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ON EXTENSION OF A GIVEN FINITELY ADDITIVE FIELD - VALUED, NON NEGATIVE MEASURE, ON A FINITELY ADDITIVE BOOLEAN TRIBE, TO ANOTHER TRIBE MORE AMPLE *)

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1. - The theory of measure in a Boolean tribe enables us to investigate the structure of tribes, especially when the measure is real-number valued and effective, i. e. when it vanishes on the null-element only. Now, it is known that there are finitely additive tribes which do not admit any finitely additive number valued and effective measure 2), so the idea of introducing measures whose values are taken from a general linearly ordered (algebraic) field, which may be not archimedean, seems to be promising. The present paper deals with such measures and gives a (positive) solution of the problem of extending a non-negative measure from a given tribe to another, wider one, containing it. There is a method, available in the literature, which is adequate to deal with the problem of extension, viz. the S. Banach’s method for extension linear functionals (1). This method was applied with success by Banach himself and by other authors. We shall also apply it in our problem.

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1) Composed under the grant from the National Science Foundation (U. S. A).
2) An example of such a tribe was kindly communicated by Mr. J. DIXMIER in a letter to me.
2. - Though the idea is known and rather simple, nevertheless to apply it in our problem many auxiliary items are needed, because the behaviour of general fields differs from that of the field of real or complex numbers. General smaller or larger auxiliary theories shall be developed to master the situation, and since our arguments must have a rather subtle character, clarification of some known basic notions will be also needed. Some of the mentioned theories will serve as a tool not only to solve our problem, but they will be also useful for the subsequent papers by the author. Therefore the auxiliaries are exposed in a more general way, than needed for the proof of the result the present paper is aiming at, and with more details, just for purpose of future references. Some of these auxiliary topics imply new methods which, we believe, will be useful in the capacity of a new technique suitable for various branches of mathematics, some ones clarifying known but confused topics.

3. - An abstract of the present paper has been published in four C. R.-notes (2). However, the present paper contains arguments which differ slightly from those in these notes, and in addition to that, it constitutes a much more explicite setting. The auxiliaries, mentioned above, are exposed in six first chapters entitled: § I. Endings and their ordering (there are many sections labelled § I A, B, C, D, E, F, G, H, K). Their purpose is to overcome the difficulty in linearly ordered non-archimedean fields which consists of the fact that a bounded set may not admit a supremum and infimum. § 2 gives a precise setting of partitions in a Boolean tribe. § 3 deals with « aggregates » and constitutes a particular case (adapted to the main purpose of the present paper) of the general theory sketched in the C. R. notes by the author on « functionoids » (3). § 4 introduces the exterior and interior ending-valued Jordan measure in Boolean tribes. § 5 deals with a kind of linear functionals and their ending-valued norm. The aim of § 6 is to prove the possibility of extension of linearly ordered fields (which may be not archimedean), by placing a new element in a
gap which may be a true or not true Dedekindian section. The proof of the corresponding existence-theorem is relied on the known theory of reel-closed fields (4), which in turn is based on the Steinitz-theorem (5) on the existence of the algebraic closure of a field, (see also (6)).

Now, the last theorem has not yet been proved correctly in the spirit of theory of types (7), and it is known that mere considering of a class whose elements have not the some logical type invariably leads to contradiction 2).

Our § 6 supplies a correct proof which is even simpler than the original proof by Steinitz. The necessity of discrimination of logical types does not allow any « identification » of isomorphic elements, neither a merely formalistic approach to polynomials and operations on them. In addition to that the notion of single element extension of a field, which is rather confused in the literature, needed clarification. Having this all in mind, it seemed necessary to the author, to give a thorough refoundation of the algebra of fields. Especially new definitions were necessary which separated some similar notions from one another. Concerning ordinals and cardinals, we refer to the abstract (8) by the author and Mrs. Stanisława Nikodým, where these notions are defined in accordance with the necessity of discrimination of types. The definitions in the auxiliary chapter, are stated carefully, but many theorems, whose proofs do not require special technique, are stated without proof, in the belief that the reader will be able to supply them. Especially § 1 on « endings » contains a great quantity of « small » theorems, whose proofs, sometimes not too short, if given explicitely (as they are in the manuscript of the author), would increase many times the volume of the present article. Also proofs of the known theorems in the theory of fields are omitted.

2) Eg. If we shall take into consideration the set whose only elements are the number 1 and the class composed of the numbers 1 and 2, a contradiction will result. It will however not result, if the class is composed of the set composed of 1 and of the set composed of 1 and 2.
5. Preliminaries and Notations. - Since our topic is rather subtle, and since it is generally true, that in abstract things nothing is "obvious", the following preliminaries will be in order — to avoid misunderstanding. Indeed, mathematical understanding of a topic differs from the intuitive common place-understanding.

We shall deal with various theories, as e.g. with chains and tribes. In each theory there is notion of equality of its elements, with respect to which the operations and relations considered are invariant, and which conditions the notion of sets of elements, that of correspondences and the notion of uniqueness of the element, satisfying a given condition. If the theory is abstract, the notion of equality is axiomatized or defined, and if the theory is constructed, the notion of equality is defined or taken from another theory. We shall call it equality governing the theory. If $A$ denotes the theory, its equality will be denoted by $\equiv_A$, though the letter may be omitted if no misunderstanding is to be feared of.

A set $E$ of elements of the theory $A$ must be $\equiv_A$-invariant, which means that if $a \in E$ and $a' \equiv_A a$, then $a' \in E$. E.g. in the theory of measurable subsets of $[0, 1)$ the notion of their governing equality is "equality almost everywhere", $\equiv_{\text{a.e.}}$; hence a class of sets must be $\equiv_{\text{a.e.}}$-invariant, and so must be the operations on them. E.g. if $a \equiv b'$, $b \equiv b'$, $c \equiv c'$, and $a \cup b \equiv c$, then $a' \cup b' \equiv c'$. We say that $a$ is the $\equiv_{\text{a.e.}}$-unique element satisfying a condition $\varphi(x)$ (invariant with respect to $\equiv_{\text{a.e.}}$), whenever $\varphi(a)$, and if $\varphi(a)$ and $\varphi(a')$, then $a \equiv a'$.

Let $M$ be a not empty subset of a set $N$, (taken from a theory) on which $\equiv$ is the governing equatity. If we do not analyse this equality, it may be termed "identity". Suppose that, for some purpose, we change the notion of equality, on $M$, into another one $\equiv_M$, satisfying, of course, the conditions of symmetry, transitivity and reflexiveness 4). This is e.g. the case when the elements of $N$ are sets of

4) New equalities are termed "equivalences", though, really, there is no logical difference between them and equalities.
real numbers and $M$ constitutes the measurable subsets of $(0, 1)$, $\equiv$ being the equality « almost everywhere ». We must discriminate between $N$-sets and $M$-sets. Instead of the equality $\equiv$ we can consider equivalence classes which are $N$-sets and operate on them through representatives. However, many times it is more comfortable to consider the $M$-equality, than identity of the corresponding equivalence classes; e.g. in abstract algebra, in the theory of Hilbert space whose vectors are square-summable functions, and in the theory of ordinary fractions.

In the literature the notion of equality is usually, with few exceptions, not taken into account. Its importance in the theory of Boolean tribes is, however, emphasized by the author (10), (11). E.g. the difference between free vectors, gliding vectors and bound vectors in the euclidean geometry derives from different kinds of equalities.

6. - The « relations » of Russell and Whitehead (7) will be termed correspondences or mappings. We consider correspondences, in accordance with these authors, as belonging to propositional functions of two variables, and not, as is now fashionable, as classes of ordered couples. A couple $(a, A)$ will be understood as the correspondence $x, y \mapsto a, y \mapsto A$. The domain of the correspondence $R$ will be denoted by $\mathcal{C} R$ and the range by $\mathcal{D} R$. They may be of different types. If the type is the same, $\mathcal{C} R \cup \mathcal{D} R$ is meaningful: it will be termed campus of $R$ and denoted by $\mathcal{O} R$. The correspondence $R$ must be equality invariant in its domain and equality invariant in its range (where these equalities may have a different character). This means that if $aRA, a \mapsto a_1, A \mapsto A_1$, then $a_1 R A_1$. We shall say $R$ is $\mapsto$ -invariant. Instead of $aRA$ we shall also write $a \rightarrow A$. Isomorphisms, as correspondences shall fit this requirement. By a function we shall understand a pluri-one correspondence. If $E \subseteq \mathcal{C} R$, by $E \mathcal{R} R$ we shall understand the correspondence $\{x, y \mid x \in E, xRy \}$, i.e. the correspondence $R$ restricted to $A$, (7).

If speaking of isomorphisms or homomorphisms we shall
say which relations and operations should be maintained by them. E. g. We shall say « order-addition-multiplication -zero-and unit-isomorphism », if these items are preserved through it. Set-operations will be denoted by Bourbaki-symbols \( \cup, \cap, - \), \( \cup \), \( \cap \) and the complementary by \( \co \). We avoid « empty » unions and intersections for reasons explained in (10). There will be, in § 1E, some other operations on sets: they will be defined there. If the elements of the sets \( E, F \) have the type \( \alpha, \beta \) respectively, and \( R \) is a correspondence with \( \cap R = E, \cap R = F \), the type of \( R \) will be denoted by \( \{ \alpha; \beta \} \). The type of a class of elements of the type \( \alpha \) will be denoted by \( \cl \alpha \).

7. - If a set \( A \) is organized into a structure by introducing some operations or relations, the structure will be denoted by \( (A) \) or \([A]\), or even \([\{A]\), for more clarity, if needed. The following structures will be considered:

Orderings (usually referred to as « partial ordering » (12), (13). An ordering is defined as a not empty correspondence \( R \), (abstract or constructed) mainly denoted by « \( \leq \) », such that 1) if \( a \in \cap R \), then \( aRa \), 2) if « \( aRb, bRc \) » then \( aRc \), 3) if \( a, b \in \cap R \) then « \( a = b \) » is equivalent to « \( aRb \) and \( bRa \) ». By a chain (linear ordering) we understand an ordering \( R \) which satisfies the additional condition: if \( a, b \in \cap R \), then either \( aRb \) or \( bRa \). By a tribe \(^5\) (Boolean tribe, Boolean lattice, Boolean algebra), we shall understand an ordering which is a distributive and complementary lattice, (12), (13). A tribe \( B \) is said to be not trivial whenever its unit \( 1_B \) differs from its zero \( 0_B \). The elements of a tribe will be termed elements or somata \(^6\). Somatic operations will be denoted by \( +, \cdot, \co, \Sigma, \Pi, - \), the algebraic addition \( (a - b) + (b - a) \) which organizes (together with the multiplication) the tribe into a Stone’s-ring (17) will be denoted by \( \vdash \). A measure on \( B \) will be supposed to be equality-invariant. If the values of a measure are

\(^5\) Term borrowed from (16).
\(^6\) Term borrowed from (15).
taken from a linearly ordered field, and are non negative, the measure will be termed effective whenever it vanishes only on $0_B$.

8. - The term ring and field are understood in the usual way with the exception that if the structure is confined to a single element (which must be the zero), we shall call the ring or field trivial. We shall consider abelian groups and semigroups only. Their « identity » will be denoted by 0 (with an index, if needed), and the group-operation by $\dagger$. A structure $\mathcal{A}$ with operations $\circlearrowleft, \ldots$ and correspondences $\sim, \ldots$ will be said to be $(\circlearrowleft, \ldots, \sim, \ldots)$-genuine substructure of $\mathcal{B}$ whenever, for elements $a, b, c, \ldots$ of $\mathcal{A}$ the following are equivalent: (10)

1) $a \circlearrowleft b \overset{\mathcal{A}}{=} c$ and $a \circlearrowleft b \overset{\mathcal{B}}{=} c, \ldots$
2) $a \sim b$ and $a \sim' b$.

If in addition to that the equality $\overset{\mathcal{A}}{=} \mathcal{A}$ is identical with $(\overset{\mathcal{A}}{=})_\mathcal{B}$ i.e. with the restriction of $\overset{\mathcal{A}}{=}$ to $\mathcal{B}$, we shall call $\mathcal{A}$ $(\overset{\mathcal{A}}{=}, \ldots, \sim, \ldots)$-genuine strict substructure of $\mathcal{A}$.

9. - References to former theorems and subsections will be stated as e.g. [§ 3; 4], but in references to the actual section the number, e.g. § 3, will be omitted, e.g. [4]. The numbers in bold parentheses, as (2), refer to the literature at the end of the article.

10. - The main theorem on extension of measure, proved in this paper, is stated at the end of § 8.

§ 1. - Endings and their ordering.

1A. Left endings.

1. - Let $(M)$ be a chain, denoted $\preceq$. It will be kept fixed through the whole present § 1. We shall consider only not empty subsets $E, F, \ldots$ of $M$.

Given $E, F$, we define $E \preceq F$ (also written $F \succeq E$) as « for every $y \in F$ there exists $x \in E$ such that $x \leq y$. $E \preceq F$
(also written $F \succ \cdot E$) will mean « for every $x \in E$ there exists $y \in F$ such that $x < y$ ».

The following properties hold true: $E \cdot \leq E$. If $E \cdot \leq F$, $F \cdot \leq G$, then $E \cdot \leq G$. For any two not empty sets $E$, $F$ we have either $E \cdot \leq F$ or $F \cdot \leq E$.

2. - The correspondence $\cdot \leq$ induces a notion of equality $E \cdot = F$ defined as « $E \cdot \leq F$ and $F \cdot \leq E$ ». It possesses the formal properties of identity. The notion $\cdot \leq$ is invariant with respect to the equality $\cdot = ;$ thus the notion $\cdot \leq$ organizes the class of all not empty subsets of $M$ into a chain ($\cdot \leq$) with ($\cdot =$) as governing equality. The notion ($\cdot =$) coincides, for chains, with the notion by J. W. Tukey of coinitial similarity (18).

3. - If $E \subseteq F$, then $F \cdot \leq E$.

4. - If $\{ a \}$ is the set composed of the single element $a$, where $a \in M$, let us agree to write $a$ instead of $\{ a \}$, when no ambiguity will be feared of. We have: if $x \in E$, then $E \cdot \leq x$.

5. - By left ending of $E$, $\mathcal{L}(E)$, we shall understand the class of all sets $F$ such that $F \cdot = E$. By a left ending $\ast$ we shall understand any $\mathcal{L}(E)$ where $E \subseteq M$, $E \neq 0$. Left endings will be denoted by greek letters provided with a star: $\ast x$, $\ast \beta$, ... If $\mathcal{L}(E)$ is denoted by $E$, we call $E$ representative of $\ast x$. Every set belonging to the class $\ast x$ is a representative of $\ast x$. The set $E$ is a representative of $\mathcal{L}(E)$.

6. - If $\ast x$, $\ast \beta$ are left endings, we define the correspondence $\ast x \leq \ast \beta$ by « there exist representatives $E, F$ of $\ast x$, $\ast \beta$ respectively, such that $E \cdot \leq F$ ». The notion of (\leq) for left endings does not depend on the choice of representa-

\footnote{This notion is a particular case of J. W. Tukey's « coinitial type » (18).}
tives. The following are equivalent: I. \( \alpha \leq \beta \), II. for all representatives \( E, F \) of \( \alpha, \beta \) respectively we have \( E \cdot \leq F \).

7. - If \( \alpha, \beta \) are left endings, we define \( \alpha = \beta \) by \( \alpha \leq \beta \) and \( \beta \leq \alpha \). The notion of equality of left endings is the identity of classes of sets.

The following are equivalent: I. \( \alpha = \beta \), II. there exist representatives \( E, F \) of \( \alpha, \beta \) respectively, such that \( E \cdot = F \), III. for all representatives \( E, F \) of \( \alpha, \beta \) respectively we have \( E \cdot = F \). IV. The classes \( \alpha, \beta \) of sets coincide.

The notion of equality (\( = \)) of left endings satisfies all conditions of identity, and the notion (\( \leq \)) for left endings is invariant with respect to it.

8. - The following properties are true: \( \alpha \leq \gamma \); if \( \alpha \leq \beta \) and \( \beta \leq \gamma \), then \( \alpha \leq \gamma \). The class of all left endings in \( (M) \) is organized into a chain (\( \leq \)) with (\( = \)) as governing equality.

9. - If \( E \) possesses the minimum \( a \), then \( E \cdot = a \). If \( a \in M \) and \( E \cdot = a \), then \( a \) is a minimum in \( E \).

A left ending whose one of the representatives is the set \( \{ a \} \) composed of a single element \( a \) of \( M \), will be termed point left ending. To simplify writing we agree to denote such an ending by \( a \).

If \( E \) is a representative of \( \alpha \) and \( E \) has the minimum \( a \), then \( \{ a \} \) is a representative of \( \alpha \) and \( \alpha = a \).

If \( \alpha = a \), and \( E \) is a representative of \( \alpha \), then \( E \) has the minimum \( a \) and \( a \) is a representative of \( \alpha \).

10. - If \( E \neq 0 \) is a subset of \( M \), and \( a \in M \), then the following are equivalent: I. \( a \cdot \leq E \), II. for every \( y \in E \) we have \( y \leq a \).

If \( E \) is a representative of \( \alpha \) and \( a \in M \), then the following are equivalent: I. \( a \leq \alpha \), II. for every \( x \in E \) we have \( a \leq x \).

11. - If \( F \neq 0 \) is a subset of \( M \), \( a \in M \), then the following are equivalent: I. \( E \cdot \leq a \), II. there exists \( y \in E \) such that
y ≤ a. If E is a representative of *a, a ∈ M, then the following are equivalent: I. *a ≤ a, II. E · ≤ a, III. there exists e ∈ E such that e ≤ a, IV. there exists an element p of M such that 1) there exists e ∈ E with e ≤ p, and 2) for every e ∈ E with e ≤ p we have e ≤ a.

12. - If a, b ∈ M, then the following are equivalent: I. a ≤ b, II. a < b, and the following are equivalent: I. a = a, II. a = a.

If E has at least two different points and a is not the minimum of E, a ∈ E, then E = { x | x ∈ E, x < a }.

If E is a representative of *a, a ∈ E, a is not the minimum of E, then the set E = { x | x ∈ E, x < a } is also a representative of *a.

If E ⊆ M, E ≠ 0, a ∈ E, then E = { x | x ∈ E, x < a }.

If E is a representative of *a, a ∈ E, then E = { x | x ∈ E, x < a } is also a representative of *a.

13. - If *a has a representative composed of the single point a and F is any set, (F ⊆ M) with minimum a, then F is a representative of *a.

14. - The following are equivalent:
I. £E(E) ≤ £E(F), II. either there exists a ∈ E such that for every b ∈ F we have a ≤ b, or £E(E) = £E(F).

If *a, *β have representatives E, F respectively, then the following are equivalent: I. *a ≤ *β, II. either *a = *β or there exists a ∈ E such that for every b ∈ F we have a ≤ b.

1B. Right endings.

1. - Given E, F, we define E ≤ · F as for every x ∈ E there exists y ∈ F such that x ≤ y. We have: E ≤ · E; if E ≤ · F, F ≤ · G, then E ≤ · G.

For any two sets E, F (not empty) we have either E ≤ · F or F ≤ · E.

2. - The correspondence ≤ · induces a notion of equality E = · F, defined as « E ≤ · F and F ≤ · E ». The notion ≤ ·
is invariant with respect to the equality $=\cdot$. Consequently the notion $\leq\cdot$ organizes the class of all not empty subsets of $M$ into a chain $(\leq\cdot)$ with $(\equiv\cdot)$ as governing equality.

3. - If $E \subseteq F$, then $E \leq\cdot F$.

4. - If $x \in E$, then $x \leq\cdot E$.

5. - By the right ending of $E$, $\mathcal{R}\mathcal{E}(E)$ we shall understand the class of all sets $F$ such that $F =\cdot E$.

By a right ending we shall understand any $\mathcal{R}\mathcal{E}(E)$ where $E \subseteq M$, $E \neq 0$.

Right ending will be denoted by greek letters provided with a star $\alpha^*, \beta^*, ...$

If $\mathcal{R}\mathcal{E}(E)$ is denoted by $\alpha^*$, we call $E$ representative of $\alpha^*$.

Every set belonging to the class $\alpha^*$ is a representative of $\alpha^*$.

6. - If $\alpha^*, \beta^*$ are right endings, we define the correspondence $\alpha^* \leq \beta^*$ by «there exist representatives $E, F$ of $\alpha^*, \beta^*$ respectively, such that $E \leq\cdot F ».$

The notion of $\leq$ for right endings does not depend on the choice of representatives.

The following are equivalent: I. $\alpha^* \leq \beta^*$. II. for all representatives $E, F$ of $\alpha^*, \beta^*$ respectively we have $E \leq\cdot F$.

If $\alpha^*, \beta^*$ are right endings, we define $\alpha^* \equiv \beta^*$ by «$\alpha^* \leq \beta^*$ and $\beta^* \leq \alpha^* ».$

Thus, this notion is the identity of classes of sets. There are properties similar to those of left endings, stated in [§ 1A, 7 and 8].

7. - If $E$ possesses the maximum $a$, then $E =\cdot a$. If $a \in M$ and $E =\cdot a$, then $a$ is the maximum in $E$.

A right ending whose one of the representatives is the set $\{ a \}$ composed of a single element $a$ of $M$, will be termed point-right ending. To simplify writing we agree to denote this ending by $a$.

8. - The notions related to right endings have properties analogous to those, stated in [§ 1A, 9, 10, 11, 12, 13, 14].
1C. Inequalities in endings.

1. The relation

\[ *\alpha \leq *\beta, \ (\beta \geq \alpha) \]

will mean that there exist representatives \( E, F \) of \( \alpha, \beta \) respectively such that for every \( x \in E \) and every \( y \in F \) we have \( x \leq y \).

2. The relation

\[ \alpha^* \leq \beta^* \]

will mean that there exist representatives \( E, F \) of \( \alpha^*, \beta^* \) respectively such that for every \( x \in E \) and every \( y \in F \) we have \( x \leq y \).

3. The following are equivalent: I. \( \alpha^* \leq \beta^* \), II. for every representatives \( E, F \) of \( \alpha^*, \beta^* \) respectively we have: for all \( x \in E \) and \( y \in F \), we have \( x \leq y \).

4. For every \( E \subseteq M, E \neq 0 \) we have

\[ \mathcal{L}(E) \leq \mathcal{R}(E) \]

If \( \mathcal{R}(E) \leq \mathcal{L}(E) \), then \( E \) is composed of a single point.

If \( \alpha^* \leq \beta^* \) and \( \beta^* \leq \alpha^* \), then there exists a point \( a \in M \), such that \( \{a\} \) is a representative of both \( \alpha^* \) and \( \beta^* \). If \( \alpha^*, \beta^* \) have both a representative \( \{a\} \), then \( \alpha^* \leq \beta^* \) and \( \beta^* \leq \alpha^* \).

If \( a \) is the maximum of \( E \) and, at the same time, the minimum of \( F \), then \( \mathcal{R}(E) \leq \mathcal{L}(F) \) and \( \mathcal{L}(F) \leq \mathcal{R}(E) \).

5. The class of all endings, right and left makes up a chain. This can be established by the following steps:

- If \( \alpha_1^* \leq \alpha^* \), \( \alpha^* \leq \beta^* \), then \( \alpha_1^* \leq \beta^* \),
- If \( \alpha^* \leq \beta^* \), \( \beta^* \leq \beta^* \), then \( \alpha^* \leq \beta^* \).
- If \( \alpha \leq \beta^*, \beta^* \leq \gamma^* \), then \( \alpha \leq \gamma^* \).
- If \( \alpha^* \leq \beta^*, \beta^* \leq \gamma^* \), then \( \alpha^* \leq \gamma^* \).

These properties prove that the correspondence \( \leq \), which is defined for all endings, is transitive. This correspondence is also reflexive.
The equalities $a^* = \beta^*$ and $\alpha = \beta$ were already defined in [§ 1A and § 1B].

Now define $a^* = \beta$ as $a^* \leq \beta$ and $\beta \leq a^*$ and define $\alpha = \beta$ as $\beta \leq \alpha$ and $\alpha \leq \beta$.

The following are equivalent: I. $a^* = \beta$, II. If $E$ and $F$ are representative of $a^*$ and $\beta$, then $E$ has a maximum and $F$ has a minimum. (Both coincide).

The notion of equality of endings is reflexive, symmetric and transitive. The notion of $\leq$ for endings is invariant with respect to the notion $\equiv$ of equality of endings. Given any two endings $\varphi, \psi$, we have either $\varphi \leq \psi$ or $\psi \leq \varphi$.

The above properties prove that the notion of $\leq$ organizes the class of all endings, right and left, into a chain on which the equality $\equiv$ of endings is the governing equality.

Concerning point-endings, there is no need to make discrimination between right point endings and left point endings. We shall call them shortly point-endings and denote, as before, by the quantities of $M$, determining them.

Thus if $a \in M$, $a$ can be conceived as a right point ending or as a left point ending.

1D. Additional properties of endings.

1. - The following properties of endings can be proved without any requirement of special technique:

We define for endings $\varphi, \psi$ the inequality $\varphi \prec \psi, (\psi \succ \varphi)$ as $\varphi \prec \psi$ but $\varphi \equiv \psi$.

2. - The following are equivalent: I. $a < \mathcal{A}(F)$, II. for every $y \in F$ we have $a < y$.

The following are equivalent: I. $\mathcal{A}(E) < b$, II. there exists $x \in E$ with $x < b$.

3. - The following are equivalent: I. $\mathcal{A}(E) < \mathcal{A}(F)$, II. there exists $x \in E$ with $x < \mathcal{A}(F)$, III. there exists $x \in E$ such that for every $y \in F$ we have $x < y$.

If $\mathcal{A}(E) < \mathcal{A}(F)$, then for every $y \in F$ we have $\mathcal{A}(E) < y$, (but not conversely).
The following are equivalent: I. for every \( y \in F \) we have \( \mathcal{L}(E) < y \), II. for every \( y \in F \) there exists \( x \in E \) with \( x < y \).

4. - The following are equivalent: I. \( a \leq \mathcal{L}(F) \), II. for every \( y \in F \) we have \( a \leq y \).

The following are equivalent: I. \( \mathcal{L}(E) \leq b \), II. there exists \( x \in E \) with \( x \leq b \).

5. - Consider the statements: I. \( \mathcal{L}(E) \leq \mathcal{L}(F) \), II. there exists \( x \in E \) with \( x \leq \mathcal{L}(F) \), III. there exists \( x \in E \) such that for every \( y \in F \) we have \( x \leq y \).

From II follows I, but from I does not follow II. II and III are equivalent.

All three following statements are equivalent: I. \( \mathcal{L}(E) \leq \mathcal{L}(F) \), II. for every \( y \in F \) we have \( \mathcal{L}(E) \leq y \), III. for every \( y \in F \) there exists \( x \in E \) such that \( x \leq y \).

6. - The following are equivalent: I. \( a < *\beta \), II. for every representative \( F \) of \(*\beta\) and for every \( y \in F \) we have \( a < y \), III. there exists a representative \( F \) of \(*\beta\) such that for every \( y \in F \) we have \( a < y \).

The following are equivalent: I. \( *\alpha < \beta \), II. for every representative \( E \) of \(*\alpha\) there exists \( x \in E \) with \( x < b \), III. there exists a representative \( E \) of \(*\alpha\) such that there exists \( x \in E \) with \( x < b \).

7. - The following are equivalent: I. \( *\alpha < *\beta \), II. for every representative \( E \) of \(*\alpha\) there exists \( x \in E \) such that \( x < *\beta \), III. there exists a representative \( E \) of \(*\alpha\) such that there exists \( x \in E \) with \( x < *\beta \), IV. for every representative \( E \) of \(*\alpha\) and every representative \( F \) of \(*\beta\) there exists \( x \in E \) such that for every \( y \in F \) we have \( x < y \), V. there exist representatives \( E, F \) of \(*\alpha\) and \(*\beta\) respectively such that there exist \( x \in E \) such that for every \( y \in F \) we have \( x < y \).

