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TERM RANK OF 0, 1 MATRICES

Memoria (*) di Robert Morton Haber (ad Urbana, Ill.)

1. Introduction.

Let $A$ be a matrix with $n$ rows and $m$ columns, all of whose entries are 0's and 1's. Let the sum of row $i$ of $A$ be denoted by $r_i$ ($i = 1, \ldots, n$), and let the sum of column $j$ be denoted by $s_j$ ($j = 1, \ldots, m$). With the matrix $A$ we associate the row sum vector

$$R = (r_1, \ldots, r_n)$$

and the column sum vector

$$S = (s_1, \ldots, s_m).$$

Let $\mathbf{\delta}_i = (1, \ldots, 1, 0, \ldots, 0)$ be a vector of $m$ components with 1's in the first $r_i$ positions and 0's elsewhere. A matrix of row sum vector $R$ of the form

$$A = \begin{pmatrix} \mathbf{\delta}_1 \\ \vdots \\ \mathbf{\delta}_n \end{pmatrix}$$

is called maximal. Throughout the discussion

$$R' = (r'_1, \ldots, r'_m)$$

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designates the column sum vector of $A$. Similarly let

$$\varepsilon_i = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

be a vector of $n$ components with 1’s in the first $s_i$ positions and 0’s elsewhere. Then

(1.5) \quad A^* = (\varepsilon_1, \ldots, \varepsilon_m)

has column sum vector $S$. The notation

(1.6) \quad S' = (s_1', \ldots, s'_m)

designates the row sum vector of $A^*$. Note that

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{m} r_i'$$

are conjugate partitions. Also

$$\sum_{i=1}^{m} s_i = \sum_{i=1}^{n} s_i'$$

are conjugate partitions. Moreover, the components of $R'$ and $S'$ always appear in descending order.

Let $U = (u_1, \ldots, u_q)$ and $V = (v_1, \ldots, v_q)$ be two vectors with integral components. We write

$$U \triangleleft V$$

or

$$V \triangleright U$$

if

(1.7) \quad u_1 + u_2 + \ldots + u_i \leq v_1 + v_2 + \ldots + v_i \quad (i = 1, \ldots, q - 1).

(1.8) \quad u_1 + u_2 + \ldots + u_q = v_1 + v_2 + \ldots + v_q.

If, furthermore, (1.7) and (1.8) still hold when the components of $U$ and $V$ are reordered so that they are in nonincreas-
ing order, we say that $U$ is majorized by $V$, written

$$ U < V $$

or

$$ V > U. $$

We are now in a position to state the existence theorem for 0, 1 matrices having row sum vector $R$ and column sum vector $S \ [1; 3]$. We give a new proof in Section 2.

**EXISTENCE THEOREM.** Let $R = (r_1, \ldots, r_n)$ and $S = (s_1, \ldots, s_m)$ be two vectors with nonnegative integral components. Then there exists a matrix $A$ of size $n \times m$ with entries 0's and 1's with row sum vector $R$ and column sum vector $S$ if and only if

$$ S' > R. $$

The term rank $\rho$ of the 0, 1 matrix $A$ is the order of the greatest minor of $A$ with a non zero term in its determinant expansion. This integer is also equal to the minimal number of rows and columns that collectively contain all of the non zero elements of $A \ [2]$. Let $\mathcal{A}$ be the class of 0, 1 matrices with row sum vector $R = (r_1, \ldots, r_m)$ and column sum vector $S = (s_1, \ldots, s_n)$. Notationally we write

$$ \mathcal{A}(R, S) $$

In [4] Ryser has found a formula for $\bar{\rho}$, the maximal term rank for matrices in $\mathcal{A}(R, S)$. In Section 3 we derive an algorithm for finding $\tilde{\rho}$, the minimal term rank of matrices in $\mathcal{A}(R, S)$. Unfortunately a simple formula for $\tilde{\rho}$, analogous to the formula for $\rho$, does not appear to be forthcoming. In Section 4 we give a method for constructing matrices of maximal term rank $\rho$. Sections 3 and 4 comprise the main portion of our paper.

Consider the $2 \times 2$ submatrices of $A$ of the types

$$ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. $$

An interchange is a transformation of the elements of $A$ that
changes a minor of type $A_1$ into type $A_2$ or vice versa and leaves all other elements of $A$ unaltered. By a theorem of Ryser [3], if $A$ and $A^*$ are two elements in the class $\mathcal{I}(R, S)$, then $A$ is transformable into $A^*$ by a sequence of interchanges. We give a different proof of this interchange theorem in Section 2. Suppose now that an element $a_{uv} = 1$ of $A$ is such that no sequence of interchanges applied to $A$ replaces $a_{uv} = 1$ by 0. Then $a_{uv}$ is called an invariant 1 of $A$. By the interchange theorem it is an invariant 1 of the class $\mathcal{I}(R, S)$. In our concluding Section 5 we obtain a formula for finding which 1's of a class $\mathcal{I}(R, S)$ are invariant 1's.

2. Existence And Interchange Theorems.

In this section we give new proofs of the existence and interchange theorems described in Section 1. We begin with the following:

**Lemma 2.1.** Let $U \prec V$. If $U$ can be transformed into a vector with nonincreasing components by successively interchanging two adjacent components which differ by 1, then $U \prec V$.

**Proof:** We may suppose $V$ has nonincreasing components for this does not upset the hypothesis $U \prec V$. Let $U = (u_1, ..., u_q)$ and $V = (v_1, ..., v_q)$ ($v_1 \geq v_2 \geq ... \geq v_q$). Suppose that $u_j = u_{j-1} + 1$. We assert that if we interchange these, the new vector $U'$ will satisfy $U \prec V$. For if not,

(2.1) \[ u_1 + u_2 + ... + u_{j-2} \leq v_1 + ... + v_{j-1}, \]
(2.2) \[ u_1 + u_2 + ... + u_{j-1} = v_1 + ... + v_{j-1}, \]
(2.3) \[ u_1 + u_2 + ... + u_j \leq v_1 + ... + v_j . \]

Then (2.1) and (2.2) imply

(2.4) \[ u_j > u_{j-1} \geq v_{j-1}. \]

But (2.2) and (2.3) imply

(2.5) \[ u_j \leq v_j . \]
This contradicts the assertion \( V \) is nonincreasing, and Lemma 2.1 follows.

**Existence Theorem.** Let \( R = (r_1, \ldots, r_n) \) and \( S = (s_1, \ldots, s_m) \) with row sum vector \( R \) and column sum vector \( S \) if and only if

\[ S' > R. \]

**Proof:** For the necessity see [3]. We may suppose \( r_1 \geq \ldots \geq r_n \). The proof is by induction on \( m \). For \( m = 1 \) the theorem is clear. Suppose the theorem is true if \( S \) has \( m - 1 \) components. Let \( s_m = t \). Define

\[ R_1 = (r_1 - 1, \ldots, r_t - 1, r_{t+1}, \ldots, r_n), \]
\[ S_1 = (s_1, \ldots, s_{m-1}). \]

Now the number of positive components of \( R \) is \( \geq \) the number of positive components of \( S' \). For otherwise we could not have \( R < S' \). Also the number of positive components of \( S' \) equals the largest \( s_i \). This implies that \( R_1 \) has nonnegative components. Now \( S'_1 - R_1 = S' - R \) so that

\[ (2.6) \quad S'_1 > R_1. \]

Since \( R_1 \) is transformable into a vector with nonincreasing components by successively interchanging two adjacent elements which differ by 1, by Lemma 2.1, \( S'_1 > R_1 \), and the class \( \mathcal{A}(R_1, S_1) \) is nonempty by induction. Adjoining the column vector

\[
\begin{pmatrix}
1 \\
1 \\
0 \\
0
\end{pmatrix}
\]

with \( t \) 1's in the initial positions to an element of \( \mathcal{A}(R_1, S_1) \) gives an element of \( \mathcal{A}(R, S) \).

**Interchange Theorem.** Let \( A \) and \( A^* \) be two \( n \times m \) matrices
in the class \( \mathcal{A}(R, S) \). Then \( A^* \) is transformable into \( A \) by a finite number of interchanges.

