COLUMN-GENERATION IN INTEGER LINEAR PROGRAMMING

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Abstract. We present an exact method for integer linear programming problems that combines branch and bound with column generation at each node of the search tree. For the case of models involving binary column vectors only, we propose the use of so-called geometrical cuts to be added to the subproblem in order to eliminate previously generated columns. This scheme could be applied to general integer problems without specific structure. We report computational results on a successful application of this approach to a telecommunications network planning problem.

Keywords. Column-generation, integer programming, branch-and-price.

1. INTRODUCTION

The techniques of column-generation were first presented in the early sixties as part of methods to solve linear programming (\textit{LP}) problems with a huge number of variables. When the size of the problem does not allow one to store the entire

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matrix of a LP problem, columns to enter the basis in the Simplex Method are generated by means of an auxiliary problem. Of course this is only possible for some problems with a special structure. In [15] Dantzig and Wolfe introduced those techniques in the context of their very well known decomposition algorithm, developed originally to deal with the limited storage capacity computers had at that time.

In their seminal works [27, 28], Gilmore and Gomory used this approach to solve the cutting stock problem. They generate columns by solving in each step a knapsack subproblem. The original problem is an Integer Programming (IP) problem; column generation was used to solve the linear programming relaxation. This relaxation has shown to provide very good lower bounds in practice. Two decades later, Marcotte [37] showed that the optimal value is very often the rounded optimal (LP) relaxation.

Since then several other applications of column generation to obtain tighter relaxations to (IP) problems have been developed, which can be found in the literature.

Column generation methods to solve integer programming problems exactly appeared more recently. So called Branch-and-price methods combine Branch and Bound with column generation methods to solve the (LP) relaxation at each node. Some ad-hoc branching rules have been designed to keep the original structure of the pricing problem tractable, all along the branch and bound tree. Sometimes exact methods derive into heuristics when only a subset of the feasible columns in the nodes of the branch and bound tree is used or only the ones generated in the root node are kept.

Pioneering work in this sense was carried out in the eighties by Desrosiers et al. [24], Desrochers and Soumis [23], and Ribeiro et al. [48]. In these initial works they mentioned the difficulties of combining branching rules with column generation, and gave two possible ways of getting rid of some of them. One of the first Branch-and-price methods to appear in the literature was the one presented in [24] for the vehicle routing problem with time windows. Desrosiers et al. modeled the problem as a set partitioning problem where columns were generated using a modified shortest path algorithm which takes into account time windows. Branching rules that preserve this shortest path structure of the subproblem were used. When it was not possible to enumerate the entire branch and bound tree, they were able to obtain good lower bounds. A similar column generation approach to the Crew Scheduling Problem in Urban Transit that uses the set covering and constrained shortest path problems as subproblems was presented in [23].

In [48] an algorithm for a traffic assignment problem arising in a satellite switching system which can be modeled as a large scale set partitioning problem is presented. The algorithm combines column generation at each node with a ranking procedure of the columns that prevents regeneration of an already generated column at the subproblem. Optimality of the final integer solution is thus ensured. This ranking procedure is based on the particular structure of the subproblem, and it is based on an algorithm for obtaining the sequence of best assignments of
a linear assignment problem. Since then numerous applications of column generation, to these and other combinatorial problems appeared in the literature. Several other successful applications of column generation techniques to vehicle routing problems are described in [7, 21, 22, 36, 49].

Ribeiro and Soumis [49] also presented a column generation approach for solving the linear programming relaxation of the multiple depot vehicle scheduling problem. They obtained a better bound than those appearing in the literature till then.

Loebel [36] developed a column generation method for the LP relaxation of vehicle scheduling problems in public transit that provides very good solutions for huge real cases.

In their comprehensive survey on constrained scheduling and vehicle routing problems Desrosiers et al. [22] stated that optimal algorithms based on Dantzig–Wolfe decomposition and column generation schemes have been shown to be the most powerful solution methodologies for that kind of problems.