If \( *\alpha < *\beta \), then for every representative \( F \) of \(*\beta\) and every \( y \in F \) we have \( *\alpha < y \), (but not conversely).

8. - If there exists a representative \( E \) of \(*\alpha\) such that there exists \( x \in E \) with \( x \leq *\beta \), then \( *\alpha \leq *\beta \), (but not conversely).
9. - The following are equivalent: I. \( *\alpha \leq \beta \), II. there exists a representative \( E \) of \( *\alpha \) such that there exists \( x \in E \) with \( x \leq b \), III. for every representative \( E \) of \( *\alpha \) there exists \( x \in E \) such that \( x \leq b \).

The following are equivalent: I. \( a \leq *\beta \), II. there exists a representative \( F \) of \( *\beta \) such that for every \( y \in F \) we have \( a \leq y \), III. for every representative \( F \) of \( *\beta \) we have: for every \( y \in F \) we have \( a \leq y \).

10. - The following are equivalent: I. \( *\alpha \leq *\beta \), II. there exists a representative \( F \) of \( *\beta \) such that for every \( y \in F \) we have \( *\alpha \leq y \), III. for every representative \( F \) of \( *\beta \) and for every \( y \in F \) we have \( *\alpha \leq y \), IV. there exist representatives \( E, F \) of \( *\alpha, *\beta \) respectively such that for every \( y \in F \) there exists \( x \in E \) with \( x \leq y \), V. for every representatives \( E, F \) of \( *\alpha, *\beta \) respectively and for every \( y \in F \) there exists \( x \in E \) with \( x \leq y \).

11. - The following are equivalent: I. \( *\alpha < *\beta \), II. there exists a representative \( E \) of \( *\alpha \) such that for every \( x \in E \) we have \( x < *\beta \).

The following are equivalent: \( *\alpha < *\beta \), II. for every representative \( F \) of \( *\beta \) there exists \( y \in F \) with \( *\alpha \leq y \).

12. - The properties [2-11] of left endings have their analogy in the corresponding properties of right endings.

13. - Remark. The endings cannot be identified with Dedekind sections in the ordering \( M \). Indeed let \( R \) be the chain of all ordinary real numbers, and let \( a \in R \).

If we define

\[
R_a^+ = \{ x \mid x \in R, a < x \}, \quad R_a^- = \{ x \mid x \in R, x < a \},
\]

then we have

\[
\mathcal{H}(R_a^-) < a < \mathcal{L}(R_a^+),
\]

where \( a \) denotes the point ending whose representative is \( \{ a \} \).
The smallest ending is $\mathcal{C}(R)$, the greatest $\mathcal{R}(R)$. Every real number $a$ is replaced by three different endings. This circumstance may be useful e.g. in the theory of ordinary functions of bounded variation.

1E. Operations on sets in a linearly ordered semigroup.

1. This will be an auxiliary topic for operations on endings. To introduce these operations we shall suppose that the ordering $M$ is a kind of linearly ordered abelian semigroup $G$, which will be axiomatized as follows: Let $F$ be a non trivial linearly ordered commutative field. There is an addition $a + b$, for $a, b \in G$, which is always performable and yielding an element of $G$. It is supposed to be $\cong$-invariant, where $\cong$ is the equality governing on $G$, (supposed to satisfy the conditions of identity). There is a multiplication $\lambda \cdot a$ performable for every $a \in G$ and every $\lambda \in F$ where $\lambda \geq 0$. We shall write $a\lambda$ or $\lambda a$.

This operation is supposed to be invariant with respect to $\cong$ and to the equality $\equiv$ governing in $F$.

In stating the following admitted axioms we shall drop superscripts over the signs of equality.

\[
\begin{align*}
\begin{cases}
  a + b &= b + a; \\
  a + (b + c) &= (a + b) + c; \text{ for } \lambda \geq 0, \\
  \mu \geq 0 \text{ we have } (\lambda + \mu) a &= \lambda a + \mu a; \text{ if } \lambda \geq 0, \text{ then } \\
  \lambda(a + b) &= \lambda a + \lambda b; \text{ if } \lambda \geq 0 \lambda \geq 0, \text{ then } \lambda(\mu a) &= (\lambda \mu)a; \\
  1_F \cdot a &= a.
\end{cases}
\end{align*}
\]

To these axioms, which look like the usual axioms for a modulus (with exception of the cancellation law), we add the following:

There exists an element $0$ of $G$ such that for every $a \in G$ we have $a + 0 = a$.

[We prove that $0$ is $\cong$-unique]. We shall denote it by $0_G$.

We also admit that for every $a \in G$, we have $0_F \cdot a \cong 0_G$. Thus $G$ is a kind of semigroup with zero.

We suppose that there is a correspondence $\subseteq$ with domains $G$ and range $G$, which makes up a chain, and is $\subseteq$-invariant.
We admit the axioms: \( a = b \) is equivalents to \( a \leq b \), \( b \leq a \);
- if \( a \leq b \), then \( a + c \leq b + c \);
- if \( a \leq b \), \( \lambda \geq 0 \), then \( \lambda a \leq \lambda b \);
- if \( a \geq 0 \), \( 0 \leq \lambda \leq \mu \), then \( \lambda a \leq \mu a \);

We prove that if \( a \leq b \), \( a' \leq b' \), then \( a + a' \leq b + b' \).

We may call \( G \) linearly ordered semigroup with nonnegative multipliers taken from a linearly ordered commutative field.

2. - \( G \) will be a linearly ordered abelian group over \( F \), when 1) \( \lambda \cdot a \) is performable for any \( \lambda \in F \), and the condition in (1) of nonnegativeness of factors is omitted, 2) the cancellation law: « if \( a + b = c + \), then \( b = b' \) » is admitted.

If \( G \) is supposed to be a semigroup only, we shall write \( G^{(w)} \).

3. - We shall consider subsets of \( G^{(w)} \) (or \( G \)) but only not empty ones. They will be denoted by \( E, F, ... \)

We define:
\[
E + F \text{ def } \{ x + y \mid x \in E, y \in F \}
\]
and
\[
\lambda \cdot E \text{ def } \{ \lambda x \mid x \in E \},
\]
(for \( G^{(w)} \) we suppose that \( \lambda \geq 0 \), \( \lambda \in F \).

4. - The following are equivalent: I. \( z \in E + F \), II. there exist \( x \in E \) and \( y \in E \), such that \( z = x + y \).

The following are equivalent: (for any \( \lambda \in F \) for \( G \), and for \( \lambda \geq 0 \) for \( G^{(w)} \)) I. \( z \in \lambda E \), II. there exists \( x \in E \), such that \( z = \lambda x \).

5. - We have \( E + F = F + E \), \( E + (F + G) = (E + F) + G \), \( 1_F \cdot E = E \).

6. - \( (\lambda + \mu)E \subseteq \lambda E + \mu E \) (for any \( \lambda, \mu \) for \( G \), and for \( \lambda \geq 0, \mu \geq 0 \) for \( G^{(w)} \).

Remark. Notice that, even for \( \lambda > 0, \mu > 0 \) and \( G \), the relation \( \lambda E + \mu E \subseteq (\lambda + \mu)E \) may be not true.
If for $G$ and for every $\lambda, \mu \in F$ we have $(\lambda + \mu)E = \lambda E + \mu E$, then $E$ is composed of a single point—and conversely.

7. $\lambda(E + F) = \lambda E + \lambda F$ (for any $\lambda$ for $G$ and for $\lambda \geq 0$ for $G^{(\omega)}$).

8. $\lambda(\mu E) = (\lambda \mu)E$ for any $\lambda, \mu$ for $G$ and for $\lambda \geq 0$, $\mu \geq 0$ for $G^{(\omega)}$.

9. - For the linearly ordered groups $G$ we define $E - F = \{x - y \mid x \in E, y \in F\}$, $-E = \{0\} - E$, $+E = \{0\} + E$, $0 = 0 + E$, (The last also for $G^{(\omega)}$).

10. - For $G$ the following are equivalent: I. $a \in E - F$, II. there exist $x \in E$ and $y \in F$ such that $a = x - y$, and the following are equivalent: I. $a \in -E$, II. there exists $x \in E$ with $a = -x$.

11. - We have for $G^{(\omega)}$ and $G$: $+E = E$.

12. - For $G$ we have $E - F = E + (-F)$, $-E = (-1_F) \cdot E$, $-(\lambda E) = (-\lambda) \cdot E$, $-(E + F) = (-E) + (-F) = -E - F$. $\lambda(-E) = -\lambda E$, $(\lambda - \mu)E \leq \lambda E - \mu E$, $\lambda(E - F) = \lambda E - \lambda F$, $-(-E) = E$.

13. - We define for $G$ and $G^{(\omega)}$, $E \leq F$ as «for every $x \in E$ and every $y \in F$ we have $x \leq y$'', and $E < F$ as «for every $x \in E$ and every $y \in F$, $x < y$''.

We define $a \pm E$ as $\{a\} \pm E$ and $E \pm a$ as $E \pm \{a\}$.

1F. Operations on endings.

1. - We suppose that the chain $M$ in which endings will be considered, is a linearly ordered abelian semi-group or group. Multipliers will be taken from a linearly ordered non trivial commutative field $F$.

2. - If $E \cdot \leq E'$, $F \cdot \leq F'$, then $E + F \cdot \leq E' + F'$.

If $E \cdot = E'$, $F \cdot = F'$, then $E + F \cdot = E' + F'$. 
3. - If \( *\alpha, *\beta \) are left endings, \( E, E' \) are representatives of \( *\alpha \) and \( F, F' \) are representatives of \( *\beta \), then \( E + F = E' + F' \) and \( \mathcal{L}(E + F) = \mathcal{L}(E' + F') \).

Definition. If \( *\alpha, *\beta \) are left endings, then by \( *\alpha + *\beta \) we shall understand the left ending \( \mathcal{L}(E + F) \), where \( E, F \) are representatives of \( *\alpha \) and \( *\beta \) respectively. This notion of addition of left endings does not depend on the choice of representatives.

If \( *\gamma = *\alpha + *\beta \), \( G, E, F \) are representatives of \( *\gamma, *\alpha, *\beta \) respectively, then \( G = E + F \). Conversely, if \( G = E + F \) and \( G, E, F \) are representatives of \( *\gamma, *\alpha, *\beta \) respectively, then \( *\gamma = *\alpha + *\beta \).

The addition \( *\alpha + *\beta \) is invariant with respect to the equality of all endings.

4. - If \( E \cdot \leq F, \lambda > 0 \), then \( \lambda E \cdot \leq \lambda F \). If \( E \cdot = F, \lambda > 0 \), then \( \lambda E \cdot = \lambda F \).

5. - Definition. If \( \lambda > 0 \), then by \( \lambda \cdot *\alpha \) we shall understand the left ending \( \mathcal{L}(\lambda \cdot E) \) where \( E \) is a representative of \( *\alpha \). This notion of multiplication does not depend on the choice of the representative \( E \) of \( *\alpha \).

If \( E, F \) are representatives of \( *\alpha, *\beta \) respectively and \( \lambda > 0 \), then the following are equivalent: I. \( \lambda E \cdot = F \), II. \( \lambda \cdot *\alpha = *\beta \).

The product \( \lambda \cdot *\alpha \) is invariant with respect to the equality of all endings.

6. - We have \( *\alpha + *\beta = *\beta + *\alpha \), and \( (*\alpha + *\beta) + *\gamma = *\alpha + (*\beta + *\gamma) \).

7. - If \( \lambda \geq 0 \), then \( \lambda \cdot (*\alpha + *\beta) = \lambda \cdot *\alpha + \lambda \cdot *\beta \). 1. \( *\alpha = *\alpha \); if \( \lambda \geq 0, \mu \geq 0 \), then \( \lambda(\mu \cdot *\alpha) = (\lambda\mu) \cdot *\alpha \).

8. - If \( \lambda \geq 0, \mu \geq 0 \), then \( (\lambda + \mu) \cdot *\alpha = \lambda \cdot *\alpha + \mu \cdot *\alpha \).

Proof. Let \( E \) be a representative of \( *\alpha \). Then \( \lambda E, \mu E, (\lambda + \mu)E \) are representatives of \( \lambda \cdot *\alpha, \mu \cdot *\alpha \) and \( (\lambda + \mu) \cdot *\alpha \) respectively. By \([§\ 1E;\ 6]\), we have \( (\lambda + \mu)E \subseteq \lambda E + \mu E \). Hence, by \([§\ 1A;\ 3]\), \( \lambda E + \mu E \cdot \leq (\lambda + \mu)E \), and then, by
[Def. § 1A; 6], $\mathcal{S}(\lambda E + \mu E) \cdot (\lambda + \mu) \cdot *\alpha$, i.e., [Def. § 1F; 3] and [Def. § 1F; 5], $\lambda \cdot *\alpha + \mu \cdot *\alpha \leq (\lambda + \mu) \cdot *\alpha$ .... (1).

Now, let $y \in \lambda E + \mu E$. There exist $x', x'' \in E$ such that $y = \lambda x' + \mu x''$.

We shall prove that there exists $y' \in (\lambda + \mu)E$ such that $y' \leq y$. We may suppose that $x' \leq x''$. We get $\mu x' \leq \mu x''$ and then $\lambda x' + \mu x' \leq \lambda x' + \mu x'' = y$.

Hence $(\lambda + \mu)x' \leq y$. We put $y' \mu = (\lambda + \mu)x'$.

Consequently, for every $y \in \lambda E + \mu E$ there exists $y' \in (\lambda + \mu)E$ such that $y' \leq y$.

Hence $(\lambda + \mu)E \cdot (\lambda + \mu)E$, which gives $(\lambda + \mu) \cdot *\alpha \leq \lambda \cdot *\alpha + \mu \cdot *\alpha$ .... (2).

From (1) and (2) the theorem follows.

9. - $0 \cdot *\alpha = 0$, i.e. the point-left ending whose representative is the set composed of the single element $0 \in M$. We also have for every $*\alpha$ the equality $*\alpha + 0 = *\alpha$.

10. - If $*\alpha \leq *\beta$, then $*\alpha + *\gamma \leq *\beta + *\gamma$;
    If $*\alpha \leq *\beta$, $\lambda > 0$, then $\lambda \cdot *\alpha \leq \lambda \cdot *\beta$,
    If $*\alpha \geq 0$, $0 \leq \lambda \leq \mu$, then $\lambda \cdot *\alpha \leq \mu \cdot *\alpha$.

11. - The properties [1F, 6-10] show that the left endings in a linearly ordered abelian semigroup with zero or in a linearly ordered abelian group, make up another linearly ordered abelian semigroup with zero and with nonnegative multipliers taken from $F$. This allow to consider endings and operations on them in the chain of left endings. A similar behaviour show the right endings in $M$.

12. - Till the end of this subsection § 1F, we shall suppose that the chain $M$, under consideration, is a linearly ordered abelian group with arbitrary multipliers taken from a linearly ordered commutative, non trivial field $F$.

13. - The following are equivalent: I. $E \cdot \leq F$, II. $(-F) \leq \cdot (-E)$.

The following are equivalent: I. $E \leq \cdot F$, II. $(-F) \cdot \leq (-E)$.

If $E \leq F$, $\lambda \leq 0$, then $\lambda \cdot F \leq \cdot \lambda \cdot E$. 

If $E \leq \cdot F$, $E' \leq \cdot F'$, then $E + E' \leq \cdot F + F'$.
If $E \leq \cdot F$, $\lambda \geq 0$, then $\lambda E \leq \cdot \lambda F$, and if $E \leq \cdot F$, $\lambda \leq 0$, then $\lambda F \cdot \leq \lambda E$.
If $E = \cdot F$, $E' = \cdot F'$, then $E + E' = \cdot F + F'$.
If $E = \cdot F$, $\lambda \leq 0$, then $\lambda E = \cdot \lambda F$, and if $E = \cdot F$, $\lambda \leq 0$, then $\lambda F \cdot = \lambda E$.
If $E = \cdot F$, $\lambda \leq 0$, then $\lambda F \cdot = \lambda E$.

The following are equivalent: I. $E \cdot = F$, II. $(-F) = \cdot (-E)$, and the following are equivalent: I. $E = \cdot F$, II. $(-F) \cdot = (-E)$.

14. - Definition. We define the addition of right endings as follows: By $\alpha^* + \beta^*$ we shall understand the right ending $\mathcal{R}\mathcal{E}(E + F)$, where $E$, $F$ are representatives of $\alpha^*$, $\beta^*$ respectively.
This notion does not depend on the choice of representatives.

15. - Definition. We define the multiplication of right endings by non negative quantities as follows: If $\lambda \geq 0$, then by $\lambda \cdot \alpha^*$ we shall understand the right ending $\mathcal{R}\mathcal{E}(\lambda E)$, where $E$ is a representative of $\alpha^*$. This notion does not depend on the choice of the representative $E$ of $\alpha^*$.

The above operations $[\S \, 1 F; \, 14, \, 15]$ are invariant with respect to the equality of all endings.

16. - The addition and multiplication of right endings by non negative factors behaves exactly as the same operations performed on left endings. Thus laws similar to $[\S \, 1 F, \, 2-10]$ are valid for right endings.

17. - Now we are going to introduce further operations on endings.
Definition. If $\lambda \leq 0$, we define $\lambda \cdot \alpha$ as the right ending $\mathcal{R}\mathcal{E}(\lambda \cdot E)$, where $E$ is a representative of $\alpha$. If $\lambda \leq 0$, we define $\lambda \cdot \alpha^*$ as the left ending $\mathcal{L}\mathcal{E}(\lambda E)$ where $E$ is a representative of $\alpha^*$.
These notions are independent of the choice of representatives.
18. - If \( \lambda = 0 \) we can apply the definition [17] as well as [5] for \( \lambda \cdot *\alpha \). Both yield the same, namely the point-ending 0. The same holds for \( \lambda \cdot \alpha^* \) if \( \lambda = 0 \).

19. - The notions [17] are invariant with respect to the equality of all endings.

20. - Definition. By \(- *\alpha\) we shall understand the right ending \( \mathcal{R}\mathcal{E}(-E) \) where \( E \) is a representative of \(*\alpha\).

By \(- \alpha^*\) we shall understand the left ending \( \mathcal{L}\mathcal{E}(-E) \) where \( E \) is a representative of \( \alpha^* \).

These notions do not depend on the choice of the representative \( E \). They are invariant with respect to the equality of all endings.

21. - For any \( E \neq 0 \) we have

\[
\mathcal{R}\mathcal{E}(E) = - \mathcal{L}\mathcal{E}(E); \quad \mathcal{L}\mathcal{E}(E) = - \mathcal{R}\mathcal{E}(E)
\]

\[
- \mathcal{R}\mathcal{E}(E) = \mathcal{L}\mathcal{E}(-E); \quad - \mathcal{L}\mathcal{E}(E) = \mathcal{R}\mathcal{E}(-E).
\]

22. - The following are equivalent: I. \(*\alpha = - \beta^*\); II. \( \beta^* = - *\alpha \).

We have

\[
- ( - *\alpha ) = *\alpha, \quad - ( - \alpha^* ) = \alpha^*, \quad ( - 1 ) \cdot *\alpha = - *\alpha;
\]

\[
( - 1 ) \cdot \alpha^* = - \alpha^*.
\]

23. - \((\alpha^* + \beta^*) = ( - \alpha^* ) + ( - \beta^* ); \quad - ( *\alpha + *\beta ) = ( - *\alpha ) + ( - *\beta ).\)

24. - For any \( \lambda \in F \) we have:

\[
- (\lambda \cdot *\alpha ) = ( - \lambda ) \cdot *\alpha ; \quad - (\lambda \cdot \alpha^* ) = ( - \lambda ) \cdot \alpha^*.
\]

25. - For any \( \lambda, \mu \in F \) we have:

\[
\lambda (\mu \cdot *\alpha ) = (\lambda \mu) \cdot *\alpha ; \quad \lambda (\mu \cdot \alpha^* ) = (\lambda \mu) \cdot \alpha^*.
\]

For any \( \lambda \in F \) we have

\[
\lambda (\alpha^* + \beta^* ) = \lambda \cdot \alpha^* + \lambda \cdot \beta^* ; \quad \lambda ( *\alpha + *\beta ) = *\alpha + \lambda \cdot \beta^*.
\]

26. - Remark. The equalities \((\lambda + \mu) \cdot *\alpha = \lambda \cdot *\alpha + \mu \cdot *\alpha\);

\((\lambda + \mu) \cdot \alpha^* = \lambda \cdot \alpha^* + \mu \cdot \alpha^*\), hold true for \( \lambda \geq 0, \mu \geq 0 \), and also for \( \lambda \leq 0, \mu \leq 0 \), but not for \( \lambda > 0, \mu < 0 \) or \( \lambda < 0, \mu > 0 \).
27. - We have defined the addition for homogeneous endings only; we do not add heterogenous endings, but we define subtraction for heterogenous endings only.

Definition. We define \( \*\alpha - \*\beta \) as \( \*\alpha + (\*\beta) \) and \( \alpha* - \*\beta \) as \( \alpha* + (\*\beta) \).

If one of two given endings is a point-ending, say \( \alpha \), the expressions \( \*\alpha \pm \*\beta \), \( \alpha* \pm \*\beta \), \( \*\alpha \pm \alpha* \), \( \alpha \pm \alpha* \) are meaningful, since a point-ending can be considered as a right ending, as well as a left ending.

We have \( \*\alpha + 0 = \*\alpha \), \( \alpha* + 0 = \alpha* \).

28. - We have some theorems concerning inequalities in endings:

If \( \*\alpha \leq \*\beta \), \( \lambda \leq 0 \), then \( \lambda \cdot \*\beta \leq \lambda \cdot \*\alpha \). If \( \*\alpha \leq \*\beta \), \( \lambda \leq 0 \), then \( \lambda \cdot \*\beta \leq \lambda \cdot \*\alpha \).

29. - If \( \lambda \leq \mu \), and \( \lambda \), \( \mu \) are any quantities of \( F \), and \( \*\alpha \geq 0 \), then \( \lambda \cdot \*\alpha \leq \mu \cdot \*\alpha \).

30. - If \( \*\alpha \geq 0 \), \( 0 \leq \lambda \leq \mu \), then \( \lambda \cdot \*\alpha \leq \mu \cdot \*\alpha \).

31. - The following are equivalent: I. \( \*\alpha \leq \*\beta \), II. \( - \*\beta \leq - \*\alpha \), and the following are equivalent: I. \( \*\alpha \leq \*\beta \), II. \( - \*\beta \leq - \*\alpha \).

1G. Supplementary theorems on endings.

1. - The following rules have a lemmatic character, since they will be only used in the discussion of extending measure.

2. - The statements I. \( \*\alpha \leq \alpha + \*\beta \), II. \( \*\alpha - \*\beta \) are equivalent, and so are I. \( \*\alpha \leq \alpha + \*\beta \), II. \( \*\alpha - \*\beta \) are equivalent.

3. - If \( \alpha \leq \*\alpha + \*\beta \), then \( \alpha - \*\alpha \leq \*\beta \).

4. - If \( \alpha - \*\alpha \leq b + \*\beta \), then \( \alpha - b \leq \*\alpha + \*\beta \), and conversely.

5. - If \( \*\alpha < \*\beta \), then \( \*\alpha + c < \*\beta + c \).
We notice that if \( \alpha < \beta \) and \( \gamma \) is any, we cannot say that \( \alpha + \gamma < \beta + \gamma \).

The cancellation law: \( \alpha + \gamma = \alpha + \gamma \), then \( \gamma = \gamma \), is not valid.

6. - We have for any chain \( M \): if \( E \leq F \) then \( \mathcal{E}(E) \leq \mathcal{E}(F) \).

7. - For any chain \( M \) the following statements are valid: If for every \( x \in E \) we have \( x \leq \alpha \), then \( \mathcal{E}(E) \leq \alpha \). If for every \( y \in F \) we have \( \alpha \leq y \), then \( \alpha \leq \mathcal{E}(F) \).

1H. Endings in a superchain of a given chain.

1. - Let \( M, M' \) be not empty \( N \)-subsets of \( N \) and suppose that \( M, M' \) are organized into chains \( \leq, \leq' \), denoted by \( (M), (M') \) respectively (see [Preface 5]). Let \( \equiv, \equiv' \) be the corresponding equalities governing on them. These equalities may differ from \( \equiv \) and from one another.

2. - Definition. According to [Preface 83] we say that \( (M) \) is an order-genuine subchain of \( (M') \) whenever for \( N \)-sets we have

1) \( M \subseteq M' \) in \( N \)
2) if \( a, b \) belong to the \( N \)-set \( M \), then the following are equivalent: I. \( a \equiv b \), II. \( a \equiv' b \).

It follows that also the following are equivalent for \( a, b \in M \): I. \( a \equiv b \), II. \( a \equiv' b \). A further consequence is this:

If the \( \langle \rangle \)-subset \( E \) of \( M \) is also an \( \langle \rangle \)-subset of \( M \), there exists a \( \langle \rangle \)-subset \( E' \) of \( M \) such that \( E \subseteq E' \) in \( N \). Thus the \( \equiv \)-equivalence classes \( A \) are \( N \)-contained in the corresponding \( \equiv' \)-equivalence classes \( A' \).

Denote by \( s \) the correspondence which attaches to every \( \equiv \)-equivalence class (which is an \( \langle \rangle \)-set) the \( \equiv' \)-equivalence class \( A' \) (which is also an \( \langle \rangle \)-set) such that \( A \subseteq A' \) in \( N \). We call \( s \) natural correspondence. (\( s \) is the class of all \( \equiv \)-equivalence classes and \( A \) is contained in the class of all \( \equiv' \)-equivalence classes. \( s \) is one-to-one. If \( E \) is an
\( M \)-subset of \( M \), the correspondence \( s \) generates the corresponding \( M' \)-subset \( E' \) of \( M' \).

We shall write \( E' \leftrightarrow s E \). For sake of simplicity we shall sometimes use the same symbol \( E \) for both \( E \) and \( E' \), but, if needed, we shall emphasize the discrimination by saying \( E \) in \( (M) \) and \( E \) in \( (M') \).

3. - Definition. Let \( (M) \) be an order-genuine subchain of \( (M') \). If the equality \( M \) coincides with the equality \( M' \) if restricted to \( M \), i.e. if \( a \in M \), then the two statements \( a \leq M b \), \( a \leq M' b \) are equivalent, we say that \( (M) \) is an order-genuine strict subchain of \( (M') \), [Preface; 8]. In this case \( s \) is identity of \( N \)-sets, (restricted to \( N \)-equivalence classes).

If \( (M) \) is an order-genuine subchain of \( (M') \), \( (M) \) can be modified so as to obtain an order genuine strict subchain of \( (M') \).

Indeed, let us replace \( M \) by the \( N \)-set \( M_1 \) of all elements \( y \), such that there exists \( x \in M \) with \( x \equiv y \), i.e. instead of \( \equiv \)-equivalence classes we consider the \( s \)-corresponding \( \equiv \)-equivalence classes. We organize \( M_1 \) into a chain \( (M_1) \), by defining \( a_1 \leq M_1 b_1 \), by « there exist \( a, b \in M \) such that \( a \equiv a_1 \), \( b \equiv b_1 \) and \( a \leq M b \) ». The ordering \( (M_1) \) is a chain which is order-isomorphic with \( (M) \) and, at the same time, \( (M_1) \) is an order-genuine strict subchain of \( (M') \). \( (M_1) \) is order-isomorphic with \( (M) \) through the isomorphism \( s \).

4. - Definition. If \( (M), (M') \) are chains, then, [Preface; 8], \( (M) \) is an order genuine subchain of \( (M') \) through the isomorphism \( t \), whenever:

1) \( t \) is a one-to-one correspondence, with domain \( M \), and is \( M \)-invariant in it.

2) \( t \) transforms \( (M) \) into a chain \( (M_2) \) which is an order-genuine subchain of \( (M') \). \( t \) may be not \( M' \)-invariant in its region. \( t \) is invariant, in its region, with respect to \( M' \).