**Proof:** Let \( r_1 \geq \ldots \geq r_n, s_1 \geq \ldots \geq s_m \). By a finite sequence of interchanges, 1's may be shifted to the left in the first row until they occupy the first \( r_1 \) positions. Now applying the same argument to the matrix with sum vector \( R_1 = (r_2, \ldots, r_n) \) and column sum vector \( S_1 = (s_1 - 1, \ldots, s_{r_1} - 1, s_{r_1+1}, \ldots, s_m) \), we can put 1's in the \( r_2 \) columns where \( S_1 \) has the largest components. Continuing in this manner we see that there are two sequences of interchanges, one taking \( A \) into a matrix \( Z \) and the other taking \( A^* \) into \( Z \). Suppose that the intermediate matrices taking \( A \) into \( Z \) are \( A_1, \ldots, A_q \). Then since there is an interchange taking \( Z \) into \( A_q \) and one taking \( A_q \) into \( A_{q-1} \), etc., there is a sequence of interchanges taking \( Z \) into \( A_1 \). Hence there is a sequence of interchanges taking \( Z \) into \( A \) and a sequence taking \( A^* \) into \( A \).

### 3. An Algorithm for \( \tilde{p} \).

Let \( \mathcal{A} = \mathcal{A}(R, S) \) be the class of 0,1 matrices with row sum vector \( R = (r_1, \ldots, r_n) \) and column sum vectors \( S = (s_1, \ldots, s_m) \). Let \( A \) be in \( \mathcal{A} \) and let

\[
A = \begin{pmatrix}
A_i^u \\
A_i^l
\end{pmatrix}
\]

where \( A_i^u \) denotes the upper \( i \) rows of \( A \) and \( A_i^l \) denotes the lower \( n-i \) rows of \( A \). Let the column sum vector of \( A_i^u \) be denoted by \( S_{i}^u \), the row sum vector of \( A_i^u \) by \( R_i^u \), and similarly \( S_{n-i}^l \) and \( R_{n-i}^l \) will denote respectively the column sum vector and the row sum vector of \( A_{n-i}^l \).

**Lemma 3.1.** Let \( A \) be an element of \( \mathcal{A}(R, S) \) and let

\[
A = \begin{pmatrix}
A_i^u \\
A_{n-i}^l
\end{pmatrix}
\]

Then \( S_i^u < (R_i^u)' \), \( S_{n-i}^l < (R_{n-i}^l)' \), and \( S_i^u + S_{n-i}^l = S \). Conversely let \( S_i^u < (R_i^u)' \), \( S_{n-i}^l < (R_{n-i}^l)' \), and \( S_i^u + S_{n-i}^l = S \), where the components of the vectors are nonnegative integers. Then
there exists an $A_j^u$ with row sum vector $R_j^u$, column sum
vector $S_j^u$, and an $A_{n-j}^i$ with row sum vector $R_{n-j}^i$, column
sum vector $S_{n-j}^i$ such that

$$A = \begin{pmatrix} A_j^u \\ A_{n-j}^i \end{pmatrix}$$

is an element of $\mathcal{A}$.

**Proof:** This is an immediate consequence of the existence theorem.

**Lemma 3.2.** If $(a_1, \ldots, a_r) < (b_1, \ldots, b_r)$, $(c_1, \ldots, c_s) < (d_1, \ldots, d_s)$, and $U = (a_1, \ldots, a_r, c_1, \ldots, c_s)$, $V = (b_1, \ldots, b_r, d_1, \ldots, d_s)$, then $U < V$.

**Proof:** We may assume of course that $a_1 \geq \ldots \geq a_r$, $b_1 \geq \ldots \geq b_r$, $c_1 \geq \ldots \geq c_s$, $d_1 \geq \ldots \geq d_s$. Clearly

$$\sum_{i=1}^r a_i + \sum_{i=1}^s c_i = \sum_{i=1}^r b_i + \sum_{i=1}^s d_i.$$

Suppose the $h$ largest components of $U$ are $a_1, \ldots, a_x, c_1, \ldots, c_\beta$, and the $h$ largest of $V$ are $b_1, \ldots, b_\gamma, d_1, \ldots, d_\delta$. Here

$h = \alpha + \beta = \gamma + \delta$ and $h = 1, \ldots, r + s - 1$. Then

$$\sum_{i=1}^\alpha a_i + \sum_{i=1}^\beta c_i < \sum_{i=1}^\gamma b_i + \sum_{i=1}^\delta d_i.$$  \tag{3.2}

Hence $U < V$.

**Lemma 3.3.** If $(a_1, \ldots, a_r) < (b_1, \ldots, b_r)$, then $(a_1, \ldots, a_r, c_1, \ldots, c_s) < (b_1, \ldots, b_r, c_1, \ldots, c_s)$.

**Proof:** This is a special case of Lemma 3.2.

**Lemma 3.4.** If $a \geq b+j$ and $j \geq 0$, then $(a - j, b+j) < (a, b)$.

**Proof:** This is immediate from the definition of $\prec$.  

**Theorem 3.1.** Let $A$ be an element of $\mathcal{A}(R, S)$. We assume

$s_1 \geq \ldots \geq s_m$, but we do not assume any ordering for $R$. Suppose

$$A = \begin{pmatrix} A_j^u \\ A_{n-j}^i \end{pmatrix}.$$
Then there is an $*A$ which is an element of $\mathcal{A}(R, S)$, and such that if

$$*A = \begin{pmatrix} *A_1^{n-i} \\ *A_{n-i}^{n-1} \end{pmatrix},$$

then $*S_{n-i}^i$ has the same components (in a different order) as $S_{n-i}^i$ and furthermore the components of $*S_{n-i}^i$ are in nonincreasing order.

**Proof:** Let $S_{n-i}^i = (h_1, \ldots, h_m)$. Suppose $h_j > h_k$ with $j > k$. Define $\mathcal{A}_{n-i}^{*}$ to be a matrix with row sum vector the same as $A_{n-i}^i$ and column sum vector $\mathcal{S}_{n-i}^i$ the same as $S_{n-i}^i$ but with $h_j$ and $h_k$ interchanged, e.g., take $\mathcal{A}_{n-i}^{*}$ the same as $A_{n-i}^i$ with the $j-$th and $k-$th columns interchanged. Let $\mathcal{S}_{n-i} = S_{n-i}^i \mathcal{S}_{n-i}$. Now $\mathcal{S}_{n-i}^i$ and $\mathcal{S}_{n-i}$ agree except for two positions. These two in $\mathcal{S}_{n-i}^i$ are $(s_h - h_j, s_j - h_k)$ and in $\mathcal{S}_{n-i}$ are $(s_h - h_k, s_j - h_h)$. Now by hypothesis $j > k$ implies $s_h \geq s_j$.

Thus

(3.3) \hspace{1cm} 0 \leq s_j - h_j \leq s_k - h_j < s_k - h_k,

and

(3.4) \hspace{1cm} 0 \leq s_j - h_j < s_k - h_k \leq s_k - h_k.

But (3.3) and (3.4) imply $(s_h - h_j, s_j - h_k) < (s_k - h_k, s_j - h_j)$, and by Lemma 3.3, $\mathcal{S}_{n-i}^i < \mathcal{S}_{n-i}$. By Lemma 3.1, $\mathcal{S}_{n-i}^i < (R^i)^t$, whence $\mathcal{S}_{n-i}^i < (R^i)^t$. Hence there exists an $\mathcal{A}_{n-i}^{*}$ with row sum vector $R_{n-i}^{i}$ and column sum vector $\mathcal{S}_{n-i}^i$. Now clearly

$$\begin{pmatrix} \mathcal{A}_{n-i}^{*} \\ \mathcal{A}_{n-i}^{*} \end{pmatrix}$$

is an element of $\mathcal{A}(R, S)$.

Continuing in this manner we obtain the desired $*A$.