Bramel and Simchi–Levi [7] intend to formalize an empirical conclusion that comes out of applications to VRPTW (vehicle routing problem with time windows) problems: when column generation techniques are applied to problems modeled as set covering ones, the method works well when the gap between the (LP) relaxation and the (IP) solution is small. They show that with some assumptions about the distribution of the customers, in the case of VRPTW this gap decreases to zero as the number of customers increases. In [57] Vance et al. present a decomposition algorithm for airline crew scheduling problem that improved bounds provided in previous works. Gamache et al. [26] present also a column generation heuristic method for solving a aircrew rostering problem. This method was able to obtain very good results on real large scale problems.

Vance et al. [56] implemented an exact algorithm for the one dimensional binary cutting stock problem. They propose branching rules that keep the structure of the subproblem tractable at each node (those rules had been previously proposed by Ryan and Foster [47]). In [55] Vance compares two algorithms for the same problem based on two different formulations of the master problem, and presents appropriate branching rules for each one.

Carvalho presents in [16] a different approach for the general, not necessary binary, case of the cutting stock problem. An arc-flow formulation with side constraints is proposed and solved by column generation techniques. The author claims that the algorithm works well when the gap between the first (LP) relaxation and the optimal integer value is less than one. He also concludes that it is sensitive to the width of the rolls. In [58] Vandeberck shows results of a column generation algorithm for the bin packing and cutting stock problems.

Mehrotra and Trick [41] developed a method for solving the graph coloring problem using the independent set formulation, which avoids some extended symmetry. They use customized branching rules that are similar to the ones in [56]. This approach leads to a heuristic when columns are generated only in the root node.
In [33] column generation was used in order to produce heuristics based on linear programming relaxations for the graph partitioning problem.

Bourjolly et al. [5] obtain lower bounds for the maximum stable set problem by means of column generation in the framework of a branch and bound method that does not use linear programming explicitly.

Savelsbergh [50] presents a branch-and-price algorithm to obtain optimal integer solutions for the generalized assignment problem using again a set partitioning formulation. He also shows that truncating the search tree in his procedure yields very good approximation algorithms.

In [60] Van Den Akker et al. describe a column generation method for solving the (\textit{LP}) relaxation of a parallel machine scheduling problem. They obtain a very good lower bound and are able to solve some problems to optimality. Hansen et al. [29] propose primal and dual algorithms for mixed integer programming and their use to solve the probabilistic maximum satisfiability problem.

Barnhart et al. [3] present an overview of column generation techniques for solving integer problems to optimality and a review of several classes of problems that were successfully solved in this way. They intend to generalize ideas that were successful in solving special problems by column generation. Vanderbeck and Wolsey [54], and [61], present an exact method that combines branch and bound with column generation. They develope an ad-hoc branching scheme and test the algorithm in three types of problems. In [52] the same approach is used to solve a problem arising in the design of telecommunications networks. In [59] Vanderbeck goes further in this direction and proposes a Dantzig–Wolfe decomposition based on the discretization of the integer polyhedron associated to a group of constraints. Furthermore, he proposes appropriate branching schemes. He tests his ideas on the cutting stock and cutting strip problems.

Although the list of articles on column generation mentioned above is by no means exhaustive, it is representative enough to conclude, as several authors already did [3, 42, 61], that most successful applications of column generation techniques happen in (\textit{IP}) problems which can be modeled as set partitioning (or set covering) ones. In most of the quoted examples columns of the set partitioning problem have a well defined structure and it has been possible to develop pricing algorithms (exact or heuristic) to identify them. This formulation also enables one usually to handle good branching rules compatible with pricing algorithms and keeping the search tree balanced.

Some articles are also found in the literature that report methods that combine column generation with branch and cut. In [44] Nemhauser and Park use a matching formulation of the edge coloring problem and propose an algorithm which combines a simple separation routine for recognizing odd circuit constraints with a pricing algorithm for the weighted matching problem.

In [2] a multicommodity flow problem is solved by combining column and row (cuts) generation. An algorithm is described which combines a pricing algorithm, branching rules and cut generation that are mutually compatible along the branch
and bound tree. Column generation and branching rules stem from a generalization of previous works, and lifted cover inequalities are added to the \((LP)\) in each node.

In this work we present a general scheme that proved to be successful in solving two problems appearing in network design, and can be applied to other combinatorial problems for which the subproblem has no particular structure. We propose a special way of getting rid of columns that we do not want to enter the basis when working with bounded variables, and we show how the columns generated at the previous nodes of the branch and bound tree can be reused.