5. - We have devoted quite much room to the above discussion, because of the subtlety of subsequent topics which, otherwise, could be confused.
If \((M)\) is an order-genuine subchain of \(M'\), and \(E, F \subseteq M\), then the following are equivalent:

1. \(E \cdot \subseteq F\) in \((M)\),
2. \(E \cdot \subseteq F\) in \((M')\).

(Here \(E, F\) denote in I. the \(\subseteq\)-subsets of \(M\) and in II., the \(s\)-corresponding \(\subseteq\)-subsets of \(M'\)).

The following are also equivalent:

1. \(E \cdot \subseteq F\) in \((M)\), and II. \(E \cdot \subseteq F\) in \((M')\),
2. \(E \cdot \subseteq F\) in \((M)\), and II. \(E \cdot \subseteq F\) in \((M')\),
3. \(E \cdot \subseteq F\) in \((M)\), and II. \(E \cdot \subseteq F\) in \((M')\).

6. - The left ending \(\mathcal{G}_M(E)\) is the class of all \(M\)-sets \(F \subseteq M\) such that \(F \cdot \subseteq E\), and the left ending \(\mathcal{G}_M'(E)\) is the class of all \(M'\)-sets \(F' \subseteq M'\) such that \(F' \cdot \subseteq E\).

It may happen that these classes are different: the class of all those \(F^'\) may be «larger» than that of the sets \(F\) — even if \((M)\) is a strict subchain of \((M')\). Nevertheless, we can define a natural order isomorphism \(N_t\) between the \(M\)-left endings and the corresponding \(M'\)-left endings, as follows: Let \(*\alpha\) be an \(M\)-left ending and \(E_M\) its representative. Let \(E_M'\) be the \(s\)-corresponding subset of \(M'\). By \(n_t(*\alpha)\) we shall understand the left ending \(\mathcal{G}_M'(E_M')\).

The correspondence \(n_t\), thus defined, does not depend on the choice of the representative \(E_M\) of \(*\alpha\). The domain of \(n_t\) is the class of all \(M\)-left endings, and its region is contained in the class of all \(M'\)-left endings. \(n_t\) is invariant, in its domain, with respect to the equality \(\subseteq\) of left endings in \(M\), and is invariant, in its range, with respect to the equality \(\subseteq\) of left endings in \(M'\).

The correspondence \(n_t\) preserves the ordering, i.e. if \(*\alpha, *\beta\) are left endings in \(M\), then the following are equivalent: I. \(*\alpha \leq *\beta\), II. \(n_t(*\alpha) \leq n_t(*\beta)\).

\(n_t(*\alpha)\) will be termed natural image of \(*\alpha\) in \(M'\). Many times we can identify, without any harm, the left ending \(*\alpha\) with its natural image. However, we shall denote them differently if discrimination will be relevant.

7. - We proceed in a similar way with right endings,
defining their *natural images* in $M'$ by means of the analogous correspondence, denoted by $n_r$. The joint correspondence $n$ of $n_l$ and $n_r$ transforms $M$-endings into $M'$-endings in a one-to-one way, preserving the ordering of endings.

Indeed we have the following properties:

If $E_M$ is a representative of the ending $\gamma$ in $M$, then $\phi(E_M) = E_{M'}$ is also a representative of $n(\gamma)$ in $M'$. If $\gamma$ is a point-ending in $M$, then $n(\gamma)$ is also a point-ending in $M'$, and conversely if $\gamma$ is an ending in $M$, and $n(\gamma)$ is a point-ending in $M'$, then $\gamma$ is a point-ending in $M$.

The point ending in $M$, generated by the element $a$ may be denoted by the same letter $a$, which will be also used to denote its $n$-corresponding point-ending.

The above can be proved by transforming $(M)$ into a strict subchain of $(M')$, as indicated in [§ 1H; 3]. We have the properties:

If $*\alpha_1, *\beta_1, \alpha_1^*, \beta_1^*$ denote $n$-images of the $M$-endings $*\alpha, *\beta, \alpha^*, \beta^*$ respectively, the following are equivalent:

\[
\begin{align*}
(1) & \quad *\alpha \leq *\beta \text{ and } *\alpha_1 \leq *\beta_1, \\
(2) & \quad *\alpha = *\beta \text{ and } *\alpha_1 = *\beta_1, \\
(3) & \quad *\alpha \leq *\beta \text{ and } *\alpha_1 \leq *\beta_1, \\
(4) & \quad *\alpha = *\beta \text{ and } *\alpha_1 = *\beta_1,
\end{align*}
\]

Thus to the chain of all $M$-endings there corresponds, through $n$, a chain of $M'$-endings, which is an order-genuine strict subchain of the chain of all $M'$-endings.

8. The natural isomorphism of endings in $(M)$ and $(M')$ preserves the operations on left endings and those on right endings.

1K. The chain of endings.

1. - The given chain $(M)$ is an order-genuine subchain of the chain $(M)$ of all endings through the order-isomor-
phism which makes corespond to the element $a \in M$ the point-ending $a$.

If $A$ is a non empty set of $M$-endings, then by the supremum of $A$ in $M$ we shall understand an ending $\beta$ such that

1) for every $a \in A$ we have $a \leq \beta$, 2) if for every $a \in A$ we have $a \leq \beta'$, then $\beta \leq \beta'$.

In a similar way we define the infimum of $A$ in $M$. If we consider the chain $M_r$ of right endings, we define analogously the supremum and infimum in $M_r$ of a set of right endings.

2. - Theorem. If $A \neq \emptyset$ is a set of right endings, then $A$ admits in $M$ a supremum. This supremum is a right ending $\alpha^*$. This ending $\alpha^*$ is also a supremum of $A$ in $M_r$.

If for every $\xi^* \in A$, we have $\xi^* \leq \beta'$, then $\alpha^* \leq \beta'$.

Proof. To prove that we shall rely on the lemma:

If $\lambda^* < \mu^*$, then there exists $y \in M$ such that $\lambda^* < y \leq \mu^*$ in the ordering of $M$-endings.

The theorem is true if we suppose that $A$ admits a maximum.

Suppose that $A$ does not admit any maximum.

The following general theorem holds true for any chain $P$ which does not admit a maximal element: There exists a distinguished well ordering $p_1 < p_2 < \ldots < p_\alpha < \ldots$ of elements of $P$ which is cofinal with $P$, i.e. for every $q \in P$ there exists $\alpha$ such that $q \leq p_\alpha$.

2a. - Let

\[
\epsilon_1^* < \epsilon_2^* < \ldots < \epsilon_\alpha^* < \ldots
\]

be a distinguished well ordering cofinal with $A$, where $\epsilon_\alpha^* \in A$. To apply the lemma, let us well order the set $M$ in

3) See (8). An ordinal is termed distinguished when it is infinite and the smallest among all ordinals with the same power.
the form of a distinguished well order-sequence.

(2) \( a_1, a_2, \ldots, a_\beta, \ldots \)

Given an index \( \alpha \) from (1), find in (2) the element with the smallest index — denote it by \( y_\alpha \) — such that \( e_\alpha^* < y_\alpha \leq e_{\alpha+1}^* \) in \( M \). The sequence

(3) \( y_1 < y_2 < \ldots < y_\alpha < \ldots \)

is a well ordering whose ordinal is that of (1).

Denote by \( Y \) the set of all elements of (3) and put

(4) \( \beta^* \overset{\text{def}}{=} \mathcal{R}_G(Y) \) in \( M \).

If we put \( Y(\alpha) = \{ y_\alpha' \mid \alpha' > \alpha \} \) for \( \alpha = 1, 2, \ldots \), the set \( Y(\alpha) \) is for every \( \alpha \) a representative of \( \beta^* \), by \([\S\ 1A; 12]\).

Applying \([\S\ 1D; 11]\), we get \( e_\alpha^* < \beta^* \) for every \( \alpha \), which proves that \( \beta^* \) is an upper bound of \( Y \), and then also for \( A \), because (1) is cofinal with \( A \).

Now let \( \gamma^* \) be a right ending in \( M \) such that \( \gamma^* < \beta^* \). We shall prove that \( \gamma^* \) is not an upper bound of \( A \). Let \( C \) be a representative of \( \gamma^* \). On account of \([\S\ 1D; 7]\) there exists \( y_\alpha \) such that for every \( z \in C \) we have \( z < y_\alpha \). We have \( z < e_{\alpha+1}^* \) for every \( z \in C \), and hence, by \([\S\ 1D; 10]\) we get \( \gamma^* \leq e_{\alpha+1}^* < e_{\alpha+2}^* \), which proves that \( \gamma^* \) is not an upper bound of \( Y \), and then neither of \( A \).

Consequently, if \( \beta_\alpha^* \) has the property that for every \( \alpha \) we have \( e_\alpha^* < \beta_\alpha^* \), then we cannot have \( \beta_\alpha^* < \beta^* \). Consequently \( \beta^* \leq \beta_1^* \), which proves that \( \beta^* \) is the supremum of \( A \) in \( M_r \).

2b. Suppose that for every \( \xi^* \in A \) we have \( \xi^* \leq *\beta^* \). Hence we have \( y_\alpha \leq *\beta^* \) for every \( \alpha \).

Let \( B \) be a representative of \(*\beta^* \). By \([\S\ 1D; 9]\), for every \( z \in B \) we have \( y_\alpha \leq z \) for all \( \alpha \). Since \( Y \) is a representative of \( \beta^* \), it follows

\[ \beta^* \leq *\beta^* \]

2c. It remains to prove that \( \beta^* \), defined in (4) is the supremum of \( A \) with respect to \( M \). This, however, follows from what has been proved. Indeed, if \( A \leq \gamma \) where \( \gamma \) is a
right ending, we have $\beta^* \leq \gamma$, and if $\gamma$ is a left ending, we also have $\beta^* \leq \gamma$. The theorem is proved.

3. - Theorem. If $A \neq 0$ is a set of left endings, then $A$ admits in $M$ an infimum. This infimum is a left ending $\alpha$. It is also an infimum of $A$ in $M_1$.

If, for every $*\xi \in A$, we have $*\xi \geq \beta^*$, then $\beta^* \leq \alpha$. Proof similar to the preceding one.

4. - Theorem. If $A \neq 0$ is a set of right endings in $M$, $B \neq 0$ is a set of left endings in $M$, and for every $\alpha^* \in A$ and every $\beta^* \in B$ we have $\alpha^* \leq \beta^*$, then

$$\sup A \leq \sup B.$$

5. - Theorem. If $A \neq 0$ is a set of right endings and does not admit a minimum, then $A$ admits an infimum in $M$, which is a left ending $\beta^*$. We also have:

1) If, for every $\xi^* \in A$, we have $\gamma^* \leq \xi^*$, then $\gamma^* < \beta^*$.
2) If, for every $\xi^* \in A$, we have $\delta^* \leq \xi^*$, then $\delta^* \leq \beta^*$.
3) We have $\beta^*_a = \sup \{ \gamma^* \mid \gamma^* \leq A \} < \beta^*$.
4) $\beta^*_a$ is the infimum of $A$ in $M_r$.

Proof. We shall rely on the following lemma:

If $y \leq \alpha^*$, and $E$ is a representative of $\alpha^*$, then the set $\{ x \mid x \in E, x \geq y \}$ is also a representative of $\alpha^*$.

5a. - There exists a distinguished well ordering

(1) $e^* > e_2^* > \ldots > e_n^* > \ldots$

coinitial with $A$, where $e^*_a \in A$ for all $a$. By lemma [§ 1K; 2] for every $a$ there exists $y \in M$ such that

(2) $e^*_{a+1} < y \leq e^*_a$.

Let $a_1, a_2, \ldots, a_\beta, \ldots$

be a distinguished well ordering with domain $M$. For every $1 \leq a$
take, in (3), the element with the smallest index satisfying (2), and denote it by $y_\alpha$, so that

$$e_{x+1}^* < y_\alpha \leq e_x^*.$$  \hfill (4)

Denote by $Y$ the set (1), and put

$$*\beta \overset{\text{df}}{=} \mathcal{C}(Y) \text{ in } M.$$

Let $F_1, F_2, \ldots, F_\alpha, \ldots$ be representatives of the endings (1) respectively, and put

$$F'_\alpha \overset{\text{df}}{=} \{ z \mid z \in F_\alpha, y_\alpha \leq z \}.$$

By the lemma (in this proof) $F'_\alpha$ is a representative of $e_\alpha^*$. We get

$$*\beta < e_\alpha^* \text{ for all } \alpha, \text{ so } *\beta \text{ is a lower bound of } A \text{ in } M.$$  \hfill (5)

If we suppose that $*\beta < *\gamma$ holds true, we can prove, by [§ 1D; 7] that there exists $\alpha$ such that $e_{x+1}^* < *\gamma$. Hence $*\beta$ is among all left endings the greatest lower bound of $A$.  

5b. - Suppose that for a right ending $\gamma^*$ we have $\gamma^* \leq A$. Hence $\gamma^* \leq e_\alpha^*$ for all $\alpha$.

It follows that $\gamma^* < y_\alpha$ for all $\alpha$. Let $C$ be a representative of $\gamma^*$.

We have: for every $x \in C$, $x < y_\alpha$ for all $y$. Since $Y$ is a representative of $*\beta$, it follows that $\gamma^* < *\beta$, [§ 1C; 3]. I say that the equality

$$\gamma^* = *\beta$$  \hfill (6)

is impossible. Indeed, if (6) were true, $*\beta$ would be a point ending, and then the set $Y$ would have a minimum, which is excluded. Thus

$$\gamma^* < *\beta.$$  

5c. - Consider the set of all right endings $\gamma^*$ such that $\gamma^* \leq A$. This set admits a supremum, by [§ 1K; 2], which
is a right ending, say $\beta_*^*$. We have

$$\beta_*^* \leq A;$$

hence, by what has been proved in [5b],

$$\beta_*^* < \beta.$$ 

Thus we have proved the thesis 3).

5d. We have proved that if $\gamma$ is a right ending with $\gamma \leq A$, then $\gamma \leq \beta^*$, and if $\gamma$ is a left ending with $\gamma < A$, then $\gamma < \beta$. It follows that $\beta$ is the infimum of $A$ in $M$. Now, $\sup \{ \gamma^* \mid \gamma^* \leq A \}$ is the infimum of $A$ in $M$, so 4) is proved. The theorem is established.

6. **Theorem.** If $A \neq 0$ is a set of left endings and $A$ does not admit a maximum, then $A$ admits the supremum in $M$ which is a right ending $\beta^*$. We have also:

1) If, for every $\xi \in A$, we have $\gamma \leq \xi$, then $\gamma > \beta$;
2) if, for every $\xi \in A$, we have $\delta \geq \xi$, then $\delta \geq \beta$;
3) we have $\beta_* \geq \inf \{ \gamma \mid \gamma \geq A \} > \beta*$;
4) $\beta_*$ is the supremum of $A$ in $M$.

Proof analogous to the preceding one.

7. **Theorem.** If $A \neq 0$ is a set of left endings then $A$ admits a supremum in $M$ which is a left ending whenever $A$ admits a maximum, and which is a right ending in the opposite case. An analogous theorem holds true for non empty set of right endings.

8. **Theorem.** If $A \neq 0$ is a set of any endings, then $A$ admits a supremum in $M$ and an infimum in $M$.

**Proof.** We shall confine us to the proof of the existance of the supremum in the case where $A$ does not admit any maximum. There exists a distinguished well ordering

$$\gamma_1 < \gamma_2 < \ldots < \gamma_\alpha < \ldots$$

of endings, cofinal with $A$, where $\gamma_\alpha \in A$. Let

$$\alpha_*^* < \alpha_*^* < \ldots < \alpha_*^* < \ldots$$
be the subsequence of (1) composed of all right endings, (point endings included) and

\[(3) \quad *\beta_1 < *\beta_2 < \ldots < *\beta_\mu < \ldots\]

the subsequence of (1) composed of all left endings (point endings excluded).

Of course there may exist only one of these sequences, or one of them may be finite. In the case one of these sequences is finite or has a maximum, the theorem follows from the foregoing theorem. Suppose that both sequences (2), (3) are infinite, both without any maximum. It suffices to suppose that both sequences (3) and (2) are cofinal with (1).

The set (2) admits in \(M\) a supremum \(\xi^*\) and the set (3) admits in \(M\) a supremum \(\eta^*\). We have \(A \leq \xi^*, A \leq \eta^*\).

By [§ 1K; 2] we have \(\xi^* \leq \eta^*\) and, by theor. [§ 1K; 6], \(\eta^* \leq \xi^*\).

Consequently \(\xi^* = \eta^*\). The theorem is proved.

9. - The above theorems are apt to overcome the main difficulty in dealing with non-archimedean linearly ordered fields, where a non empty set of quantities, even if bounded, may not admit any supremum or infimum.

10. - If \(M\) is a chain, then its right endings make up a new chain \(M_r\) and the elements of \(M_r\) satisfy the axioms of a linearly ordered semigroup, given in [§ 1E; 1], (see [§ 1F; 11]). Hence we can consider right and left endings in \(M_r\). To these « double » endings we can apply the operations of additions and multiplication.

A similar behaviour shows \(M_l\). Each \(M\)-ending \(*\alpha\) generates an \(M\)-point ending whose one of \(M\)-representatives is the set \(\{ *\alpha \}\) composed of the single element \(*\alpha\).

If \(\Phi\) is a set of \(M\)-endings \(\alpha^*\), and \(\Phi'\) the set of corresponding \(M\)-point endings \(\mathcal{B} \Phi (\{ \alpha^* \})\), then the supremum of \(\Phi'\) in the ordering of \(M\)-endings can be smaller than the supremum of \(\Phi\) in the ordering of \(M\)-endings. Indeed, the chain \(M\) is « larger » than \(M\).
§ 2. - Partitions in a tribe.

1. - Partitions are usually considered as an easy topic which does not require any detailed setting. Since the present paper requires precision, we shall devote some care to correct and not ambiguous definitions.

Let $B$ be a finitely additive (abstract or constructed) non trivial tribe. By partition in $B$ we shall understand any finite sequence $a_1, a_2, \ldots, a_n$, $(n \geq 1)$, $a_i \in B$, such that $a_i \cdot a_j = 0_B$ for $i \neq j$, and $a_1 + \ldots + a_n = 1_B$.

It will be denoted by $\{a_i\}$.

Expl. The sequences $\{0, 1\}$; $\{0, 1, 0, 0, 0\}$; co $a$, $a'$, $\{1\}$ are partitions.

2. - If $\{a_i\}$ is a partition, there exists $j$ such that $a_j \neq 0$. A partition $\{a_i\}$ is said to be equal to the partition $\{b_j\}$,

$$\{a_i\} \equiv \{b_j\},$$

whenever for every $i$ with $a_i \neq 0$ there exists $j$ such that $a_i = b_j$ and for every $j$ with $b_j \neq 0$ there exists $i$ such that $b_j = a_i$.

The notion $\equiv$ of equality of partitions is invariant with respect to the equality $\equiv_B$, governing in $B$. The equality $\equiv$ obeys the usual formal rules of identity.

3. - The following are equivalent: I. $\{a_i\} \equiv \{b_j\}$, II. $\{a_i \mid a_i \neq 0\} \equiv \{b_j \mid b_j \neq 0\}$.

Let $E$ be a non empty, finite set of mutually disjoint somata, such that

$$\sum_{a \in E} a = 1,$$

let $n \geq 1$ be the number of elements of $E$, and let $a(i)$ be a function with $\bigcup a$ equal to the set $(1, 2, \ldots, n)$ and with $\Delta a = E$. Then $\{a(i)\}$ is a partition in $B$. Let $a_1, \ldots, a_n$ be a partition, and $T$ a one-to-one mapping of the set $1, 2, \ldots, n$ into itself, (permutation). If we put $b_j = a_{T(j)}$ for $j = 1, 2, \ldots, n$. 


then \{b_j\} is also a partition, and we have \{a_i\} \equiv \{b_j\}. If \(a_1, \ldots, a\) is a partition, then \(a_1, \ldots, a_n, 0\) is also a partition which is \(\equiv\)-equal to the former one. If \(a_1, \ldots, a_n, 0\) is a partition, so is \(a_1, \ldots, a_n\), and the last is equal to the former one. To every partition \{a_i\} there exists a partition \{b_j\} such that all \(b_j \neq 0\) and \(\{a_i\} \equiv \{b_j\}\). The set \{b_j\} is unique.

4. - The above properties show that the partition is, up to equality, well determined by the set of its somata which differ from the null-soma. The presence or absence of a finite number of null-somata in the sequence \{a_1, \ldots, a_n\} does not matter. The permutation of elements of the sequence does neither. Thus, from the point of view of formal logic, a partition is, with respect to \(\equiv\), neither a set of somata nor a sequence. But, with respect to \(\equiv\), a partition could be defined \(^9\) as a suitable class of sequences of somata of \(B\).

5. - A partition \(\{a_i\}\) is said to be a subpartition of \(\{b_j\}\), \(\{a_i\} \subseteq \{b_j\}\), whenever for every \(i\) there exists \(j\) such that \(a_i \equiv b_j\).

If \(\{a_i\} \equiv \{b_j\}\), \(\{b_j\} \equiv \{c_k\}\), then \(\{a_i\} \equiv \{c_k\}\). We have \(\{a_i\} \equiv \{a_i\}\).

The following are equivalent: I. \(\{a_i\} \equiv \{b_j\}\). II. \(\{a_i\} \equiv \{b_j\}\) and \(\{b_j\} \equiv \{a_i\}\).

The notion \(\equiv\) is invariant with respect to the equality \(\equiv\) of partitions, and to the equality \(\equiv\), governing in \(B\). If \(\{a_i\}, \{b_k\}\) are partitions in \(B\), then the double sequence \(\{a_i, b_k\}\), if ordered in any way into a single sequence, is a subpartition of \(\{a_i\}\) and \(\{b_k\}\). If \(\{b_k\}\) is a subpartition of \(\{a_i\}\), then there exists a partition \(\{a_{ik}\}\) such that 1) \(\Sigma_k a_{ik} = a_i\) for all \(i\), 2) \(\{b_k\} \equiv \{a_{ik}\}\).

6. - Given two partitions \(\{a_i\}, \{b_j\}\), there exists a \(\equiv\)-unique partition \(\{c_k\}\), such that 1) \(\{c_k\} \equiv \{a_i\}\), \(\{c_k\} \equiv \{b_j\}\); 2) if \(\{d_i\} \equiv \{a_i\}\) and \(\{d_i\} \equiv \{b_j\}\), then \(\{d_i\} \equiv \{c_k\}\).

\(^9\) differently from our setting.
This partition \( \{ c_k \} \) equals \( \{ p_{ij} \} \), where \( p_{ij} \overrightarrow{a_i} b_j \) for all \( i \) and \( j \).

We call \( \{ a_i b_j \} \), i.e., \( \{ c_k \} \), product of \( \{ a_i \} \) and \( \{ b_j \} \):

\[
\{ a_i \} \cdot \{ b_j \}.
\]

The product of two partitions is invariant with respect to \( \pi \) and \( B \). We have \( \{ a_i \} \cdot \{ a_i \} \overrightarrow{\pi} \{ a_i \} \); the product is commutative and associative. The relation \( \{ a_i \} \overrightarrow{\pi} \{ b_j \} \) is equivalent to \( \{ a_i \} \cdot \{ b_j \} \overrightarrow{\pi} \{ a_i \} \).

The maximal partition is the sequence \( \{ 1_B \} \) composed of the single element \( 1_B \).

It contains every partition. Of course, for any \( a \in B \) the sequence \( a, co a ! \) is a partition.

If \( \{ a_i \} \overrightarrow{\pi} \{ b_i \} \), then there exists a partition \( \{ c_k \} \) such that \( \{ a_i \} \overrightarrow{\pi} \{ b_i \} \cdot \{ c_k \} \).

However, we notice that if \( \{ a_i \} \cdot \{ b_k \} \overrightarrow{\pi} \{ a_i \} \cdot \{ b_i \} \), this does not imply the equality \( \{ b_k \} \overrightarrow{\pi} \{ b_i \} \), so the cancellation law does not hold true.

7. - The above properties show that the set of all partitions constitutes an ordering which admits finite meets and a unit. It makes up a down-stream (directed set).

Remark. In a similar way we can define partition of a given, fixed soma \( a \in B \), where \( a \neq 0 \).

§ 3. - Aggregates.

1. - Let \( B \) be a finitely additive, non trivial tribe and \( F \) a non trivial, commutative ring with unit. By a BF-aggregate we shall understand a finite sequence of ordered couples

\[(a_1, \lambda_1), (a_2, \lambda_2), ..., (a_n, \lambda_n), \quad (n \geq 1),\]

where \( a_1, ..., a_n \) is a partition in \( B \) (see [§ 2]) and \( \lambda_i \in F \).

The aggregates will be written

\[\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_n a_n \] or \[\overline{\sum_{i=1}^{n} \lambda_i a_i}, \quad [\Sigma^* \lambda_i a_i] \] or \[[\Sigma^* a_i \lambda_i] \]
where $\dot{,}$, $\Sigma'$ and $[\cdot]$ are mere symbols (unqualified). The notion is $B$ and $F$ invariant. All $a_i$ are disjoint and $\sum_i a_i = 1_B$.

2. The aggregate $[\Sigma_i^\lambda a_i]$ is said to be equal to the aggregate $[\Sigma_j^\mu b_j]$

\[ [\Sigma^\lambda a_i] \underbrace{\oplus}_{A} [\Sigma^\mu b_j], \]

whenever for every $i$ and $j$ with $a_i \cdot b_j = 0_B$ we have $\lambda_i = \mu_j$.

The notion of equality of aggregates is invariant with respect to $B$ and $F$.

If $|a_i| = |b_j|$, [see § 2], and if for every $i$, $j$ with $a_i = b_j$, $a_i \neq 0$ we have $\lambda_i = \mu_j$, then

\[ [\Sigma_i^\lambda a_i] \underbrace{\oplus}_{A} [\Sigma_j^\mu b_j]. \]

This theorem expresses a kind of invariance of equality of partitions.

If $\{a_i\}$ is a partition in $B$, $(i = 1, 2, ..., n)$, $(n \geq 1)$, $T$ is one-to-one correspondence with domain and range $1, 2, ..., n$, and $\lambda_1, ..., \lambda_n \in F$, then $[\lambda_1 a_1 + \ldots + \lambda_n a_n] = [\lambda_1 a_{1}(a_{1}) + \ldots + \lambda_n a_{n}(a_{n})]$

It follows that for any aggregate we have

\[ [\Sigma_i^\lambda a_i] = [\Sigma_j^\mu b_j], \]

and if

\[ [\Sigma_i^\lambda a_i] = [\Sigma_j^\mu b_j], \]

then

If $a_i = a_i' + a_i''$, $a_i' \cdot a_i'' = 0$, then $[\lambda_1 a_1 + \ldots + \lambda_n a_n] = [\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n]$.

We have for any $\lambda_n \in F$

\[ [\lambda_1 a_1 + \ldots + \lambda_n a_n] = [\lambda_1 \cdot a_1 + \ldots + \lambda_n a_n]. \]

Thus we see that permutation of terms and absence or presence of a term, involving a null-sort, does not change the aggregate up to $A$.

3. $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ are partitions in $B$, and

\[ [\Sigma_i^\lambda a_i] = [\Sigma_j^\mu b_j] \]

then

\[ [\Sigma_i^\lambda a_i b_j c_k] = [\Sigma_j^\mu a_i b_j c_k]. \]
If \(|a_i|, |b_j|, |c_k|\) are partitions in \(B\), and
\[
[\sum_{ij} \lambda_i(a_ib_jc_k)] = [\sum_{ij} \mu_j(a_ib_jc_k)],
\]
then \(\sum_i \lambda_i a_i = \sum_j \mu_j b_j\).

This allows to prove the transitivity of the equality \(\equiv\). Thus the equality \(\equiv\) of aggregates satisfies the formal conditions of identity. The above theorems imply that the presence or absence of a finite numbers of terms with null-soma, permutation of terms, and splitting of somata into a finite number of disjoint parts do not affect the aggregates—up to \(\equiv\).

4. - If \(|b_j|\) is a partition in \(B\), then \([\sum_{i} \lambda_i a_i] = [\sum_{i} \lambda_i (a_i b_j)]\).