**Corollary.** In addition to the hypotheses of Theorem 3.1, let $R = (r_1, \ldots, r_n)$, where $r_1 \geq \ldots \geq r_n$. Let $A$ be an element of $\mathcal{A}$ such that rows $i_1, \ldots, i_t$ and columns $j_1, \ldots, j_u$ exhaust all 1's. Then there is an $A^*$ in $\mathcal{A}$ such that rows 1, ..., $i$ and columns 1, ..., $u$ exhaust all 1's.

**Proof:** By Theorem 3.1 there is an $A$ in $\mathcal{A}$ such that
rows \(i_1, \ldots, i_t\) and columns \(1, \ldots, u\) exhaust all 1's. Consider \(A_i^T\) (i.e. the transpose of \(A_i\)). \(A_i^T\) need not, of course, be an element of \(\mathcal{C}\). In \(A_i^T\) columns \(i_1, \ldots, i_t\) and rows \(1, \ldots, u\) exhaust all 1's. Again by Theorem 3.1 there exists an \(A_2\) with \(A_2^T\) in \(\mathcal{C}\) and where columns \(1, \ldots, t\) and rows \(1, \ldots, u\) exhaust all 1's of \(A_2\). Then in \(A_2^T\) rows \(1, \ldots, t\) and columns \(1, \ldots, u\) exhaust all 1's and \(A_2^T\) is the required \(A^*\) of the corollary.

The preceding corollary gives the following canonical form for a matrix \(A_{\tilde{\rho}}\) in \(\mathcal{C}\) with minimal term rank.

\[ A_{\tilde{\rho}} = \begin{pmatrix} W & X \\ Y & 0 \end{pmatrix}, \]

where \(W\) is of size \(e \times f\) and \(\tilde{\rho} = e + f\).

We now proceed to develop an algorithm for determining \(\tilde{\rho}\).

Let \(U = (u_1, \ldots, u_m)\), where the \(u\)'s are integers as usual. Let \(k_1\) be the smallest subscript (if any) such that there exists an \(l < k_1\) satisfying \(u_{k_1} > u_l + 1\). That is \(u_{k_1}\) is the first component with a component as much as two smaller to the left of it. With this fixed \(k_1\) let \(l_1\) be the largest of the subscripts \(l\). That is \(u_{l_1}\) is the component as far to the right as possible but still to the left of \(u_{k_1}\) which satisfies \(u_{l_1} + 1 < u_{k_1}\).

Define

\[ (3.5) \quad \sigma U = (u_1, \ldots, u_{l_1} + 1, u_{l_1+1}, \ldots, u_{k_1} - 1, u_{k_1+1}, \ldots, u_m). \]

If no \(k_1\) exists define \( \sigma U = U \). \(\sigma\) is then a "smoothing" operator. \(\sigma^i\) will denote \(\sigma\) applied \(i\) times. We write

\[ (3.6) \quad \sigma(U) = (\sigma(u_1), \ldots, \sigma(u_m)). \]

For clarity we consider the following example. Let \(U = (5, 3, 4, 5, 1, 7)\). Then \(u_{k_1} = u_4 = 5\), \(u_l = u_2 = 3\), so that \(\sigma U = (5, 4, 4, 4, 1, 7)\) and \(\sigma^2 U = (5, 4, 4, 4, 2, 6)\).

**Lemma 3.5.** \(\sigma U < U\).

**Proof:** Lemma 3.3 and Lemma 3.4.

**Lemma 3.6.** \(\sigma U \nless U\), or equivalently, \(C \nless U\) implies \(\sigma C \nless U\).

**Proof:** Immediate from definition.
Lemma 3.7. If \( U = (u_1, \ldots, u_n) \) and \( u_1 \geq \ldots \geq u_n \), then \( C > U \) implies \( C > U \).

Proof: Immediate from definition.

Lemma 3.8. Let \( H = (h_1, \ldots, h_\mu, \ldots, h_\nu, \ldots, h_m) \). Suppose \( \sigma(h_\mu) = h_\mu + 1, \sigma(h_\nu) = h_\nu - 1 \). Then \( \mu \leq \lambda < \nu \) implies \( h_\mu + 1 > h_\lambda \).

Proof: Suppose to the contrary, \( h_\mu + 1 < h_\lambda \). Then \( h_\lambda \geq h_\mu + 2 \). But then \( \sigma(h_\nu) = h_\nu \) contrary to hypothesis.

Lemma 3.9. With the same hypothesis as in Lemma 3.8, \( \mu < \lambda < \nu \) implies \( h_\mu \leq h_\lambda - 1 \).

Proof: For suppose \( h_\mu \geq h_\lambda \). Then \( \sigma(h_\mu) = h_\mu \) which is contrary to assumption.

Lemma 3.10. Let \( H = (h_1, \ldots, h_m) \). Suppose \( \sigma(h_\mu) = h_\mu + 1, \sigma(h_\nu) = h_\nu - 1 \). Then \( \mu < \lambda < \nu \) implies \( \sigma(h_\mu) = h_\mu + 1 = h_\lambda \) or equivalently

\[
\mu \leq \lambda \leq \rho < \nu \quad \text{implies} \quad \sigma(h_\lambda) = \sigma(h_\rho) \leq \sigma(h_\nu).
\]

Proof: Lemma 3.8 and Lemma 3.9.

Lemma 3.11. Let \( H = (h_1, \ldots, h_m) \). Let \( \sigma^t H = (h_1^t, \ldots, h_m^t) \). Let \( j_1^t, \ldots, j_s^t \) \((j_1^t \leq j_2^t \leq \ldots \leq j_s^t)\) be the subscripts for which

\[
(3.7) \quad \sum_{v=1}^{j_k^t} h_e^t = \sum_{v=1}^{j_k^t} h_e
\]

(Note that we always have \( \sum_{v=1}^{j_s^t} h_e^t \geq \sum_{v=1}^{j_s^t} h_e \).

If \( j_i^t \) and \( j_{i+1}^t \) are two consecutive subscripts for which equality holds, then

\[
(3.8) \quad j_i^t < \lambda < j_{i+1}^t \quad \text{implies} \quad h_\lambda^t \leq h_{\lambda+1}^t.
\]

Proof: The proof is by induction on \( t \). The lemma is certainly true if \( t = 0 \) and is true for \( t = 1 \) by Lemma 3.10. Suppose then it is true for \( t = 1 \). Suppose that after the next application of \( \sigma \) component \( \alpha \) is increased by 1 and component \( \beta \) is decreased by 1. (If \( \sigma \) has no effect we are, of course, through.) Let \( j_i^{t-1} \) be the largest subscript of the \( j_i^{t-1} \) such that \( j_i^{t-1} < \alpha \), if such exists. Let \( j_s^{t-1} \) be the smallest subscript
of the $j_{i-1} \geq \beta$. In this case one always exists since $m$ is among the $j_{i-1}$. Now the subscripts for which (3.7) holds are

$$j_{i-1}, \ldots, j_{\gamma}, j_{\delta}, j_{\delta+1}, \ldots, j_{k-1}. \tag{3.9}$$

Now by the induction hypothesis the only ones to worry about are $j_{\gamma-1}^t$ and $j_{\delta-1}^t$ and again there is nothing more to prove if $j_{\gamma-1}^t$ does not appear.

Let $j_{\gamma-1}^t < \lambda < j_{\delta-1}^t$. Consider first the case

$$\alpha \leq \lambda < \beta. \tag{3.10}$$

Then by Lemma 3.10, $h_{i}^t \leq h_{i+1}^t$. Consider next the case

$$j_{\gamma-1}^t < \lambda < \alpha \leq j_{\gamma+1}^t. \tag{3.11}$$

Then by the induction hypothesis, $h_{i}^t \leq h_{i+1}^t$. Consider finally the case

$$j_{\delta-1}^{t-1} < \beta \leq \lambda < j_{\delta}^{t-1}. \tag{3.12}$$

By the induction hypothesis, $h_{i}^t \leq h_{i+1}^t$.