We also propose, for the binary case, a geometric cut to be added to the subproblem to avoid a column to be regenerated in one of the branches of the tree when the classical 0-1 branching rule is used. Although this approach could be used to solve any binary problem it looks particularly suitable for a problem for which there are not apparent branching rules compatible with the subproblem structure, or for problems where we cannot generate the \(k\)-best solutions of the subproblem easily, as it is done in [48]. This cut has been implemented in branch-and-price methods for a network design problem [40] and for the Steiner tree packing problem [51]. In the next section we describe how to use a column-generation technique in the framework of the simplex algorithm for linear programming problems with bounding variable constraints following the ideas of the COLGEN algorithm presented in [39]. Section 2 presents a linear inequality that can be used, when the matrix of the linear problem is binary, to avoid forbidden columns to be generated. A branch-and-bound scheme using this column-generation algorithm is presented in Section 3. In Section 4 we briefly report computational results of the network design problem we solved with this method. The last section is devoted to some conclusions.

2. A COLUMN-GENERATION METHOD FOR LINEAR PROGRAMMING WITH BOUNDED VARIABLES

In order to solve IP problems by a branch and bound method, we need to solve a LP problem at each node of the branch and bound tree. We assume that this LP problem is suitable to be solved using column generation. We consider the following linear programming problem:

\[
(LP) : \text{minimize } z = \sum_{j=1}^{p} c_j x_j
\]

subject to:

\[
\sum_{j=1}^{p} a_j x_j = b, \quad (2)
\]

\[0 \leq x_j \leq d_j, \quad j = 1, 2, ..., p \quad (3)\]
where \( a_j \in \mathbb{R}^m, \, b \in \mathbb{R}^m, \, b \geq 0, \, d = (d_1 \ldots d_p)^T \in \mathbb{R}^p, \, c_j \in \mathbb{R} \). When all \( c_j, \, a_j \)
and \( d_j \) are known \textit{a priori}, this problem can be solved using the very well known upper bounding method proposed by Dantzig [10, 13, 14, 35]. We consider that \( a_j \)
belongs to a finite set \( \mathcal{K} \), and \( |\mathcal{K}| = p \). To write explicitly all this \( (LP) \)
problem can be prohibitive for problems with a huge integer number \( p \) of columns. As an example we can consider \( \mathcal{K} \) the set of 0-1 vectors associated with all spanning trees of a complete graph with \( n \) vertices, in this case \( |\mathcal{K}| = n^{n-2} \). In this case we would like to solve \( (LP) \) using column-generation techniques in the context of the simplex method. For that we have to suppose \( c_j = f(a_j) \) and \( d_j = g(a_j) \), where \( f : \mathcal{K} \to \mathbb{R} \) and \( g : \mathcal{K} \to \mathbb{Z}_+ \). Following the usual notation, see [10], let \( B = (a_{B(1)} a_{B(2)} \ldots a_{B(m)}) \) be an \( m \) by \( m \) matrix with \( \det(B) \neq 0 \) and \( N \) a \( m \) by \( p - m \) matrix formed with the \( a_j \) columns which are not in \( B \). Let \( x_B = (x_{B(1)} x_{B(2)} \ldots x_{B(m)})^T \) and \( x_N \) be the vector associated with the columns of \( N \). Thus (2) can be written as follows: \( Bx_B + Nx_N = b \), then we have

\[
x_B = B^{-1}b - B^{-1}Nx_N. \tag{4}
\]

Now let \( N_2 \) be the matrix whose \( a_j \) columns of \( N \) are associated with \( x_j = d_j \), and \( N_1 \) the matrix whose columns of \( N \) are associated with \( x_j = 0 \). We can present (4) as follows:

\[
x_B = B^{-1}b - B^{-1}N_1x_{N_1} - B^{-1}N_2x_{N_2}. \tag{5}
\]