Any two aggregates \([\sum_{i} \lambda_i a_i], [\sum_{j} \mu_j b_j]\) can be represented in the form involving the same partition: \(\sum_{i} \lambda_i (a_i b_j), \sum_{j} \mu_j (a_i b_j)\).

5. - By the sum of two aggregates, \([\sum_{i} \lambda_i a_i] + [\sum_{j} \mu_j b_j]\) we shall understand the aggregate \(\sum_{ij} (\lambda_i + \mu_j) \cdot a_i b_j\).

The notion of sum is invariant with respect to \(\equiv, \sim\) and to the equality \(\equiv\) of aggregates. We have
\[
[\sum_{i} \lambda_i a_i] + [\sum_{j} \mu_j a_i] \equiv [\sum_{i} (\lambda_i + \mu_j) a_i].
\]

6. - If \(\lambda \in F\), then by the product \(\lambda\), \([\sum_{i} \lambda_i a_i]\) we shall understand the aggregate \([\sum_{i} \lambda \lambda_i a_i]\).

This notion of multiplication of an aggregate by a «scalar» is invariant with respect to \(\equiv, \sim\) and \(\equiv\).

7. - The addition of aggregates is commutative and associative. If \(X, Y, Z\) are aggregates and \(X + Y \equiv X + Z\), then \(Y \equiv Z\).

We have \(\lambda(X + Y) \equiv \lambda X + \lambda Y\) for any aggregates \(X, Y\) and any \(\lambda \in F\).

The following holds true:
\[
\lambda(\mu X) \equiv (\lambda \mu) X, \ (\lambda + \mu) X \equiv \lambda X + \lambda Y, \ 1_F \cdot X \equiv X.
\]

Consequently the class of all \(BF\)-aggregates is organized by \(\equiv\), addition and multiplication into a linear vector-space with multipliers taken from \(F\).
The aggregate \([0_F \cdot 1_B]\) is the zero-vector \(0_A\) of the space. It is \(A\)-unique.

The subtraction can be defined, and the difference \(X - Y\) of two aggregates is \(A\) invariant. We define \(+ X \equiv 0_A + X, -X \equiv 0_A - X\).

We have \(X - Y \equiv X + (-1)Y\), where \((-1) \in F\).

8. - We have for any aggregate

\[\sum_i \lambda_i a_i \equiv \lambda_1 [1_F \cdot a_1 + 0_F \cdot \co a_1] + \lambda_2 [1_F \cdot a_2 + 0_F \cdot \co a_2] + \ldots\]

If we put, in general, for \(b \in B : S b \equiv [1_F \cdot b + 0_F \cdot \co b]\), we get \([\sum_i \lambda_i a_i] \equiv \sum_i \lambda_i Sa_i\).

Now, we can simplify orthography by writing \(a_i\) instead of \(Sa_i\), getting the following manner of writing aggregates \(\sum_i \lambda_i a_i\), where the sum means summation of aggregates. In the sequel we shall use only this manner of writing, because no ambiguity will be caused by this licence.

9. - If the ring \(F\) is linearly ordered, so that the absolute value \(|\lambda|\) of \(\lambda \in F\) is available, we define the absolute value of \(X \equiv \sum \lambda_i a_i\) as

\[|X| \equiv \sum_i |\lambda_i| a_i\]

The notion of \(|X|\) is \(A\)-invariant, hence it does not depend on the representation of \(X\) as sum.


1. - Let \(b\) be a finitely additive, non trivial tribe. \(\mu(a) > 0\) a finitely additive measure on \(b\), with values taken from a linearly ordered not trivial field \(f\). We suppose that \(\mu(1_b) > 0\).

Let \(b'\) be a finite operations-zero — and unit — genuine strict supertribute of \(b\). This means that 1) the equality \(\equiv\) is the restriction of \(\equiv\) to \(b\), i.e. if \(a \in b\), then the following are equivalent: I. \(a \equiv b\), II. \(a \equiv b\),
2) if \( a, b, c \in \mathfrak{b} \), then the relations \( a \mathfrak{b} b = c \) and \( a \mathfrak{b'} b = c \) are equivalent, and so are \( a \mathfrak{b} a = c \) and \( a \mathfrak{b'} a = c \); 3) \( 0_{\mathfrak{b}} = 0_{\mathfrak{b'}} \). (Compare (11) and [Preface 8]). We have \( \mathfrak{b} \subseteq \mathfrak{b'} \).

The operations, zero, unit and the equality governing on \( \mathfrak{b} \) are just taken from \( \mathfrak{b'} \). The somata of \( \mathfrak{b} \) will be denoted by small latin letters and those of \( \mathfrak{b'} \) by latin capitals. \( \mu(a) \) is supposed to be equality invariant.

2. - By a \( \mathfrak{b} \text{-}(Jordan) \) covering of \( A \in \mathfrak{b} \) we shall understand any non empty set \( \mathfrak{v} \) composed of a finite number of somata of \( \mathfrak{b} \): \( a_1, a_2, \ldots, a_n, \ (n \geq 1) \), such that \( A \subseteq a_1 + \ldots + a_n \). We put \( J(\mathfrak{v}) \overset{\mathfrak{d}}{=} \mu(a_1) + \ldots + \mu(a_n) \) and \( E(A) \overset{\mathfrak{d}}{=} \{ \mathfrak{v} \mid J(\mathfrak{v}) \} \) is a \( \mathfrak{b} \text{-}Jordan covering of \( A \) \).

By \( \mathfrak{b} \text{-}exterior measure of \( A \) with respect to \( \mu \) and \( f \) we shall understand the left-ending

\[ *\mu_{\mathfrak{b} f}(A) \overset{\mathfrak{d}}{=} \mathfrak{d}(E(A)). \]

By \( \mathfrak{b} \text{-}interior measure of \( A \) with respect to \( \mu \) and \( f \) we shall understand the right-\( f \)-ending:

\[ \mu_{\mathfrak{b} f}(A) \overset{\mathfrak{d}}{=} \mu_{\mathfrak{b} f}(1_{\mathfrak{b}}) - *\mu_{\mathfrak{b} f}(\co A). \]

If no misunderstanding is feared, we shall write simplier \( \mu_{\mathfrak{b}}(A), \mu_{\mathfrak{f}}(A) \) or even \( \mu(A) \) and a similar agreement is admitted for \( \mu^* \).

3. - Concerning \( \mu \), we have \( \mu(0_{\mathfrak{b}}) = 0_{\mathfrak{f}} \), and for any \( a, b \in \mathfrak{b} \) we have \( \mu(a + b) \leq \mu(a) + \mu(b) \). If \( a \subseteq b \), then \( \mu(a) \leq \mu(b) \).

**Theorem.** The following are equivalent:

I. \( A \in \mathfrak{b} \); II. \( *\mu_{\mathfrak{b} f}(A) = \mu_{\mathfrak{b} f}(A) = \mu(A) \).

**Proof.** Let. I., i.e. \( A \in \mathfrak{b} \). Put \( a \overset{\mathfrak{d}}{=} A \). Let \( a_1, \ldots, a_n, \ (n \geq 1) \) be a covering of \( A \). We have \( \mu(a) \leq \mu(a_1) + \ldots + \mu(a_n) \).

Hence the set \( E(A) \) has the minimum \( \mu(a) \); indeed, \( a \) is a covering of \( A \).

It follows [§ 1A : 9]

\[ (1) \quad *\mu(A) = \mu(a). \]
We have \( \text{co } A = \text{co } a \in b \). Hence, by what has been proved for somata of \( b \), \( \ast \mu(\text{co } A) = \mu(\text{co } a) \); indeed \( \text{co } A \in b \).

Since \( a + \text{co } a = 1 \), \( a \cdot \text{co } a = 0 \), it follows, by additivity of \( \mu \), \( \mu(1) = \mu(a) + \mu(\text{co } a) \). Hence

\[
(2) \quad \mu^*(A) = \mu(1) - \ast \mu(\text{co } A) = \mu(a).
\]

From (1) and (2) follows \( \ast \mu(A) = \mu^*(A) \), i.e. II.

Let \( 1 \), i.e. \( \mu^*(A) = \ast \mu(A) \). By [§ 1C; 4], there exists a quantity \( \lambda \in f \) such that \( \{ \lambda \} \) is a representative of both these endings. Since \( E(A) \) is a representative of \( \ast \mu(A) \), it has the minimum \( \lambda \), [§ 1A; 9]. Hence there exists a covering \( a_1, \ldots, a_n \) of \( A \) such that \( \lambda = \mu(a_1) + \ldots + \mu(a_n) \), \( (n \geq 1) \).

Now, the set, composed of the single soma \( b \overline{\text{def}} a_1 + \ldots + a_n \), is also a covering of \( A \), and we have

\[
\mu(b) \leq \mu(a_1) + \ldots + \mu(a_n).
\]

Hence \( \mu(b) \leq \lambda \). Since \( \lambda \) is a minimum, we get \( \mu(b) = \lambda \).

We also have

\[
(3) \quad A \subseteq b.
\]

The set \( \{ \lambda \} \) is a representative of \( \mu^*(A) \), i.e. of [§ 1F; 27],

\[
\mu(1) - \ast \mu(\text{co } A) = \mu(1) + (- \ast \mu(\text{co } A)).
\]

Hence \( \lambda = \mu(1) + (- \ast \mu(\text{co } A)) \), and then, by [§ 1G; 2],

\[
\lambda - \mu(1) = - \ast \mu(\text{co } A).
\]

Hence [§ 1F; 22] \( - (\lambda - \mu(1)) = \ast \mu(\text{co } A) \),

\[
- \lambda + \mu(1) = \ast \mu(\text{co } A).
\]

Hence \( \ast \mu(\text{co } A) \) is a point-ending. By similar argument we get a soma \( b' \) such that

\[
(3.1) \quad \text{co } A \subseteq b'.
\]

Hence, by (3), \( \text{co } b' \subseteq A \subseteq b \). It follows

\[
(4) \quad b \cdot b' = 0.
\]

Now, as \( A \subseteq b \), \( \text{co } A \subseteq b' \), it follows that \( A + \text{co } A \subseteq b + b' \subseteq 1 \), and then

\[
(5) \quad b + b' = 1.
\]
From (4) and (5) we obtain

\[ b' = \text{co } b. \]

We have, from (3) \( A \subseteq b \) and, from (3.1) and (6), \( \text{co } A \subseteq \text{co } b, \) i.e. \( b \subseteq A. \)

Thus we get \( A = b \) and, consequently \( A \in b, \) i.e. we get I. The theorem is proved.

4. - Theorem. For any \( A \in b' \) we have

\[ *\mu(b)(A) = \mathcal{L} \{ \mu(a) \mid A \subseteq a \}, \quad \mu_{b'}(A) = \mathcal{K} \{ \mu(a) \mid a \subseteq A \}. \]

**Proof.** Let \( A \subseteq a. \) Since \( \{ a \} \) is a covering of \( A, \) we have \( \mu(a) \in E(A); \) hence, by [§ 1A ; 4], \( E(A) \cdot \leq \mu(a). \) Since \( E(A) \) is a representative of \(*\mu(A), \) it follows, [§ 1A ; 11],

\[ \mu^*(A) \leq \mu(a). \]

The inequality (1) holds for every \( a \) with \( A \subseteq a. \) Hence

\[ [§ 1G ; 7] \]

\[ *\mu(A) \leq \mathcal{L} \{ \mu(a) \mid A \subseteq a \}. \]

The inequality (2) can be inverted. Indeed, let \( y \in \{ J(\varphi) \mid \varphi \) is a \( b^-\)Jordan covering of \( A \} = E(A). \)

There exists a covering \( \{ a_k \} \) of \( A \) such that \( y = \Sigma_k \mu(a_k). \) Put \( a \bar{\varphi} \Sigma_k a_k. \) We have \( \mu(a) \leq \Sigma_k \mu(a_k), \) \( x \bar{\varphi} \mu(a) \in \{ \mu(b) \mid A \subseteq b \}; \)

\[ x \leq y. \]

Hence, for every \( y \) belonging to \( E(A) \) there exists \( x \in \{ \mu(b) \mid A \subseteq b \} \) with \( x \leq y. \)

This gives, [§ 1A ; 1], \( \{ \mu(b) \mid A \subseteq b \}. \leq E(A), \) and then, [§ 1A ; 6],

\[ \mathcal{L} \{ \mu(b) \mid A \subseteq b \} \leq *\mu(A). \]

From (2) and (3) it follows

\[ *\mu(A) = \mathcal{L} \{ \mu(a) \mid A \subseteq a \}. \]

To prove the second formula of the thesis, notice that

\[ *\mu(A) = \mu(1) - *\mu(\text{co } A) = \mu(1) - \mathcal{L} \{ \mu(b) \mid \text{co } A \subseteq b \}, \] by (4).
Hence $\mu^*(A) = \mu(1) + \{- \mathcal{G} \{ \mu(b) \mid \text{co } A \subseteq b \} \}$, and then, by
[§ 1G; 2], $\mu^*(A) = \mu(1) = - \mathcal{G} \{ \mu(b) \mid \text{co } A \subseteq b \}$, [§ 1F; 21],
$= \mathcal{G} \{ - \mu(b) \mid \text{co } A \subseteq b \}$. Hence, by [§ 1G; 2],

(5) \[ \mu^*(A) = \mu(1) + \mathcal{G} \{ - \mu(b) \mid \text{co } A \subseteq b \}. \]

\{ \mu(1) \} is a representative of the right ending $\mu(1)$ and
\{- $\mu(b)$ | co $A \subseteq b$ \} is a representative of its right ending.
Hence

(6) \[ \{ \mu(1) \} + \{- \mu(b) \mid \text{co } A \subseteq b \} = \{ \mu(1) - \mu(b) \mid \text{co } A \subseteq b \} \]
is, [§ 1F; 3], a representative of the ending (5), on the right side of the equation.

Now (6) is equal to \{ $\mu$($co$ $b$) | $co$ $A \subseteq b$ \}. Since the inclusion $co$ $A \subseteq b$ is equivalent to $co$ $b \subseteq A$, we have

\{ $\mu$($co$ $b$) | $co$ $A \subseteq b$ \} = \{ $\mu$($co$ $b$) | $co$ $b \subseteq A$ \} = \{ $\mu(c) \mid c \subseteq A$ \}.

Hence

\[ \mu^*(A) = \mathcal{G} \{ \mu(c) \mid c \subseteq A \}, \]
which proves the second part of the theorem.

5. - Theorem. For every $A \in \mathcal{U}$ we have $\mu^*_f(A) \leq *\mu_f(A)$.

Proof. Take $a, b$ with $a \subseteq A \subseteq b$. We have $\mu(a) \leq \mu(b)$.
If we vary $a$ and $b$, we get by [§ 1E, 13]

\[ \{ \mu(a) \mid a \subseteq A \} \leq \{ \mu(b) \mid A \subseteq b \}; \]
hence, by [1F; 3],

\[ \mathcal{G} \{ \mu(a) \mid a \subseteq A \} \leq \mathcal{G} \{ \mu(b) \mid A \subseteq b \}. \]

Applying theor. [4] we obtain

\[ \mu^*(A) \leq *\mu(A). \] Q. E. D.

6. - Theorem. If $A \in \mathcal{U}$, \{ $a_i$ \} is a partition in $\mathcal{U}$ (see
[§ 2], \{ $a_{ik}$ \} is a subpartition of \{ $a_i$ \}, [§ 2; 3], $\alpha$ is the set of all $i$ for which $a_i \cdot A \neq 0$ and $\beta$ is the set of all couples
$(i, k)$ for which $a_{ik} \cdot A \neq 0$, then

\[ *\mu_f(A) \leq \sum_{(i, k) \in \beta} \mu(a_{ik}) \leq \sum_{i \in \alpha} \mu(a_i). \]
If \( A \in b' \), \( \{ a_i \} \) is a partition in \( b \), \( \{ a_{ik} \} \) a subpartition of \( \{ a_i \} \), \( \alpha' \) is the set of all \( i \) for which \( a_i \subseteq A \) and \( \beta' \) is the set of all \((i, k)\) for which \( a_{ik} \subseteq A \), then

\[
\mu_{\beta'}(A) \geq \sum_{(i, k) \in \beta'} \mu(a_{ik}) \geq \sum_{i \in \alpha'} \mu(a_i)
\]

7. - Theorem. If \( A, B \in b' \), then

\[
*\mu_{\beta'}(A + B) \leq *\mu_{\beta'}(A) + *\mu_{\beta'}(B).
\]

Proof. The sets \( E \overset{\text{df}}{=} \{ \mu(a) \mid A \subseteq a \} \), \( F \overset{\text{df}}{=} \{ \mu(b) \mid B \subseteq b \} \), \( G \overset{\text{df}}{=} \{ \mu(c) \mid A + B \subseteq c \} \) are representatives of \( *\mu(A) \), \( *\mu(B) \) and \( *\mu(A + B) \) respectively. Hence, by \([\S 1F ; 4]\) \( E + F \) is a representative of \( *\mu(A) + *\mu(B) \).

Let \( u \in E + F \). There exist \( x \in E \), \( y \in F \) with \( u = x + y \). There exist \( a, b \) with \( A \subseteq a \), \( B \subseteq b \) such that \( x = \mu(a) \), \( y = \mu(b) \); hence \( u = \mu(a) + \mu(b) \). We have \( \mu_{\beta'}(a + b) \leq \mu(a) + \mu(b) = x + y = u \). We also have \( v \in G \), because \( A + B \subseteq \alpha + b \).

Consequently for every \( u \in E + F \) there exists \( v \in G \) such that \( v \leq u \).

Hence, by \([\S 1A ; 1]\) \( G \cdot \leq E + F \), and then

\[
*\mu(A + B) \leq *\mu(A) + *\mu(B).
\]

Q. E. D.

§ 5. - Linear functionals of aggregates and the norm of an aggregate.

1. - Hypothesis: \( b' \) is a non-trivial finitely additive tribe, \( b \) is its finite operations—zero and unit strict subtribe, (see \([\S 4 ; 1] \) and \([\text{Preface} 8]\)), i.e. with the same zero, unit and finite operations, and with the same equality, restricted to \( b \). We have \( b \subseteq b' \).

The somata of \( b \) will be denoted by \( a, b, c, \ldots \), and those of \( b' \) by \( A, B, C, \ldots, M(a) \) is a finitely additive, \( \mathcal{B} \) invariant measure on \( b \); \( M(a) \geq 0 \); \( M(1_b) > 0 \); the values of \( M \) are taken from a given, not trivial, linearly ordered field \( f \). We shall consider \( b'f \)-aggregates, (see \([\S 3]\)).
2. - Lemma. If \( \Sigma_i \lambda_i A_i = \Sigma \mu_i A_i \) are \( b'f' \)-aggregates, and if \( a \in b \), then
\[
\{ \lambda_i \mid aA_i = 0 \} = \{ \mu_i \mid aA_i = 0 \}.
\]

Lemma. If \( 1. \Sigma_i \lambda_i A_i = \Sigma i \mu_i B_{ki} \) are \( b^f \)-aggregates, where \( B_{ki} = A_k B_i \).

2. \( \{ B_i \} \) is a partition in \( b' \) (see \([\S 2]\)).

3. \( a \in b \),
then
\[
\{ \lambda_i \mid aA_i = 0 \} \equiv \{ \mu_i \mid aB_{ki} = 0 \}.
\]

These two lemmas yield the following

Lemma. If \( \Sigma_i \lambda_i A_i = \Sigma i \mu_i B_i \) are \( b'f' \)-aggregates, and \( a \in b \), then
\[
\{ \lambda_i \mid aA_i = 0 \} = \{ \mu_j \mid aB_i = 0 \},
\]
and then
\[
\max \{ |\lambda_a| \mid a \cdot A_x = 0 \} = \max \{ |\mu_\beta| \mid aB_\beta = 0 \}.
\]

3. - Let \( \{ a_1, \ldots, a_m \} \), \( m \geq 1 \) be a partition in \( b \) and \( X = \frac{\Sigma_i \lambda_i A_i}{\Sigma_j \mu_j B_i} \) be \( b'f' \)-aggregates. We have
\[
\sum_{k=1}^m M(a_m) \max \{ |\lambda_a| \mid a_k \cdot A_x = 0 \} = \sum_{k=1}^m M(a_k) \max \{ |\mu_\beta| \mid a_k B_\beta = 0 \}.
\]

This quantity of \( f' \), depending only on \( X \) and on \( \{ a_k \} \), will be denoted by
\[
\mathcal{G}(\{ a_k \}, X)
\]
This quantity is invariant with respect to \( b', b, \pi, f' \), and \( \frac{\cdot}{\cdot} \).

If \( \{ b_i \} \) is a subpartition of \( \{ a_k \} \), and both are partitions in \( b \), and \( X \) is a \( b'f' \)-aggregate, then
\[
\mathcal{G}(\{ a_k \}, X) \geq \mathcal{G}(\{ b_i \}, X).
\]

4. - Definition. Given a \( b'f' \)-aggregate \( X \), consider the set \( \mathcal{G}_X \) of all quantities \( \mathcal{G}(\{ b_k \}, X) \) where \( \{ b_k \} \) are all partitions in \( b \) (see \([\S 2]\)). Its left ending in \( f' \) will be termed \( b'M \)-norm of \( X \) and denoted by \( *p_bM(X) \). We have
\[
*p_bM(X) = \mathcal{G}_X.
\]
The norm is defined for all $b'f'$-aggregates $X$.

5. If $X = \sum \lambda_i a_i$ is a $b'f'$-aggregate, then its $bM$-norm, $*p(X)$, is a point-ending, and we have $*p(X) = \sum M(a_i) \cdot |\lambda_i|$. To prove that, we first show that for every subpartition $\{ b_j \}$ in $b$ of $|a_i|$ we have

$$\mathcal{G}[\{ b_j \}, X] = \text{const} = \sum M(a_i) \cdot |\lambda_i|,$$

and then we go over to general partitions, proving a similar equality.

Remark. The converse is not true. $*p(X)$ can be a point-ending, even if $X$ is not a $b'f'$-aggregate.

6. - Theorem. If $X, Y$ are $b'f'$-aggregates, then $*p_{bM}(X+Y) \leq *p_{bM}(X) + *p_{bM}(Y)$.

Proof. We shall prove it by proving that $\mathcal{G}X + \mathcal{G}Y \leq \mathcal{G}X + \mathcal{G}Y$, (see § 1A; 3, § 1F; 3).

Let $\xi \in \mathcal{G}X + \mathcal{G}Y$, [§ 1E]. There exist

$$\xi \in \mathcal{G}Y, \eta \in \mathcal{G}X \text{ with } \zeta = \xi + \eta.$$

There exist partitions $\{ a_i \}, \{ b_k \}$ in $b$ such that, [3],

$$\xi = \mathcal{G}[\{ a_i \}, X], \eta = \mathcal{G}[\{ b_k \}, Y].$$

Take them. Since $\{ a_i b_k \}$ is a subpartition of $\{ a_i \}$ and $\{ b_k \}$, we have

$$\mathcal{G}[\{ a_i b_k \}, X] \leq \mathcal{G}[\{ a_i \}, X], \mathcal{G}[\{ a_i b_k \}, Y] \leq \mathcal{G}[\{ b_k \}, Y].$$

Hence, by (1) and (2)

$$\mathcal{G}[\{ a_i b_k \}, X] + \mathcal{G}[\{ a_i b_k \}, Y] \leq \xi + \eta = \zeta.$$

Now $X, Y$ have representatives with a same partition in $b'$:

$$X = \sum_{\alpha} \lambda_{\alpha} A_{\alpha}, \quad Y = \sum_{\alpha} \mu_{\alpha} A_{\alpha},$$

(see [§ 3; 4]), so

$$X + Y = \sum_{\alpha} (\lambda_{\alpha} + \mu_{\alpha}) A_{\alpha}, \quad [§ 3; 5].$$
We have
\[ \mathcal{G}[\{ a_i b_j \}, X] = \sum_{ik} M(a_i b_k) \cdot \max |\lambda_x|, \]
\[ \mathcal{G}[\{ a_i b_k \}, Y] = \sum_{ik} M(a_i b_k) \cdot \max |\mu_x|. \]
\[ \mathcal{G}[\{ a_i b_k \}, X + Y] = \sum_{ik} M(a_i b_k) \max |\lambda_x + \mu_x|, \]
all these maxima being taken for all \( \alpha \) for which
\[ A_x \cdot a_i \cdot b_k = 0. \]

Now we have for these \( \alpha \)
\[ \max |\lambda_x + \mu_x| \leq \max |\lambda_x| + \max |\mu_x|, \]
so, since \( M \geq 0 \), we get, taken (3) into account:
\[ \delta \mathcal{G}[\{ a_i b_k \}, X + Y] \leq \mathcal{G}[\{ a_i b_k \}, X] + \mathcal{G}[\{ a_i b_k \}, Y] \leq \zeta. \]

As \( \delta \in \mathcal{S}_X + \mathcal{S}_Y \), we get: For every \( \zeta \in \mathcal{S}_X + \mathcal{S}_Y \) there exists \( \delta \in \mathcal{S}_X + \mathcal{S}_Y \), such that \( \delta \leq \zeta \). Hence \( \mathcal{S}_X + \mathcal{S}_Y \leq \mathcal{S}_X + \mathcal{S}_Y \), which gives
\[ *p(X + Y) \leq *p(X) + *p(Y). \]

7. - Theorem. If \( X \) is a \( b'f \)-aggregate, \( \lambda \geq 0 \), \( \lambda \in f \), then for the \( bM \)-norm we have
\[ *p_{bM}(\lambda X) = \lambda \cdot *p_{bM}(X). \]

Proof. Take a partition \( \{ a_i \} \) in \( b \), and let \( X = \sum \lambda_x A_x \).
We get \( \mathcal{G}[\{ a_i \}, \lambda X] = \lambda \cdot \mathcal{G}[\{ a_i, X \}] \), and then \( \mathcal{L}(\lambda \cdot \mathcal{G}(X)) = \mathcal{L}(\lambda \cdot \mathcal{G}(X)). \)
Hence, by [1F; 5],
\[ *p(\lambda X) = \lambda \cdot *p(X). \]

8. - Theorem. If \( X \) is a \( b'f \)-aggregate, then \( *p_{bM}(-X) = \)
\[ *p_{bM}(X). \]

9. - If \( X = 1_b \cdot 0_f \), then \( *p(X) = 0_f \). We have for any \( b'f \)-aggregate \( X \) the equality \( *p(X) = *(|X|). \)

10. - Theorem. If \( X, Y \) are \( b'f \)-aggregates and \( |X| \leq |Y|, \)

then for every partition \(| b_k |\) in \( b \) we have \( \mathcal{G}[| b_k |, X] \leq \mathcal{G}[| b_k |, Y] \) and then

\[ *p(X) \leq *p(Y). \]

We rely on \([\S \ 1A ; 1]\) and \([\S \ 1A ; 6]\).

11. - By the characteristic aggregate of a soma \( A \in b' \) we understand the aggregate \( \Omega_A \) defined by \( \Omega_A \overset{df}{=} 1_f \cdot A + 0_f \cdot \text{co} A \).

**Theorem.** If \( A \in b' \), then \( *p_{b_M}(\Omega_A) = *M_{bf}(A) \), where \( *M(A) \) is the exterior measure of \( A \) with respect to \( b \) and \( f \) (see \([\S \ 4]\)).