**Lemma 3.12.** Under the same hypothesis as Lemma 3.11,

$$1 \leq \lambda < j_{1}^t \text{ implies } h_{i}^t \leq h_{i+1}^t. \tag{3.13}$$

**Proof:** The proof is by induction on $t$. The theorem is valid for $t = 0$ and $t = 1$. Suppose the theorem valid for $t - 1$. Suppose that after the next application of a component $\alpha$ is increased by 1 and component $\beta$ is decreased by 1. If $\alpha > j_{i-1}^t$, then $j_{i-1}^t = j_{i}^t$ and the result follows by the induction hypothesis. Suppose $\beta \leq j_{i-1}^t$. Then once again $j_{i-1}^t = j_{i}^t$ and the result follows from the induction hypothesis and Lemma 3.10. Suppose then that

$$\alpha \leq j_{i-1}^t < \beta.$$

Let $j_{\delta}^{t-1}$ be the first subscript among the $j_{k}^{t-1}$ such that $j_{\delta}^{t-1} \geq \beta$. Now $j_{\delta-1}^{t-1} < \beta$ and we have $j_{1}^{t} = j_{\delta}^{t-1}$. Let

$$1 \leq \lambda < j_{1}^{t} = j_{\delta}^{t-1}. \tag{3.14}$$
Now if
\[ 1 \leq \lambda < \alpha - 1, \]
the conclusion follows by the induction hypothesis. If
\[ \alpha \leq \lambda \leq \beta - 1, \]
the conclusion follows by Lemma 3.10. If
\[ j_{s-1}^t < \beta \leq \lambda < j_s^t, \]
then the conclusion follows by Lemma 3.11.

**Lemma 3.13.** Let \( H \triangleleft G \). Suppose \( \sigma^{t+1}H = \sigma^tH \). Then \( \sigma^tH \triangleleft G \).

**Proof:** We may suppose \( G \) has nonincreasing components \( g_1 \geq g_2 \geq \ldots \geq g_n \). Let \( \gamma_1, \ldots, \gamma_s \) be the subscripts \( j_1^t, \ldots, j_s^t \) of Lemma 3.11 for which equality holds in (3.7). Now define
\[
\rho_j = \sum_{i=1}^{j} g_i - \sum_{i=1}^{j} h_i = \sum_{i=1}^{j} g_i - \sum_{i=1}^{j} h_i,
\]
and note that
\[
\rho_j \geq 0 \quad (j = 1, \ldots, s)
\]

Since \( \sigma^{t+1}H = \sigma^tH \) by Lemmas 3.11 and 3.12, the components between two \( \gamma_i \) differ by at most 1. Hence
\[
(h_1^t, \ldots, h_{\gamma_1-1}^t) \triangleleft (g_1, \ldots, g_{\gamma_1-1}, g_{\gamma_1} - \rho_1) = G_1
\]
\[
(h_{\gamma_1+1}^t, \ldots, h_{\gamma_2}^t) \triangleleft (g_{\gamma_1+1} + \rho_1, g_{\gamma_1+2}, \ldots, g_{\gamma_2-1}, g_{\gamma_2} - \rho_2) = G_2
\]
(3.16)
\[
(h_{\gamma_2+1}^t, \ldots, h_{\gamma_3}^t) \triangleleft (g_{\gamma_2+1} + \rho_2, \ldots, g_{\gamma_3-1}, g_{\gamma_3} - \rho_3)
\]
\[
\vdots
\]
\[
(h_{\gamma_{s-1}+1}^t, \ldots, h_m^t) \triangleleft (g_{\gamma_{s-1}+1} + \rho_{s-1}, g_{\gamma_{s-1}+2}, \ldots, g_m) = G_s
\]
Thus
\[
\sigma^tH \triangleleft (G_1, \ldots, G_s) \triangleleft G
\]
Since \( \sigma \) has no effect on \( \sigma^tH \), \( \sigma^tH \) can be made to have nonincreasing components by interchanging adjacent elements which differ by 1. Hence by Lemma 2.1,
\[
\sigma^tH \triangleleft G.
\]
Suppose $H \triangleleft G$. Let $\mathcal{L} = \mathcal{L}(H, G)$ be the class of vectors $V$ with integral components satisfying

\begin{equation}
H \triangleleft V < G.
\end{equation}

Note that $G$ is in $\mathcal{L}$, and by Lemma 3.13, there is a $t$ with $\sigma^t H$ in $\mathcal{L}$. Let $V = (v_1, \ldots, v_m)$ and $H = (h_1, \ldots, h_m)$. Suppose that

\begin{align*}
v_1 &= h_1, & v_{t+1} &= h_{t+1}, & v_{t+2} &= h_{t+2}, & \ldots, & v_m &= h_m.
\end{align*}

Then define

\begin{equation}
n(V) = m - t.
\end{equation}

**Lemma 3.14.** Suppose $V$ is in $\mathcal{L}$. Then $\sigma V$ is in $\mathcal{L}$ and, moreover, $n(V) \geq n(\sigma V)$.

**Proof:** $\sigma V$ is in $\mathcal{L}$ by Lemma 3.5 and Lemma 3.6. Suppose $n(V) = \alpha$. Let $\beta = m - \alpha$. Then $h_\beta = v_\beta$, $h_{\beta+1} = v_{\beta+1}$, $\ldots$, $h_m = v_m$. Since $V \triangleright H$,

\begin{equation}
\sum_{i=0}^\alpha h_{i+\beta} \geq \sum_{i=0}^\alpha v_{i+\beta}
\end{equation}

so that

\begin{equation}
h_\beta > v_\beta.
\end{equation}

Now suppose that $n(\sigma V) > n(V)$. Then $\sigma(v_\gamma) = h_\gamma > v_\gamma$. Hence there is a $\gamma > \beta$ such that $\sigma(v_\gamma) < v_\gamma = h_\gamma$. This implies $n(\sigma V) \leq m - \gamma < m - \beta = \alpha$, which is a contradiction.

**Lemma 3.15.** Suppose $H \triangleleft G$. Suppose $\sigma^{t-1} H \not< G$ but $\sigma^t H < G$. Then $\sigma^t H \in \mathcal{L}(H, G)$ and $n(\sigma^t H)$ satisfies

\begin{equation}V \in \mathcal{L}(H, G) \text{ implies } n(V) \leq n(\sigma^t H).
\end{equation}

**Proof:** Suppose that $U \in \mathcal{L}$ with $n(U) = \alpha$ maximal. We apply $\sigma$ as often as possible to the first $m - \alpha$ components of $H$. These, by the definition of $\sigma$, are truly the first applications of $\sigma$ to all of $H$. Suppose this takes $\lambda$ applications of $\sigma$. We assert

\begin{equation}\sigma^\lambda H < G.
\end{equation}

For let $\sigma^\lambda H = (h_1, \ldots, h_m)$ and let $U = (u_1, \ldots, u_m)$. Let $\beta = m - \alpha$. Now

\begin{equation}(u_1, \ldots, u_\beta) \triangleright (h_1, \ldots, h_\beta),
\end{equation}
and $\sigma^{i+1}(h_1, \ldots, h_\beta) = \sigma^i(h_1, \ldots, h_\beta)$. so by Lemma 3.13, $\sigma^i(h_1, \ldots, h_\beta) < (u_1, \ldots, u_\beta)$. Hence by Lemma 3.2,

$$\sigma^j H < U < G.$$  

Now of course $\lambda \geq t$ and since by Lemma 3.14, $i < j$ implies $n(\sigma^j H) \geq n(\sigma^i H)$, we have

$$n(\sigma^i H) \geq n(\sigma^j H) \geq \alpha.$$  

**Lemma 3.16.** $h_i \leq h_{i+1}$ implies $\sigma(h_i) \leq \sigma(h_{i+1})$.

**Proof:** There are six easy cases to dispose of.

**Case 1.** $\sigma(h_{i+1}) = h_{i+1} + 1$.

Then $\sigma(h_i) \leq h_i + 1 \leq h_{i+1} + 1 = \sigma(h_{i+1})$.