Let \( \bar{x}_B = B^{-1}b \), and \( \bar{x}_B = \bar{x}_B - B^{-1}N_2\bar{x}_{N_2} \), where \( \bar{x}_{N_2} = d_{N_2}^T \). If \( 0 \leq \bar{x}_{B(i)} \leq d_{B(i)}, \, i = 1, 2, \ldots, m \), then \( B \) is a feasible basic solution to \( (LP) \). Let \( u = c_B^T B^{-1} \), where \( c_B = (c_{B(1)} c_{B(2)} \ldots c_{B(m)})^T \in \mathbb{R}^m \), \( z_j = u a_j \), and \( \bar{z}_j = z_j - c_j \). We will assume that \( \bar{z}_j \geq 0 \), for such \( j \) associated with \( x_j = d_j \). It is always possible to have this situation if \( (LP) \) has a not empty solution set. If we have \( \bar{z}_j < 0 \) for \( x_j = d_j \), then solving \( (LP) \) without the columns in \( N_1 \), we obtain a new \( N_2 \) for which \( \bar{z}_j \geq 0 \) for all \( x_j = d_j \).

We know that at each iteration of the simplex method, we need to solve the following pricing subproblem (oracle or pricing algorithm) in order to recognize an optimal basic solution of \( (LP) \), or to determine a column to enter the basis:

\[
(SP) : \begin{align*}
\text{maximize} & \quad u a - f(a) \\
\text{subject to} & \quad a \in \mathcal{K}.
\end{align*}
\]

Using this scheme we may find not only the optimal solution to this subproblem but the second one, third one, and so on, because we do not want columns in \( N_2 \) to be chosen. Let \( a_k \) be an optimal solution for \( (SP) \), that is \( \text{val}(SP) = u a_k - f(a_k) \), where \( \text{val}(\cdot) \) denotes the optimum value of the objective function in \( (\cdot) \). We will need to be able to compute the \( s \)-best solution of \( (SP) \) for \( s = 1, 2, \ldots \) such that \( \text{val}(s\text{-best}) \geq \text{val}([s+1]\text{-best}) \). Thus \( \text{val}(1\text{-best}) = \text{val}(SP) \).
Let $I = \{1, 2, \ldots, p\}, I_{N_1} = \{j \mid a_j \in N_1\}, I_{N_2} = \{j \mid a_j \in N_2\}$, and $I_B = \{j \mid a_j \in B\}$. We will solve our LP problem by means of solving iteratively:

$$(LP, B, N_2) : \text{minimize } ub - \sum_{j \in N_2} (z_j - c_j)x_j$$

subject to:

$$x_B = \bar{x}_B - \sum_{j \in N_2} y_j x_j,$$

$$0 \leq x_{B(i)} \leq d_j, \quad i = 1, 2, \ldots, m,$$

$$0 \leq x_j \leq d_j, \quad j \in N_2,$$

where $u = c^T B^{-1}$, $z_j = ua_j$, $c_j = f(a_j)$, $d_j = g(a_j)$ and $y_j = B^{-1}a_j$.

So given a feasible basic solution $B$ for $(LP)$, the COLGEN algorithm for this problem can stated as:

BEGIN

solve $(LP, B, I_{N_2})$;

$s := 0$;

1

$s := s + 1$;

t := val(s - best) := u\bar{a} - f(\bar{a});$ (oracle)

if $\bar{a} \in N_2$, goto 1;

if $t \leq 0$, an optimal solution found STOP;

$\bar{a} \in I_{N_1}$ will enter $B$ or $N_2$; (simplex pivot rule)

solve $(LP, B, I_{N_1})$;

$s := 0$;

goto 1;

END.

3. An Inequality to Avoid Regenerating Forbidden Columns When Matrix $A$ is Binary

When the set of all $a$ in $K$ can be described by:

$$Ka \leq h,$$  \hspace{1cm} (6)

$$a \in \{0, 1\}^m,$$  \hspace{1cm} (7)

where $K \in \mathbb{R}^{q \times m}$ is a known matrix and $h \in \mathbb{R}^q$ is a given vector, we propose a special way to avoid to generate a forbidden column to enter the basis.

We will use a linear inequality that eliminates a given point of $\{0, 1\}^m$. Let $\bar{a} \in \{0, 1\}^m$, we wish to construct an inequality to eliminate just $\bar{a}$ in (6) and (7).