**Proof.** Put \( X \overset{df}{=} \Omega_A = 1_f \cdot A + 0_f \cdot \text{co} A \) and \( \lambda_1 \overset{df}{=} 1, \lambda_2 \overset{df}{=} 0, A_1 \overset{df}{=} A, A_2 \overset{df}{=} \text{co} A \). We have

\[ X = \lambda_1 A_1 + \lambda_2 A_2. \]

Take a \( b \)-partition \(| b_k |\). We have

(1) \[ \mathcal{G}[| b_k |, X] = \Sigma_k M(b_k) \cdot \max |\lambda_{x} | \text{ for } b_k \cdot A_{x} = 0. \]

Consider the sets of indices

\[ K_1 \overset{df}{=} \{ k \mid b_k \subseteq A \}, \quad K_2 \overset{df}{=} \{ k \mid b_k \subseteq \text{co} A \}, \quad K_3 \overset{df}{=} \{ k \mid b_k \cdot A = 0 \text{ and } b_k \cdot \text{co} A = 0 \}. \]

If \( k \in K \), we have

\[ \max_{b_k \cdot A_{x} = 0} |\lambda_{x} | = 1; \]

if \( k \in K_2 \), we have

\[ \max_{b_k \cdot A_{x} = 0} |\lambda_{x} | = 0; \]

and if \( k \in K_3 \), we have

\[ \max_{b_k \cdot A_{x} = 0} |\lambda_{x} | = 1. \]

Hence for all \( k \in K_1 \cup K_3 \), the coefficient by \( M(b_k) \), in (1), is \( 1 \), and for all \( k \in K_2 \), the coefficient is \( = 0 \).

The sets \( K_1, K_2, K_3 \) are mutually disjoint and their union is the set of all indices \( k \). Thus

(2) \[ \mathcal{G}[| b_k |, X] = \Sigma_{k \in K_1 \cup K_3} M(b_k). \]
The soma $\overline{\delta_{k}} \sum_{k \in K, \cup K} b_{k}$ is the «smallest covering» of $A$ taken from the partition $\{ b_{k} \}$, and we have $\mathcal{G}[\{ b_{k} \}, X] = M(b)$.

On the other hand we know, [§ 4; 4] that

$$* M(A) = \mathcal{G} \{ M(b) \mid A \subseteq b \}.$$

Now, if $A \subseteq b$, the sequence $b$, co $b$ makes up a $b$-partition. Applying the obtained result (2) to this partition, we see that

$$M(b) = \mathcal{G}[\{ b, \text{ co } b \}, X].$$

It follows that $* M(A)$ is the left ending of the set of all $\mathcal{G}[\{ b_{k} \}, X]$; hence $\mathcal{G}\mathcal{G}\mathcal{G}X = * M(A)$, i.e.

$$* \mathcal{P}(A) = * M(A).$$

12. - **Theorem.** If $M(a)$ is an effective, $f$-valued, non-negative measure on $b$, $X$ is a $b'$-aggregate, $* p_{b M}(X) = 0$, then $X = 0_{f} \cdot 1_{b}$.

**Proof.** By hypothesis, $* p(X)$ is a point ending. Hence [§ 1A; 9] there exists a partition $\{ b_{k} \}$ in $b$ such that $\mathcal{G}[\{ b_{k} \}, X] = 0$. We have

$$\mathcal{G}[\{ b_{k} \}, X] = \sum_{k} M(b_{k}) \cdot \max_{A_{x} \cdot b_{k} = a} | \lambda_{x} | = 0. \tag{1}$$

Since all terms in (1) are non-negative, we get for all $k$:

$$M(b_{k}) \cdot \max_{A_{x} \cdot a_{k} = a} | \lambda_{x} | = 0.$$

Let $b_{k} \cdot A_{x} = 0$. We have $b_{k} = 0$, and then, by the effectiveness of $M$, $M(b_{k}) = 0$. Hence $\max | \lambda_{x} | = 0$.

Thus for every $\alpha$ with $b_{k} \cdot A_{x} = 0$ we have $\lambda_{x} = 0$. Since $b_{k}$ is a partition, therefore, given $\alpha$, there exists $k'$ with $A_{x} \cdot b_{k'} = 0$. Hence $\lambda_{x} = 0$. Thus $X = \sum_{x} 0_{f} \cdot A_{x}$, and then $X = 0_{f} \cdot 1_{b}$.

13. - **Remark.** It may happen that there exists a $b'$-aggregate $X$ such that

$$* p(X) = \mathcal{G} \{ a \mid a \in f, a > 0 \}.$$

14. - By [§ 3; 7] the $b'$-aggregates make up a linear vector-space $V$ with multipliers taken from $f$. 
Define for all $bf$-aggregates $X : f(X) = \sum_{i=1}^{n} M(a_i) \cdot \lambda_i$, where $X = \sum_1^n \lambda_i \cdot a_i$.

The function $f(X)$ is invariant with respect to the $A$-equality $bf$-aggregates.

$\forall f = \forall, \Rightarrow f \subseteq f$. We have $f(\lambda X) = \lambda f(X)$ for all $X \in V$ and all $\lambda \in f$, and we also have

$$f(X + Y) = f(X) + f(Y) \text{ for all } X, Y \in V.$$  

Thus $f(X)$ is an $f$-valued linear functional in $V$ with multipliers taken from $f$.

15. - Theorem. We have $f(X) \leq \lambda_p M(X)$ for all $bf$-aggregates $X$.

In the sequel linear functionals, and especially this one will play an important role.

16. - Theorem. If $Y$ is a $bf$-aggregates, $f(X)$ is the linear functional defined in (14), then

$$\lambda_p M(Y) = \{ f(Z) \mid Z \in V, \mid Y \mid \leq Z \}.$$

§ 6. - Fields.

Polynomials:

1. - Let $\Phi$ be a non empty set of elements which may be abstract or defined and endowed with a notion of equality $\Phi$. Let $E$ be an abelian half-group $^{10}$) which we define as a structure having an addition of its elements $a, b, \ldots$, always performable, yielding an element of $E$, and satisfying the conditions: 1) $a + b$ is invariant with respect to the equality $E$ governing on $E$, 2) it is associative and commutative, 3) possesses a neutral element $0_E$, and 4) obeys the cancellation law i.e. if $a + b = a + b'$, then $b = b' \quad^{11}$. 

$^{10}$) It differs from the semigroup considered in (§ 1E).

$^{11}$) The half-group $E$ can be extended, through an addition-isomorphism, to an abelian group $G$ by means of ordered couples $(a, b)$, where $a, b \in E$. We define $(a, b) \equiv (a', b')$ as $a + b = a' + b$, and we define the sum so as to have $(a, b) + (c, d) \equiv (a+c, b+d)$. The defined addition is $G$-invariant. We have $0_G \equiv (a, a)$. 

Choose an element of \( \Phi \), denote it by \( 0_\Phi \) and keep it fixed. By an \( E\Phi \)-function we shall understand any function \( f[s] \) from \( E \) into \( \Phi \), where \( f[s] = 0_\Phi \) for all \( s \in E \), excepting perhaps a finite number of values of \( s \). The class of all \( E\Phi \)-functions will be denoted by \( P \). We define \( f \overset{P}{\equiv} g \) as \( f[s] \equiv g[s] \) for all \( s \in E \).

2. - For farther purposes it is needed to admit that \( \Phi \) is organized into a commutative ring or even field \( F \) with unit \( 1_F \) and zero \( 0_F \) where \( 1_F \neq 0_F \). The elements of \( F \) will be termed quantities, and these of \( E \) exponents. The elements of \( P \) will be termed \( EF \)-polynomials (9). We define the null-polynomial \( 0_P \) as such that \( 0_F[s] = 0_F \) for all \( s \in E \). Given two \( EF \)-polynomials \( f, g \) we define their sum \( f + g \) as the polynomial \( h \) such that \( h[s] = f[s] + g[s] \) for all \( s \in E \). This addition is \( (F) \)-invariant, associative, commutative, and admitting subtraction, so it is an abelian group with \( 0_P \) as group-identity.

To define multiplication in \( P \), we need some auxiliary notions. Let \( f \) be an \( EF \)-polynomial; by a set fitting \( f \) we shall understand a not empty, finite subset \( A \) of \( E \) such that if \( s \in E - A \), then \( f[s] = 0_F \). Given two polynomials \( f, g \) and sets \( A, B \) fitting them respectively we define \( (A, B) \) as the set of ordered couples:

\[
\{ (s, t) \mid s \in A, t \in B \},
\]

and we put

\[
\overline{A, B} \iff \{ u \mid \text{there exist } s \in A \text{ and } t \in B \text{ such that: } u = s + t \}.
\]

Given two \( EF \)-polynomials \( f, g \), we define their product \( f, g \) as the function \( h[s] \) such that:

1) if \( u \in \overline{A, B} \), we put \( h[u] = \sum f[s] \cdot g[t] \), where the summation is extended over all couples \( (s, t) \) for which \( s \in A, t \in B, s + t = u \), the letters \( A, B \) denoting sets fitting \( f, g \) respectively and where each couple \( (s, t) \) is taken \( \equiv \) once only;

2) if \( u \in \overline{A, B} \), we put \( h[u] = 0_F \). The function \( h[s] \) is an \( EF \)-polynomial.
One can prove that the above notion does not depend on the choice of the sets $A, B$.

The multiplication is equality invariant, it admits $0_F$ as annihilator, it is associative and commutative, and admits as neutral element the unit-polynomial $1_F$, defined by $1_F(0_E) \equiv 1_F, 1_F(s) \equiv 0_F$ for all $s \equiv 0_E$. We have $0_F P \equiv 1_P$.

The distributive law holds true. Thus, under the operations of addition and multiplication, $P$ is organized into a commutative ring with $0_P \equiv 1_P$ as zero and unit.

If we define the correspondence $b$ with $b \subset P$ as such one which is $F$-invariant and carries the quantity $a \in F$ into the polynomial $b(a) \equiv a*[s]$ defined by $a*[0_E] = a, a*[1] = 0_F$ for all $s \in E$, then $b$ is an addition multiplication-zero-and-unit-isomorphism from the ring $F$ into the ring $P$. The set $\{a*|a \in F\}$ is organized within $P$ into a commutative ring $F^*$ with $0_P$ and $1_P$ as zero $0_{F^*}$ and unit $1_{F^*}$.

3. - Let us suppose that $E$ is linearly ordered; this will mean that there exists a correspondence $s < t$ with domain and range $E$, which is $(E)$-invariant, and obeys the conditions: 1) if $s < t, t < r$, then $s < r$, 2) if $s < t$, then $s + r < t + r$, 3) given any exponents $s, t$, we have disjointedly either $s < t$, or $t < s$, or $t \equiv s$.

**Theorem.** If $E$ is a linearly ordered abelian half group and $F$ is a commutative ring with $0_F \equiv 1$, then the ring $P$ of $EF$-polynomials is an integrity-domain i.e. if $f \cdot g \equiv 0_P$, then either $f \equiv 0_P$ or $g \equiv 0_P$.

4. - In what will follow we shall admit that $F$ is a commutative ring with $0_F \equiv 1_F$, and that $E$ is the linearly ordered half group $\{0, 1, 2, ...\}$ of all non negative integers ordered as usually.

Then the ring $P$ of $EF$-polynomials is a commutative integrity domain with $0_P \equiv 1_P$. The ring $P$ can be conceived as extension of $F^*$ by adjunction of the single element $\xi = \xi[s]$ defined by $\xi[1] = \xi[1_E] = 1_F$ and $\xi[s] = 0_F$ for all $s \equiv 1$.

Indeed let us define $\xi^0 \equiv 1_P, \xi^{n+1} \equiv \xi^n \cdot \xi$ for $n = 0, 1 ...$
We have for \( n = 0, 1, 2, ... \) \( \xi^n[n] = 1_F \) and \( \xi^n[s] = 0_F \) for \( s \neq n \).

Hence, if \( f \) is an \( EF \)-polynomial and \( A = [s_1 < s_2 < ... < s_k] \) a set fitting \( f \), we get

\[
f = f[s_1] \cdot \xi^{s_1} + ... + f[s_k] \cdot \xi^{s_k}.
\]

Since \([0, 1, ..., s_k]\) is also a set fitting \( f \) we can write

\[
f = \sum_{i=0}^{s_k} \xi^i \cdot a_i^*
\]

where \( a_i^* = f[i], \ (i = 0, ..., s_k) \).

Given \( f \neq 0 \), we always have the representation of \( f \) in the shape (1):

\[
f = \sum_{i=0}^{n} \xi^i a_i^* \quad \text{with} \quad a_n \neq 0_F \quad \text{for some} \quad n \geq 0.
\]

For a given \( f \) this representation is \( F \)-unique, i.e. if

\[
f = \sum_{i=0}^{P} \xi^i a_i^* + ... + a_n^* \xi^n = \sum_{i=0}^{P} \xi^i b_i^* + ... + b_m^* \xi^m,
\]

then \( n = m \) and \( a_i^* \neq b_i^* \) for all \( i = 0, ..., n \).

5. - Theorem. The correspondence \( b \) which attaches to each polynomial

\[
f[s] = f_0^* \cdot \xi^0 + ... + f_n^* \cdot \xi^n, \quad (n = 0, 1, ...)
\]

the function

\[
f(x) = f_0 x^0 + ... + f_n x^n,
\]

where \( x \) is a variable with domain \( F \) and \( x^0 \overset{df}{=} 1 \) for all \( x \in F \), in an operation–zero–and unit homomorphism.

It is from \( P \) onto the ring of all above polynomial–functions, where addition and multiplication of functions are the usual ones, the zero and unit–functions being constant functions with values \( 0_F \) and \( 1_F \) respectively.

Remark. The above homomorphism may be not an isomorphism. E.g., If \( F \) is the field of all integers modulo 3, the polynomial function \( \sum_{i=0}^{5} x^i \) vanishes for all \( x \in F \).
This circumstance does not allow to define polynomials as functions of the variable quantity of the ring. On the other hand, a definition of polynomials as expressions $\sum a_x^x$ where $x$ is just a mere symbol or a variable with indetermined domain is logically not clear—hence incorrect.

6. - If $f \neq 0_P$ is a polynomial, we define its order, ord $f$, as the maximal $s$ for which $f[s] \neq 0_F$. The null-polynomial does not admit any order. The polynomials of order 0 are the polynomials $a^P* a^* \xi^0$ where $a \neq 0_F$. Those polynomials and $0_P$ will carry the common name constant polynomials, sometimes denoted by $c_P$. We have ord $(f \cdot g) = \text{ord } f + \text{ord } g$, and if ord $f < \text{ord } g$, then ord $(f \pm g) = \text{ord } g$. We have ord $f = \text{ord } (c_P \cdot f)$, where $c_P \neq 0_P$. The polynomials of order $n \geq 0$ can be represented in a $\sim$-unique way as $a_n^* \xi^n + \ldots + a_0^* \xi^0$, where $a_n \neq 0_F$.

7. - We shall consider ideals in the ring of $EF$-polynomials. By an ideal we understand, as usually, any non empty subset $J$ of $E$ with the properties: 1) if $f, g \in J$, then $f - g \in J$, 2) if $f \in J, p \in P$, then $f \cdot p \in J$.

The sets $P$ and $\{0_P\}$ are ideals. $0_P \in J$ for every ideal $J$. We define the equivalence modulo $J, f \equiv g$ as $f - g \in J$. By the equivalence class modulo $J$ represented by the polynomial $f$ we shall understand the set $\{p \in P, p \equiv f\}$. $f$ is termed representative of the equivalence class.

The equivalence class modulo $\{0_P\}$ represented by $f$ is $\{f\}$, and that modulo $P$ is $= P$. Every element of an equivalence class is its representative. The intersection of any non empty collection of ideals is an ideal.

If we change the equality $\sim$ governing on $P$ into $\equiv$,

$1^2)$ The so called - identification - of isomorphic elements whose type is different can be applied only with caution; in a treatise where the logical structure is emphasized, the identification is logically disorderly and leads to contradiction.
then the ring operations, zero and unit will be invariant with respect to \( J \). E.g. If \( f \equiv f', \ g \equiv g' \), then \( f + g \equiv f' + g' \). Thus an ideal organizes \( P \) into another commutative ring \( P_J \), termed equivalence ring modulo \( J \) of \( P \). If we consider the set of all equivalence classes modulo \( J \) and define, for equivalence classes, the operations of addition and multiplication, zero and unit in the usual way by means of representatives, we get — as is well known — the quotient ring, \( P/J \), which is operation — zero — and unit isomorphic with \( P_J \) \(^4\)). Given a non empty set \( M \) of polynomials, there exists the smallest (unique) ideal containing \( M \); it will be termed the ideal spanned on \( M \). By a principal ideal we shall understand an ideal which is spanned on the set composed of a single polynomial, say \( f \). It will be termed the principal ideal determined by \( f \). The principal ideal determined by \( 1_P \) is \( P \). The principal ideal determined by \( 0_P \) is \( \{0_P\} \). The principal ideal determined by \( f \) coincides with the set 
\[
\{ f \cdot p \mid p \in P \}.
\]

In what will follow we admit that \( F \) is a commutative field with \( 1_F \equiv 0_F \) and \( E = \{0, 1, 2 \ldots \} \). If \( c_P \equiv 0_P \) is a constant polynomial, then the principal ideal determined by \( f \cdot c_P \) coincides with the principal ideal determined by \( f \). We have the theorem: If \( \text{ord } f \geq 1 \), \( J \) is the principal ideal determined by \( f \), and \( S \) an equivalence class modulo \( J \), \( S \equiv J \), then \( S \) contains a polynomial of order \(< f \). By its use we prove that the division algorithm holds for polynomials, i.e.: If \( 1 \leq \text{ord } q \leq \text{ord } p \), then there exists a \(-unique \) decomposition \( p = A \cdot q + B \), where either \( B = 0_P \) or \( \text{ord } B < \text{ord } q \). From that it follows that every ideal in \( P \) is a principal ideal. The following are equivalent: I. \( J \) is the ideal determined by \( f_1 \) and also the ideal determined by \( f_2 \). II. there exists a constant polynomial \( c_P \equiv 0_P \) such that \( f_1 = c_P f_2 \).

---

\(^4\)) \( P_J \) and \( P/J \) are usually «identified». This identification will be avoided in what follows.
The following are equivalent: I. $J$ is the ideal determined by $f$. II. $f$ is a polynomial $\neq 0_F$ of the lowest order among those belonging to $J$.

8. $f, g$ being polynomials, both $\neq 0_F$, we say that $f$ is divisible by $g$ whenever $f = g \cdot h$ for some $h \in P$. Given two polynomials, $f, g$ both $\neq 0_F$, there exists a greatest common divisor (i.e., with the greatest order); it is unique up to a multiplicative constant polynomial $\neq 0_F$. Two polynomials $f, g$, both $\neq 0_F$ are said to be prime to one another if their greatest common divisor is a constant polynomial $\neq 0_F$.

The following are equivalent for non-null polynomials $f, g$: I. $f, g$ are prime to one another, II. there exist polynomials $p, q$ such that $1_F = p \cdot f + q \cdot g$.

If $p, q$ are non-null polynomials, $p \cdot q$ is divisible by $d$, $q$ and $d$ are prime to one another, then $p$ is divisible by $d$.

If $a \in F$, $b \in F$, $a \neq 0_F$, then $a^* \neq 1_F$. If $a, b \in F$, $a \neq b$, then $\xi - a^*$ is prime to $\xi - b^*$.

9. - If

$$f = f_0^* \cdot \xi^0 + ... + f_n^* \cdot \xi^n, \quad \text{ord } f \geq 1, a \in F,$$

$$f_0 \cdot a^0 + ... + f_n a^n \equiv 0_F \quad (\text{where } a^0 \equiv 1_F),$$

then $f$ is divisible by $\xi - a^*$.

If

$$\text{ord } \xi \geq 1, f = f_0^{*\xi^0} + ... + f_n^{*\xi^n}, \; a_1, a_2, ..., a_s \in F, \; (s \geq 1)$$

are all different,

$$f_0 a_1^0 + ... + f_n a_s^n \equiv 0_F \quad \text{for } i = 1, 2, ..., s,$$

then $f$ is divisible by

$$(\xi - a_1^*) \cdot (\xi - a_2^*) \cdot ... \cdot (\xi - a_s^*).$$

\text{\textsuperscript{14}} We do not define divisibility and indivisibility involving null-polynomials. The notion of *prime* or not *prime* will be not defined for null-polynomials.
If in addition to that we suppose that \( s \geq n + 1 \), then \( f_0 = \ldots = f_s = 0_F \).

10. - A polynomial \( f \) with \( \text{ord } f \geq 1 \) is said to be irreducible whenever there do not exist \( g, h \in P \) such that \( \text{ord } g > \text{ord } f \), \( \text{ord } h < \text{ord } f \), \( f = g \cdot h \). A polynomial \( f \) with \( \text{ord } f = 1 \) is irreducible \(^{13} \).

If a polynomial \( f \) is irreducible, it is divisible only by \( c_P \cdot f \) and by \( c_P \), where \( c_P \neq 0_P \) is a constant polynomial. Conversely, if \( f \) is divisible only by \( c_P \neq 0_P \) and by \( c_P \cdot f \) and if \( \text{ord } f \geq 1 \), then \( f \) is irreducible.

If \( f \) is irreducible, \( f = f_0 \xi^0 + \ldots + f_n \xi^n \), \((n \geq 1)\) and there exists \( a \in F \) such that \( f_0 a^0 + \ldots + f_n a^n = 0_F \), then \( \text{ord } f = 1 \).

If \( f \) is reducible, then \( f = g_1 \cdot g_2 \cdot \ldots g_s \cdot c_P \), \( s \geq 2 \), \( \text{ord } g_i \geq 1 \), where all polynomials \( g_i \) have the coefficient \( 1_P \) at the term with highest power of \( \xi \) and where all \( g_i \) are irreducible with \( \text{ord } g_i \geq 1 \). This decomposition is unique up to permutation of factors.

11. - The ideal \( J \) is termed maximal (indecomposable) whenever 1) \( J \neq P \) and 2) when for every ideal \( J' \), such that \( J \subseteq J' \subseteq P \), we have either \( J = J' \) or \( J' = P \). The following are equivalent: I. \( J \) is a maximal ideal, II. \( J \) is the ideal generated by an irreducible polynomial.

Remark. Notice that a prime ideal \( J \), i.e. such that \( P_J \) is an integrity domain, may not be a maximal ideal e.g. put \( J = \{ 0_P \} \).

If \( J \) is a maximal ideal, then \( P_J \) (and \( P/J \)) are commutative fields, and conversely.

If \( J \) is determined by the irreducible polynomial \( f = a_0^* \xi^0 + \ldots + a_n^* \xi^n \), \( \text{ord } f = n \geq 1 \), then the elements of the field \( P_J \) have the form \( b_0^* \xi^0 + \ldots + b_{n-1}^* \xi^{n-1} \), \( b_i \in F \). This follows from the theorem on the division-algorithm.

\(^{13} \) We do not define irreducibility and reducibility for constant polynomials.
12. - Given a polynomial

\( f = a_0 \xi^0 + \ldots + a_n \xi^n, \ (n \geq 1), \)

by its derivative \( f' \) we shall understand the polynomial

\[
1_F a_1 \xi^0 + 2_F a_1 \xi^1 + \ldots + n_F a_n \xi^{n-1}
\]

(where \( 2_F = 1_F + 1_F, \ (n + 1)_F f + 1_F \) etc.).

The notion of derivative does not depend on the choice of the representation (1) of \( f \). For a constant polynomial \( f \) we have \( f' = 0_p \). We define for \( r = 0, 1, 2, \ldots \) \( f^{(r)} = f^{(r)} \) \( f \), \( f^{(r)} \) \( f' \), i.e. the « higher derivatives » of \( f \).

The following formulas are valid:

\[
(f \pm g)' = f' \pm g', \quad (c_F f)' = c_F \cdot f', \quad (f \cdot g)' = f'g + fg'.
\]

If \( \text{ord } f = n \), then

\[
f^{(n+s)} = 0_p \quad \text{for } s = 1, 2, \ldots
\]

If

\[
f = a_0 \xi^0 + \ldots + a_n \xi^n, \ 1 \leq \text{ord } f \leq n, \ h \in F,
\]

then

\[
a_0 (\xi + h^*)^0 + \ldots + a_n (\xi + h^*)^n = f^{(0)} +
\]

\[
+ \frac{h^*}{1!} f' + \frac{(h^*)}{2!} f'' + \ldots + \frac{(h^*)^n}{n!} f^{(n)}
\]

and for \( \text{ord } f = 0 \) we have \( a_0 (\xi + h^*)^0 = f^{(0)}. \) (Here \( 1!, \ 2!, \ldots \) stand for \( (1_F !), \ (2_F !), \ (3_F !), \ldots \)).

**Extension of a non trivial commutative field.**

13. - Let \( F, G \) be two non trivial fields. \( (F) \subseteq (G) \) will signify that \( F \) is an all operations–genuine strict subfield of \( G. \) (see [Preface 8]). We shall also call \( F \) partial field of \( G, \) and \( G \) superfield of \( F. \)

If \( A, B \) are correspondences, their composition will be written \( BA \) or \( A | B, \) (7).

It is defined as \( | x, z \), there exists \( y \) with \( xAy \) and \( yAz \). Both above manners of writing are useful.
If $E$ is a set, then $\text{card } E$ will mean the power of $E$, i.e. the cardinal number of $E$.

We also shall use the sign $\equiv$ which will mean equality of classes, (sets).

In the development of the theory of fields the field $P_J$ is more convenient that $P/J$, though in some topics, $P/J$ is more useful. The field $P_J$ is an all operations - zero - and unit - genuine extension of $F$ through isomorphism $h'$ which is $F - J$ - invariant and carries $a \in F$ into the polynomial $a^*$. We call $h'$ standard. $h'$ differs from $h$ (defined in [2]), in that $h$ is $F - P$ - invariant only. Denote by $t$ the $J - P$ - invariant isomorphism from $P_J$ onto $P/J$, defined by $f t \overline{f}$ where $\overline{f}$ is the equivalence class whose representative is $f$. The correspondence $t$ carries the polynomial $b_0^* \xi^0 + \ldots + b_m^* \xi^m$ into $\delta_0^* (\xi)^0 + \ldots + \delta_m^* (\xi)^m$.

The correspondence $t \overline{h'}$ carries $a \in F$ into $\overline{a^*}$. It is $F - P$ - invariant.

We call $P_J$ elementary algebraic (equivalence) extension of $F$ and $P/J$ elementary algebraic (quotient) extension of $F$.

14. - The ring $P$ being an integrity domain with unit, it generates another ring $Q$, whose elements are ordered couples of polynomials, in the following way:

We consider only the couples $(f, g)$ where $g \equiv 0_P$. We define the equality $\equiv$ governing in $Q$ by $\langle f_1, g_1 \rangle = \langle f_2, g_2 \rangle$ means $f_1 \cdot g_2 \equiv f_2 \cdot g_1$, and the operations by:

$$
(f_1, g_1) + (f_2, g_2) \equiv (f_1 \cdot g_2 + g_1 \cdot f_2, g_1 \cdot g_2)
$$

$$
(f_1, g_1) \cdot (f_2, g_2) \equiv (f_1 \cdot f_2, g_1 \cdot g_2).
$$

These operations are $\equiv$ - invariant. If we define $0_Q \equiv (0_P, 1_P)$, $1_Q \equiv (1_P, 1_P)$, we get zero and the unit of the ring $Q$. The ring $Q$ is a field; we call it quotient field of $P$ (couple form). If we consider saturated classes of mutually $\equiv$ - equal couples, (so called $\equiv$ - equivalence classes), and define the operations of addition and multiplication by means of representatives, we get another field $\tilde{Q}$, called quotient-field of $P$ (class-form). $Q$ and $\tilde{Q}$, are isomorphic.
with one another through the correspondence \( n \) which is \( \mathcal{Q} \equiv \alpha \) -invariant and which attaches to the couple \((f, g)\) the equivalence class whose representative is \((f, g)\). Denote by \( c \) the correspondence which attaches to the polynomial \( f \) the couple \( f^x \frac{x}{df} (f, 1_F) \), and consider \( c \) as \( \frac{F}{\mathcal{Q}} \) -invariant.

The correspondence \( b \mid c = cb \) carries \( a \in F \) into the couple \((a^*, 1_F)\) and is \( \frac{F}{\mathcal{Q}} \) -invariant.

The field \( \mathcal{Q} \) is an all operations - zero - and unit genuine extension of \( F \) through the isomorphism \( cb \). The field \( \tilde{\mathcal{Q}} \) is also so through the isomorphism \( scb \). We call \( \mathcal{Q} \) elementary transcendental extension of \( F \) (couple form), and \( \tilde{\mathcal{Q}} \) elementary transcendental extension of \( F \) (class form).