**Case 2.** $\sigma(h_{i+1}) = h_{i+1}$ and $\sigma(h_i) = h_i + 1$.

In this case $h_{i+1} > h_i$ so $\sigma(h_i) = h_i + 1 \leq h_{i+1} = \sigma(h_{i+1})$.

**Case 3.** $\sigma(h_{i+1}) = h_{i+1}$ and $\sigma(h_i) \leq h_i$.

Then $\sigma(h_i) \leq h_i \leq h_{i+1} = \sigma(h_{i+1})$.

**Case 4.** $\sigma(h_{i+1}) = h_{i+1} - 1$ and $\sigma(h_i) = h_i$.

In this case $h_{i+1} > h_i$ so $\sigma(h_i) = h_i \leq h_{i+1} - 1 = \sigma(h_{i+1})$.

**Case 5.** $\sigma(h_{i+1}) = h_{i+1} - 1$ and $\sigma(h_i) = h_i + 1$.

In this case $h_{i+1} \geq h_i + 2$ so $\sigma(h_i) = h_i + 1 \leq h_{i+1} - 1 = \sigma(h_{i+1})$.

**Case 6.** $\sigma(h_i) = h_i - 1$.

Then $\sigma(h_i) = h_i - 1 \leq h_{i+1} - 1 \leq \sigma(h_{i+1})$.

**Lemma 3.17.** $h_i > h_{i+1}$ implies $\sigma(h_i) - \sigma(h_{i+1}) \leq h_i - h_{i+1}$.

**Proof:** $h_i > h_{i+1}$ implies $\sigma(h_{i+1}) \geq h_{i+1}$ and $\sigma(h_i) \leq h_i$.

Hence $\sigma(h_i) - \sigma(h_{i+1}) \leq h_i - h_{i+1}$.

**Lemma 3.18.** Let $S$ be a vector with nonincreasing integral components. If $S - H$ is nonincreasing, then $S - \sigma^j H$ is non-increasing.

**Proof:** $S - H$ is nonincreasing so that

$$(3.23) \quad s_i - h_i \geq s_{i+1} - h_{i+1}.$$
Then if \( h_i \leq h_{i+1} \), by Lemma 3.16, \( \sigma(h_i) \leq \sigma(h_{i+1}) \) so that

(3.24) \[ s_i - \sigma(h_i) \geq s_i - \sigma(h_{i+1}) \geq s_{i+1} - \sigma(h_{i+1}). \]

If \( h_i > h_{i+1} \), by Lemma 3.17, \( \sigma(h_i) - \sigma(h_{i+1}) \leq h_i - h_{i+1} \), so that

(3.25) \[ s_i - s_{i+1} \geq h_i - h_{i+1} \geq \sigma(h_i) - \sigma(h_{i+1}) \]

and

(3.26) \[ s_i - \sigma(h_i) \geq s_{i+1} - \sigma(h_{i+1}). \]

Hence \( S - \sigma H \) is nonincreasing, and repeating the proof, gives the desired result.

Let \( \mathcal{A} \) be the class of 0,1 matrices with row sum vector \( R = (r_1, \ldots, r_n) \) \((r_1 \geq \ldots \geq r_n)\) and column sum vector \( S = (s_1, \ldots, s_m) \) \((s_1 \geq \ldots \geq s_m)\). Let

\[
A = \left( \begin{array}{c} A_i^u \\ A_{n-i}^l \end{array} \right),
\]

where \( A_i^u \) has row sum vector \((r_1, \ldots, r_i)\) and column sum vector \( S_i^u \), and where \( A_{n-i}^l \) has row sum vector \((r_{i+1}, \ldots, r_n)\) and column sum vector \( S_{n-i}^l \). Here

\[
S = S_i^u + S_{n-i}^l.
\]

Let \( G \) be the family of vectors \( S_i^u \), where \( S_i^u \) is the column sum vector of some \( A_i^u \) and where \( S - S_i^u = S_{n-i}^l \) is nonincreasing. Let \( \psi_i(S_i^u) \) equal the number of final components of \( S_i^u \) equal to the corresponding components of \( S \). Define

(3.27) \[ \psi_i = \max_{S_i^u \in G} \psi_i(S_i^u). \]

Let

(3.28) \[ \psi = \min_{1 \leq i \leq n-1} (i + m - \psi_i). \]

Then by Theorem 3.1 and its corollary,

(3.29) \[ \bar{\rho} = \min \{ m, n, \psi \}. \]
We proceed to evaluate $\psi_i$, and thereby $\phi$, and $\tilde{\rho}$. Let $S^t_i$ be in $G$. Define

\begin{align*}
T &= R' - S, \\
H_i &= S - (R^t_{n-i})' = (R^t_i)' - T.
\end{align*}

Then since

$$S_{n-1} < (R^t_{n-i})',$$

we must have

\begin{equation}
S^t_i = S - S_{n-i} \supset S - (R^t_{n-i})' = H_i.
\end{equation}

But (3.32) implies

\begin{equation}
H_i < S^t_i < (R^t_i)',
\end{equation}

whence

\begin{equation}
G \subseteq \mathcal{L}(H_i, (R^t_i')).
\end{equation}

Now let $V_i$ be in $\mathcal{L}(H_i, (R^t_i'))$. Then $H_i = (h_1, \ldots, h_m) \lhd V_i = (v_1, \ldots, v_m)$, so that

\begin{equation}
\sum_{j=0}^{\gamma} v_m-j \leq \sum_{j=0}^{\gamma} h_m-j \quad (\gamma = 0, \ldots, m-1).
\end{equation}

Moreover, by (3.31), every component of $H_i$ is less than or equal to the corresponding component of $S$, so that

$$h_i \leq s_i.$$

Now if $v_m = s_m$, then $h_m = s_m$. If also $v_{m-1} = s_{m-1}$, then $h_{m-1} = s_{m-1}$, and so on. Thus if $\psi_i(V_i)$ equals the number of final components of $V_i$ equal to the corresponding components of $S$, then

\begin{equation}
n(V_i) \geq \psi_i(V_i).
\end{equation}

Now let $V_i$ have $n(V_i)$ maximal for the vectors in $\mathcal{L}(H_i, (R^t_i'))$. Then $V_i$ also has $\psi_i(V_i)$ maximal for the vectors in $\mathcal{L}(H_i, (R^t_i'))$. Let $t$ be such that $\tau^{t-1}H_i \lhd (R^t_i')$ but $\tau^tH_i < (R^t_i')$. Then
by Lemma 3.15, $\sigma^tH_i$ is in $\mathcal{L}(H_i, (R_i^m))$. By Lemma 3.15 and (3.27), (3.34),

\begin{equation}
\psi_i \leq \psi_i(\sigma^tH_i).
\end{equation}

We next assert that $\sigma^tH_i$ is in $G$. This will give us an effective procedure to calculate the $\psi_i$ defined by (3.27), and thereby, $\rho$. To show that $\sigma^tH_i$ is in $G$, we must show that $\sigma^tH_i \prec (R_i^m)'$, $S - \sigma^tH_i$ is nonincreasing, and $S - \sigma^tH_i \prec (R_n^1)'$.

Now

$\sigma^tH_i \notin \mathcal{L}(H_i, (R_i^m))$,

so that $\sigma^tH_i \prec (R_i^m)'$. Moreover,

$S - \sigma^tH_i \prec S - H_i = (R_n^1)'$.

Thus we need only show that $S - \sigma^tH_i$ is nonincreasing. But $S - H_i$ is nonincreasing, so the last conclusion follows by Lemma 3.18.

**Example 1.**

Let $\mathcal{A}$ be the class of 0,1 matrices of order 11 with row sum vector $\mathbf{R} = (9, 9, 9, 5, 1, 1, 1, 1, 1, 1, 1)$ and column sum vector $\mathbf{S} = (8, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)$. Then

$T = (3, -1, 0, 0, 1, -1 -1, -1, 2, -1, -1)$.