That is given $B = \{0, 1\}^m - \{\bar{a}\}$ how to find a facet of the convex hull of $B$, which keeps out $\bar{a}$?. All points of a sphere centered in $\bar{a}$ with radius equal to 1 satisfy the following equation: $\sum_{j=1}^m (a_j - \bar{a}_j)^2 = 1$. All the neighbors of $\bar{a}$ in the cube
whose vertices are all points of \( \{0,1\}^m \) belong to the surface of this sphere. If we want to consider all points of \( B \) we will write:

\[
\sum_{j=1}^{m} (a_j - \bar{a}_j)^2 \geq 1.
\]

(8)

As we have \( a_j^2 = a_j \),

\[
\sum_{j=1}^{m} (a_j - \bar{a}_j)^2 \geq 1
\]

implies

\[
\sum_{j=1}^{m} (a_j^2 - 2\bar{a}_j a_j + a_j^2) \geq 1
\]

then

\[
\sum_{j=1}^{m} (1 - 2\bar{a}_j) a_j \geq 1 - \sum_{j=1}^{m} \bar{a}_j.
\]

(9)

It is easy to see that (9) is a facet of the convex hull of \( B \), which keeps out \( \bar{a} \).

We include this inequality in the COLGEN algorithm which in this case can be presented as follows:

BEGIN

1. solve\((LP, B, I_{N_2})\);
   solve\((SP)\), including \( \sum_{j=1}^{m} (1 - 2\bar{a}_j) a_j \geq 1 - \sum_{j=1}^{m} \bar{a}_j, \forall \bar{a} \in N_2; \)
   \( \text{val}(SP) := u \hat{a} - f(\hat{a}); \) (oracle)
   if \( \text{val}(SP) \leq 0 \), an optimal solution is found STOP;
   \( \hat{a} \) will enter \( B \) or \( N_2; \) simplex pivot rule
   goto1;

END.

4. SOLVING INTEGER PROGRAMMING WITH COLGEN

We define an integer program as follows:

\[
(IP): \text{ minimize } \sum_{j=1}^{p} c_j x_j
\]

subject to:

\[
\sum_{j=1}^{p} a_j x_j = b,
\]

\[
0 \leq x_j \leq d_j, \ j = 1, 2, \ldots, p,
\]

\[
x_j \in \mathbb{Z}, \ j = 1, 2, \ldots, p,
\]
where \( c_j, a_j, b \) and \( d_j \) are defined as in (1) to (3). We solve \((IP)\) using branch-and-bound techniques, see [12] and [45] and at each node of the enumerative scheme the column-generation algorithm will be applied.

We consider \((LP)\) the linear relaxation of \((IP)\), that is, we do not consider in \((LP)\) constraints \( x_j \in \mathbb{Z}, j = 1, 2, ..., p. \)

At the initial node we solve \((LP)\) if its optimal solution is integer then we have solved \((IP)\), otherwise we have a \( k \) such that \( \hat{x}_{B(k)} \) is not integer. Thus we will branch this node solving two linear programming problems:

- \((LP) \cap \{ x_{B(k)} \leq \lfloor \hat{x}_{B(k)} \rfloor \} \), and
- \((LP) \cap \{ x_{B(k)} \geq \lfloor \hat{x}_{B(k)} \rfloor + 1 \} \),

where \( \lfloor r \rfloor \) is the largest integer \( \leq r. \)

We suppose to be at node \( i \) of the branch-and-bound tree and its associated linear programming problem is under the following form

\[
(LP_i) : \text{minimize } \sum_{j=1}^{p} c_j x_j
\]

subject to:

\[
\sum_{j=1}^{p} a_j x_j = b,
\]

\[
\alpha_j \leq x_j \leq \beta_j, \ j \in S_i,
\]

\[
0 \leq x_j \leq d_j, \ j \in I - S_i,
\]

where \( S_i \) is the set of the indices associated with all generated columns along the current branch, including the columns in \( B \) at node \( i \). We can have \( \alpha_j = 0 \) or \( \beta_j = d_j \) for some \( j \in S_i \). An initial feasible basic solution for \((LP_i)\) is a matrix \( B \) for which \( \alpha_{B(j)} \leq \bar{x}_{B(j)} \leq \beta_{B(j)}, \ j = 1, 2, ..., m, \bar{x}_j = \alpha_j \) or \( \bar{x}_j = \beta_j, \ j \in S_i - I_B, \) and \( \bar{x}_j = 0, \ j \in I - S_i. \)