We have

\[
(a_0^* \xi^0 + \ldots + a_n^* \xi^n, b_0^* \xi^0 + \ldots + b_m^* \xi^m)_{0_{\mathcal{Q}}} = \frac{a_0^* \sigma(\xi^0) + \ldots + a_n^* \sigma(\xi^n)}{b_0^* \sigma(\xi^0) + \ldots + b_m^* \sigma(\xi^m)},
\]

\[n \geq 0, m \geq 0, b_m \neq 0,
\]

where on the right the operations are those in \( \mathcal{Q} \).

15. - Let \((F) \subseteq (G)\) be two non trivial fields, and let \( u \in G \). Denote by \([F, u]\) the smallest partial field of \( G \) containing \( F \) and \( u \). It does not depend on the choice of the superfield \( G \), provided that \((F) \subseteq (G)\) and \( u \in G \). We call \([F, u]\) single element extension of \( F \) in superfield. (In a similar way we define \([F, u_1, u_2, \ldots, u_d]\)). Under these circumstances there can occur two and only two following cases:

1) Algebraic case: There exists an irreducible \( EF\)-polynomial

\[(1) \quad a_0^* \xi^0 + \ldots + a_n^* \xi^n, \quad n \geq 1, \quad a_n \neq 0, \quad a_i \in F, \]

such that \( a_0 u^0 + \ldots + a_n u^n \equiv 0_G \), (where \( u^0 \equiv 1_F \)).

This polynomial is unique up to a multiplicative constant polynomial \( \equiv 0_P \).

Let \( J \) be the maximal ideal determined by (1). Then the \( J \equiv G \) -invariant correspondence \( \varphi^J \) which carries \( b_0^* \xi^0 + \ldots + b_n^* \xi^{n-1} \) into \( b_0 u^0 + \ldots + b_{n-1} u^{n-1} \) for \( b_i \in F \) is isomorphism from \( P_J \) onto \([F, u]\). We have \( \xi u' u \). The corres-
pondence \( t^{-1} \alpha \) carries the quantities of \( P/J \) onto those of \( [F, u] \). The correspondences \( \alpha \) and \( t^{-1} \alpha \) will be termed standard; they are well determined by \( u \). We call \( [F, u] \) single element algebraic extension of \( F \) within superfields.

Such an extension is called apparent if \( [F, u] = F \); if \( u \in G - F \), it is termed proper. The quantity \( u \), in both cases is termed algebraic with respect to \( F \), in superfields.

2) Transcendental case: There does not exist any \( EF \)-polynomial

\[ a_0^* \xi^0 + \ldots + a_n^* \xi^n, \quad a_n \neq 0_F, \quad n \geq 1, \quad a_i \in F, \]

such that

\[ c_0 u^0 + \ldots + a_n u^n \neq 0_G. \]

We term \( [F, u] \) single element transcendental extension of \( F \) within superfields.

The quantity \( u \) of \( G \) is termed transcendental with respect to \( F \) in superfields.

It never happens that \( [F, u] = F \), so the extension is always proper. The \( \mathcal{Q} \mathcal{G} \)-invariant mapping \( \alpha \), defined by

\[ (b_0^* \xi^0 + \ldots + b_n^* \xi^n ; c_0^* \xi^0 + \ldots + c_m^* \xi^m) \rightarrow \frac{b_0 u^0 + \ldots + b_n u^n}{c_0 u^0 + \ldots + c_m u^m}, \]

for \( b_i, c_j \in F, \quad c_m = 0_F, \quad n \geq 0, \quad m \geq 0, \)

is an isomorphism from \( Q \) onto \( [F, u] \). The \( \mathcal{G} \mathcal{G} \)-invariant correspondence \( d \mathcal{G}^{-1} \) is an isomorphism from \( \tilde{Q} \) onto \( [F, u] \).

They both are called standard, and are well determined by \( u \).

16. - A field \( F \) can be isomorphic with its algebraic single element extension. Exple.

Let \( R \) be the field of all ordinary rational numbers. Put \( F \mathcal{G} \mathcal{R}[\pi] \). The field \( [F, \pi] \) is an algebraic extension of \( F \) and is isomorphic to \( F \).

A field \( F \) can be isomorphic with its transcendental single element extension. Expl. Let \( \alpha_1, \alpha_2, \ldots \) be an infinite sequence of ordinary transcendental numbers, such that there
does not exist any algebraic relation between $\alpha_1 \ldots \alpha_n$, whatever $n$ may be. Put $F = [R, \alpha_2, \alpha_3, \ldots]$. $[F, \alpha_i]$ is a transcendental extension of $F$ and is isomorphic to $F$.

However, if $(F) \subseteq (G)$, $u, v \in G$, and we are only interested in isomorphism $m$ for which $ma = a$ for all $a \in F$, $(F$-rigid isomorphism), then $F$ is never isomorphic to any of its proper single element extension. Under the above requirement $[F, u]$ and $[F, v]$ are isomorphic if and only if either $u, v$ are both transcendental or are both algebraic with the same ideal $J$. This can be proved by considering various standard mappings.

Concerning algebraic extension of $F$ in superfields $G$, given an irreducible polynomial of order $n > 2$, there can exist in $G$ at most $n$ different quantities $u_1, \ldots, u_n$ such that $[F, u_1], \ldots, [F, u_n]$ are mutually $F$-rigid isomorphic.

17. - If $(F) \subseteq (G) \subseteq (H)$ we say that $(G)$ is an extension of $F$ within $H$ (in superfield). $G$ is termed algebraic in superfield whenever every quantity $u \in G$ is algebraic with respect to $F$. It is known that if $G$ is an algebraic extension of $F$ and $v \in H$ is algebraic with respect to $G$, then $v$ is also algebraic with respect to $F$. If $(F) \subseteq (G)$ and $f$ is an $EF$-polynomial, then $G$ is termed splitting field of $f$, if $f$ can be decomposed into linear $EG$-polynomials in the $EG$-ring. $(G)$ is termed algebraic closure of $F$ whenever 1) every quantity of $G$ is algebraic with respect to $F$, 2) $G$ is a splitting field of every $EF$-polynomial with order $\geq 2$.

Our next purpose is to prove the existence of the algebraic closure. To do this we shall give a proof of the existence of a splitting field for a given $EF$-polynomial. The proof of the main theorem will need only some supplementary remarks. We shall need few preparations.

18. - We shall perform many elementary algebraic extension of $F$, applied one after another. Now, such an extension changes the logical type of quantities considered. Indeed, let us fix the type of the numbers 0, 1, 2, ..., denoting it by $v$. Let $\alpha$ be the logical type.
of quantities of the field \( F \). An EP-polynomial, being a function, is a correspondence between the elements of type \( v \) and elements of type \( a \). It has the type which we may denote by \( \{ v, a \} \) (see [Preface 6]). The elements of \( P/J \) are classes of polynomials, so their type may be denoted by \( \mathrm{cl} \{ v; a \} \). The standard isomorphism from \( F \) into \( P/J \) is \( \theta \) and its type is \( \{ a; \mathrm{cl} \{ v; a \} \} \), so the inverse correspondence, \( (\theta')^{-1}t^{-1} \), has the type \( \{ \mathrm{cl} \{ v; a \}; a \} \).

19. - Let \( \Phi \) be a set of any elements (abstract or constructed) provided with an equality \( \Phi \). Suppose that \( \mathrm{card} \Phi \geq \mathrm{card} F \). There exists one - to - one \( = \) - correspondence which maps the quantities of \( F \) into \( \Phi \). Choose such a correspondence \( m \) and put \( F_0 \equiv \mathrm{m}(F) \). The set \( F_0 \) can be organized through \( m \) into a commutative field by defining the operations:

\[
p + q m(m^{-1}d + m^{-1}q), \quad p \cdot q m(m^{-1}p \cdot m^{-1}q),
\]

and by defining

\[
1_{F_0} \equiv m1_F, \quad 0_{F_0} \equiv m0_F.
\]

Thus, the given field can be isomorphically transformed into another one with elements having a type given in advance, satisfying only a condition concerning the power. This device will be now applied.

20. - Let \( F \) be a non trivial commutative field, \( \Phi \) a set as in [19], such that \( \mathrm{card} \Phi > \max \{ \mathrm{card} F, \aleph_0 \} \). If \( F \) is finite, the \( P = P(F) \) is denumerable, and if \( F \) is infinite, \( \mathrm{card} P = \mathrm{card} F \). Thus in any case \( \mathrm{card} P < \Phi \). Transport \( F \) through an isomorphism, as in [19] into a field \( F_0 \) such that \( F_0 \subset \Phi \). We shall operate only on \( F_0 \). Let \( J \) be a maximal ideal in the \( EF \)-ring \( P(F_0) \) and put

\[
G' \equiv P(F_0)/J.
\]

Let

\[
b(F_0)a_0 \cdot \xi^0 + \ldots + b(F_0)a_n \cdot \xi^n, \quad a_0 \equiv 0_{F_0}, \quad a_i \in F_0, \quad n \geq 1
\]
be an irreducible $EF_0$-polynomial generating $J$. Since $\text{card}(G' - \text{card} F_0) < \text{card}(\Phi - F_0)$ we can choose a $1 \rightarrow 1$ correspondence $n$ which carries $G'$ into a subset $G \subseteq \Phi$ in such a way that $t b'n = m$. We organise $G$ into a field $(G)$ by means of $n$ as in [19]. We get $(F_0) \subseteq (G)$.

Let $u \overset{df}{=} n t \xi$. Then we have

$$a_0 u^0 + \ldots + a_n u^n = 0,$$

and the $EG$-polynomial (1) which can be written as

$$b^{(G)} a_0 \cdot \xi^0 + \ldots + b^{(G)} a_n \cdot \xi^n$$

is divisible by $\xi - b^{(G)} u$. Every quantity of $G$ has the shape $b_0 u^0 + \ldots b_{n-1} u^{n-1}$, so $G = [F_0, u]$. Thus we have extended $F$ through isomorphism to $G$. Of course the extension may be apparent. Now we can proceed with $G$ in the same way as we did with $F_0$, and by the choice of an ideal in the ring of $EG$-polynomials, extend $G$, getting a field $H$ where $(F) \subseteq (G) \subseteq (H)$, $H \subset \Phi$. We call the construction, sketched above, $[F_0, J, n]$-construction. It can be performed because the cardinal of $\Phi$ is sufficiently high.

21. - Now, we are going to prove the existence of the splitting field of a polynomial.

Let $F, F_0, n, \Phi$ be as before in [20] and let $f$ be an $EF_0$-polynomial with order $\geq 1$. Let $\delta$ be the type of elements of $\Phi$, and $\nu$ - that of the numbers $0, 1, 2, \ldots$. Consider the class of all $1 \rightarrow 1$ correspondences which carry a set of elements of the type $\{\nu; \delta\}$ into $\Phi$, and order well this class, getting the well ordering

(1) $n_1, n_2, \ldots, n_\alpha, \ldots$

Consider all functions defined on finite subsets of $E = \{0, 1, 2, \ldots\}$ and with values in $\Phi$. Order them well

(2) $\pi_1, \pi_2, \ldots, \pi_\beta, \ldots$

It may happen that $f$ has the shape

$$c_F(\xi - c_1) \ldots (\xi - c_k), k \geq 1$$
where $c_P$ is a constant polynomial $\neq 0_P$ and $c_i \in F_0$. In this case we stop the process and put

$$H = \overline{F_0}.$$ 

$H$ is the splitting field of $f$.

In the remaining case we have

$$(3) \quad f = f_1^{s_1} \cdots f_m^{s_m} \cdot (\xi - c_1^*) \cdots (\xi - c_k^*) \cdot c_P$$

where $s_1 \geq 1$, ..., $s_m \geq 1$, $m \geq 1$, $k \geq 0$, $c_P = 0_P$, and where $f_i$ are different irreducible polynomials of order $\geq 2$ with the coefficient $= 1^*$ at the highest power of $\xi$.

The representation (3) is unique up to permutation of factors. In general, if $g$ is an $EF$-polynomial, say $b_0 \xi^n + \cdots + b_n \xi^n$, $b_n = 0$, $n \geq 1$, there corresponds to it a well defined function $\pi$ written in (2) which attaches to $0, 1, \ldots, n$ the quantities $b_0, b_1, \ldots, b_n$ respectively. This having noticed, choose among $f_1, \ldots, f_m$ the polynomial $f_i$ for which the corresponding function $\pi$ in (2) has the smallest index $\beta$. The polynomial $f_i$ is irreducible. Let the corresponding ideal be $I$. Apply the construction $[F_0, J, n_\alpha]$ by using the correspondence $n$ in (1) with the smallest index $\alpha$.

21a. - We get a field $[F_0, u] = \overline{F_1}$ in which the polynomial $f_i$ is divisible by $\xi - b^{(F_0)} u_1$. Considering $f$ as an $EF_1$-polynomial, we get the decomposition

$$(1) \quad f = g_1^{k_1} \cdots g_m^{k_m} (\xi - d_1^*) \cdots (\xi - d_k^*) \cdot c_P \quad \text{where} \quad k_1 > k \geq 0.$$ 

If $p = 0$, the process stops and $F_1$ is a splitting field $H$ of $f$.

This process can carried through by ordinary induction until it stops. It must stop, because $0 < k < k_1 < k_2 < \ldots \leq n$.

Thus we get a very definite sequence of nested fields

$$(2) \quad (F_0) \subseteq (F_1) \subseteq (F_2) \subseteq \ldots,$$

where the last, called $H$, is a splitting field of $f$. The theorem on the existence of a splitting field is established. We underscore that given $F$, $\Phi$, $m$, $f$ and the well order-
ings (1) and (2), the constructed splitting field is well determined. We denote it by $H(F_0, f)$.

23. - Sketch of a proof of the existence of the algebraic closure of a given field $F$.

We take over the entities $F, m, F_0, \Phi$ from [21] and the well orderings $\{ \alpha \}, \{ \pi \}$. Let us order well all different ideals of the ring of $EF$-polynomials:

(3) \[ J_1, J_2, \ldots, J_\gamma \ldots \]

The power of (3) is $\text{card} \Phi$. Let

(4) \[ f_1, f_2, \ldots, f_\gamma, \ldots \]

be the corresponding irreducible polynomials, which we suppose to have the coefficient $1_\Pi$ at the highest power of $\xi$. Thus the sequence (4) is well determined. Determine the splitting fields $H_1 = H(F_0, f_1), H = H(H_1, f_\alpha), \text{etc. as in [21]}. They are well determined and we have

\[ (F_0) \subseteq (H_1) \subseteq (H_2) \subseteq \ldots \]

Indeed, suppose we have already defined all $H_\gamma$ for $\gamma < \alpha$. If $\alpha = 1$ exists, we define $H_\alpha \overset{df}{=} [H_{\alpha-1}, f_\alpha]$, and if $\alpha$ is a limit ordinal, we define

\[ H_\alpha \overset{df}{=} \bigcup_{\gamma < \alpha} H_\gamma. \]

The construction can be carried through, because card $\Phi$ is enough high. We can prove that all $H_\alpha$ are algebraic extensions of $F_0$. We have: if $\alpha_1 < \alpha_2$, then $(H_{\alpha_1}) \subseteq (H_{\alpha_2})$. The union of all $H_\alpha$ is the required algebraic closure of $F$ through the isomorphism $m$. The theorem is established.

23. - Linearly ordered fields. Let $F$ be a non trivial field and $P \neq 0$ a subset of $F$, satisfying the following conditions: 1) if $a \in F$, then one and only one of the relations $a \in P, -a \in P, a = 0$ holds true; 2) if $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$. We call the quantities of $P$ positive quantities of $F$. We define the correspondence $a \leq b$ as $< b - a \in P$.
or \( b - a = 0 \). The correspondence \( \leq \) is \( F \)-invariant and it organizes \( F \) into a chain. A non trivial field endowed with thus defined ordering is termed \textit{linearly ordered field}.

We define \( a < b \) as \( b - a \in \mathbb{P} \), \( b > a \) as \( a < b \), and \( b \geq a \) as \( a \leq b \). We define the \textit{absolute value} \( |a| \) of \( a \) as this one among \( a, -a \) which is \( \geq 0 \). All usual formal rules of ordinary algebra of real numbers, with \( \leq \), hold true in linearly ordered fields.

24. - A linearly ordered field \( F \) has the characteristic 0 i.e. its smallest partial field is isomorphic with the field of ordinary rational numbers. The power of \( F \) is infinite, therefore a polynomial \( \sum_{i=0}^{n} a_i \xi^i \), such that \( \sum a_i c^i = 0 \) for all \( c \in F \), is necessarily the null polynomial (compare [5]). It follows that there exists an all operations–zero–and–unit–isomorphism between the polynomials \( \sum_{i=0}^{n} a_i \xi^i \) and the polynomial–functions \( \sum_{i=0}^{n} a_i \dot{x}^i \) where \( \dot{x} \) is a variable with domain \( F \). This fact allows us to replace polynomials \( p \) by polynomial–functions \( p(x) \).

In the sequel, dealing with linearly ordered fields, we shall use polynomial–functions \( p(x) \) only. The polynomial \( \xi \) will be replaced by polynomial–function \( \dot{x} \). The derivative of the polynomial \( p \) can be denoted by \( \frac{dp(x)}{dx} \), defined not as the limit of the quotient of increments, but in the pure formal way as \( \sum_{i=1}^{n} i a_i \dot{x}^{i-1} \), with \( \frac{da}{dx} = 0 (\dot{x} = 0) \) (see [12]). The Taylor formula

\[
p(x + h) = p(x) + \frac{h}{1!} p'(x) + \ldots + \frac{h^n}{n!} p^{(n)}(x)
\]

holds true for any \( x, h \in F \), and polynomial \( p \).

25. - If \( a > 0, b > 0 \), then there exists \( h_0 > 0 \) such that for every \( |h| < h_0 \) we have \( |bh| < a \). Let \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( (n \geq 0) \); there exists \( M > 0 \) such that for all \( x \) with \( |x| \leq 1 \) we have \( |f(x)| < M \). If \( a_1, a_2, \ldots, a_n \in F \), \( (n \geq 1), b > 0 \), then
there exists $h > 0$ such that if $|x| \leq h$, then $|ax + \ldots + anx^n| < b$.

If $a_s \neq 0$, $n \geq 1$, $b > 0$, $s \geq 1$, $P(x) \equiv a_s x^s + \ldots + a_{s+n} x^{s+n}$, then there exists $h > 0$ such that if $x \neq 0$, $|x| < h$, then $|a_{s+1} x^{s+1} + \ldots + a_{s+n} x^{s+n}| < |a_s x^s| \cdot b$ and $|x| \leq h$, even for $x = 0$.

If $P(x) \equiv a_n x^n + \ldots + a_0$, $P(x_0) > 0$, $a_0 \neq 0$, $n \geq 0$, then there exists $h > 0$, such that if $|x - x_0| \leq h$, then $P(x) > \frac{P(x_0)}{2}$.

The following «continuity-property» holds true: If for a polynomial $P(x)$ we have $P(x_0) = 0$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| \leq \delta$, then $|P(x)| < \varepsilon$.

26. - If $P(x)$ is a polynomial and $P'(x_0) > 0$, then there exists $h > 0$ such that if $x_0 - h \leq x_1 < x_2 \leq x_0 + h$, then $P(x_1) < P(x_2)$. The proof uses the identity $h_2^r - h_1^r = (h_2 - h_1) \cdot (h_2^{r-1} + h_2^{r-2} h_1 + \ldots + h_1^{r-1})$.

If $P'(x_0) = 0$, $P''(x_0) > 0$, then there exists $\delta > 0$ such that if $x_0 - \delta \leq x_1 < x_2 \leq x_0$, then $P(x_1) > P(x_2)$, and if $x_0 \leq x_1 < x_2 < x_0 + \delta$ then $P(x_1) < P(x_2)$. The proof is based on the inequality $h_2^r + h_2^{r-1} h_1 + \ldots + h_1^r \leq (h_2 + h_1)^r$, valid for non negative $h_1, h_2$.

27. - If for polynomial functions the Rolle’s theorem holds true, then the Lagrange’s and Cauchy’s mean value theorems are valid. Consequently we get also the Taylors theorem «with remainder»:

$$P(x_0 + h) = P(x_0) + \frac{h}{1!} P'(x_0) + \ldots + \frac{h^r}{r!} P^{(r)}(x_0) + \frac{h^{r+1}}{(r + 1)!} P^{(r+1)}(x_0 + \theta h)$$

where $0 < \theta < 1$. Under this condition, if we suppose that

$$P(x_0) = P'(x_0) = \ldots = P^{(r)}(x_0) = 0, P^{(r+1)}(x_0) \geq 0,$$

then 1) if $r$ is even, there exists $\delta > 0$ such that if $x_0 - \delta \leq x_1 < x_2 \leq x_0 + \delta$, then $P(x_1) \geq P(x_2)$, and 2) if $r$ is odd, then there exists $\delta > 0$ such that if $x_0 - \delta \leq x_1 < x_2 \leq x_0$, we have $P(x_1) \geq P(x_2)$, and if $x_0 \leq x_1 < x_2 \leq x_0 + \delta$, then $P(x_1) \leq P(x_2)$. 

If for polynomial functions $P(x)$ the Weierstrass Null-
theorem holds true, i.e. if $a, b \in F$, $a < b$, $P(a) < 0$, $P(b) > 0$, then there exists $c \in F$ such that $a < c < b$, $P(c) = 0$, then the Rolle’s theorem can be proved for polynomial functions.

28. - Formally real fields of Artin and Schreier, (4), (6).

Let $F$ be a non trivial field. It is termed formally-real whenever $(-1) \in F$ cannot be represented as the sum of a finite number of squares of quantities of $F$. The field $F$ is termed algebraically closed formally real (acfr) if 1) it is formally real, 2) there does not exist any proper algebraic extension of $F$ which would be formally-real. A linearly ordered field always is formally real.

An acfr can always be linearly ordered and in a unique way. If $F$ is an acfr, then the Weierstrass Null-theorem holds true for polynomials, and consequently the Rolle’s theorem too.

Let $F$ be a linearly ordered field, and, $G$ its algebraic closure through isomorphism $m$. Put $F_1 = m F \subseteq G$. Then there exists at least one linearly ordered superfield $G_1$ of $F$, within $G$, such that $G_1$ is an acfr and all operations, - zero - unit - and order - genuine strict extension of $F$, within $G$. $G_1$ will be termed algebraic real closure of $F$ through isomorphism. If we suppose that $F \subseteq G$ and that $G$ is the algebraic closure of $F$, then there may exist many real algebraic closure $F_1$ of $F$ within $G$, such that $(F) \subseteq (G_1) \subseteq (G)$ with preservation of order, but they all are all operations -zero-unit - and order - $F$-rigid-isomorphic. In the sequel we shall rely on theorems taken from the theory of Artin and Schreier.

Extension of linearly ordered fields.

29. - In what follows we shall suppose that the given linearly ordered field $F$ is a subfield of its algebraic closure $F^A$. We select among all real algebraic closures of $F$ within $F^A$ a single one, we shall keep it fixed and denote it by $\bar{F}$. So we have $(F) \subseteq (\bar{F}) \subseteq (F^A)$.

By a gap in $(F)$ we shall understand a couple $(A, B)$
of subsets of $F$ such that 1) $A \neq 0$, $B \neq 0$, 2) $A \cap B = 0$, $A \cup B = F$, 3) $A \subseteq B$, i.e., for every $x \in A$ and every $y \in B$ we have $x < y$ in $(F)$. A gap is a kind of Dedekind-section of $(F)$.

If $(A, B)$ is a gap in $(F)$ then the following is true:

if $a \in A$, $a' \leq a$, then $a' \in A$; if $b \in B$, $b' \geq b$, then $b' \in B$.

If $(A, B)$ is a gap in $(F)$, then it cannot happen that $A$ admits a maximum and, at the same time, $B$ admits a minimum. The expressions $\mathcal{R}(A)$ and $\mathcal{L}(B)$ are meaningful. We put $\alpha^* \overset{df}{=} \mathcal{R}(A)$, $\beta^* \overset{df}{=} \mathcal{L}(B)$. We call $\alpha^*$, $\beta^*$ endings determined by the gap $(A, B)$. We have $\alpha^* < \beta^*$ in the ordering of $F$-endings.

29. - As we mentioned in [24], we shall consider polynomials-functions instead of polynomials. The symbol $p_{F}(x)_{F'}$, where $F$, $F'$ are fields, will denote a polynomial function whose coefficients belong to $F$ but whose range of the variable $x$ is $F'$. We shall drop the indices and the dot if no misunderstanding can occur, so e.g. $p(x)$ can be understood as $p_{F}(x)_{F'}$. We shall call $p_{F}(x)_{F'}$ polynomial in $F$ with domain $F'$.

Let $M \subseteq F$. We define $p(M) \overset{df}{=} \{ p(x) \mid x \in M \}$. We have $p(M) \subseteq F$. The following are equivalent: I. $M \neq 0$, II. $p(M) \neq 0$.

If $p = c$ is a constant polynomial in $F$, then $p(M)$ is composed of the single quantity $c$, for all $M \neq 0$.

30. - We have supposed that $F$ is embedded in its algebraic real closure $\overline{F}$ with preservation of all operations and order. A gap $(A, B)$ in $F$ will be termed apparent whenever either $A$ admits a maximum, say $a_0$, or $B$ admits a minimum, say $b_0$.

In this case the single elements extensions $[E; a_0]$ or $[F; b_0]$ within any superfield of $F$ coincide with $F$. These extensions are apparent, and so are the corresponding elementary extensions.

31. - In the remaining case $(A, B)$ will be termed proper gap in $F$. Such a proper gap will be termed algebraic
whenever there exists in \( \bar{F} \) a quantity \( z \) with 
\[ A < z < B. \]

This is equivalent to \( \alpha^* < z < \beta^* \) in the ordering of \( F \)-endings and also to \( \alpha^* < z < \beta^* \) in the ordering of \( \bar{F} \)-endings, where \( \alpha^*, \beta^* \) are the natural correspondents of \( \alpha^*, \beta^* \) in \( F \) (see [§ 1G; 2]). The single element extension \([F; z]\) of \( F \) in \( \bar{F} \) is algebraic and so is the corresponding elementary extension of \( F \). The notion of an algebraic proper gap does not depend on the choice of the real closed algebraic extension \( \bar{F} \) of \( F \).

32. - Suppose that the proper gap \((A, B)\) is not algebraic.
Hence there does not exist any quantity \( z \in \bar{F} \) with \( A < z < B \) in \( \bar{F} \).

Suppose that \((A', B')\) is a gap in \( \bar{F} \), such that \( A \subseteq A' \), \( B \subseteq B' \). We shall prove that
\[ \alpha^* = \mathcal{R}(B), \quad \beta^* = \mathcal{L}(B'). \]

We have by [§ 1A; 3]
\[ \alpha^* \leq \mathcal{R}(A'), \quad \mathcal{L}(B') \leq \beta^*. \]

Suppose that \( \alpha^* < \mathcal{R}(A) \). Then there exists \( a' \in A \) such that \( \alpha^* < a' \) [§ 1D; 3].

Since \( a' \in A' \), \( A' < B \) and hence \( A' < B \), it follows that \( a' < \mathcal{L}(B) \), i.e. \( a' < \beta^* \). Hence \( \alpha^* < a' < \beta^* \) and then \( A < a' < B \), i.e. \((A, B)\) is an algebraic gap in \( F \), which is a contradiction. Consequently \( \alpha^* = \mathcal{R}(A') \). In a similar way we prove that \( \beta^* = \mathcal{L}(B') \).

Def. We call \((A', B')\) the gap in \( \bar{F} \) corresponding to \((A, B)\).