We have the following table:

<table>
<thead>
<tr>
<th>$H_i = (R_i^m') - T$</th>
<th>$(R_i^m)'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 2, 1, 1, 0, 2, 2, 2, -1, 1, 1)</td>
<td>(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0)</td>
</tr>
<tr>
<td>(-1, 3, 2, 2, 1, 3, 3, 3, 0, 1, 1)</td>
<td>(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0)</td>
</tr>
<tr>
<td>(0, 4, 3, 3, 2, 4, 4, 4, 1, 1, 1)</td>
<td>(3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 0, 0)</td>
</tr>
<tr>
<td>(1, 5, 4, 4, 3, 4, 4, 4, 1, 1, 1)</td>
<td>(4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 0, 0)</td>
</tr>
<tr>
<td>(2, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)</td>
<td>(5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 0, 0)</td>
</tr>
<tr>
<td>(3, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)</td>
<td>(6, 5, 5, 5, 3, 3, 3, 3, 3, 3, 0, 0)</td>
</tr>
<tr>
<td>(4, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)</td>
<td>(7, 5, 5, 5, 3, 3, 3, 3, 3, 3, 0, 0)</td>
</tr>
<tr>
<td>(5, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)</td>
<td>(8, 5, 5, 5, 3, 3, 3, 3, 3, 3, 0, 0)</td>
</tr>
<tr>
<td>(6, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)</td>
<td>(9, 5, 5, 5, 3, 3, 3, 3, 3, 3, 0, 0)</td>
</tr>
<tr>
<td>(7, 6, 5, 5, 4, 4, 4, 4, 1, 1, 1)</td>
<td>(10, 5, 5, 5, 3, 3, 3, 3, 3, 3, 0, 0)</td>
</tr>
</tbody>
</table>
Thus \( \tilde{\rho} = 7 \). We note for computational purposes once an entry in column \( \sigma^t(H_i) \) is the same as the corresponding entry in column \( H_i = (R_i^m)' - T \), then this is true for every subsequent entry of column \( \sigma^tH_i \).

**Example 2.** Let \( \mathcal{A}_{rs} \) be the class of 0,1 matrices of size \( n \times m \), with \( r \) 1's in every row and \( s \) 1's in every column. Suppose that \( r \geq s \). Then since \( rn = ms \), \( m \geq n \). It is well known that for \( \mathcal{A}_{rs} \), \( \tilde{\rho} = n \). We can use the algorithm to obtain this result. We may assume \( 1 < r < n \) and \( 1 < s < m \).

Let \( R = (r, \ldots, r) \) (\( n \) components \( r \)) and let \( S = (s, \ldots, s) \) (\( m \) components \( s \)). Then \( R' = (n, \ldots, n, 0, \ldots, 0) \) (\( r \) components \( n \)) and \( T = R' - S = (n - s, \ldots, n - s, -s, \ldots, -s) \) (\( r \) components \( n - s \) and \( m - r \) components \( -s \)). Now \( R_i^m = (r, \ldots, r) \) (\( i \) components), so that \( (R_i^m)' = (i, \ldots, i, 0, \ldots, 0) \) (\( r \) components \( i \)). Then \( H_i = (R_i^m)' - T = (i - n + s, \ldots, i - n + s, s, \ldots, s) \) (\( r \) components \( i - n + s \) and \( m - r \) components \( s \)).

Now we must apply \( \sigma \) to \( H_i \) until it is majorized by \( (R_i^m)' \).

**Case 1.** Suppose \( s > i > 0 \). Then \( \psi_i(\sigma^tH_i) = 0 \), and \( i + m - \psi_i(\sigma^tH_i) = i + m \geq n \).

**Case 2.** Suppose \( s \leq i < n \) and \( -n + s + i \geq 0 \). Then \( H_i < (R_i^m)' \), so that \( H_i = \sigma^tH_i \) and \( \psi_i(\sigma^tH_i) = m - r \). Then \( i + m - \psi_i(\sigma^tH_i) = r + i \geq s + i \geq n \).
CASE 3. Suppose $s \leq i < n$ and $-n + s + i \leq 0$. In this case we need not have $H_i \leq \langle R_i \rangle'$. Now
\[
(i - n + s)r + s(n - r - i) = ir - nr + sr + sn - sr - si
= s(n - i) - r(n - i)
= (s - r)(n - i) \leq 0.
\]
Thus we must smooth at least $n - r - i$ of the $s'$s in $H_i$ in order to obtain $\sigma^t H_i \leq \langle R_i \rangle'$. Hence
\[
\psi_i(\sigma^t H_i) \leq m - r - (n - r - n - r - i)
= m - n + i,
\]
and
\[
i + m - \psi_i(\sigma^t H_i) \geq i + m - m + n - i = n.
\]
Hence we conclude $\tilde{\rho} = n$.

4. Constructions.

(I) Construction of a matrix in $\mathcal{A}$.

Let the class $\mathcal{A}$ have row sum vector $R = (r_1, \ldots, r_n)$ and column sum vector $S = (s_1, \ldots, s_m)$, with $s_1 \geq \ldots \geq s_m$. We may place 1's in row 1 and in the 1st $r_1$ columns. This follows upon noting that since column sums are nonincreasing, 1's may be shifted to the left by interchanges until they occupy the 1st $r_1$ position. Now applying the same argument to the class $\mathcal{A}_1$ with row sum vector $R_1 = (r_2, \ldots, r_n)$ and column sum vector $S_1 = (s_1 - 1, \ldots, s_{r_1} - 1, s_{r_1} + 1, \ldots, s_m)$, we can put 1's in the $r_2$ columns where $S_1$ has the largest components. Continuing in this way we construct an $A$ in $\mathcal{A}$. We remark that the proof of the existence theorem in Section 2 uses this construction with respect to columns. We have been unable to determine the term rank of the matrix $A$ constructed by this device in the general case. This would be a matter of some interest.

EXAMPLE 3. Let $A$ have row sum vector $R = (3, 1, 2, 2)$ and column sum vector $S = (3, 3, 2)$. Then following our construction
we have:

\[ S = (3, 3, 2), \quad R = (3, 1, 2, 2), \]
\[ S_1 = (2, 2, 1), \quad R_1 = (1, 2, 2), \]
\[ S_2 = (1, 2, 1), \quad R_2 = (2, 2), \]
\[ S_3 = (0, 1, 1), \quad R_3 = (2). \]

Thus we construct

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\]

(II) Construction of a matrix \( A_{\tilde{\varphi}} \) in \( \mathcal{A} \) with minimal term rank.

The algorithm of Section 3 enables one to get \( A_{\tilde{\varphi}} \) in the form

\[
A_{\tilde{\varphi}} = \begin{pmatrix}
A_{\tilde{\varphi}}^w \\
A_{\tilde{\varphi}}^t
\end{pmatrix},
\]

where the row and column sum vectors of the submatrices are determined by the algorithm. Furthermore, these row and column sum vectors determine \( \tilde{\varphi} \). Thus we need only construct a matrix in each class determined by each of the submatrices. This can be done by the preceding construction.

**Example 4.** Let \( \mathcal{A} \) be the class of Example 1. Then \( A_5^w \) determines the class with row sum vector \((9, 9, 9, 5, 5)\) and column sum vector \((3, 5, 5, 5, 4, 4, 4, 4, 1, 1)\). \( A_5^t \) determines the class with row sum vector \((1, 1, 1, 1, 1, 1)\) and column sum vector \((5, 1, 0, 0, 0, 0, 0, 0, 0, 0)\). Hence we construct

\[
A_5^w = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
and, thereby,

\[ A^* \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

(III) Construction of a matrix \( A^* \) in \( \mathcal{A} \) with maximal term rank.

Lemma 4.1. Let \( A \) be the class of square 0,1 matrices with row sum vector \( R = (r_1, \ldots, r_n) \) \( (r_1 \geq \ldots \geq r_n) \) and column sum vector \( S = (s_1, \ldots, s_n) \) \( (s_1 \geq \ldots \geq s_n) \). Suppose that \( p = n \). Then there exists an \( A^* \) in \( \mathcal{A} \) with \( n \) 1's on the diagonal from the top right to the lower left, which we will call the off diagonal.