We consider now a feasible basic solution of \((LP_i)\) associated with an optimal basic solution of

\[
(LXP_i) : \text{minimize } \sum_{j \in S_i} c_j x_j
\]

subject to:

\[
\sum_{j \in S_i} a_j x_j = b,
\]

\[
\alpha_j \leq x_j \leq \beta_j, \ j \in S_i,
\]

where \( \bar{x}_j = 0, \ j \in I_B; \bar{x}_j = 0 \) for \( \bar{x}_j = \alpha_j \) and \( \bar{x}_j \geq 0 \) for \( \bar{x}_j = \beta_j \) and \( j \in S_i - I_B. \)

If \((LP_i)\) is a leaf of the branch-and-bound tree then we resort to backtracking, otherwise we choose a \( k \in I_B \) such that \( \hat{x}_{B(k)} \) is not integer, and we branch as follows:

- \((LP_{i+1}) := (LP_i) \cap \{ x_{B(k)} \leq \lfloor \hat{x}_{B(k)} \rfloor \}, \) and
- \((LP_{i+2}) := (LP_i) \cap \{ x_{B(k)} \geq \lfloor \hat{x}_{B(k)} \rfloor + 1 \}. \)
We can take linear programming problems considering just the columns generated up to reach node $i$.

- $(LP_{i+1}) := (LP_i) \cap \{ x_{B(k)} \leq \lceil \hat{x}_{B(k)} \rceil \}$, and
- $(LP_{i+2}) := (LP_i) \cap \{ x_{B(k)} \geq \lfloor \hat{x}_{B(k)} \rfloor + 1 \}$.

It is important to note that an optimal basic solution of $(LP_i)$ is still a dual feasible basic solution for $(LP_{i+1})$ and $(LP_{i+2})$.

Starting the (dual) simplex algorithm using an optimal basic solution of $(LP_i)$ we can solve easily $(LP_{i+1})$ and $(LP_{i+2})$, see [10, 17] or [38]. Unfortunately $(LP_{i+1})$ can be empty and $(LP_{i+1})$ is not. The same can be assumed for $(LP_{i+2})$ with respect to $(LP_{i+2})$.

When $(LP_{i+1})$ is not empty we have an optimal basic solution for it, and this optimal solution will be the initial feasible basic solution for $(LP_{i+1})$. Then we use COLGEN algorithm to solve $(LP_{i+2})$, from an optimal basic solution for $(LP_{i+2})$.

If $(LP_{i+1})$ has no feasible solution we have to solve $(LP_{i+1})$ starting the simplex algorithm using an artificial solution. We introduce an artificial variable $r$ in the $k$-row as follows:

$$x_{B(k)} + r = \bar{x}_{B(k)} - \sum_{j \in I_{N_1}} y_{kj} x_j - \sum_{j \in I_{N_2}} y_{kj} x_j.$$ 

We put $x_{B(k)} = \lceil \hat{x}_{B(k)} \rceil$, then

$$\bar{r} = \bar{x}_{B(k)} - \sum_{j \in I_{N_1}} y_{kj} \alpha_j - \sum_{j \in I_{N_2}} y_{kj} \beta_j - \lfloor \hat{x}_{B(k)} \rfloor > 0.$$ 

As $a_{B(k)}$ is in the basis $B$ we have $y_{B(k)} = B^{-1} a_{B(k)} = e_k$. If we consider $I_{N_2} := I_{N_2} \cup \{ B(k) \}$, and $\beta_{B(k)} = \lfloor \hat{x}_{B(k)} \rfloor$, then we can solve the following linear programming problem using COLGEN algorithm:

$$(AP_1) : \text{minimize} \ r$$

subject to:

$$\sum_{j=1}^{p} a_j x_j + a_{B(k)} r = b,$$

$$\alpha_j \leq x_j \leq \beta_j, \ j \in S_i,$$

$$0 \leq x_j \leq d_j, \ j \in I - S_i, \text{ and } r \geq 0.$$ 

Let $B$ a primal feasible basic solution for $(AP_1)$. If $\text{val}(AP_1) = 0$ we have an initial feasible basic solution for $(LP_{i+1})$. Otherwise $(LP_{i+1})$ is empty and the $(i+1)$-node is closed, and backtracking takes place.