33. - Theorem. If 1) \((A, B)\) is proper but not a proper algebraic gap, 2) \( p_F(\bar{x}) \) is a polynomial function \( \neq 0 \), then there exist \( a' \in A' \), \( b' \in B' \) and \( s > 0 \), \( s \in \bar{F} \) such that either 16) \( p(a', b') > s \) or \( p(a', b') < -s \), where \((A', B')\) is the gap

16) \( (a', b')_{\bar{F}} \) means \(| x | \ x \in \bar{F}, a' \leq x \leq b' \).
Proof. The thesis is true for constant polynomials. Suppose that \( p \) is not constant. The polynomial–function \( p_F(x)_F \) and its algebraic derivative \( \left( \frac{dp(x)}{dx} \right)_F \) may have roots in \( F \) or not. Suppose that neither \( p \) nor \( \frac{dp}{dx} \) has. Then by the Weierstrass null–theorem \( p \) is either \( > 0 \) in \( F \) or \( p < 0 \) in \( F \), and the same holds for \( \frac{dp}{dx} \). It follows, by the mean value theorem [28], that \( p \) is in \( F \) either constantly decreasing or constantly increasing. Take any \( a' \in A' \), \( b' \in B' \). We have \( a' < b' \) and \( p \) is in \( F_{a', b'} \) decreasing or increasing and has the same sign.

Suppose \( p > 0 \). The polynomial–function \( p \) admits as minimum in \( F_{a', b'} \) either the quantity \( p(a') \) or \( p(b') \). Thus we have, denoting by \( 2s \) this minimum, \( p(x) > s \) in \( F_{a', b'} \), \( s \in F \) so, in this case the theorem is proved. Similar proof is in the case where \( p < 0 \). We get an inequality

\[
p(x) < -s' \text{ in } F_{a', b'}, \text{ where } s' > 0, s' \in F.
\]

Now suppose that the set of quantities of \( F \) which are either roots of \( p \) or of \( \frac{dp}{dx} \) is not empty. Let \( z_0 < z_2 < \ldots < z_n \), \((n \geq 1)\) be all these roots.

Using the Weierstrass null–theorem and the mean–value theorem we prove that in each open interval:

\[
(-\infty, z_1), (z_1, z_2), \ldots, (z_n, +\infty)_F
\]

the polynomial \( p_F(x)_F \) has a constant sign and is either decreasing or increasing, (being, of course, \( \neq 0 \)). The inequality

\[
\alpha^* < z_k < \beta^*
\]

is not true, whatever \( k \) may be. Indeed \( z_k \in F \), and \( (A, B) \) is a proper gap which is not algebraic. The equalities \( \alpha^* = z_k \) and \( \beta^* = z_k \) are also not true. Indeed, if \( \alpha^* = z_k \),
we would have $\alpha^* = z_k$ too, and then, $A$ would have a maximum. In a similar way we prove that the equality $\beta^* = z_k$ does not take place. Consequently one and only one of the following inequalities takes place:

$$\alpha^* < \beta^* < z_1, \ z_i < \alpha^* < \beta^* < z_{i+1}, \ z_n < \alpha^* < \beta^* \quad (i = 1, \ldots, n - 1).$$

Suppose that

$$z_i < \alpha^* < \beta^* < z_{i+1}.$$

There exist $a, b \in \overline{F}$ such that

$$z_i < a < \alpha^* < \beta^* < b < z_{i+1}$$

in the ordering of $\overline{F}$-endings. Since, in $(z_i, z_{i+1})$, the polynomial $p$ is either decreasing or increasing and has a constant sign, so it is also in $(a, b)_{\overline{F}}$, and $p$ admits there its minimum and maximum at the endpoints. Using a similar argument, to that applied at the beginning of the proof, we establish the thesis.

34. - We resume the notations of [33]. A polynomial $p$ is said to be positive, $p \text{ pos}$, with respect to $(A, B)$ whenever there exist $a, b \in \overline{F}$ and $s > 0$ such that $a < b$, $a \in A'$, $b \in B'$, $s \in \overline{F}$ and such that for the corresponding polynomial-function $p_F(x)_{\overline{F}}$ we have $p((a, b)) > s$.

The notion of $p$ being positive does not depend on the choice of the algebraic real closure $\overline{F}$ of $F$. Indeed, any two real-algebraic closures of $F$ are operation, order $F$-rigid-isomorphic.

Given a polynomial $p$ in $F$, we have disjointedly either $p \text{ pos}$ or $(-p) \text{ pos}$ or $p = 0$. The notion $p \text{ pos}$ is invariant with respect to equality of polynomials. If $p \text{ pos}$, $q$ pos, then $(p + q) \text{ pos}$ and $(p \cdot q) \text{ pos}$.

Having this, consider ordered couples of $EF$-polynomials $(p, q)$ with $q \equiv 0$, and define their equality $\equiv$ and operations as in [14]. The collection of couples, considered in thus organized field $Q$, constitutes an elementary transcendental extension of $F$ through the isomorphism $\epsilon b$ which carries $a \in F$ into the couple $(a^*, 1_F)$. 
From [14] and [15] we see that \( Q \) is a single element transcendental extension of \( F^* \) within \( Q \), namely \([F^* ; \xi^*]\). The quantities of \( Q \) are

\[
a_0^*x(\xi^*)^0 + \ldots + a_m^*x(\xi^*)^m
\]

\[
b_0^*x(\xi^*)^0 + \ldots + b_m^*x(\xi^*)^m
\]

where

\[a_i^*, b_j \in F, \; n \geq 0, \; m \geq 0, \; b_m \neq 0, \; \xi^x \in F(\xi, 1_F), a_i^*x = cb(a_i) = (a_i^* ; 1_F), b_j^*x = cb(b_j) = (b_j^* ; 1_F).\]

Now we shall define an ordering on \( Q \). If \( p \neq 0 \) is a polynomial, define \( \text{sign } p \) as \( +1 \) or \(-1\) accordingly to whether \( p \) pos or \((-p) \) pos. Having the couple \((p, q)\) of \( EF\)-polynomials with \( q \neq 0 \), we call \((p, q)\) positive, \((p, q) \pos\), whenever \( \text{sign } p \cdot \text{sign } q = +1 \). We prove that the notion \((p, q) \pos\) is \( F \)-invariant, and does not depend on the choice of the real algebraic closure \( F \) of \( F \). If \((p, q) \pos\) and \((p', q') \pos\), then \([p, q] + [p', q'] \pos\) and \([p, q] \cdot [p', q'] \pos\). If \( q \neq 0 \) and \( p \) is any \( EF\)-polynomial, then either \((p, q) \pos\), or \([- (p, q)] \pos\) or \((p, q) \notin 0_Q\), these alternatives excluding one another.

It follows that \( Q \) is organized into a linearly ordered field \( (Q) \). The correspondence \( cb \) transforms \( a \in F \) into the couple \((a^* ; 1_F)\) of polynomials, and the following are equivalent: I. \( a^* \) is positive, II. \((a^* ; 1_F) \) is positive. Consequently the correspondence \( cb \) transforms the linearly ordered field \( (F) \) into an order-genuine linear partial field of \( (Q) \), so \((Q) \) is an extension of \((F) \) with preservation of order—through the isomorphism \( cb \).

If \( a \in A, b \in B \), we have \( a^* < x < b^* \), so if we denote by \( A^*x, B^*x \) the \( cb \)-correspondents of the sets \( A, B \), we have \( A^*x < \xi^x < B^*x \). Thus we get, through isomorphism, a linearly ordered transcendental extension of \( F \), which introduces a new element into the gap \((A, B)\) of \( F \). The above discussion yields a proof of the following theorem.

34. - Theorem. If 2. \( F \) is a linearly ordered field, 2. \((A, B)\) is a gap in \( F \), [31], which is proper but not alge-
bradic, 3. \( Q \) is the elementary transcendental extension of \( F \),
4. We define on \( Q \) the linear ordering by considering the
couples \((p, q)\) of \( EF\)-polynomials with \( q \neq 0 \) as positive
whenever for the corresponding polynomial functions \( p_F(x)_F, \)
\( q_F(x)_F \), where \( F \) is a real closed algebraic extension of \( F \),
there exist \( a' \in A', b' \in B', a' < b' \) such that \( p(a', b')_F, \)
\( q(a', b')_F \) are both > 0 or both < 0. \((A', B') \) is the gap in \( F \) corre-
sponding to \((A, B), [32]\), then the linearly ordered field \( Q \)
is an operation and order-genuine linear extension of \( F \)
through the isomorphism \( cb \).

If we put for \( a \in F, a \overset{df}{=} \xi a = a^{**} \), where \( f \overset{df}{=} \xi f \)
for \( EF\)-polynomials \( f \), the elements of \( Q \) are

\[
\frac{a_0(x)^0 + \ldots + a_n(x)^m}{b_0(x)^0 + \ldots + b_m(x)^m}, \quad b_m \neq 0, \quad n \geq 0, \quad m \geq 0,
\]

and in the linear ordering \( \langle Q \rangle \) we have \( cb(A) < \xi < cb(B) \),
so \( Q \) is the single element \( \xi \)-transcendental extension
of \( cb(F) \) within \( Q \), i.e. \([CB(F); \xi]\).

The gap spoken of in [theor. 34] yields a well determined
linearly ordered extension of \( F \). We call it elementary
linear transcendental extension of \( F \) determined by the
gap \((A, B)\).

35. - The isomorphism \( s \), in [14], transforms the couple-
form field \( Q \) into the class - form - quotient field \( \tilde{Q} \). If we
define through \( s \), the order in \( \tilde{Q} \), we get a linearly ordered
field \( \langle \tilde{Q} \rangle \). Thus \( \langle \tilde{Q} \rangle \) is an elementary extension of \( F \)
through the isomorphism \( scb \) which preserves operations, zero, unit
and order.

The type \( \psi(a) \) of quantities of \( \tilde{Q} \) is that of a class of
couples of polynomials. Now, by [18], the type of a poly-
nomial is \( |v; \alpha| \), where \( \alpha \) is the type of elements of \( F \).
Hence

\[
(1) \quad \psi(a) = \text{cl} \{ |v; \alpha|; |v; \alpha| \}.
\]

36. - In [30-35] we have proved the existence of elemen-
tary linear extensions of a linearly ordered field \( F \), by
placing a new element into any gap of $F$. We have obtained three categories of extensions: apparent, algebraic and transcendental. The apparent extension can be considered as an algebraic extension, so we have obtained two different kinds of extensions, if the type of elements is only considered: the algebraic, apparent or not, where the elements of the extended field have the type

\[ \varphi(\alpha) = \text{cl} \{ v; \alpha \}, \quad [18], \]

and the transcendental, where the type of elements is $\psi(\alpha)$, $[35, (1)]$.

Now since we shall need to perform many times these extensions, we shall unify the types of elementary extensions, as follows:

Denote, in general, by $O(\rho)$ the empty set whose elements have the type $\rho$. Let $M$, $N$ be sets whose elements have the type $\lambda$, $\mu$ respectively. Then the ordered couples

\[(M, O(\mu)), (O(\lambda), N)\]

have the same type $\{ \text{cl} \lambda; \text{cl} \mu \}$. Applying this remark to extensions of a linearly ordered field $F$, whose quantities have the type $\alpha$, consider the elements $x$ of an extension $F/J$ and the elements $y$ of the extension $\bar{Q}$. Let us make correspond, to every $x$, the ordered couple

\[ (\text{set composed of } x \text{ only}, O(\psi)), [35]. \]

This one-to-one correspondence organizes the class of all couples (2) into a linearly ordered field, as in [19], which is isomorphic to $F/J$.

Analogously let us make correspond to every $y$ the couple

\[ (0(\varphi), \text{set composed of } y \text{ only}), [36; (2)]. \]

We get a field isomorphic to $\bar{Q}$. Now the type of (3) and (4) is the same, viz,

\[ \sigma(\alpha) \overline{\text{df}} \{ \text{cl} \varphi, \text{cl} \psi \} = \{ \text{cl} \text{cl} \{ v; \alpha \}; \text{cl} \text{cl} \{ v; \alpha \}; \{ v; \alpha \} \}. \]
Thus we have obtained for any elementary gap-extension a field whose elements have the same type (5).

37. - Let us remark that the real algebraic closure $\overline{F}$ of $F$ had only the capacity of a tool for proving the existence of a gap-extension. The type of $\overline{F}$ does not influence the type of extensions, since it can be always performed through a suitable isomorphism.

38. - Thus we have proved the following

**Theorem.** If 1. $F$ is a non trivial, linearly ordered field whose elements have the type $\alpha$,
2. $(A, B)$ is a gap in $F$,
then there exists an elementary linear extension $G$ of $F$, (induced by the gap), through isomorphism, such that
1) the type of elements of $G$ is $\sigma(\alpha)$, [36; (5)],
2) if we denote by $\psi$ this isomorphism, then there exists $\rho \in G$ such that
$$\psi(A) \leq \rho \leq \psi(B).$$

39. - Notice that, if $F$ is given, and we need to perform many elementary linear gap-extension, step by step, (as we shall do in § 7), we always can arrange so as to have, for all these extension, the same type $\beta$ of elements. To do this, we can use the method indicated in [19]. This is possible whenever the cardinal number of the set of all elements of type $\beta$ is sufficiently great.

§ 7. - Construction of a well ordering of fields.

1. - All necessary preparations having been made, we shall proceed to the problem of extension of measure.

Let $B$ be a non trivial finitely additive Boolean tribe and $B'$ its all operations, zero and unit genuine strict extension. So we have $1_B = 1_{B'}$, $0_B = 0_{B'}$, $B \subseteq B'$. 
We suppose that $B \equiv B'$. Let $\varphi_0$ be a linearly ordered not trivial field, and $\mu(a)$ finitely additive measure on $B$, such that $\mu(a) \in \varphi_0$ and $\mu(a) \geq 0$ for every $a \in B$.

We shall prove that the measure can be extended, through isomorphism, to $B'$. Of course $\varphi_0$ may not suffice for the extension, so $\varphi_0$ should be extended to a sufficiently wide linearly ordered field. This will be made by stepwise extension of $\varphi_0$, so we shall construct a suitable well ordering of more and more wide linearly ordered fields. Each step will be a proper or apparent elementary linear extension. Since a proper elementary extension of an abstract field can be made only through isomorphism, therefore the elements of the amplier field will be of a logical type differing from that of the elements of the given field. Now, there may occur, in the general case, that an infinity of proper extensions will be needed, so we may obtain an infinity of fields of different types. Finally, we must construct a field which would contain them all through isomorphism; hence we shall make a union of these fields. Consequently we shall apply results and methods, shown in § 6, concerning unification of logical types of various elementary extensions.

2. - A set $\varphi$ organized into an ordering, field or linearly field will be sometimes, for more clarity denoted by $(\varphi)$, $[\varphi]$ or $[(\varphi)]$ respectively. In accordance with [§ 6; 13], and [Preface 8], $(\varphi) \subseteq (\psi)$ will mean that $(\varphi)$ is an order-genuine strict subordering of $(\psi)$. If $[\varphi]$, $[\psi]$ are fields, then $[\varphi] \subseteq [\psi]$ will mean that $[\varphi]$ is an all-operations-genuine strict subfield of $[\psi]$, (partial field of $[\psi]$). $[(\varphi)] \subseteq [(\psi)]$ will mean that the linearly ordered field $[(\varphi)]$ is an order - and operation - genuine strict subfield of the linearly ordered field $[(\psi)]$.

3. - Let

\begin{equation}
Q_1, Q_2, ..., Q_\lambda, ...
\end{equation}

be a distinguished well ordering of different elements of any kind, with power $\aleph$, where $\aleph > \max (\text{card } \varphi_0, \text{card } B')$. 21*
Its domain will be denoted by \(|Q_\lambda|\). Let 
\[ s_1, s_2, \ldots, s_\beta, \ldots \]
be a well ordering of different somata with domain \(B'\).

4. - Let \(v_\circ\) be the linear space composed of all 
\[ B_{\varphi_0}\text{-aggregates} \] (see \([\S\ 3]\)).

We define the functional \(g_\circ(x)\) of the variable \(B_{\varphi_0}\text{-aggre-}
gate x\), as, (see \([\S\ 5; 14]\))
\[ g_\circ(x) = \sum_{i=1}^{\infty} \mu(a_i) \cdot q_i, \]
whenever
\[ x = \sum_{i=1}^{n} a_i q_i \text{ with } a_1, \ldots, a_n \in B, q_i \in \varphi, \]
\(n \geq 1\), where \(a_1 + \ldots + a_n = 1\) and where \(a_1, \ldots, a_n\) are disjoint. We know that this definition is independent of the choice of the representation (5) of \(x\). We also know that \(g_\circ(x)\) is a linear functional with multipliers taken from \(\varphi_0\). It is \(\varphi_0\) — valued and its domain is composed of all \(B_{\varphi_0}\) — aggregates.

5. - If \(X = \sum_{k=1}^{m} \lambda_k\), (\(m \geq 1\)), where \(A_k \in B'\) and where \(\lambda_k\) are quantities of a given linearly ordered field \(\psi\), denote by \(N^\psi(X)\) (see \([\S\ 5]\)) the \(B_\mu\)-norm of \(X\) in \(\psi\). This is a left ending in \((\psi)\).

We know that
\[ g_\circ(x) \leq N^\varphi x \text{ for all } x \in v_\circ, \ [\S\ 5; 14]. \]

6. - We shall construct, as in [19], a linearly ordered field \(((\Phi))\) whose quantities belong to \(|Q_\lambda|\) and which is order — and operations — isomorphic to \(((\varphi_0))\).

Let \(\Phi_0\) be the smallest ideal (segment) in the well ordering (1), such that \(\text{card } \Phi_0 = \text{card } \varphi_0\). Choose a \(1 \rightarrow 1\) mapping \(\alpha_0\) of \(\varphi_0\) onto \(\Phi_0\). This correspondence induces on \(\Phi_0\) a
chain ordering \((\Phi_0)\), defined by
\[
\nu' \leq \nu'' \iff a_0^{-1}\nu' \leq a_0^{-1}\nu''.
\]

It also organizes \(\Phi_0\) into a field \([\Phi_0]\) by means of the definitions
\[
\nu' + [\Phi_0] \nu'' a_0^{-1}\nu' + [\Phi_0] a_0^{-1}\nu'',
\]
thus \(a_0\) gives a linearly ordered field \([\Phi_0]\) isomorphic to \([\Phi]\) through \(a_0^{-1}\). Its quantities belong to \([Q_1]\).

6.1. - The correspondence \(a_0\) induces another one, also denoted by \(a_0\), which transforms the aggregates \(x = \Sigma a_iq_i\), where \(a_i \in B, q_i \in \varphi_0\) into \(X = \Sigma a_0a_0(q_i) = a_0(X)\), which are \(B\Phi_0\)-aggregates. Their collection makes up a linear space, denoted by \(V_0\) with multipliers taken from \(\Phi_0\). It also transforms the functional \(g_0(x)\) into \(G_0(X)\) defined by
\[
G_0(X) = G_0(a_0(x)) = \Sigma a_0(\mu(a_i))a_0(q_i) \text{ whenever } x = \Sigma a_iq_i.
\]

The quantity \(a_0(\mu(a))\) for \(a \in B\) is a finitely additive \(\Phi_0\)-valued measure on \(B\). \(G_0(X)\) is a \(\Phi_0\)-valued linear functional defined on \(V_0\) with multipliers taken from \(\Phi_0\).

6.2. - We have for all \(X \in V:\)
\[
G_0(X) \leq N^\Phi_0(X).
\]

We have here the \(B - a_0(\mu)\) norm of \(X\) in \(\Phi_0\). It is a left ending in \((\Phi_0)\).

7. - Let us denote by \(\alpha\) the logical type of the elements \(Q_1\). In \([\S 6; 36]\) we have shown that an elementary proper or apparent extension of \([\Phi_0]\) can be made so as to have the type \(\sigma(\alpha)\) for the quantities of the amplified field, this type being the same whatever be the extension, transcendent, proper algebraic or apparent. Now, consider all chains \((C)\) whose domains are composed of elements with type \(\alpha\), and also consider all chains \((C')\) whose domains are composed of elements with type \(\sigma(\alpha)\). Having two chains \((C), (C')\) and
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supposing that card $C = \text{card } C'$, there may exists an order-
isomorphism $\mu$ from $(C)$ onto $(C')$. Such an isomorphism is
a correspondence between elements of type $\alpha$ and elements
of type $\sigma(\alpha)$. Consider the class of all order–isomorphism
having the above logical type. Let us order it well, getting
a well ordering

$$(7) \quad m_1, m_2, \ldots, m_r, \ldots$$

where all elements of $(7)$ are different.

8. - LEMMA. If 1. $C_1', C_2'$ are fields with elements of
type $\sigma(\alpha)$, 2. $(C_1') \subseteq (C_2')$, 3. card $C_2' < \aleph$, 4. $([C_1])$ is a field
with $C_1 \subseteq \{ Q_1 \}$, 5. $t_1$ is an order – and operation–iso-
morphism from $(C_1')$ onto $(C_1)$, then there exists $C_2$ with
$C_2 \subseteq \{ Q_1 \}$, and an order and operation–isomorphism $t_2$ from
$C_2'$ onto $C_2$ such that

$$(C_1) \subseteq (C_2), \quad t_1 = C_1' \, t_2.$$  

In addition to that $t_1^{-1}$ and $t_2^{-1}$ belong to $(7)$.

Proof. It suffices to suppose that $(C_1) \neq (C_2)$. As card
$\{ Q_1 \} = \aleph$ and card $C_1 = \text{card } C_1' \leq \text{card } C_2' < \aleph$, it follows
that $\{ Q_1 \} - C_1$ has the power $\aleph$. Since card $(C_2' - C_1') \leq$
$\text{card } C_2' < \aleph$, there exists in $\{ Q_1 \} - C_1$ a set $D$ such that
card $D = \text{card } (C_2' - C_1')$. Putting $D' \equiv C_2' - C_1'$, there exists
a $1 \rightarrow 1$ correspondence $s$ from $D'$ onto $D$. Put $C_2 \equiv C_1 \cup D$. We have
$C_1 \cap D = 0$.

Let $t_2$ be the correspondence, defined by $C_1' \, t_2 = t_1$,
$D' \, t_2 = s$.

$t_2$ is a one-to-one – mapping carrying $C_2'$ into $C_2$, and $C_1'$
into $C_1$; $t_2$ induces on $C_2$ an ordering $(C_2)$ and it reproduces
on $C_1$ the given ordering $(C_1)$. We have $(C_2) \subseteq (C_2)$. We also
have $([C_1]) \subseteq ([C_2])$. Since $t_0^{-1}$ and $t_0^{-1}$ are order isomorphisms
between chains with elements of type $\alpha$ and chains with
elements of type $\sigma(\alpha)$, therefore $t_1^{-1}$ and $t_2^{-1}$ are elements
elements of the well ordering $(m_r)$.

9. - Let

$$(7a) \quad W_1, W_2, \ldots, W_\gamma, \ldots$$
be well ordering of different elements, each of which being
a linear superfield of \([(\Phi_0)]\), with elements belonging to
\(\{Q_\alpha\}\).

Starting with the entities \([(\Phi_0)]\), \(V_\alpha\), \(G_\alpha(X)\), defined in [6]
and [6.1], we shall construct a well ordering

\[(\Phi_0), (\Phi_1), \ldots, (\Phi), \ldots\]

of linearly ordered, more and more ample fields with do-
 mains \(\subseteq \{Q_\lambda\}\). The construction will be based on the corres-
pondence \(a_0\) from [6], on the well ordering \(\{Q_\lambda\}\), on \((s_8)\)
from [3], on \((m_\gamma)\) from [7], and on \([9; (7a)]\) which were
chosen arbitrarily. The construction will yield well deter-
mined entities.

Let \(\alpha \geq 1\) be an ordinal where \(\text{card } \alpha < \aleph\).

Suppose that we have already constructed the fields
\([(\Phi_\beta)]\), the spaces \(V_\beta\) and the linear functionals \(G_\beta(X)\) for
every \(\beta < \alpha\), \(\beta \geq 0\), and suppose that the following hypotheses
are satisfied for every \(\beta < \alpha\), \(\beta \geq 0\): 1) \(\Phi_\beta \subseteq \{Q_\lambda\}\); 2) if \(\beta' \leq \beta'' \leq \beta\),
then \([(\Phi_{\beta'})] \subseteq [(\Phi_{\beta''})] \); 3) \(V_\beta\) is a linear space with multipliers
taken from \(\Phi_0\) and composed of some \(B'\Phi_0\)-aggregates; 4) if
\(\beta' \leq \beta'' \leq \beta\), then \(V_{\beta'} \subseteq V_{\beta''}\); 5) \(G_\beta(X)\) is a linear \(\Phi_\beta\)-valued
functional, defined on \(V_\beta\) with multipliers taken from \(\Phi_0\); 6) if \(\beta' \leq \beta'' \leq \beta\), and \(X \in V_{\beta'}\), then \(G_{\beta'}(X) = G_{\beta''}(X)\);
7) for every \(X \in V_\beta\) we have \(G_\beta(X) \leq N^{\Phi_\beta}(X)\), the \(B, a_\alpha(\mu)\)-
norm of \(X\) in \(\Phi_\beta\). 8) \(\text{card } \Phi_\beta < \aleph\).

We shall construct \([(\Phi_\alpha)]\), \(V_\alpha\) and \(G_\alpha(X)\) and prove that
for \(\beta = \alpha\) the conditions 1) - 8) are satisfied.

10. - Notice that from 1) - 8) the following properties
are resulting:

2) if \(\beta' < \beta'' < \alpha\), then \([(\Phi_{\beta'})] \subseteq [(\Phi_{\beta''})] \);
4) if \(\beta' < \beta'' < \alpha\), then \(V_{\beta'} \subseteq V_{\beta''}\);
6) if \(\beta' < \beta'' < \alpha\), then for \(X \in V_{\beta'}\) we have \(G_\beta(X) = G_{\beta''}(X)\).

11. - First we shall define the auxiliary entities
\([(\Phi_\alpha^\bullet)]\), \(V_\alpha^\bullet\), \(G_\alpha^\bullet(X)\)
as follows: The type of all sets \( \Phi_\beta \) being the same, [Hyp. 1] and since, by \( \alpha \geq 1 \), there exists \( \beta \) with \( 0 \leq \beta \leq \alpha \), the union

\[
\Phi_\alpha \overset{df}{=} \bigcup_{\beta < \alpha} \Phi_\beta
\]

is meaningful. We shall organize \( \Phi_\alpha \) into a chain by the definition: If \( Q' \), \( Q'' \in \Phi_\alpha \), then there exist \( \beta' \), \( \beta'' \) with \( \beta' < \alpha \), \( \beta'' < \alpha \) such that

\[
Q' \in \Phi_{\beta'}, \quad Q'' \in \Phi_{\beta''}.
\]

Supposing \( \beta' < \beta'' \), we have \( \Phi_{\beta'} \subseteq \Phi_{\beta''} \), [§ 7; 10, 2°]. Now, if \( Q' \supseteq Q'' \) in \( (\Phi_{\beta''}) \), we put \( Q' \supseteq Q'' \) in \( (\Phi_{\alpha}) \).

This definition does not depend on the choice of \( \beta' \), \( \beta'' \) satisfying (9), and it organizes \( \Phi_\alpha \) into a chain \( (\Phi_\alpha) \). We have for all \( \beta < \alpha \)

\[
(\Phi_\beta) \subseteq (\Phi_\alpha).
\]

We shall organize \( \Phi_\alpha \) into a field as follows: Admitting (9) and \( \beta' \leq \beta'' \), we define \( Q' + \Phi_\alpha Q'' \) and \( Q' \cdot \Phi_\alpha Q'' \) as \( Q' + Q'' \) and \( Q' \cdot Q'' \) in \( (\Phi_{\beta''}) \).

