems. However, as it is shown above, H2 can be used for any metric space, whenever a finite set of possible values for $R^*$ is known. Then H2 is also a 2-approximation heuristic, since RANGE output $S$ with $|S| \leq p$ for a given $R$ if there exists a partition $\alpha$ with $R_a \leq R$ (see [25]), which implies $R_a \leq 2R_0 \leq 2R^*$ where $R_0$ is the least value found by RANGE.
3. A NEW HEURISTIC METHOD

We propose a new heuristic method for the problem in any metric space. The method is based on a lower bound of $R(M_j)$ given in Dearing and Francis (5) by:

$$B(M_j) = \max \left\{ \frac{w_iw_kd(P_i, P_k)}{w_i+w_k} : P_i, P_k \in M_j \right\}$$

$$j = 1, \ldots, p.$$

First, let $R_{ik}$ denotes the value $w_iw_kd(P_i, P_k)/(w_i+w_k)$ for each $i, k = 1, \ldots, m,$ $i \neq k.$

**Heuristic H3**

1. Choose an initial partition $\alpha = \{M_1, M_2, \ldots, M_p\}$ and calculate $R_0 = \max \left\{ B(M_1), B(M_2), \ldots, B(M_p) \right\}.$

2. Make a list $L$ arranging all the distinct values $R_{ik} \leq R_0$ into an increasing sequence.

3. If $|L| > 1$ take any $R \in L$ such that $R < R_0,$ make unlabelled all points in $M$ and go to 4. Else, $L = \{ R_0 \},$ output $R_0$ and $\alpha.$

4. Choose an unlabelled point $P_i$ of maximum weight and set $M'_i = P_i \cup \{ P_i : P_i$ is unlabelled and $R_{it} \leq R \}.$ Label all points in $M'_i.$ If all points in $M$ are labelled go to 5, else go to 4.

5. Set $\alpha' = \{ M'_i : M'_i$ is generated in 4 $\}.$ If $|\alpha'| \leq p$ reset $L = \{ R_{ik} : R_{ik} \leq R \},$ $\alpha = \alpha',$ and $R_0 = R.$ Else reset $L = \{ R_{ik} : R_{ik} > R \}.$ Go to 3.

We now state some properties of H3.

**Property 1.** The complexity of H3 is $O(m^2 \log m)$.

**Proof.** Steps 1 and 2 can be performed in time $O(p \log m)$ and $O(m^2 \log m)$ respectively. Step 3 to step 5 is a binary search that find the minimum value in $L$ for which the partition $\alpha'$ generated in step 4 verifies $|\alpha'| \leq p.$ As step 4 is $O(m^2)$ and the binary search is $O(\log m), it follows that the complexity of H3 is $O(m^2 \log m).$

**Property 2.** If $R^* \in L$ then H3 is a 2-approximation algorithm and the output value $R_0$ is a lower bound of $R^*.$

**Proof.** Let $C'_i$ denotes an optimal solution to the 1-center problem associated with $M'_i$ for each $M'_i$ generated in step 4. It is verified that

$$R(M'_i) = \max \left\{ w_id(P_i, C'_i) : P_i \in M'_i \right\} \leq \max \left\{ w_id(P_i, P) : P_i \in M'_i \right\}.$$
As $R_{it} \leq R$ and $w_i \leq w_t$ then $w_i d(P_i, P_t) \leq (w_i + w_t) R/w_t \leq 2 R$ for each $P_t \in M_i'$. Therefore $R(M_t') \leq 2 R$ and $R_\alpha = \text{Max} \{ R(M_t') : M_t' \text{ is generated in step 4} \} \leq 2 R$.

Let $R$ be such that there exists a partition $\alpha = \{ M_1, M_2, \ldots, M_p \}$ with $R_\alpha \leq R$. In each iteration of step 4 the chosen point $P_t$ belong to $M_j$ for some $j$. If $P_t \in M_j$ and $P_t$ is unlabelled when $P_t$ is chosen, as $R_{it} \leq B(M_j) \leq R(M_j) \leq R_\alpha \leq R$, it follows that $P_t \in M_j'$. Therefore $|\alpha'| \leq p$, where $\alpha' = \{ M_t' : M_t' \text{ is generated in step 4} \}$. Then, H3 iterates while there is some $R$ in $L$ such that $R^\ast \leq R$.

From the above results, if $R^\ast \in L$ the output value $R_0$ and the output partition $\alpha$ will verify that $R_0 \leq R^\ast$ and $R_\alpha \leq 2 R_0 \leq 2 R^\ast$.

**Property 3.** — If $R^\ast \in L$ and $w_i = w$, $i = 1, \ldots, m$, then H2 and H3 are equivalents. Besides, H1, H2 and H3 are 2-approximation heuristics.

**Proof.** — As $w_i = w$, $i = 1, \ldots, m$, $w_i d(P_i, P_t) \leq 2 R$ is equivalent to $R_{it} \leq R$ for any $R \in L$. Then, each iteration of RANGE in H2 generates the same set of labelled points that the corresponding iteration of step 4 in H3. Therefore, starting with the set $L$, H2 and H3 generate the same output partition. Since $\delta_0 = 2$, it follows that H1, H2, and H3 are 2-approximation heuristics.

From property 2 it follows that H3 is a 2-approximation algorithm when $B(M_j) = R(M_j)$, $j = 1, \ldots, p$. Necessary and sufficient conditions for this are given in (5) and (22). For instance, it happens in the following particular cases (see [5, 11, 22, 24]):

(a) $X$ is a tree in Network Location.

(b) $X = \mathbb{R}^n$ and $M$ is any set of collinear points.

(c) $X = \mathbb{R}^2$ and $d$ is given by the Rectilinear norm.

(d) $X = \mathbb{R}^n$ and $d$ is given by any weighted Tchebycheff norm.

In such cases, an interesting aspect of H3, in comparison with H1 and H2, is that H3 generates an output partition $\alpha$ for which is not necessary to use any SCS to evaluate $R_\alpha$. Furthermore, H3 gives a better approximation than H1 for weighted problems since then $\delta_0 > 2$.

In any metric space, in spite of H3 can be used, there is no guarantee that it generates a partition within 2 times $R^\ast$. A sufficient condition for this is that $R_0 \leq R^\ast$.
4. GENERAL CONSIDERATIONS

We have shown different heuristic methods for the $p$-center problem in any metric space. Heuristic methods of Type 1 can be used whenever a SCS is given. Usually, these methods are time consuming, so we recommend to use them after someone heuristic of Type 2. Besides, the quality of an output partition given by an heuristic of Type 2 can be improved by an heuristic of Type 1, taking that partition as an initial partition.

Concerning to heuristics of Type 2, H2 or H3 can be normally used for case (a) to case (d) given in section 3, and H2 for $p$-center problems on a Network. For all these cases, output partitions within $2$ times $R^*$ can be generated, and an upper bound of the related error $(R_a - R^*)/R^*$ is given by $(R_a - R_0)/R_0$. Furthermore, SCS have been given and then heuristics of Type 1 can be used to improve the quality of their output partitions. For problems in any metric space, H2 or H3 can also be used, but 2-approximation is not guaranteed unless a finite set of possible values for $R^*$ is known or $R^* \in \{ R_{ik} : i, k = 1, \ldots, m, i \neq k \}$; however, H1 can always be used as a $\delta_0$-approximation.

REFERENCES


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