Proof: Consider any \( A \) in \( \mathcal{A} \) with term rank \( n \). Clearly then there is a permutation of the rows of \( A \) which will give 1's on the off diagonal. Suppose that after this permutation row \( j \) has fewer 1's than row \( k \), with \( j < k \). We consider the \( 2 \times 2 \) submatrix of the permuted \( A \) composed of the entries from positions \( (j, n - j + 1) \), \( (j, n - k + 1) \), \( (k, n - k + 1) \), and \( (k, n - j + 1) \). The following are the possibilities for this \( 2 \times 2 \) submatrix:

\[ B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \]

If we have \( B_1 \), interchange \( B_1 \) to \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \) and permute rows \( j \) and \( k \). If \( B_4 \) occurs, permute rows \( j \) and \( k \). If \( B_2 \) occurs, then since we have assumed row \( j \) has fewer ones than row \( k \), there must be an interchange which changes \( B_2 \) to \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \). Then permute rows \( j \) and \( k \). If \( B_3 \) occurs, since \( s_1 \geq \ldots \geq s_n \), there
must be an interchange which changes $B$ to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then permute rows $j$ and $k$. The preceding manipulations still leave 1's on the off diagonal but now row $j$ has more 1's than row $k$. Continuing in this manner we obtain the desired $A$.

We remark this lemma does not hold for the main diagonal. Maximal matrices may be used to construct simple counterexamples.

Let $R = (r_1, \ldots, r_n), S = (s_1, \ldots, s_m)$, where $r_1 \geq \ldots \geq r_n > 0$ and $s_1 \geq \ldots \geq s_m > 0$. Let

$$R^* = R - Q_n,$$

$$S^* = S - Q_m,$$

where $Q_n, Q_m$ are vectors of $n$ and $m$ 1's respectively. Define

$$T^* = S - (R^*)^\gamma = (t_1^*, \ldots, t_m^*),$$

and

$$U^* = (u_1^*, \ldots, u_m^*), \quad u_k^* = \sum_{i=1}^{k} t_i^*.$$

Now let

$$M^* = \max_i (u_i^*) (i = 1, \ldots, m)$$

and

$$N^* = \max(0, M^*).$$

Then the formula for $\bar{\rho}$ of the class $\mathcal{C}(R, S)$ established in [4] is given by

$$\bar{\rho} = n - N^*.$$

Note that for $m = n$, $M^* = N^*$.

**Lemma 4.2.** Let $\mathcal{C}(R, S)$ be given with $R = (r_1, r_2, \ldots, r_n)$ ($r_1 \geq r_2 \geq \ldots \geq r_n > 0$) and $S = (s_1, \ldots, s_m)$ ($s_1 \geq s_2 \geq \ldots \geq s_m > 0$). Let $\delta_1$ and $\delta_2$ be vectors with $m$ components, $t$ of which are 1 and $m - t$ which are 0, where $t = r_j$ for some $j$ and where $S_1 = S - \delta_1$ and $S_2 = S - \delta_2$ have nonincreasing components.
Let \( R_1 = (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n) \). Suppose that \( z_1 < z_2 \). Then if \( \bar{A}(R_1, S_1) \) exists so does \( \bar{A}(R_1, S_2) \) and the maximal term rank for \( \bar{A}(R_1, S_1) \) is less than or equal to the maximal term rank for \( \bar{A}(R_1, S_2) \).

Proof: Now \( z_1 < z_2 \) implies \( S_1 \succ S_2 \). Since \( S_1 \) and \( S_2 \) have nonincreasing components, this implies \( S_1 \succ S_2 \). Thus \( R_1' \succ S_2 \) and \( \bar{A}(R_1, S_2) \) exists. Suppose \( S_1 = (s_{11}, \ldots, s_{1k}, 0, \ldots, 0) \) and \( S_2 = (s_{21}, \ldots, s_{2l}, 0, \ldots, 0) \), where \( s_{1k} > 0 \) and \( s_{2l} > 0 \). Since \( S_1 \succ S_2 \) this implies \( k \leq l \). Now define

\[
R_1^* = R_1 - Q_{n-1},
\]

\[
S_1^* = S_1 - Q_k, \quad S_2^* = S_2 - Q_l.
\]

Here \( Q_{n-1} \) has \( n-1 \) 1's, and \( Q_k \) and \( Q_l \) have \( k \) and \( l \) 1's, respectively, in initial positions and 0's elsewhere. Then

\[
\sum_{i=1}^{l} s_{1i}^* \geq \sum_{i=1}^{l} (s_{1i} - 1) \geq \sum_{i=1}^{l} (s_{2i} - 1) = \sum_{i=1}^{l} s_{2i}^*(\gamma = 1 \ldots, l).
\]

Let us now consider the classes \( \bar{A}(R_1, S_1) \) and \( \bar{A}(R_1, S_2) \), possible zero columns deleted. We may apply the \( \bar{r} \) formula described in (4.1) - (4.7) to each of these classes. Let \( U_1^* \) and \( U_2^* \) correspond to the \( U^* \) of (4.4) for the classes \( \bar{A}(R_1, S_1) \) and \( \bar{A}(R_1, S_2) \), respectively. Then (4.10) implies that the maximal component of \( U_1^* \) is \( \geq \) the maximal component of \( U_2^* \). Hence by the formula for maximal term rank, it follows that the maximal term rank of \( \bar{A}(R_1, S_1) \) is \( \leq \) the maximal term rank of \( \bar{A}(R_1, S_2) \).

Note that the lemma may also be applied with the roles of \( R \) and \( S \) interchanged.

**Lemma 4.3.** Let \( R = (r_1, \ldots, r_n) \) \((r_1 \geq \ldots \geq r_n)\) and \( S = (s_1, \ldots, s_m) \) \((s_1 \geq \ldots \geq s_m)\). Suppose the maximal term rank \( \rho \) for \( \bar{A}(R, S) \) satisfies \( \rho < m \). Then there is an \( A \) in \( \bar{A}(R, S) \) where the submatrix of the first \( m - 1 \) columns of \( A \) has maximal term rank \( \rho \).

**Proof:** Let \( A \) in \( \bar{A}(R, S) \) have maximal term rank \( \rho \).
Select 1's of $A$ accounting for the term rank $\rho$. Suppose one of these 1's occurs in column $m$ and row $j$. Suppose that column $k$ has none of these 1's. If there is a 1 in position $(j, k)$, we may use this 1 instead of the 1 in position $(j, m)$. If there is a 0, since the $s_i$ are nonincreasing, an interchange will place a 1 in position $(j, k)$ which can be used as one of the 1's accounting for $\rho$.

We now show how to construct the matrix of Lemma 4.1. We are given $\mathcal{R} = (r_1, \ldots, r_n)$ and $\mathcal{S} = (s_1, \ldots, s_n)$ nonincreasing. These vectors determine a class $\mathcal{A}$ with maximal term rank $\bar{\rho} = n$. We are to construct the matrix of order $n$ with 1's in the off diagonal positions. By Lemma 4.1 there exists a matrix in $\mathcal{A}$ of term rank $n$ with a 1 in the $(1, n)$ position. The matrix obtained by deleting the first row is of term rank $n - 1$ and determines a class $\mathcal{A}(R_1, S_1)$. By Theorem 3.1, we may obtain a matrix $A$ in $\mathcal{A}$ and such that if the first row $\delta$ of this matrix is deleted, then the resulting $(n - 1) \times n$ matrix has nonincreasing rows and columns. The $(n - 1) \times n$ submatrix also determines the class $\mathcal{A}(R_1, S_1)$ and so may be selected to be of term rank $n - 1$. Now consider a vector $\delta^*$ of $r_1$ 1's and $n - r$ 0's with a 1 in position $n$ and defined so that 1's are placed to the left as far as possible provided only

$$\mathcal{S} - \delta^*$$

is nonincreasing. We assert

$$\delta^* \triangleright \delta.$$

By Lemma 3.2, this means that there exists a matrix in $\mathcal{A}$ with first row $\delta^*$ and such that the $(n = 1) \times n$ submatrix with $\delta^*$ deleted has nonincreasing rows and columns and is of term rank $n - 1$.