For solving $(LP_{i+2})$ when $(LP_{i+2})$ is empty we proceed the same way.
5. Computational results for a network design problem

The above-described techniques were successfully applied to a problem that arises in telecommunications network planning which was presented in detail in [40, 46] and [6].

In recent years, studies about network dimensioning and survivability focused mainly on ring based and mesh based architectures. Most works treated both models separately. But several companies deploy networks based on rings, and at the same time they have many meshed networks in operation and even some currently being built. In large metropolitan centers it is easy to identify clusters of nodes generating big data traffic among themselves. Meanwhile peripheral areas of the same network may present reduced traffic. If the area under study is already served by a meshed network it is possible to deal with the growth of traffic demands by superimposing self-healing rings on the existing mesh. This is called the two-level architecture network design solution.

The mesh part of the problem can be modeled quite well as a capacitated multicommodity flow problem with expandable arc capacities. This will be the initial master problem. The other way the flow can circulate on the net is on an additional net of rings. Only flow between nodes on each ring is allowed on it.

The complete formulation of the problem is a large mixed integer linear programming one and includes a column associated with each possible ring (see [40] for details). As it is not possible to have all the columns available since the beginning, a column generation procedure (SP) was chosen to solve the problem. The problem of generating each column can be modeled as an integer programming problem.

Initially in [40], four instances of the problem were solved this way, two real problems corresponding to 7-nodes and two corresponding to 10-node networks. The program was written in C, and the XPRESS-MP library was used. All experiments have been carried on a Pentium III 450 MHz computer. These results are summarized in Tables 1 and 2; problems with 20, 30, 40 and 50 nodes were also artificially generated. The number of traffic requirements and the values of the requirements were randomly generated. The maximum rings capacities were taken as 16. Detailed results for these new problems and one of the previous 10 node problems (with capacities of rings equal to 16) are presented in Table 3. In all cases the exact optimum was found.

The characteristics of each problem, the cost of the solution of the problem that only consider the mesh architecture and the number of generated support cycles are first shown in the three tables. The next lines correspond to the Branch and Bound solution of the problem where only a subset of the rings is considered. Then the total number of generated columns, active columns (columns whose associated variables have positive values at the optimal solution), open nodes used in the complete branch-and-price procedure are then shown as well as the number of the geometric cuts proposed in Section 3 which were necessary to add in each problem. We also show the gap between the optimum and the linear and Branch and Bound solutions. Looking at Table 2, we can see that using column generation for the
linear relaxation of the problem with ring capacity 12, we generated 88 columns, and we obtain a solution cost equal to 24.403. Using only these columns we started a branch-and-bound procedure and we obtain a solution with a value of 29 that provides an upper bound for the solution of the original problem. Among these 88 generated columns, 7 were related to nonzero variables, which means that only 7 rings with capacity 12 were used in the final solution of this branch and bound procedure. Then starting from the linear relaxed solution mentioned above, a branch-and-price method as the one described in Section 4 was used.

<table>
<thead>
<tr>
<th>Ring capacity</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edges</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>Demands</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Cost of the mesh network</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>Support cycles generated</td>
<td>15</td>
<td>15</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Linear solution with Colgen</th>
<th>10.067</th>
<th>8.740</th>
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</thead>
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<td>CPU time (sec.)</td>
<td>9</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B&amp;B Integer solution</th>
<th>12</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generated columns</td>
<td>25</td>
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<tr>
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<td>2</td>
</tr>
<tr>
<td>CPU time (sec.)</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Optimum (branch-and-price)</th>
<th>11</th>
<th>10</th>
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</thead>
<tbody>
<tr>
<td>Generated columns</td>
<td>43</td>
<td>50</td>
</tr>
<tr>
<td>Active columns</td>
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<td>2</td>
</tr>
<tr>
<td>CPU time (sec.)</td>
<td>10</td>
<td>7</td>
</tr>
</tbody>
</table>

| GAP(%): Optimum and linear solution | 9.27 | 14.42 |
| GAP(%): BB Integer solution and optimum | 9.09 | 0.00 |

<table>
<thead>
<tr>
<th>Master problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of constraints</td>
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<tr>
<td>Number of columns</td>
</tr>
</tbody>
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<th>Slave problem</th>
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</thead>
<tbody>
<tr>
<td>Number of constraints</td>
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<tr>
<td>Number of columns</td>
</tr>
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Table 2. Results on the 10 node network.