This definition does not depend on the choice of \( \beta' \) and \( \beta'' \), [§ 7; 10, 2°], and organizes \( \Phi_\alpha \) into a field \( (\Phi_\alpha) \). Thus we obtain the linearly ordered field \( ([\Phi_\alpha]) \) such that for every \( \beta < \alpha \) we have

\[
([\Phi_\beta]) \subseteq ([\Phi_\alpha])\).
\]

Since \( \text{card} \alpha < \aleph_0 \), and as, by [Hyp 8], \( \text{card} \Phi_\beta < \aleph_0 \), it follows that

\[
(10.1) \quad \text{card} \Phi_\alpha < \aleph_0.
\]

12. - By [Hyp. 3]], \( V_\beta \) is composed of some \( B\Phi_\alpha \)-aggregates, and there exists \( \beta \) with \( \beta < \alpha \). Consequently the union

\[
V_\alpha \overset{df}{=} \bigcup_{\beta < \alpha} V_\beta
\]

is meaningful. \( V_\alpha \) is composed of some \( B\Phi \)-aggregates. \( V_\alpha \) is a linear space with multipliers taken from \( \Phi_\alpha \).
13. - By [Hyp. 5)] \( G_\beta(X) \) is a linear \( \Phi \)-valued functional, defined on \( \mathcal{V}_\beta \), \( (\beta < \alpha) \), with multipliers taken from \( \Phi_0 \). Hence, by (8),

\[
\mathcal{D} \subseteq \Phi_0^*.
\]

We define \( G_\alpha^*(X) \) on \( \mathcal{V}_\alpha^* \) as follows: Let \( X \in \mathcal{V}_\alpha^* \). There exists, by (11), \( \beta < \alpha \) such that \( X \in \mathcal{V}_\beta \). We put

\[
G_\alpha^*(X) = \mathcal{G}_\beta(X).
\]

This definition does not depend on the choice of \( \beta \), this by [§ 7; 10, 6°]. We have \( G_\alpha^* = \mathcal{V}_\alpha^* \). We see that \( G_\alpha^*(X) \) is a \( \Phi_\alpha^* \)-valued linear functional on \( \mathcal{V}_\alpha^* \) with multipliers taken from \( \Phi_0 \).

14. - Let \( X \in \mathcal{V}_\alpha^* \). There exists \( \beta < \alpha \) such that \( X \in \mathcal{V}_\beta \). By [Hyp 7)] we have

\[
G_\beta(X) < N^{\Phi_\beta}(X),
\]

where the right hand side is the \( B, a_0(\mu) \) — norm of \( X \) in \( \Phi_\beta \). It is a left ending in (\( \Phi_\beta, \)). Now, by [§ 7; 11, (10)],

\[
(\Phi_\beta) = (\Phi_\alpha^*).
\]

We get \( G_\alpha(X) \leq N^{\Phi_\alpha}(X) \), and then, by Def (12), \( G_\alpha^*(X) \leq N^{\Phi_\alpha}(X) \) for all \( X \in \mathcal{V}_\alpha^* \), (14) where the right hand side is a left ending in \( \Phi_\alpha^* \). It is the \( B a_0(\mu) \)-norm of \( X \) in \( \Phi_\alpha^* \).

15. - Notice that if \( \alpha - 1 \) exists, then the entities ([\( \Phi_\alpha^* \)], \( \mathcal{V}_\alpha^* \) and \( G_\alpha^*(X) \) coincide with ([\( \Phi_{\alpha-1} \)], \( \mathcal{V}_{\alpha-1} \) and \( G_{\alpha-1}(X) \) respectively.

16. - It may happen that the space \( \mathcal{V}_\alpha^* \) contains the characteristic aggregats of all somata of \( (\mathcal{B}') \). If so, the process stops through the formation of ([\( \Phi_\alpha^* \)]).

17. - Let us suppose that \( \mathcal{V}_\alpha^* \) does not contain the characteristic aggregates of all somata of \( (\mathcal{B}) \). Under this hypothesis we are going to construct the entities ([\( \Phi_\alpha \)], \( \mathcal{V}_\alpha \) and \( G_\alpha(X) \) as follows. For the sake of simplicity we shall drop the index \( \alpha \), if no ambiguity will be feared of.
18. - There exists a soma $A_0$ of (B) whose characteristic $B\Phi^0$-aggregate (see $[\S\ 5; 11]$) $X_0 = \Omega_{A_0} = A_0 \cdot 1 + \co A_0 \cdot 0$ does not belong to $V^\bullet$. (The coefficients 1,0 can be understood as taken from $\Phi$. We shall apply the known Banach device (1):

Let $x_1$, $x_2 \in V^\bullet$. We have, by (14),

$$G^\bullet(x_2 - x_1) \leq N^\bullet(x_2 - x_1) = N^\bullet([-x_1 - X_0] + (x_2 + X_0)).$$

Now, the $B\Phi^0(\mu)$-norm in $\Phi^\bullet$ has a meaning for all $B\Phi^0$-aggregates.

Since $x_1 - X_0$, $x_2 + X_0$ are so, [12], we get, by $[\S\ 5; 6]$,

$$G^\bullet(x_2 - x_1) \leq N^\bullet([-x_1 - X_0] + N^\bullet(x_2 + X_0)).$$

Hence, by [13] and $[\S\ 5; 8]$,

$$G^\bullet(x_2) - G^\bullet(x_1) \leq N^\bullet(x_1 + X_0) + N^\bullet(x_2 + X_0).$$

Applying $[\S\ 1G; 4]$, we get

$$G^\bullet(x_2) - N(x_1 + X_0) < - G^\bullet(x_2) + N(x_2 + X_0)$$

which is valid for all $x_1$, $x_2 \in V^\bullet$.

The left hand side in (15) is a right ending in $(\Phi^\bullet)$, and the right hand side is a left ending in $(\Phi^\bullet)$.

By $[\S\ 1G; 6]$ we obtain for iterated endings:

$$G^\bullet(x) \equiv \sigma \in \mathcal{E} | - G^\bullet(x) - N(x + X_0) | x \in V^\bullet | \leq$$

$$\leq \sigma \in \mathcal{E} | - G^\bullet(x) + N(x + X_0) | x \in V^\bullet | \equiv \sigma \in \mathcal{E} | - G^\bullet(x) + N(x + X_0) | x \in V^\bullet | \equiv \sigma \in \mathcal{E} | - G^\bullet(x) + N(x + X_0) | x \in V^\bullet | \equiv \sigma \in \mathcal{E} | - G^\bullet(x) + N(x + X_0) | x \in V^\bullet |.$$ 

in the chain of all $\Phi^\bullet$-endings. By $[\S\ 1K; 2, 3]$ there exists a unique smallest right $\Phi^\bullet$-ending $\gamma^*$ such that $C^\bullet \leq \gamma^*$, and there exists a unique greatest $(\Phi^\bullet)$-ending $\delta^*$, such that $*\delta^* \leq *D$. Besides we have $\gamma^* \leq *\delta^*$. Thus we get for $\Phi^\bullet$-endings the relation

$$- G^\bullet(x) - N^\Phi^\bullet(x + X_0) \leq \gamma^* \leq *\delta \leq - G^\bullet(x) + N^\Phi^\bullet(x + X_0)$$

valid for all $x \in V^\bullet$. 
19. - Concerning \( \gamma^* \), \( \delta^* \) two cases may occur. It may happen that there exists a quantity \( \rho' \in \Phi \) such that

\[
\gamma^* \leq \rho' \leq \delta^*.
\]

Suppose this be so. We take the element \( Q_\beta \) from [§ 7; 3, (1)] with the smallest index for which \( \gamma^* \leq Q_\beta \leq \delta^* \). We denote this \( Q_\beta \) by \( \rho'' \), getting

\[
\gamma^* \leq \rho'' \leq \delta^*.
\]

20. - It may happen that the quantity \( \delta' \), mentioned above, does not exist. In this case the couple \( (\gamma^*, \delta^*) \) of endings represents a gap in \( (\Phi^o) \) which is either algebraic or transcendental, (see [§6; 31, 34, 37]). To proceed with necessary correctness, take the case [§ 7; 19.1] and denote by \( \varphi_\alpha \) the linearly ordered field, with elements of type \( \sigma(\alpha) \) which constitutes the corresponding elementary linear extension of \( (\Phi^o) \) through a well determined isomorphism denoted by \( A_\alpha^{-1} \). Let \( \varphi_\alpha^o \) be the \( A_\alpha^{-1} \) image of \( \Phi_\alpha^o \), and let \( \rho'' \) (of type \( \sigma(\alpha) \)) be a quantity in \( \varphi_\alpha \) which fills up the gap, according to [§ 6; 37]. We have \( (\varphi_\alpha^o) \subseteq (\varphi_\alpha) \) and \( A_\alpha^o \) is the isomorphism from \( \varphi_\alpha^o \) onto \( \Phi_\alpha^o \), where

\[
\Phi_\alpha^o \subseteq \{ Q_\lambda \}.
\]

In the case (19) denote by \( (\varphi_\alpha) \) the linearly ordered field with elements of type \( \sigma(\alpha) \) which constitutes the isomorphic image of \( \Phi_\alpha^o \) through isomorphism denoted by \( A_\alpha^{-1} \). Put \( \varphi_\alpha^o = \varphi_\alpha \). We have, as in (19), \( (\varphi_\alpha^o) \subseteq (\varphi_\alpha) \) and \( A_\alpha^o \) is the isomorphism from \( (\varphi_\alpha^o) \) onto \( (\Phi_\alpha^o) \), where

\[
\Phi_\alpha^o \subseteq \{ Q_\lambda \}.
\]

We are in the conditions of Lemma [8]. Indeed, we have (19) and (20), and, by [11; (11.2)] \( \text{card } \Phi_\alpha^o < \aleph \), hence \( \text{card } \varphi_\alpha^o < \aleph \). In the case, considered in [19], we have \( \text{card } \varphi_\alpha < \aleph \). Now consider the case involving a true extension.

The extension is defined by polynomials in \( \Phi_\alpha^o \) or by couples of polynomials. The cardinal number of the class of all polynomial of the \( n^{th} \) order is \( (\text{card } \Phi_\alpha^o)^{n+1} = \text{card } \Phi_\alpha^o \),
hence the cardinal of the class of all polynomials is \( \aleph_0 \cdot \cdot \cdot \) \( \text{card } \Phi^\circ \). Since the fields are linearly ordered, their characteristic is 0 and they contain a partial field isomorphic with the field of all ordinary rational numbers. Hence \( \aleph_0 \leq \text{card } \Phi^\circ \). Consequently the power of the class of all ordered couples of polynomials is \( (\text{card } \Phi^\circ)^2 = \text{card } \Phi^\circ \). Consequently we have \( \text{card } \varphi \prec \aleph_0 \), whenever the case may be.

By Lemma [8] there exists an isomorphism \( m_\gamma \), taken from \([7, (7)]\) such that \( \alpha_\gamma^\circ = \varphi_\gamma^\circ m_\gamma^{-1} \) and where \( ([\Phi^\circ]) \subseteq \{ m_\gamma^{-1}(\varphi_\gamma) \} \).

Take \( m_\gamma \) with the smallest index and denote it by \( \alpha_\gamma^{-1} \). Put

\[
(20.1) \quad ([\Phi^\circ]) \overset{\text{def}}{=} \alpha_\gamma([\varphi_\gamma]).
\]

We have

\[
(21) \quad ([\Phi^\circ]) \subseteq ([\Phi^\circ]), \quad \Phi^\circ \subseteq \{ Q_\gamma \}
\]

and

\[
(22) \begin{cases}
([\varphi_\gamma]) \overset{\alpha_\gamma}{\longrightarrow} ([\Phi^\circ]), \\
([\varphi_\gamma]) \overset{\alpha_\gamma}{\longrightarrow} ([\Phi^\circ]).
\end{cases}
\]

Define

\[
(23) \quad \rho_\alpha \overset{\text{def}}{=} \alpha_\gamma \rho''.
\]

We have \( \rho_\alpha \in \Phi^\circ \).

If there is no true extension of \( \Phi^\circ \), we get back \( [\Phi^\circ]) = \equiv ([\Phi^\circ]), \) and \( \rho_\alpha = \rho'' \).

If the gap yields an apparent extension, it is well determined, and so is also the extension if the gap is transcendental. However an algebraic gap may admit many extensions with type \( \sigma(\alpha) \).

For each such extension we obtain by the above construction a well determined element

\[
\rho_\alpha = \alpha_\gamma \rho''.
\]

To have a well defined construction, take this extension of \( \Phi^\circ \) in \( \{ Q_\gamma \} \) which in \([9, (7a)]\) has the smallest index, and if, nevertheless, there will be more than one element \( \rho_\alpha \) which fills up the gap, take among them one which has in \( \{ Q_\gamma \} \) the smallest index. In this way the extension and the element \( \rho_\alpha \) will be well determined.
21. - Consider the relation (16) in [18]. Since \((\Phi_\alpha^\bullet) \subseteq (\Phi_\alpha)\), the ordering of endings in \((\Phi_\alpha^\bullet)\) constitutes an order (and operation)-genuine subordering of the ordering of endings in \(\Phi_\alpha\).

We get for the isomorphic images of endings

\[-G_\alpha^\bullet(x) - N^{\Phi_\alpha}(x + X_0) \leq \gamma^* \leq \delta \leq G_\alpha^\bullet(x) + N^{\Phi_\alpha}(x + X_0)\]

in the ordering of \(\Phi_\alpha\)-endings. Since in \(\Phi_\alpha\) we have

\[(23.1) \quad \gamma^* \leq \rho_\alpha \leq \delta, \quad \rho_\alpha \in \Phi_\alpha,\]

we obtain

\[(24) \quad -G_\alpha^\bullet(x) - N^{\Phi_\alpha}(x + X_0) \leq \rho_\alpha \leq -G_\alpha^\bullet(x) + N^{\Phi_\alpha}(x + X_0)\]

in \(\Phi_\alpha\), valid for all \(x \in \mathcal{V}_\alpha^\bullet\).

22. - Define the space \(V_\alpha\) as the set of all aggregates \(x + tX_0\), where \(x \in \mathcal{V}_\alpha\), \(t \in \Phi_0\). First we notice that if \(Y \in \mathcal{V}_\alpha\), then \(Y\) admits a unique representation

\[(24.1) \quad Y = x + tX_0\]

where

\[x \in \mathcal{V}_\alpha^\bullet, \quad t \in \Phi_0.\]

Indeed let

\[Y = x_1 + t_1X_0 = x_2 + t_2X_0,\]

where

\[x_1, x_2 \in \mathcal{V}_\alpha^\bullet; \quad t_1, t_2 \in \Phi_0.\]

We get

\[x_1 - x_2 = (t_2 - t_1)X_0.\]

If we had \(t_2 = t_1\), we would get

\[X_0 = \frac{x_1 - x_2}{t_2 - t_1} \in \mathcal{V}_\alpha^\bullet\]

which we excluded in [17], Hence \(t_2 = t_1\), and then \(x_1 = x_2\).

I say that \(V_\alpha\) is a linear space with multipliers taken from \(\Phi_0\). It is composed of some \(B^{\Phi_0}\)-aggregates. Indeed

\[(x_1 + t_1X_0) + (x_2 + t_2X_0) = (x_1 + x_2) + (t_1 + t_2)X_0,\]
where
\[ x_1 + x_2 \in \mathcal{V}_\alpha, \quad t_1 + t_2 \in \Phi_0. \]

If \( \lambda \in \Phi_0 \) we have
\[ (x + tX_0)\lambda = \lambda x + (t\lambda)X_0, \]
where
\[ \lambda x \in \mathcal{V}_\alpha \quad \text{and} \quad t\lambda \in \Phi_0. \]

Let \( x + tX_0 \in \mathcal{V}_\alpha \). Since \( x \in \mathcal{V}_\alpha \), it is a \( B'\Phi_0 \)-aggregate and since \( t \in \Phi_0 \), \( tX_0 \) is also a \( B'\Phi_0 \)-aggregate, therefore \( x + tX_0 \) is a \( B'\Phi_0 \)-aggregate.

We have
\[ \mathcal{V}_\alpha^\bullet \subseteq \mathcal{V}_\alpha. \]
Indeed if \( x \in \mathcal{V}_\alpha^\bullet \), we can write \( x = x + 0 \cdot X_0 \).

23. Define for all aggregates \( Y = x + tX_0 \in \mathcal{V}_\alpha \) the functional \( G_\alpha(Y) \), by putting
\[ G_\alpha(x + tX_0) \overset{df}{=} G_\alpha^\bullet(x) + t\rho_\alpha. \]

The definition is meaningful on account of the uniqueness of representation (24.1) of \( Y \). We see that \( G_\alpha \) is a linear functional in \( \mathcal{V}_\alpha \) with multipliers taken from \( \Phi_0 \). Indeed, let
\[ Y_1 = x_1 + t_1X_0, \quad Y_2 = x_2 + t_2X_0 \]
be elements of \( \mathcal{V}_\alpha \).

We have
\[ G_\alpha(Y_1) = G_\alpha^\bullet(x_1) + t_1\rho_\alpha, \quad G_\alpha(Y_2) = G_\alpha^\bullet(x_2) + t_2\rho_\alpha, \]
hence
\[ G_\alpha(Y_1) + G_\alpha(Y_2) = [G_\alpha^\bullet(x_1) + G_\alpha^\bullet(x_2)] + (t_1 + t_2)\rho_\alpha = G_\alpha^\bullet(x_1 + x_2) + (t_1 + t_2)\rho_\alpha = G_\alpha^\bullet(x_1 + t_1X_0 + x_2 + t_2X_0) = G_\alpha^\bullet(x_1 + t_1X_0) = G_\alpha^\bullet(x_1 + t_1X_0) + G_\alpha^\bullet(x_2 + t_2X_0) = G_\alpha(Y_1 + Y_2). \]

Let \( Y = x + tX_0 \in \mathcal{V}_\alpha \) and \( \lambda \in \Phi_0 \). We have
\[ G_\alpha(\lambda Y) = G_\alpha[\lambda x + (t\lambda)X_0] = G_\alpha^\bullet(\lambda x) + (t\lambda)\rho_\alpha = \lambda G_\alpha^\bullet(x) + \lambda(t\rho_\alpha) = \lambda G_\alpha^\bullet(x + tX_0) = \lambda G_\alpha(Y). \]
If \( Y \in V_\alpha \), then \( G_\alpha(Y) = G^\bullet_\alpha(Y) \). Indeed \( Y = Y + 0 \cdot X_0 \).
Hence \( G_\alpha(Y) = G^\bullet_\alpha(Y) + 0 \cdot \rho_\alpha \).

24. - We shall prove that if \( Y \in V_\alpha \), then

\[
G_\alpha(Y) \leq N^{\Phi_\alpha}(Y)
\]
i.e. the \( B\alpha_\alpha(\mu) \)-norm of \( Y \) in \((\Phi_\alpha)\).

**Proof.** Let \( Y = x + tX_0 \). Case 1) \( t = 0 \). We have, by [23],
\( G_\alpha(Y) = G^\bullet_\alpha(Y) \), and then, by [14, (14)] \( G_\alpha(Y) \leq N^{\Phi_\alpha}(Y) \), hence,
since [21, 10],
\( G_\alpha(Y) \leq N^{\Phi_\alpha}(Y) \).

Case 2), \( t > 0 \). We have, by [21, (24)]
\( \rho_\alpha \leq -G^\bullet_\alpha(x) + N^{\Phi_\alpha}(x + X_0) \)
for every \( x \in V_\alpha \). Since \( \frac{x}{t} \in V^\bullet_\alpha \), we get (for \( \frac{1}{t} \in \Phi_\alpha \)),
\[
\rho_\alpha \leq -G^\bullet_\alpha\left(\frac{x}{t}\right) + N^{\Phi_\alpha}\left(\frac{x}{t} + X_0\right).
\]

Hence, by [§ 5; 7],
\[
\rho_\alpha \leq -\frac{1}{t} G^\bullet_\alpha(x) + \frac{1}{t} N(x + tX_0).
\]

Hence, by [§ 1F; 7], since \( t > 0 \),
\[
\rho_\alpha \leq \frac{1}{t} [-G^\bullet_\alpha(x) + N(x + tX_0)].
\]

Hence, by [§ 1F; 10, 7],
\[
t\rho_\alpha \leq -G^\bullet_\alpha(x) + N(x + tX_0).
\]

Hence, by [§ 1G; 3],
\[
G^\bullet_\alpha(x) + t\rho_\alpha \leq N(x + tX_0),
\]
and then
\[
G_\alpha(Y) \leq N(Y).
\]
Case 3), $t < 0$. We have, by [§ 7; 21, (24)],

$$-G^\bullet(x) - N(x + X_0) \leq \rho_x$$

for all $x \in V^\bullet$. Since $\frac{x}{t} \in V^\bullet$, we get

$$-G^\bullet\left(\frac{x}{t}\right) - N\left(\frac{x}{t} + X_0\right) \leq \rho_x.$$

Applying [§ 1F; 27], we get $-G^\bullet - N \leq -0 + \rho_x$ hence, by a theorem similar to [§ 1G; 4], $0 - G^\bullet \leq N + \rho_x$ and, by [§ 1F; 27]

$$-G^\bullet \leq N + \rho_x, \quad -G^\bullet - \rho_x \leq N, \text{ i.e. } -G^\bullet\left(\frac{x}{t}\right) - \rho_x \leq N\left(\frac{x}{t} + X_0\right).$$

Hence, by [5; 8],

$$\left(-\frac{1}{t}\right)G^\bullet(x) - \rho_x \leq N\left(\frac{x + tX_0}{-t}\right).$$

Hence, by [§ 5; 7],

$$-\left(\frac{1}{t}\right)[G^\bullet(x) = t\rho_x] \leq \frac{1}{-t}N(x + tX_0), \text{ i.e. } \left(-\frac{1}{t}\right)G^\bullet(Y) \leq \frac{1}{-t}N(Y).$$

Using [§ 1F; 10], we get $G^\bullet(Y) \leq N(Y)$, so the assertion (27) is proved in all cases.

25. Having defined (\[(\Phi_\beta)\]) \cdot V_\alpha and $G_\alpha(x)$, we shall prove that all properties 1) - 8) in [9] hold true for $\beta = \alpha$. We have $\Phi_\beta \subseteq \{ Q \}$. This follows from (21).

If $\beta' \leq \beta'' \leq \alpha$, then $([\Phi_{\beta''}) \subseteq ([\Phi_{\beta'}])$. This follows from [9, 3], [11, (11.2)] and [20, (21)].

$V_\alpha$ is a linear space with multipliers from $\Phi_0$, and its elements are $B\Phi_0$-aggregates. This follows from [22].

If $\beta' \leq \beta'' \leq \alpha$, then $V_{\beta'} \subseteq V_{\beta''}$. This follows from [9, 3], [22, (25)] and [12, (11)].

$G_\alpha(x)$ is a linear $\Phi_\alpha$-valued functional, defined on $V_\alpha$ with multipliers, taken from $\Phi_0$. This follows from [23].

If $\beta' \leq \beta'' \leq \alpha$, and $X \in V_{\beta'}$, then $G_{\beta'}(X) = G_{\beta''}(X)$. This follows from [9, 6], [23] and [13].
For every $X \in V_\alpha$, we have $G_\alpha(X) \leq N^{\Phi_\alpha}(X)$. This was proved in [24].

We have $\text{card } \Phi_\alpha < \aleph$. This follows from [20], because $\text{card } \Phi_\alpha < \aleph$ and $\Phi_\alpha$ is an isomorphic image of $\varphi_\alpha$.

26. - This concludes the inductive process. We have got a well ordering

\[(28) \quad [([\Phi_0]) \subseteq ([\Phi_1]) \subseteq \cdots \subseteq ([\Phi_\alpha]) \subseteq \cdots,\]

of linearly ordered fields, a well ordering of linear spaces

\[(29) \quad V_0 \subseteq V_1 \subseteq \cdots \subseteq V_\alpha \subseteq \cdots,\]

and a well ordering of linear functionals

\[(30) \quad G_\alpha(X), G_1(X), \ldots, G_\alpha(X), \ldots,\]

defined on (29) respectively.

The elements of all spaces (29) are $B'\Phi_\alpha$-aggregates; $G_\alpha(X)$ is $\Phi_\alpha$-valued with multipliers taken from $\Phi_0$.

All fields (28) have the same logical type.

§ 8. - The extended measure.

1. - The construction process exhibited in § 7 must stop at $\Phi^\bullet_\gamma$ for a certain index $\gamma$, where $\text{card } \gamma < \aleph$. Indeed, each step of the process is conditioned by the existence of a soma $\in B'$, whose characteristic aggregate does not belong to $V^\bullet_\gamma$.

Now, since $\text{card } B < \aleph$, the well ordering must have its power $< \aleph$, Take this index, consider the field $([\Phi^\bullet_\gamma])$, the space $V^\bullet_\gamma$ and the linear functional $G^\bullet_\gamma(X)$. The space $V^\bullet_\gamma$ contains the characteristic aggregates of all somata belonging to $B'$. We have

\[(1) \quad \text{card } \gamma \leq \text{card } B'.\]

To simplify notions, put

$\Phi \overset{df}{=} \Phi^\bullet_\gamma$, $V \overset{df}{=} V^\bullet_\gamma$, $G(X) \overset{df}{=} G^\bullet_\gamma(X)$. 
2. We define for \( A \in B' \):
\[
\mu(A) \overset{\text{def}}{=} G(\Omega_A), \quad \text{where} \quad \Omega_A = A \cdot 1 + \text{co} \, A \cdot 0.
\]
We have
\[
\mu(A) \in \Phi.
\]
If \( A_1, A_2 \in B, \ A_1 \cdot A_2 = 0 \), we have \( \Omega_{A_1 + A_2} = \Omega_{A_1} + \Omega_{A_2} \), and then
\[
G(\Omega_{A_1 + A_2}) = G(\Omega_{A_1}) + G(\Omega_{A_2}), \quad \text{i.e.} \quad \mu(A_1 + A_2) = \mu(A_1) + \mu(A_2).
\]
If \( A \in B \), then
\[
\mu(A) = G(\Omega_A) = a_\phi(\mu(A)).
\]
Consequently \( \mu(A) \) represents the extended measure \( \mu \) from \( B \) over all \( B' \); it is finitely additive.

3. We have supposed that the measure \( \mu(a) \), given on \( B \), is non negative. We shall prove that \( \mu \) is also non negative. To do this, consider the formula (27)
\[
G(Y) \leq N^{\phi^*}(Y),
\]
proved in [24, (27)].

If \( Y = \Omega_A \) where \( A \in B' \), then by [§ 5; 11], we have for \( G(Y), N(Y) = \phi^*(A) \). Hence \( G(Y) \leq \phi^*(A) \), and then \( G(Y) \leq \phi^*(A) \) which gives
\[
\mu(A) = \phi(A).
\]
This is true for any \( A \in B' \). Applying (4) to \( \text{co} \, A \), we get
\[
\mu(\text{co} \, A) \leq \phi(\text{co} \, A),
\]
hence
\[
\mu(1) - \mu(A) \leq \mu(1) - \phi(A),
\]
and then
\[
\phi(A) \leq \mu(A).
\]
Since \( \phi(A) \geq 0 \) for every \( A \in B' \), it follows that
\[
\mu(A) \geq 0.
\]
We have proved the following
Theorem. If 1. \( B \) and \( B' \) are finitely additive Boolean tribes,
2. \( B \) is a finitely genuine strict subtribe of \( B' \), i.e. with equality, finite operations, zero and unit taken from \( B' \),
3. \( \varphi_0 \) is a non trivial linearly ordered field,
4. \( \mu(a) \) is an \( \varphi_0 \)-valued, finitely additive, non negative measure on \( B \),
then there exists a linearly ordered field \( \Phi \) and a \( \Phi \)-valued, finitely additive, non negative measure \( M(a) \) on \( B' \) such that
1) \( \varphi_0 \) is an operation and order genuine subfield of \( \Phi \) through isomorphism,
2) if we denote this isomorphism by \( t \), then \( M(a) = t\mu(a) \) for all \( a \in B \),
3) for every \( A \in B' \) we have
\[
\ell(\mu_{B'\mu}(A)) \leq M(A) \leq \ell(\mu_{B'\mu}(A)),
\]
where \( \mu^*, \mu^* \) are Jordan \( B\varphi_0 \)-exterior and exterior, ending valued measures on \( B' \),
4) \( \text{card } \Phi \leq \max \left( \text{card } \varphi_0, \text{card } B' \right) \).
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