Consider now the class

$$\mathcal{A}(R_1, \mathcal{S} - \delta^*).$$

This contains matrices of size $(n - 1) \times n$ and the maximal term rank is $\rho = n - 1$. By Lemma 4.3, we know that there exists a matrix in $\mathcal{A}(R_1, \mathcal{S} - \delta^*)$ such that if its last column is deleted, then the term rank of the resulting submatrix
is equal to $n - 1$. Let the last column of this matrix be $\varepsilon$. Then by Theorem 3.1 there exists a matrix $A_1$ in $\mathcal{A}(R, S - \varepsilon^*)$ such that if its last column is deleted the resulting submatrix of order $n - 1$ has nonincreasing rows and columns. It may be selected to be of term rank $n - 1$. Now define $\varepsilon^*$ to be the vector $\varepsilon$ with 1's placed to the top as far as possible provided only the transpose of $R_1$ minus $\varepsilon^*$ is a column vector with nonincreasing components. Then

$$\varepsilon^* \triangleright \varepsilon,$$

and by Lemma 4.2 there exists a matrix in $\mathcal{A}(R_1, S - \varepsilon^*)$ with last column $\varepsilon^*$. The submatrix obtained by deleting $\varepsilon^*$ is of order $n - 1$, has nonincreasing rows and columns, and is of term rank $n - 1$.

We may then proceed inductively and construct the desired matrix of Lemma 4.1.

**Example 5.** We carry out the construction for the case $R = (3, 3, 3, 3, 2, 1, 1)$ and $S = (4, 4, 4, 1, 1, 1, 1)$. Following the lemmas we get

$$
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

We now proceed to construct a matrix $A\bar{\varrho}$ of maximal term rank for an arbitrary $\mathcal{A}(R, S)$. We assume $R$ and $S$ are nonincreasing. Suppose $\bar{\varrho} < m$, and let $\varepsilon^*$ a column vector of $s_m$ 1's and $n - s_m$ 0's. Let the 1's of $\varepsilon^*$ be placed to the top as far as possible provided only the transpose of $R$ minus $\varepsilon^*$ is a column vector with nonincreasing components. Then as before, we may show that there exists a matrix in $\mathcal{A}(R, S)$ with last column $\varepsilon^*$. Then $n \times (m - 1)$ submatrix obtained by deleting $\varepsilon^*$ has nonincreasing rows and columns and is of term rank $\bar{\varrho}$. We continue filling in the last $m - \bar{\varrho}$ columns by this procedure. Then we work on the resulting $n \times \bar{\varrho}$ matrix.
If \( p \leq n \), we fill in the last \( n - p \) rows. This may be done so that we are left with a \( p \times p \) submatrix with nonincreasing rows and columns of term rank \( p \). This matrix we fill in by the procedure described previously.

The following theorem is a byproduct of our discussion.

**Theorem 4.1.** Let \( R \) and \( S \) be nonincreasing and let \( \mathcal{A}(R, S) \) have maximal term rank \( p \). Then there exists an \( A \) in \( \mathcal{A}(R, S) \) where the leading \( p \times p \) minor has 1's on its off diagonal.

### 5. Invariant 1's.

In [4] Ryser has proved that if \( A \) contains an invariant 1, then by permutations of rows and columns, \( A \) may be reduced to the form

\[
(S \quad X) \\
(Y \quad 0)
\]

where \( S \) is the matrix of 1's and contains the invariant 1 of \( A \).

It is easy to see that if the row sum vector and column sum vector have nonincreasing components, then \( A \) must be of the form (5.1) without permutations of rows or columns. It also follows from (5.1) that the class \( \mathcal{A} = \mathcal{A}(R, S) \) contains only invariant 1's if and only if \( \mathcal{A} \) is maximal. Thus only the maximal class contains a single entry.

We begin by establishing a result containing (5.1).

**Theorem 5.1.** Let \( A \) be in \( \mathcal{A}(R, S) \), where \( R \) and \( S \) have nonincreasing components. Suppose \( a_{uv} = 1 \) is invariant. Then \( A \) has the form

\[
A = (S \quad X) \\
(Y \quad 0)
\]

Here \( S \) has all 1's and is of size \( k \times j \), where \( j \geq v \) is the number of invariant 1's of row \( u \) and \( k \geq u \) is the number of rows with at least \( j \) invariant 1's.

**Proof:** Now \( a_{uv} = 1 \) an invariant 1 implies \( a_{rs} = 1 \) is invariant for \( 1 \leq r \leq u \), \( 1 \leq s \leq v \). For otherwise an interchange would contradict the invariance of \( a_{uv} = 1 \). It then follows
that we must have

\[ A = \begin{pmatrix} S & X \\ Y & W \end{pmatrix}, \]

where \( S \) is a matrix of 1's of size \( k \times j \). All 1's of \( S \) are invariant. We may assume the entry in row \( k \) and column \( j + 1 \) is 0. Suppose a 1 occurs in \( W \) in row \( t \) of \( A \). Then we may apply an interchange if necessary and assume that a 1 occurs in row \( t \) and column \( j + 1 \) of \( A \). But then all entries in row \( t \) and columns 1, ..., \( j \) of \( A \) are also 1's. Indeed, these are invariant 1's, and this is not possible. Hence \( W = 0 \).

**THEOREM 5.2.** Let \( R = (r_1, ..., r_n) \) and \( S = (s_1, ..., s_m) \) have nonincreasing components. Let \( r_1 = \lambda \). Then \( \mathcal{A}(R, S) \) has no invariant 1's if and only if

\[
(s_1, s_2 - 1, ..., s_{\lambda + 1} - 1, s_{\lambda + 2}, ..., s_m)' > (r_2, ..., r_n).
\]

**Proof:** From (5.1) it is clear that \( \mathcal{A} \) has no invariant 1's if and only if we can put a 0 in the \((1, 1)\) position of some \( A \). Then by applying interchanges \( A \) has no invariant 1's if and only if \((0, 1, ..., 1, 0, ..., 0)\) is a possibility for the first row of some \( A \) in \( \mathcal{A} \). The result now follows from the existence theorem.

**THEOREM 5.3.** Let \( R \) and \( S \) have nonincreasing components. Form \( T = (t_1, ..., t_m) = S - R' \) and

\[
U = (u_1, ..., u_m), \quad u_k = \sum_{i=1}^{k} t_i
\]

Then row \( i \) of a matrix in \( \mathcal{A}(R, S) \) has exactly \( j \) invariant 1's if and only if \( j \) is the largest subscript such that

\[ u_j = 0 \]

and

\[ r_i \geq j \]

If there is no \( i \) and \( j \) satisfying (5.2) and (5.3), then row \( i \) has no invariant 1's.

**Proof:** Suppose that row \( i \) has exactly \( j \) invariant 1's.
Then by Theorem 5.1 a matrix $A$ in $\mathcal{A}$ is of the form

$$A = \begin{pmatrix} S & X \\ Y_1 & A_1 \\ Y_2 & 0 \end{pmatrix}.$$ 

Here $S$ is a matrix of 1's of size $i$ by $j$ and $Y_1$ contains only invariant 1's. It follows that $u_j = 0$. Also $r_i \geq j$. The integer $j$ is the maximal integer with these properties.

Suppose that $u_j = 0$ and $r_i \geq j$. Then the first $j$ 1's in row $i$ are invariant. Otherwise we would deny $u_j = 0$.

In conclusion, we mention that invariant 1's are closely associated with certain properties of the integers $\tilde{\rho}$ and $\rho$. Ryser [4] has shown that if $\mathcal{A}$ is without an invariant 1 and if $\rho < m, n$, then $\tilde{\rho} < \rho$. A topic deserving further study is the determination of necessary and sufficient conditions on the class $\mathcal{A}$ in order that $\tilde{\rho} = \rho$. Such conditions could conceivably be developed by a study of the $\tilde{\rho}$ formula and the $\rho$ algorithm. Our Example 2 is an instance of a class $\mathcal{A}$ with this property.

BIBLIOGRAPHY