<table>
<thead>
<tr>
<th>Rings capacities</th>
<th>12</th>
<th>16</th>
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</thead>
<tbody>
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<td>22</td>
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<tr>
<td>Demands</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>Cost of the mesh network</td>
<td>152</td>
<td>152</td>
</tr>
<tr>
<td>Support cycles generated</td>
<td>169</td>
<td>169</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Linear solution with Colgen</th>
<th>24.403</th>
<th>21.958</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU time (sec.)</td>
<td>48</td>
<td>57</td>
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</tbody>
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<table>
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<tr>
<th>B&amp;B Integer solution</th>
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<th>27</th>
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</thead>
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<td>Generated columns</td>
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<td>78</td>
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<td>Active columns</td>
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<td>6</td>
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<td>CPU time (sec.)</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Optimum (branch-and-price)</th>
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<th>24</th>
</tr>
</thead>
<tbody>
<tr>
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<td>11104</td>
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<tr>
<td>Active column</td>
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<td>6</td>
</tr>
<tr>
<td>Open nodes</td>
<td>540</td>
<td>685</td>
</tr>
<tr>
<td>CPU time (hours)</td>
<td>2</td>
<td>2.5</td>
</tr>
<tr>
<td>Max depth in Branch Bound tree</td>
<td>39</td>
<td>48</td>
</tr>
<tr>
<td>Geometric cuts</td>
<td>154</td>
<td>466</td>
</tr>
</tbody>
</table>

| GAP(%): Optimum and linear solution | 10.64 | 9.30 |
| GAP(%): BB Integer solution and optimum | 7.41  | 12.50 |

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<tr>
<th>Master problem</th>
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<td>Number of columns</td>
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<tbody>
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<td>201</td>
</tr>
<tr>
<td>Number of columns</td>
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<td>128</td>
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</table>

6. Conclusions

The solution of combinatorial optimization problems for which the columns of the constraint matrix belong to a huge finite set can be found using the techniques we described in this paper. Solving \((SP)\) problem may be difficult. The use of \((LP_1)\) instead of \((LP)\) may accelerate the branch-and-bound implementation as was the case in the application mentioned in Section 5. We observe that, when the matrix \(A\) of the problem is binary, the inequality presented in Section 3 may be used in the context of a branch-and-price method to avoid a column to reenter the successors of the node where it has been deleted.
Table 3. Results on the 10, 15, 20, 30, 40 and 50 node networks.

<table>
<thead>
<tr>
<th>Nodes</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edges</td>
<td>22</td>
<td>25</td>
<td>40</td>
<td>58</td>
<td>60</td>
<td>80</td>
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<tr>
<td>Demands</td>
<td>36</td>
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<td>25</td>
<td>50</td>
<td>45</td>
<td>70</td>
</tr>
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<td>Rings capacities</td>
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<td>16</td>
<td>16</td>
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<td>108</td>
<td>142</td>
<td>252</td>
<td>230</td>
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<tr>
<td>Support cycles generated</td>
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<td>13694</td>
<td>12083</td>
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<td>25.74</td>
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<td>72.37</td>
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<tr>
<td>CPU (time secs)</td>
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<td>46</td>
<td>61</td>
<td>507</td>
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<td>1099</td>
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<td>B&amp;B Integer solution</td>
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<td>20</td>
<td>27</td>
<td>52</td>
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<td>225</td>
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<td>CPU (time secs)</td>
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<tr>
<td>Geometric cuts</td>
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<td>2.39</td>
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<td>3.85</td>
<td>8.33</td>
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<td>1.35</td>
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<td>B&amp;B Int. solution and optimum</td>
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<td>280</td>
<td>481</td>
<td>1414</td>
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<td>3417</td>
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References


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